2010년 2학기 선박설계자동화특강 강의자료

서울대학교 조선해양공학과

선박설계자동화연구실

이규열

CONTENTS

Naval Architecture & Ocean Engineering

Topics in ship design automation

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1. Particle Dynamics

1.1 Newton Equation

Force Vector and Newton’s Laws

The First Law:

The velocity of a particle can only be changed by the application of a force

The Second Law:

The resultant force (that is, the sum of all forces) acting on a particle is proportional to the acceleration of the particle. The factor of proportionality is the mass.

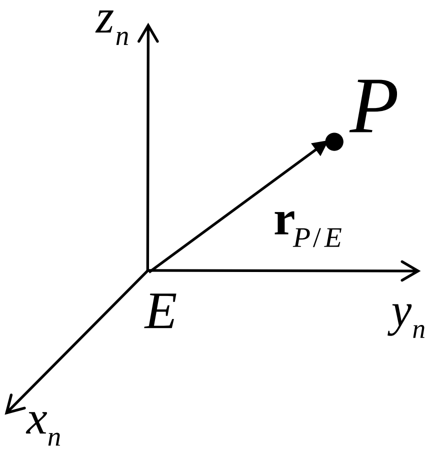
$$\boxed{m\ddot{\mathbf{r}} = \sum \mathbf{F}}$$

The Third Law:

All forces acting on a body result from an interaction with another body, such that there is another force, called a *reaction*, applied to the other body. The action–reaction pair consists of forces having the same magnitude, and acting along the same line of action, but having opposite direction.

Ginsberg, J.H., Engineering Dynamics, Georgia Institute of Technology, 2008, pp. 14

Newton's 2nd Law and Euler Equation



$x_n y_n z_n$ axis(n-frame):
Inertial reference frame

The position vector extending from origin *E* to point *P* is labeled

$$\mathbf{r}_{P/E}$$

→A start point of the position vector.

→An end point of the position vector.

which should be read as **the position vector to *P* from *E***, or equivalently, **the position of *P* with respect to *E***.

→To define this position vector, a reference frame is required.

Which frame is used to define the position vector?

Newton's 2nd Law and Euler Equation

Which frame is used to define the position vector?

We can choose any reference frame for our convenience.

When do Newton's 2nd Law and Euler equation work?

Newton's 2nd law is valid in any inertial reference frame.(1)

An inertial reference frame is one that **translates at a constant velocity**. The translation condition, by definition, means that the **coordinate axes point in fixed directions**, so that we may interpret **velocity and acceleration in the same way** as we do for a fixed reference frame.(2)

Reference 1) Ginsberg, J. H., Advanced Engineering Dynamics, 2nd edition, Cambridge University Press, 1995, p.4.

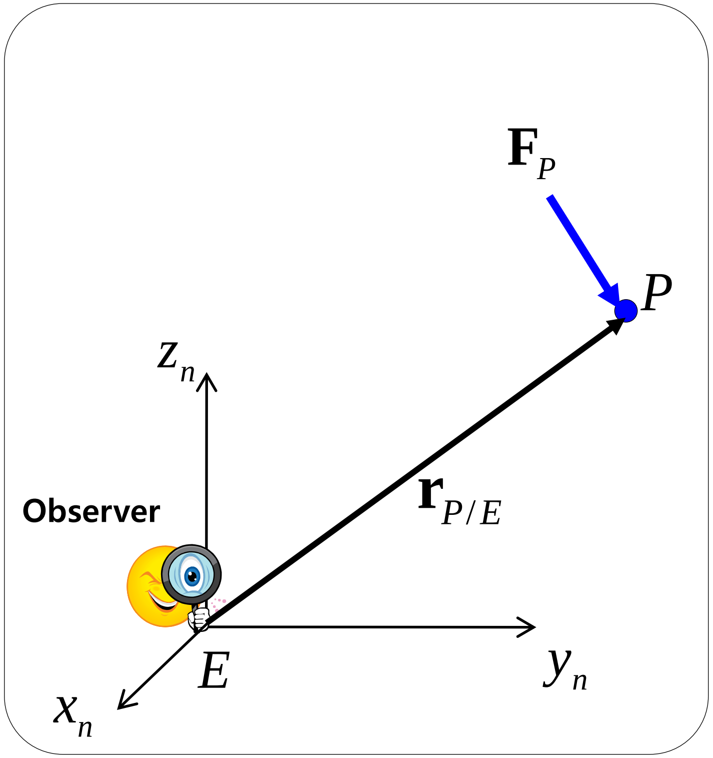
Reference 2) Ginsberg, J. H., Engineering Dynamics, Cambridge University Press, 2008, p.15.

1.2 Translational Relative Motion

Translational Relative Motion
- Motion of a particle with respect to an inertial reference frame

Newton's law is valid in any **inertial reference frame.**[1)]

$$m\ddot{\mathbf{r}} = \sum \mathbf{F}$$ **: Newton's 2nd Law**



E : The origin of the inertial reference frame (n-frame)

A particle *P* of mass m_p is observed from the origin *E* of an inertial reference frame, n-frame.

We can apply Newton's 2nd Law to the particle *P*.

$$m_P \ddot{\mathbf{r}}_{P/E} = \mathbf{F}_P$$

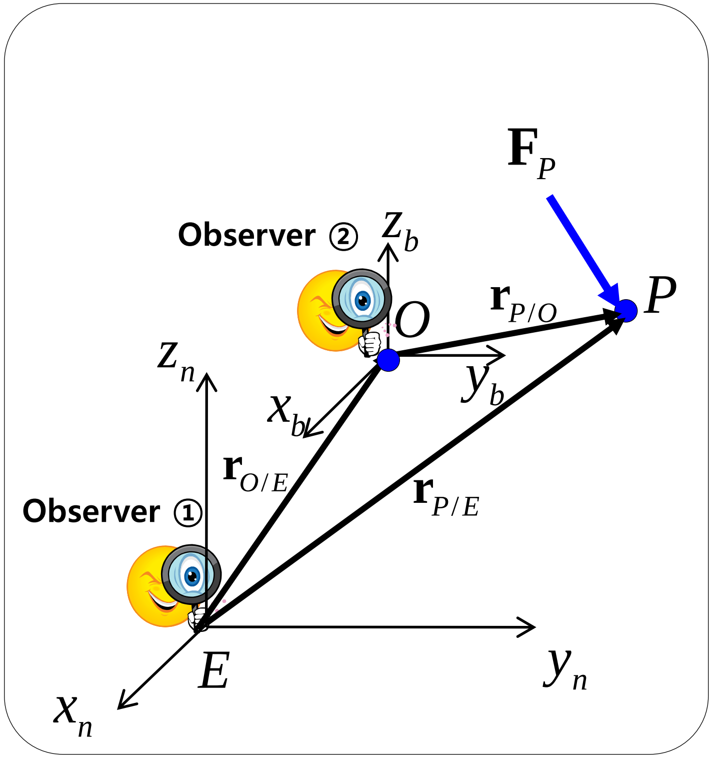
Then, what will be if the reference frame is non-inertial?

Reference 1) Ginsberg, J. H., Advanced Engineering Dynamics, 2nd edition, Cambridge University Press, 1995, p.4.

Relative Motion

- Motion of a particle with respect to a non-inertial reference frame

$$m\ddot{\mathbf{r}} = \sum \mathbf{F}$$: Newton’s 2nd Law



If we observe the point P in a non-inertial reference frame, b-frame, that ~~(which)~~ is moving with an acceleration of “a”, what is the force exerted on the point P ?

According to Newton’s 2nd Law with respect to the n-frame,

$$m_P\ddot{\mathbf{r}}_{P/E} = \mathbf{F}_P$$

By substituting the kinematic relation $\ddot{\mathbf{r}}_{P/E} = \ddot{\mathbf{r}}_{P/O} + \ddot{\mathbf{r}}_{O/E}$ acceleration leads to,

$$m_P(\ddot{\mathbf{r}}_{P/O} + \ddot{\mathbf{r}}_{O/E}) = \mathbf{F}_P$$

$$m_P\,\ddot{\mathbf{r}}_{P/O} + m_P\,\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_P$$

$$\underbrace{m_P\ddot{\mathbf{r}}_{P/O}}_{\text{Acceleration Vector relative to the non-inertial reference frame}} = \boxed{\underbrace{\mathbf{F}_P}_{\text{External Force}} - \underbrace{m_P\ddot{\mathbf{r}}_{O/E}}_{\text{Inertial force}}}$$

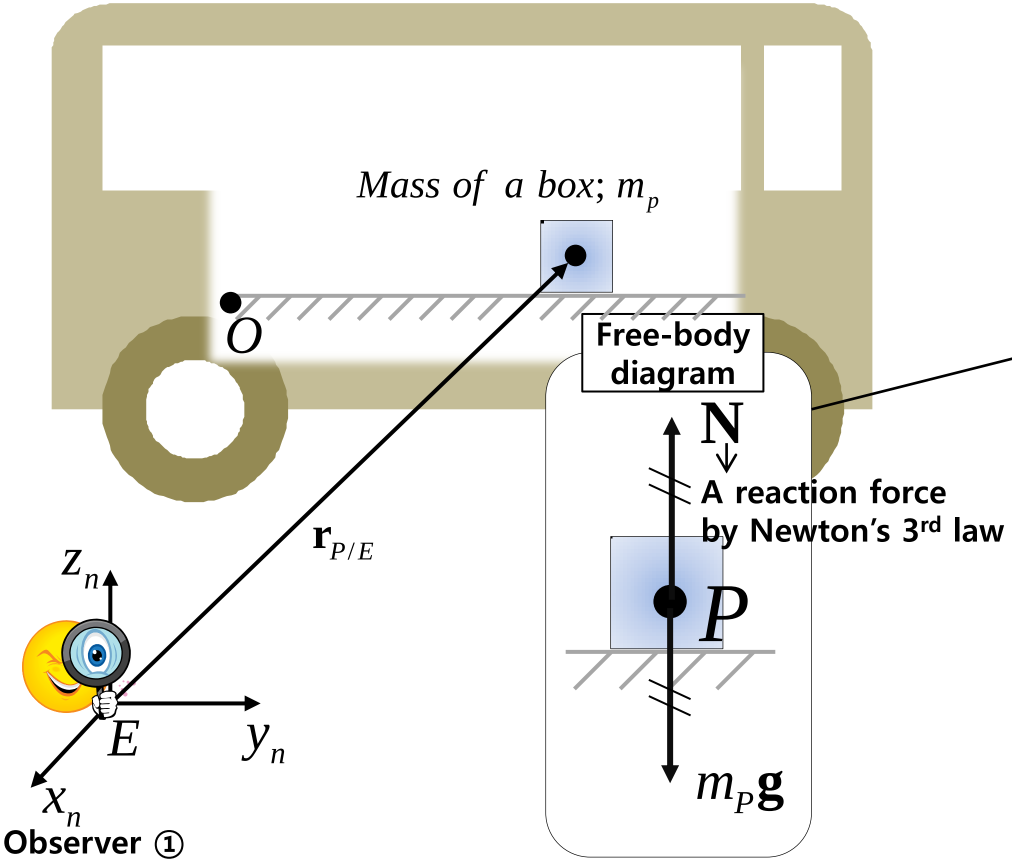
E : The origin of the inertial reference frame (n-frame)

O: The origin of the non-inertial reference frame (b-frame)

Relative Motion
- Examples of a Bus

Case #1-1

- A box is fixed on a bus which is at rest.
- Find the forces exerted on the box.



An observer① describes the force exerted on the box.

1. At first, we consider the forces exerted on the box in vertical direction. Newton's 2nd law is applied to the box in the bus.

$$m_P \ddot{\mathbf{r}}_{P/E} = \mathbf{F}$$

$$= m_P\mathbf{g} + \mathbf{N}$$

Since the box is at rest, it is in static equilibrium.

$$\ddot{\mathbf{r}}_{P/E} = 0$$

$$0 = m_P\mathbf{g} + \mathbf{N}$$

$$\Downarrow$$

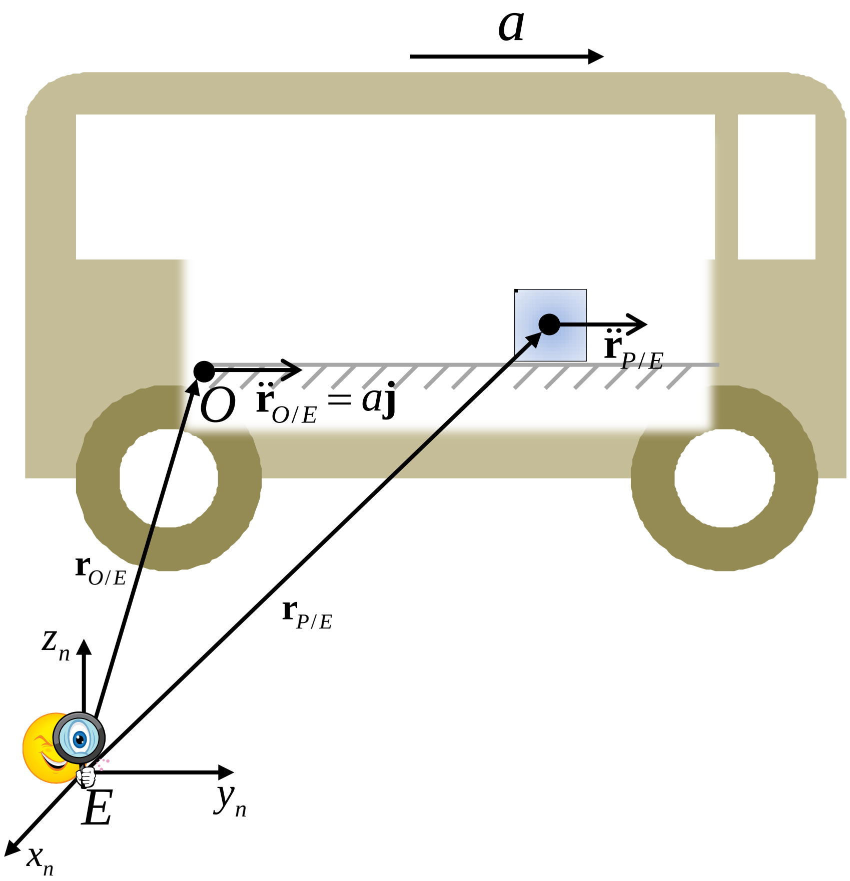
$$\boxed{\mathbf{N} = -m_P\mathbf{g}}$$

2. There is no force in horizontal direction.

Relative Motion
- Examples of a Bus

Case #2-1

- A box is fixed on a bus that ~~(which)~~ is moving with an acceleration of a in the horizontal direction.
- Find the force exerted on the box in the horizontal direction.



Observer ①

An observer① describes the force exerted on the box.

We apply Newton's 2nd law to the box in the bus.

$$m_P \ddot{\mathbf{r}}_{P/E} = \mathbf{F}_P \qquad \ddot{\mathbf{r}}_{P/E} = a\mathbf{j}$$

$$m_P a\mathbf{j} = \mathbf{F}_P$$

→ The force exerted on the box ~~is~~ in the horizontal direction is:

$$m_P a$$

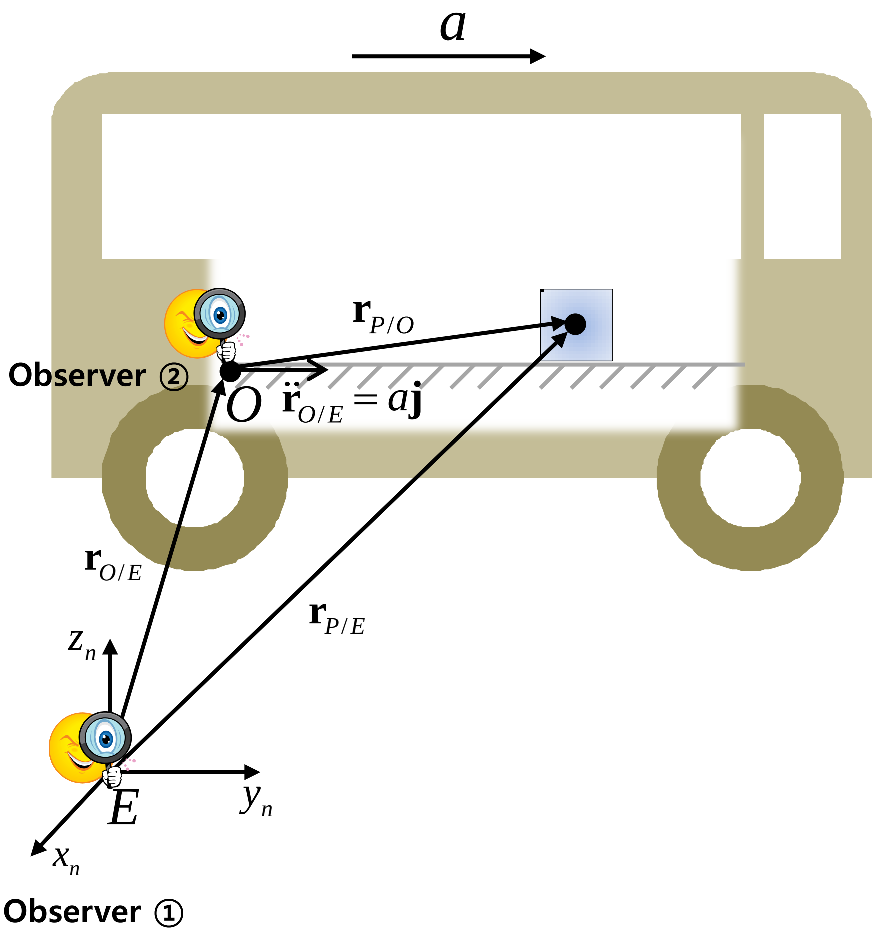
Relative Motion
- Examples of a Bus

~~Inertial force-> inertia force???~~

$$m_P\ddot{\mathbf{r}}_{P/O} = \underbrace{\mathbf{F}_P}_{\text{External Force}} \underbrace{- m_P\ddot{\mathbf{r}}_{O/E}}_{\text{Inertial force}}$$

Case #2-2

- A box is fixed on a bus that ~~(which)~~ is moving with an acceleration of a in the horizontal direction.
- Find the force exerted on the box in the horizontal direction.



An observer② in the bus describes the force exerted on the box.

Suppose an ~~the~~ observer is located at the origin of the non-inertial reference frame that ~~(which)~~ moves with an acceleration of a (observer②).

Thus, the inertial force should be considered.

$$\begin{aligned} m_P\ddot{\mathbf{r}}_{P/O} &= \mathbf{F}_P \overset{\text{inertial force}}{\boxed{- m_P\ddot{\mathbf{r}}_{O/E}}} \\ &= m_P a\mathbf{j} - m_P a\mathbf{j} \\ &= 0\mathbf{j} \end{aligned}$$

$\mathbf{F}_P = m_P a\mathbf{j}$

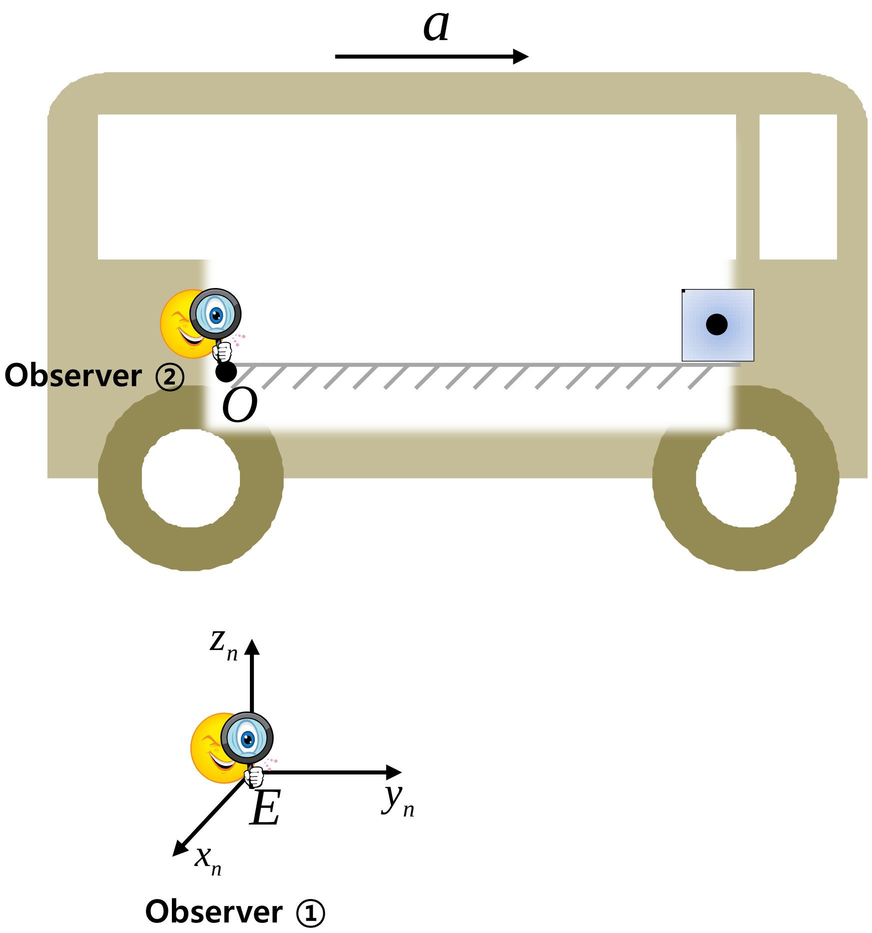
$\ddot{\mathbf{r}}_{O/E} = a\mathbf{j}$

The observer② recognizes that no force is exerted on the box.

Relative Motion
- Examples of a Bus

Case #3

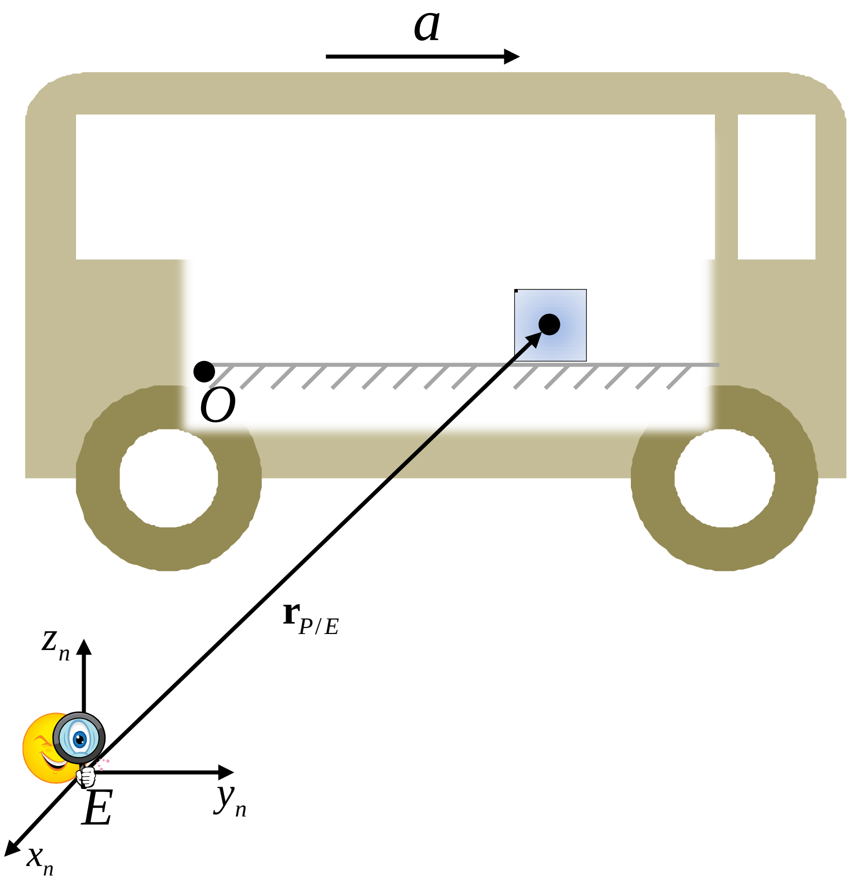
- The box is not fixed and there is no friction btw the box and the bus.
- The bus is moving with acceleration of a in the horizontal direction.
- Find the force exerted on the box in the horizontal direction.



Relative Motion
- Examples of a Bus

Case #3-1

- The box is not fixed and there is no friction btw the box and the bus.
- The bus is moving with acceleration of a in the horizontal direction.
- Find the force exerted on the box in the horizontal direction.



Observer ①

An observer① describes the force exerted on the box.

We apply Newton's 2nd law to the box in the bus.

$$m_P \ddot{\mathbf{r}}_{P/E} = \mathbf{F}_P$$

$$0 = \mathbf{F}_P$$

$$\ddot{\mathbf{r}}_{P/E} = 0$$

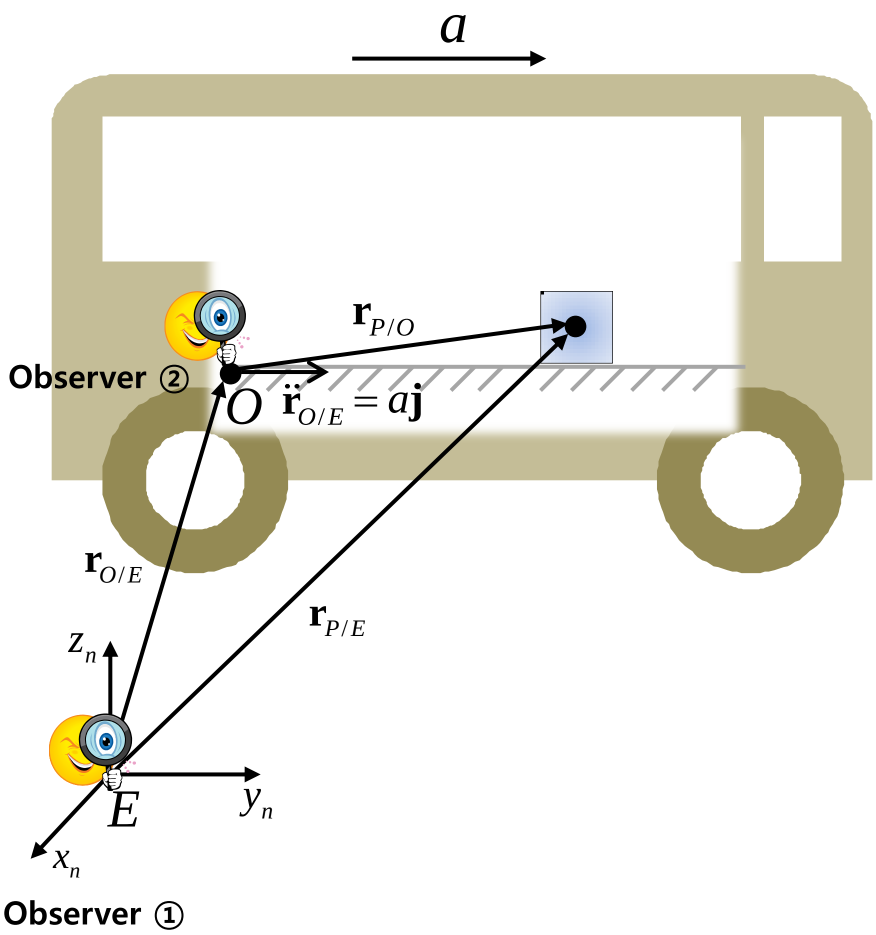
→ The force exerted on the box is zero in the horizontal direction.

Relative Motion
- Examples of a Bus

$$m_P\ddot{\mathbf{r}}_{P/O} = \underbrace{\mathbf{F}_P}_{\text{External Force}} \underbrace{- m_P\ddot{\mathbf{r}}_{O/E}}_{\text{Inertial force}}$$

Case #3-2

- The box is not fixed and there is no friction btw the box and the bus.
- The bus is moving with acceleration of a in the horizontal direction.
- Find the force exerted on the box in the horizontal direction.



An observer② in the bus describes the force exerted on the box.

The observer② is located at the origin of the non-inertial reference frame which moves with an acceleration of a.

So, the inertial force should be considered.

$$m_P\,\ddot{\mathbf{r}}_{P/O} = \mathbf{F}_P \overset{\text{inertial force}}{\boxed{-m_P\ddot{\mathbf{r}}_{O/E}}}$$

$$\mathbf{F}_P = 0\mathbf{j}$$

$$\ddot{\mathbf{r}}_{O/E} = a\mathbf{j}$$

$$= 0\mathbf{j} - m_P\,a\mathbf{j}$$

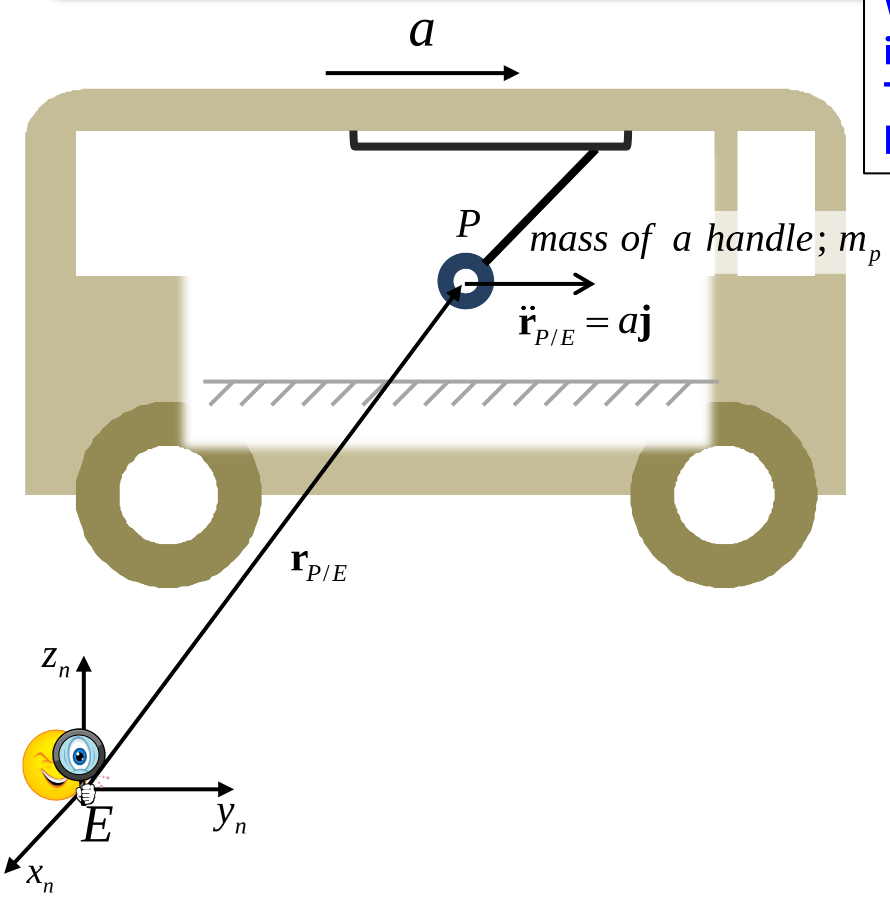
$$= -m_P\,a\mathbf{j}$$

The observer② recognizes that the negative force $-m_p a$ is exerted on the box.

Relative Motion
- Examples of a Bus

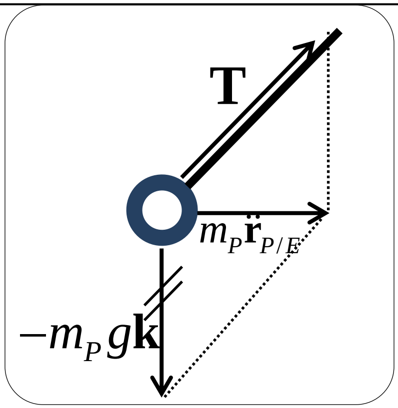
Case #4-1

- A bus is moving with an acceleration of a in the horizontal direction.
- A handle is connected to the top of the bus by the strap.
- Find the force exerted on the handle.



Observer ①

When an observer① describes the handle, the handle is accelerated in the horizontal direction.
The observer① describes the force exerted on the box as follows.



$$m_P\ddot{\mathbf{r}}_{P/E} = \mathbf{T} - m_P g\mathbf{k}$$

We apply Newton’s 2nd law to the box.

$$m_P\ \ddot{\mathbf{r}}_{P/E} = \mathbf{F}_P$$

$$= \mathbf{T} - m_P g\mathbf{k}$$

$$\therefore \mathbf{T} = m_P\ddot{\mathbf{r}}_{P/E} + m_P g\mathbf{k}$$

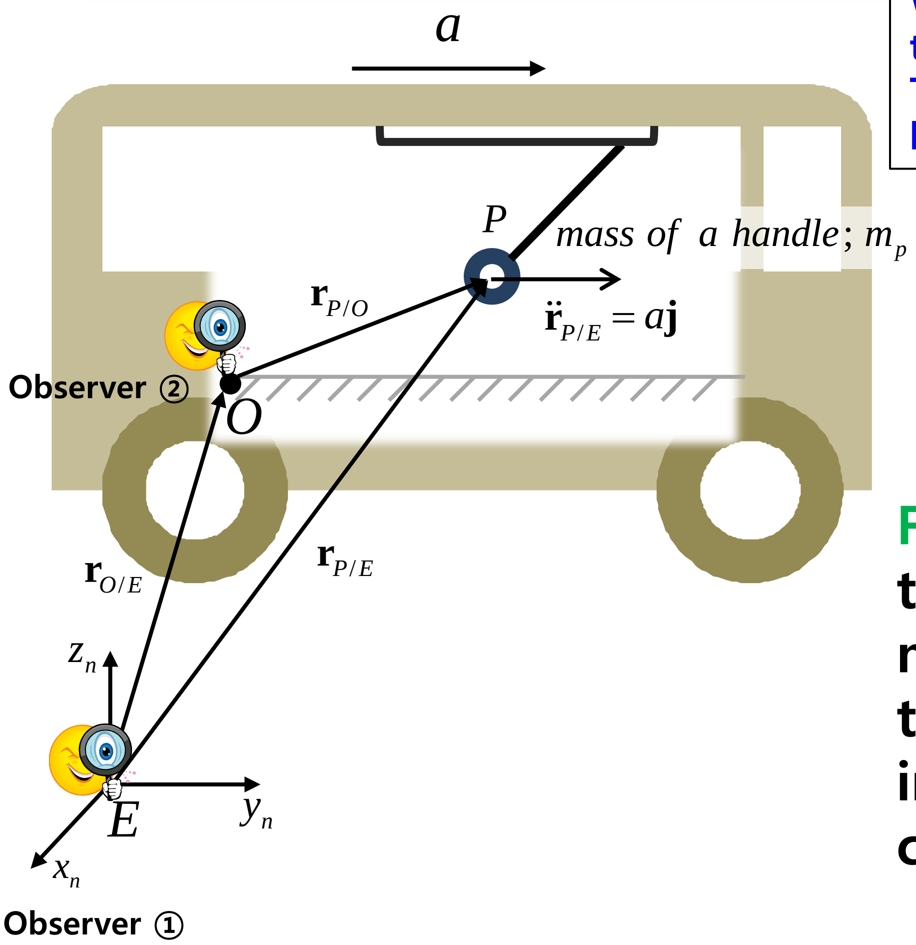
Relative Motion
- Examples of a Bus

$$m_P\ddot{\mathbf{r}}_{P/O} = \underbrace{\mathbf{F}_P}_{\text{External Force}} - \underbrace{m_P\ddot{\mathbf{r}}_{O/E}}_{\text{Inertial force}}$$

Case #4-2

- A bus is moving with an acceleration of a in the horizontal direction.
- A handle is connected to the top of the bus by the strap.
- Find the force exerted on the handle.

When an observer② in the bus describes the handle, the handle is not moving, but the strap is inclined. The observer② describes the force exerted on the box as follows.

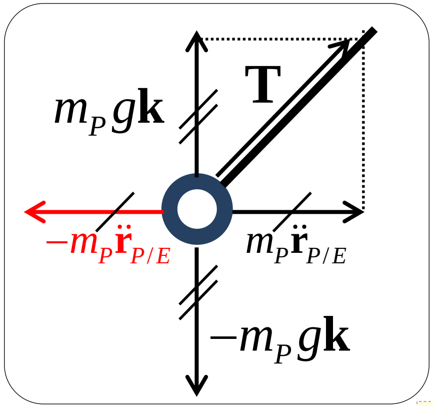


$$m_P\ddot{\mathbf{r}}_{P/O} = \underbrace{\mathbf{T} - m_P g\mathbf{k}}_{= m_P\ddot{\mathbf{r}}_{P/E}} \underbrace{\overset{\text{Inertial force}}{- m_P\ddot{\mathbf{r}}_{O/E}}}_{\underset{\text{Inertial force}}{- m_P\ddot{\mathbf{r}}_{P/E}}}$$

$$= 0$$

Observer ② and handle have the same velocity.

For the observer② the handle is not moving, since the tension and the inertial force are canceled each other.

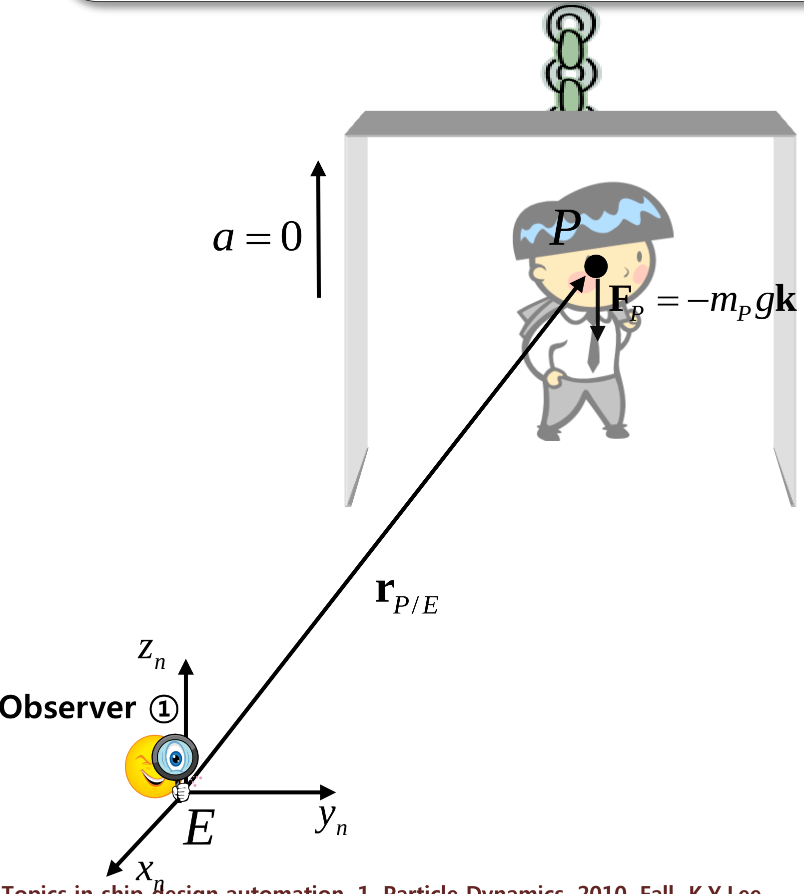


Relative Motion
- Examples of an elevator

Case #5-1 (From observer 1 's view):

A person stands in an elevator that ~~(which)~~ is at rest $(a=0)$, where ~~and~~ the bottom of the elevator is ~~not attached~~ open.

- What will happen?



From observer 1 's view:

- To understand this phenomena, we will apply Newton's 2nd law to the person in the elevator.

$$m_P\ \ddot{\mathbf{r}}_{P/E}=\mathbf{F}_P$$

$$m_P\,\ddot{\mathbf{r}}_{P/E}=-m_P g\,\mathbf{k}$$

$$\ddot{\mathbf{r}}_{P/E}=-g\,\mathbf{k}$$

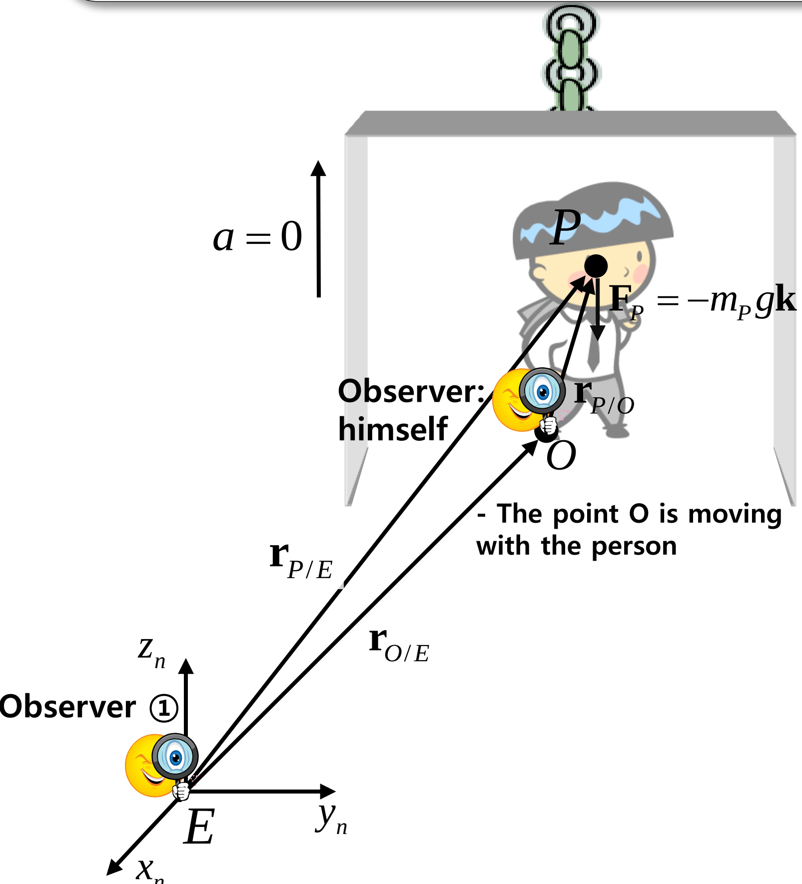
The observer 1 recognize that the person is moving with the acceleration g in the downward direction

"The person will fall down".

Relative Motion
- Examples of an elevator

$$m_P\ddot{\mathbf{r}}_{P/O} = \underbrace{\mathbf{F}_P}_{\text{External Force}} \underbrace{-m_P\ddot{\mathbf{r}}_{O/E}}_{\text{Inertial force}}$$

Case #5-2: When the falling person observes himself, what does he recognize ?



→ The person will **fall down**.

When he observes himself, **the inertial force should be considered, because he is moving with the acceleration of –*g*.**

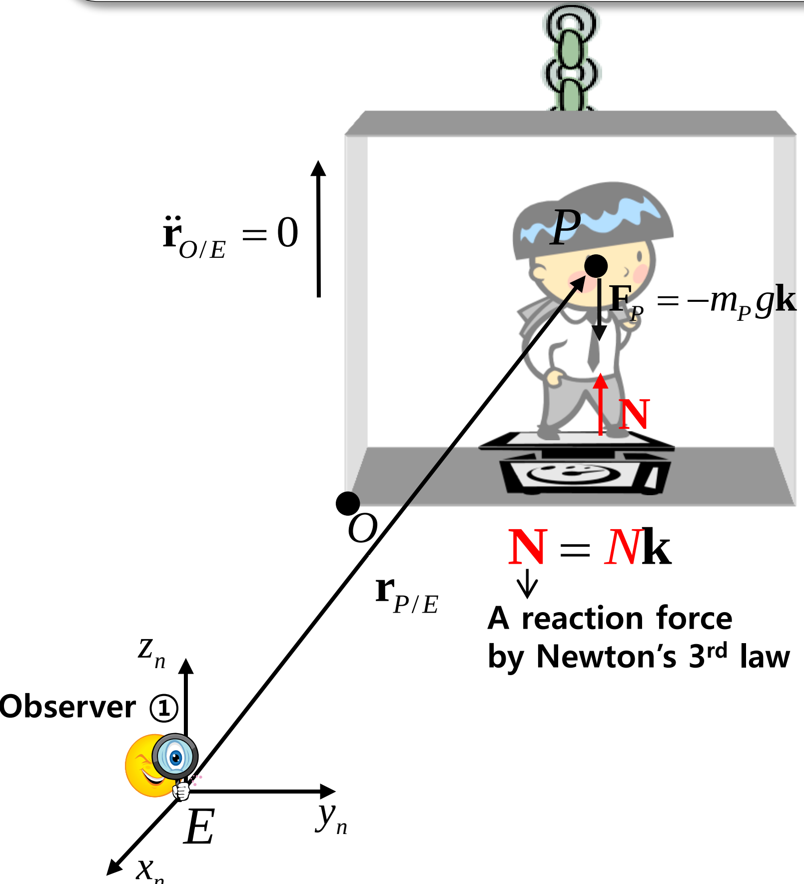
$$\begin{aligned} m_P\,\ddot{\mathbf{r}}_{P/O} &= \mathbf{F}_P \overset{\text{inertial force}}{\boxed{-m_P\ddot{\mathbf{r}}_{O/E}}} \\ &= -m_P\,g\mathbf{k} + m_P\,g\mathbf{k} \\ &= 0 \end{aligned}$$

$$\mathbf{F}_P = -m_P\,g\mathbf{k}$$
$$\ddot{\mathbf{r}}_{O/E} = -g\mathbf{k}$$

Therefore, the person(observer 2) feels(recognizes) that **he is weightless**.

Relative Motion
- Examples of an elevator

Case #6 (From observer 1's view)
- A person stands in an elevator that ~~(which)~~ is at rest $(a=0)$, and the bottom of the elevator is ~~attached~~ closed.
- How much weight does a bathroom scale indicate?



$\mathbf{N}=N\mathbf{k}$
↓
A reaction force by Newton's 3rd law

→ **The person is at rest.**

- We apply Newton's 2nd law to the person in the elevator.

$$m_P\,\ddot{\mathbf{r}}_{P/E}=\mathbf{F}_P$$
$$=-m_P g\,\mathbf{k}+N\mathbf{k}$$

Since the person is at rest, static equilibrium, $\ddot{\mathbf{r}}_{P/E}=0$

$$0=-m_P g\,\mathbf{k}+N\mathbf{k}$$

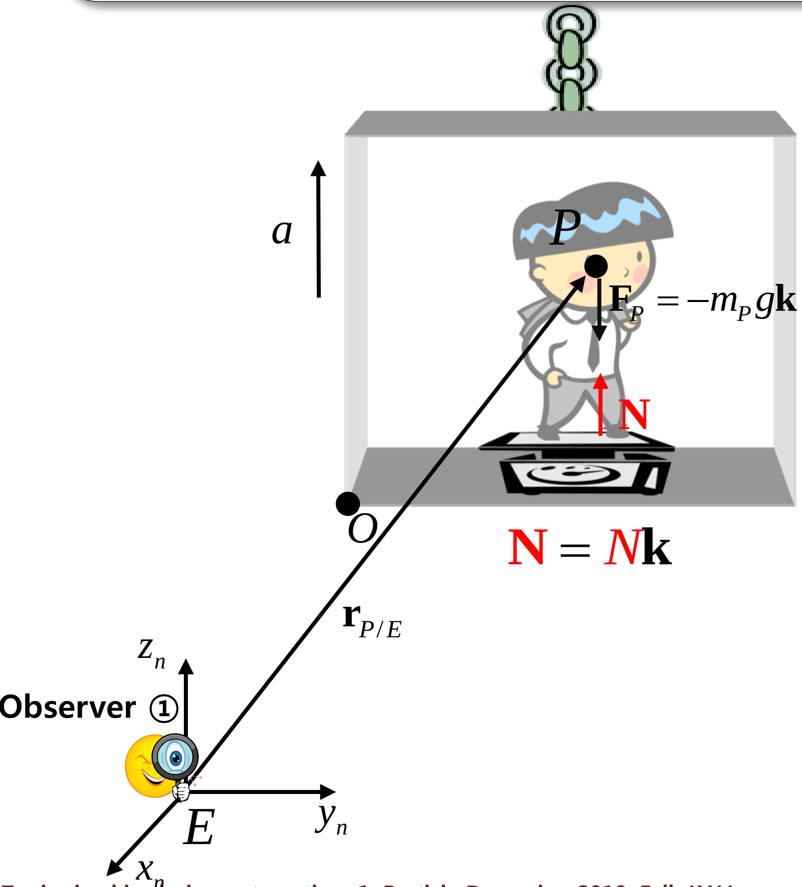
⇩

$$\boxed{N=m_P g}$$

The bathroom scale indicates $m_P g$

Relative Motion
- Examples of an elevator

Case #7-1(From observer 1's view)
- A person stands in an elevator that ~~(which)~~ is moving upward with an acceleration of *a*.
- How much weight does a bathroom scale indicate?



→ The person is moving with the elevator.

- We apply Newton's 2nd law to the person in the elevator.

$$m_P\,\ddot{\mathbf{r}}_{P/E} = \mathbf{F}_P$$

$$= -m_P g\,\mathbf{k} + N\mathbf{k}$$

$$m_P a\mathbf{k} = -m_P g\,\mathbf{k} + N\mathbf{k}$$

$$\ddot{\mathbf{r}}_{P/E} = a\mathbf{k}$$

$$\Downarrow$$

$$N = m_P\left(g + a\right)$$

- The bathroom scale indicates $m_p(g+a)$
- In other words, the forces exerted on the person is the gravitational force and the force from the bottom
- The resultant force is $m_P a\mathbf{k}$, and the person is moving upward with an acceleration of *a*

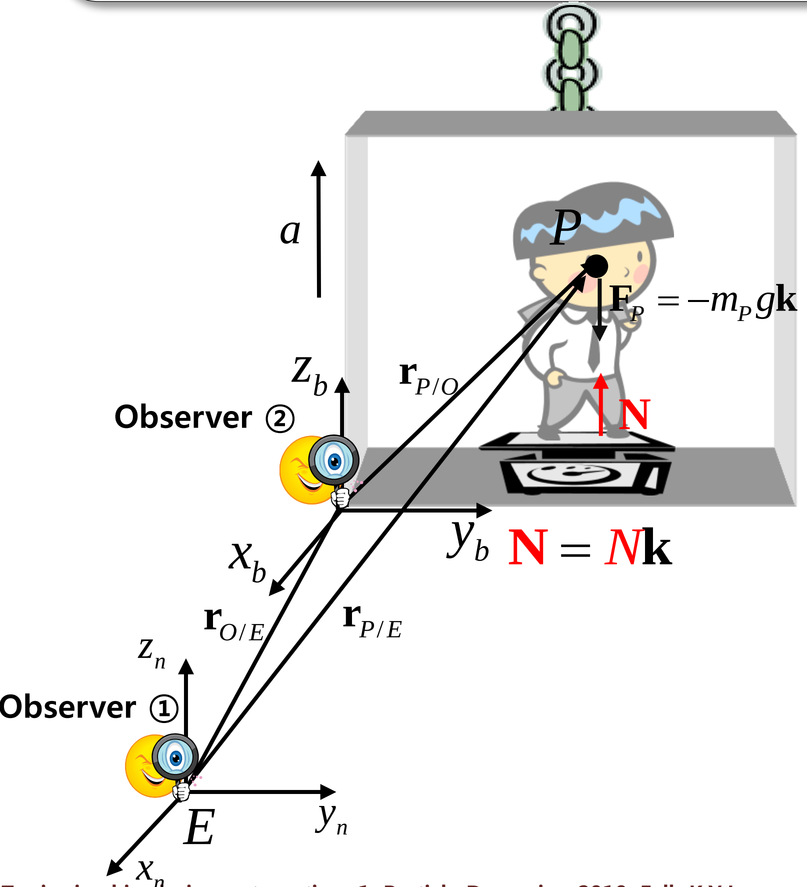
Relative Motion
- Examples of an elevator

$$m_P\ddot{\mathbf{r}}_{P/O} = \underbrace{\mathbf{F}_P}_{\text{External Force}} \underbrace{- m_P\ddot{\mathbf{r}}_{O/E}}_{\text{Inertial force}}$$

Case #7-2(From observer 2's view)

- A person stands in an elevator that ~~(which)~~ is moving upward with an acceleration of a.

- Find the exerted force on the person.



An observer② in the elevator describes the force exerted on the person.

The observer② is located at the origin of the non-inertial reference frame that ~~(which)~~ moves with an acceleration of a.

Thus, the inertial force should be considered.

$$m_P\ddot{\mathbf{r}}_{P/O} = \mathbf{F}_P \overset{\text{inertial force}}{\boxed{-m_P\ddot{\mathbf{r}}_{O/E}}}$$

$$= -m_P g\mathbf{k} + N\mathbf{k} \overset{\text{inertial force}}{\boxed{-m_P a\mathbf{k}}}$$

$$= 0\mathbf{k}$$

$\mathbf{F}_P = -m_P g\mathbf{k} + N\mathbf{k}$

$\ddot{\mathbf{r}}_{O/E} = a\mathbf{j}$

$N = m_P(g+a)$

- The observer② recognizes that the inertial force is exerted on the person.

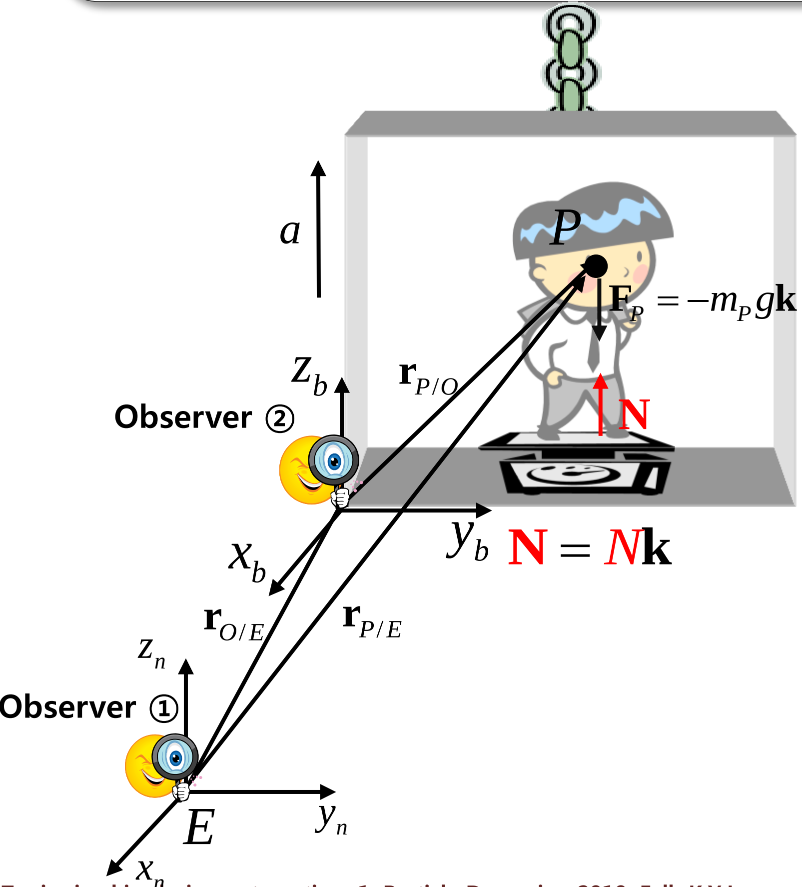
Relative Motion
- Examples of an elevator

$$m_P\ddot{\mathbf{r}}_{P/O} = \underbrace{\mathbf{F}_P}_{\text{External Force}} \underbrace{-m_P\ddot{\mathbf{r}}_{O/E}}_{\text{Inertial force}}$$

Case #7-3(From observer himself)

- A person stands in an elevator that ~~(which)~~ is moving upward with an acceleration of *a*.

- Find the exerted force on the person.



The person in the elevator describes the force exerted on himself.

The person is moving with an acceleration of *a*.

Thus, the inertial force should be considered.

$$m_P\ddot{\mathbf{r}}_{P/O} = \mathbf{F}_P \overset{\text{inertial force}}{\boxed{-m_P\ddot{\mathbf{r}}_{O/E}}}$$

$$= -m_P g\mathbf{k} + N\mathbf{k} \overset{\text{inertial force}}{\boxed{-m_P a\mathbf{k}}}$$

$$= 0\mathbf{k}$$

$$N = m_P(g+a)$$

$$\mathbf{F}_P = -m_P g\mathbf{k} + N\mathbf{k}$$

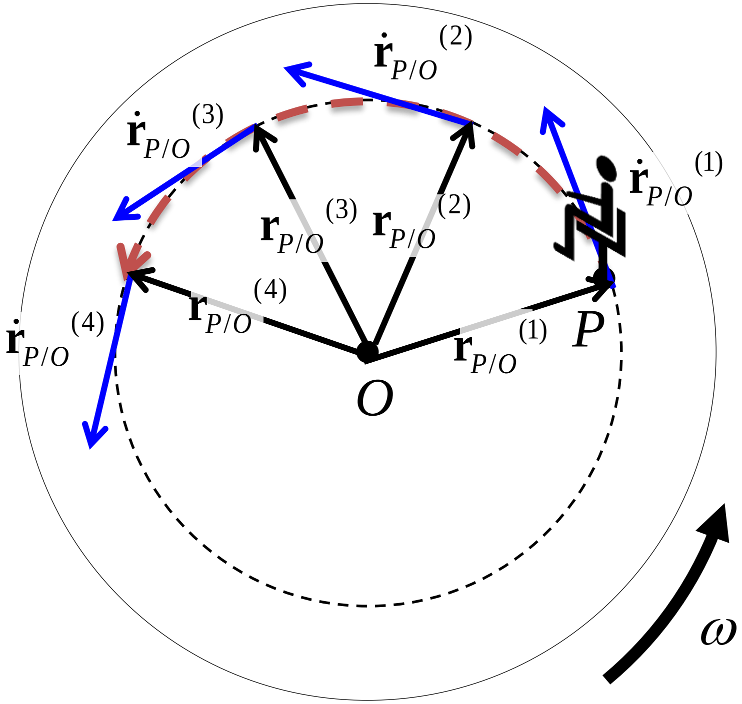
$$\ddot{\mathbf{r}}_{O/E} = a\mathbf{j}$$

- The person recognizes that the inertial force is exerted on himself.

- The person feels additional force -$m_p a$

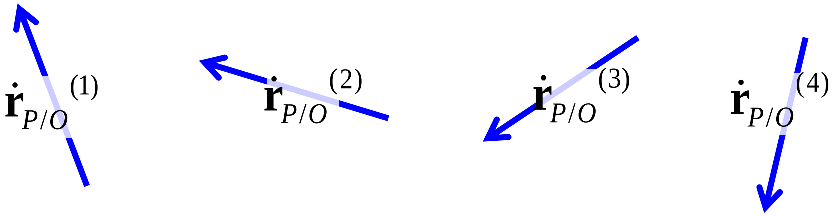
1.3 Rotational Motion

Uniform Circular Motion



Uniform circular motion
: Motion with constant speed along a circular path.

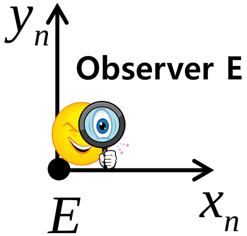
In this figure, all the velocity vectors have the same magnitude (same speed), but they differ in direction.



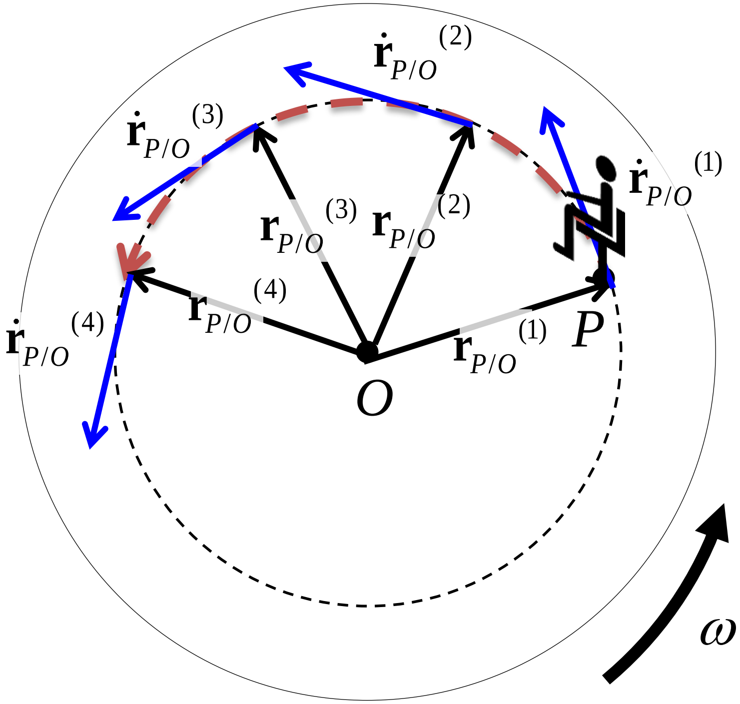
Suppose that the constant speed of the particle is v.

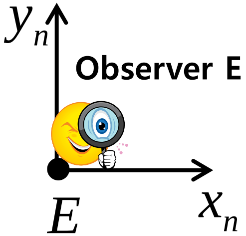
$$v = \left|\dot{\mathbf{r}}_{P/O}^{(1)}\right| = \left|\dot{\mathbf{r}}_{P/O}^{(2)}\right| = \left|\dot{\mathbf{r}}_{P/O}^{(3)}\right| = \left|\dot{\mathbf{r}}_{P/O}^{(4)}\right|$$

Because of this change of direction, uniform circular motion is accelerated motion.



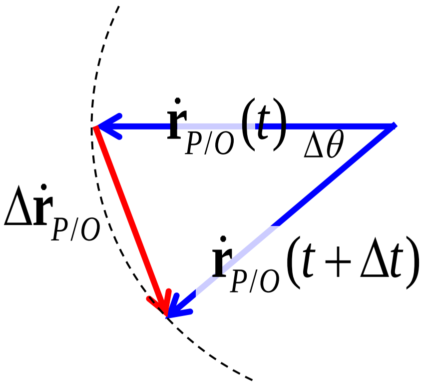
Uniform Circular Motion





- ✓ **Describe the acceleration of uniform circular motion.**

To find the value of the acceleration, we must look at velocity change in a very short time interval Δt .



$$|\mathbf{a}| = a = \frac{|\Delta\dot{\mathbf{r}}_{P/O}|}{\Delta t} \underset{\uparrow}{=} \frac{v\cdot\Delta\theta}{\Delta t}$$

$(|\Delta\dot{\mathbf{r}}_{P/O}| \simeq v\cdot\Delta\theta)$

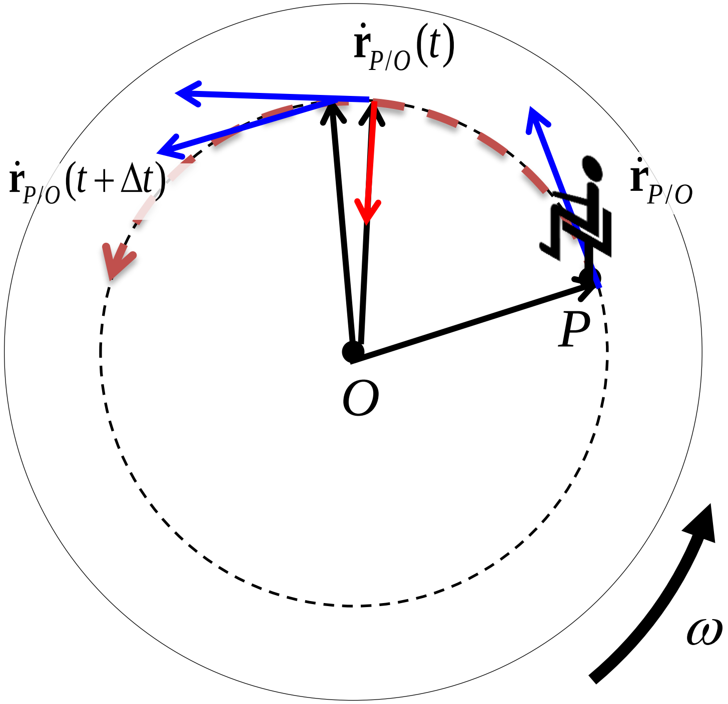
$$\underset{\uparrow}{=} \frac{v\cdot\cancel{\Delta\theta}}{r_{P/O}\cdot\cancel{\Delta\theta}/\dot{r}_{P/O}} = \frac{(v)^2}{r_{P/O}}$$

$(\Delta t = r_{P/O}\cdot\Delta\theta/v)$

$$\Delta t \to 0, \quad \boxed{a \overset{!}{=} \frac{(v)^2}{r_{P/O}}}$$

: Magnitude of the acceleration of uniform circular motion

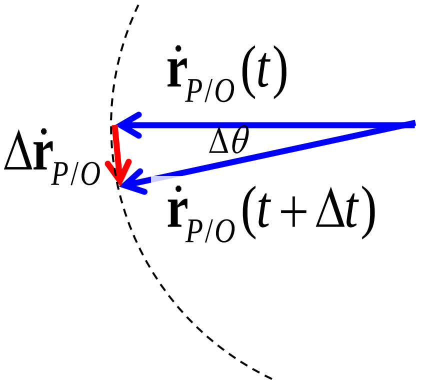
Uniform Circular Motion



- ✓ **Describe the acceleration of uniform circular motion.**

$$\boxed{a = \frac{(v)^2}{r_{P/O}}}$$

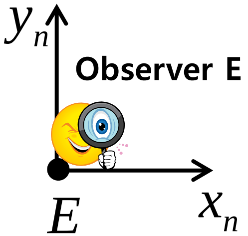
: Magnitude of the acceleration of uniform circular motion



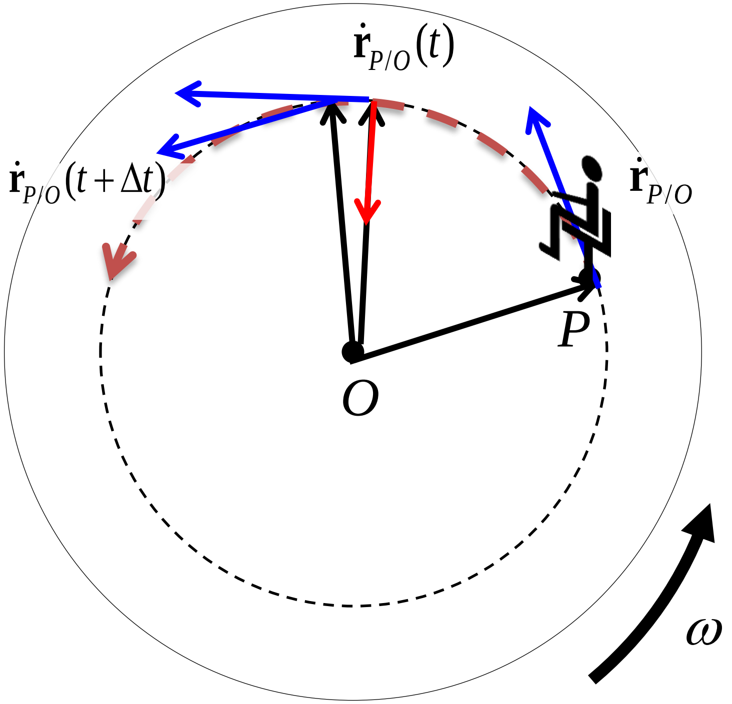
$$|\mathbf{a}| = \frac{|\Delta\dot{\mathbf{r}}_{P/O}|}{\Delta t}$$

When $\Delta t \to 0,$ the direction of $\Delta\dot{\mathbf{r}}_{P/O}$ will be perpendicular to the velocity vectors $\dot{\mathbf{r}}_{P/O}(t)$ and $\dot{\mathbf{r}}_{P/O}(t+\Delta t)$.

Hence the instantaneous acceleration vector is perpendicular to the instantaneous velocity vector.



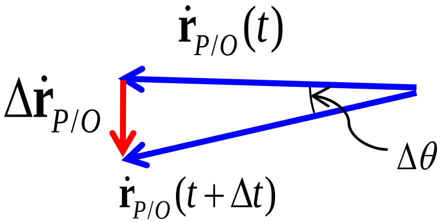
Uniform Circular Motion



- ✓ **Describe the acceleration of uniform circular motion.**

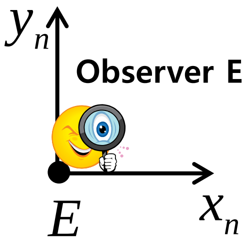
$$\boxed{a = \frac{(v)^2}{r_{P/O}}}$$

: Magnitude of the acceleration of uniform circular motion



$$|\mathbf{a}| = \frac{|\Delta\dot{\mathbf{r}}_{P/O}|}{\Delta t}$$

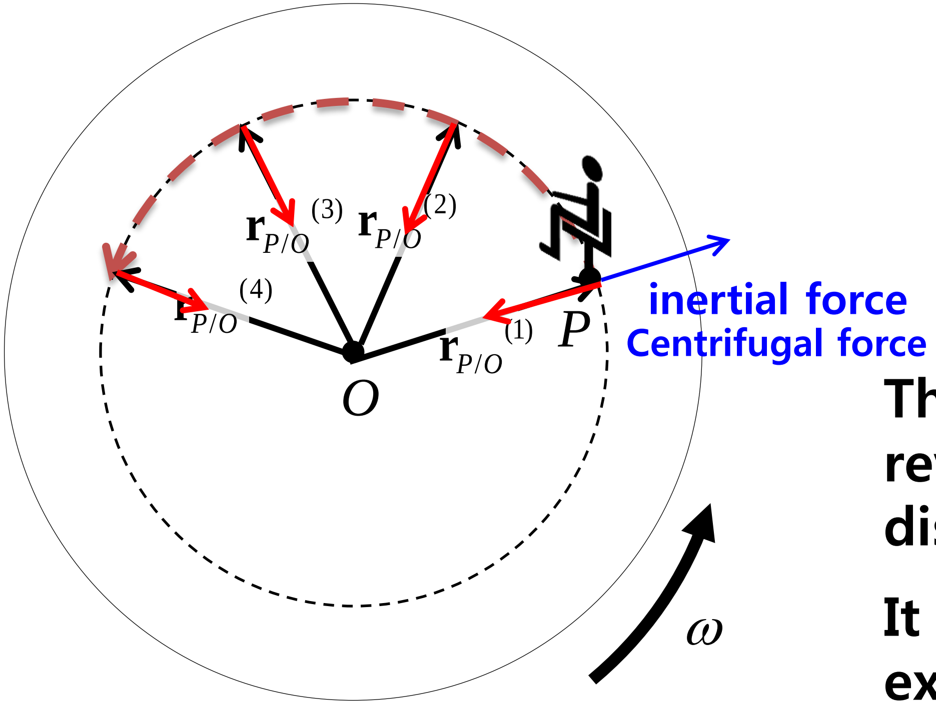
$$\mathbf{a} \perp \dot{\mathbf{r}}_{P/O}$$



Since the velocity vector corresponding to circular motion is tangential to the circle, the acceleration points along the radius, toward the center of the circle.

This acceleration is called "Centripetal Acceleration"

Uniform Circular Motion



Since the velocity vector corresponding to circular motion is tangential to the circle, the acceleration points along the radius, toward the center of the circle.

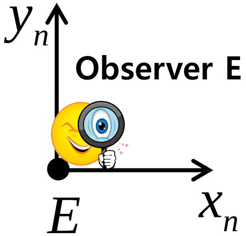
This acceleration is called "Centripetal Acceleration"

The person sitting on the chair revolves around the center of the disk.

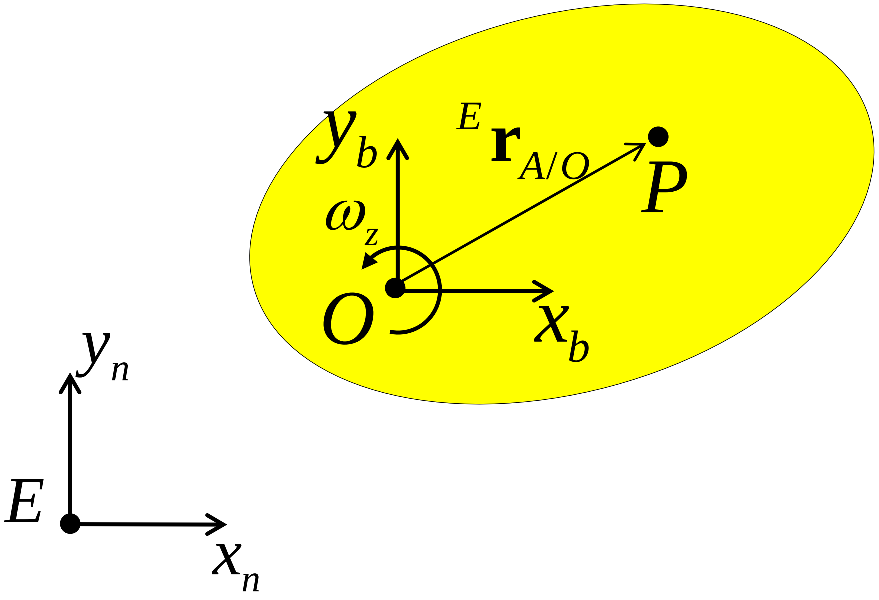
It shows that the centripetal force is exerted on the person

Description from the person sitting on the chair.

The person sitting on the chair feels centrifugal force. ← inertial force



Angular Velocity



Linear Velocity Vector of Point O

$${}^{n}\mathbf{v}_{O/E} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

Angular Velocity Vector of O-frame

$${}^{n}\boldsymbol{\omega}_{b/n} = \begin{bmatrix} 0 & 0 & \omega_{b/n,z} \end{bmatrix}^T$$

Linear Velocity Vector of Point P

$${}^{n}\mathbf{v}_{P/E} = {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{r}_{P/O}$$

n-frame: Inertial Frame

b-frame: Body Fixed Frame

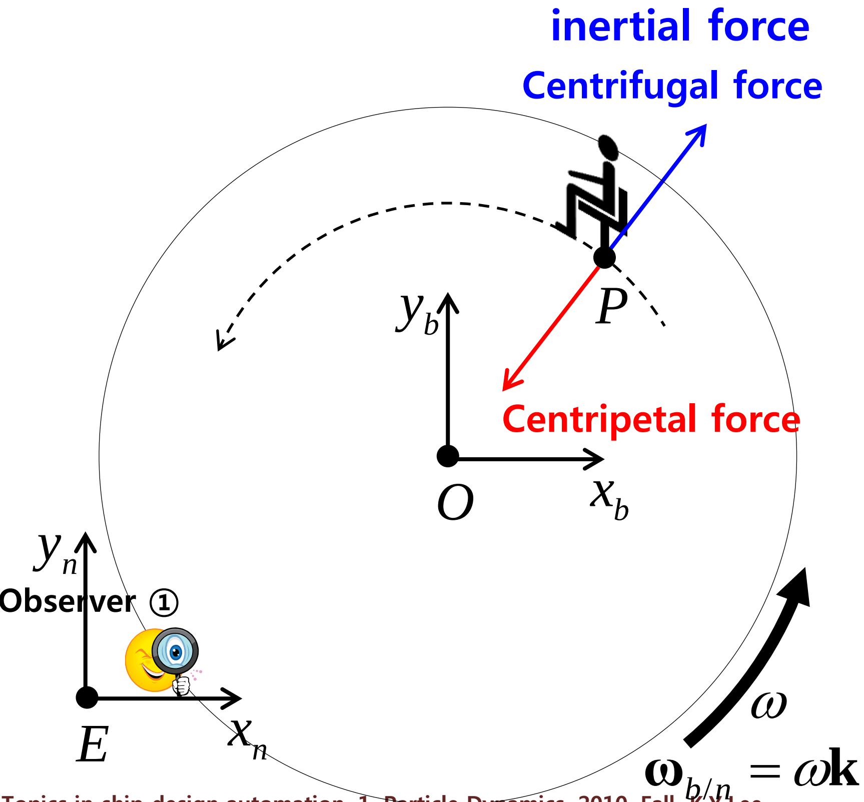
Point O: Pivot(stationary) Point

Point P: Arbitrary Point on the Rigid Body

Relative Rotational Motion : Example 1: rotating reference frame

Example 1

- A chair is fixed on a circular disk which is rotating with an angular velocity ω.
- What kind of forces does a person sitting on the chair feel?



Description from the observer ①

The person sitting on the chair revolves around the center of the disk.

It shows that the centripetal force is exerted on the person

Description from the person sitting on the chair.

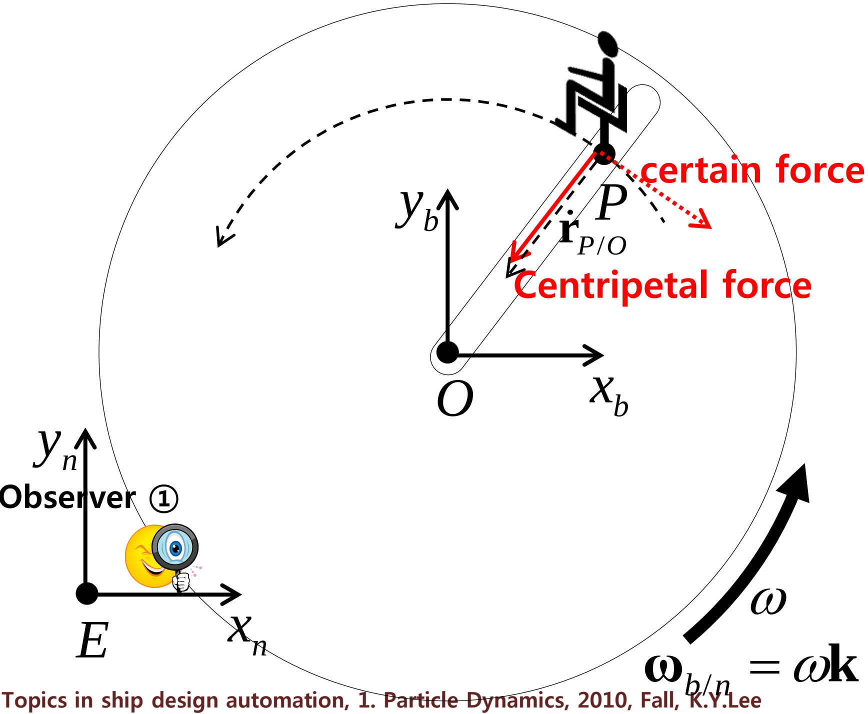
The person sitting on the chair feels centrifugal force. ← inertial force

[sen'trɪpɪtl] [sen'trɪfjʊgl]

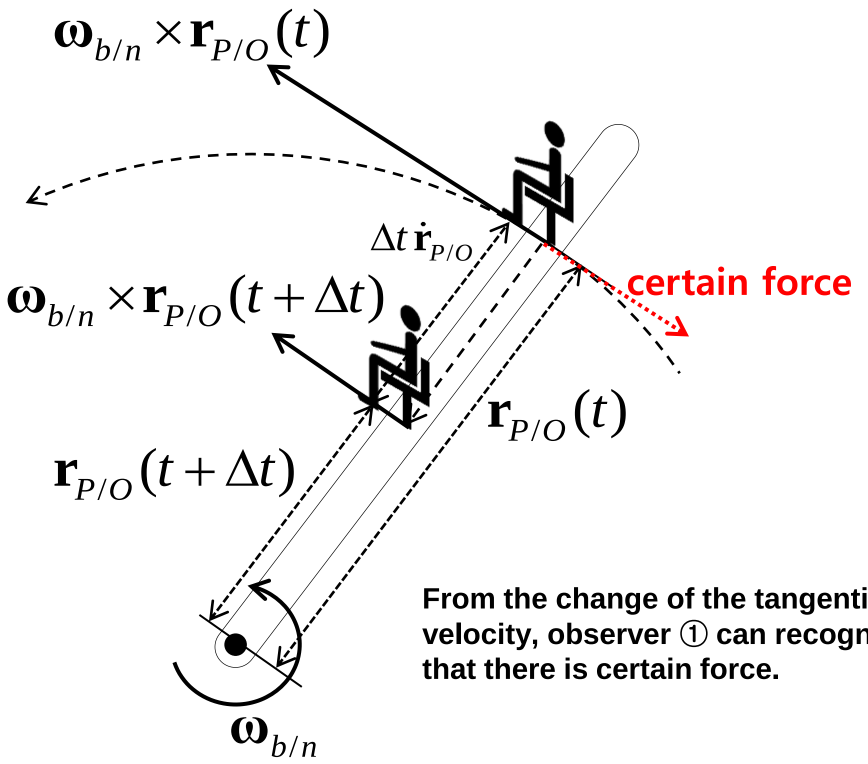
Relative Rotational Motion Example 2:

Example 2

- A chair moves with velocity v along the line on a circular disk which is rotating with an angular velocity ω.
- What kind of forces does a person sitting on the chair feel?



Description from the observer ①



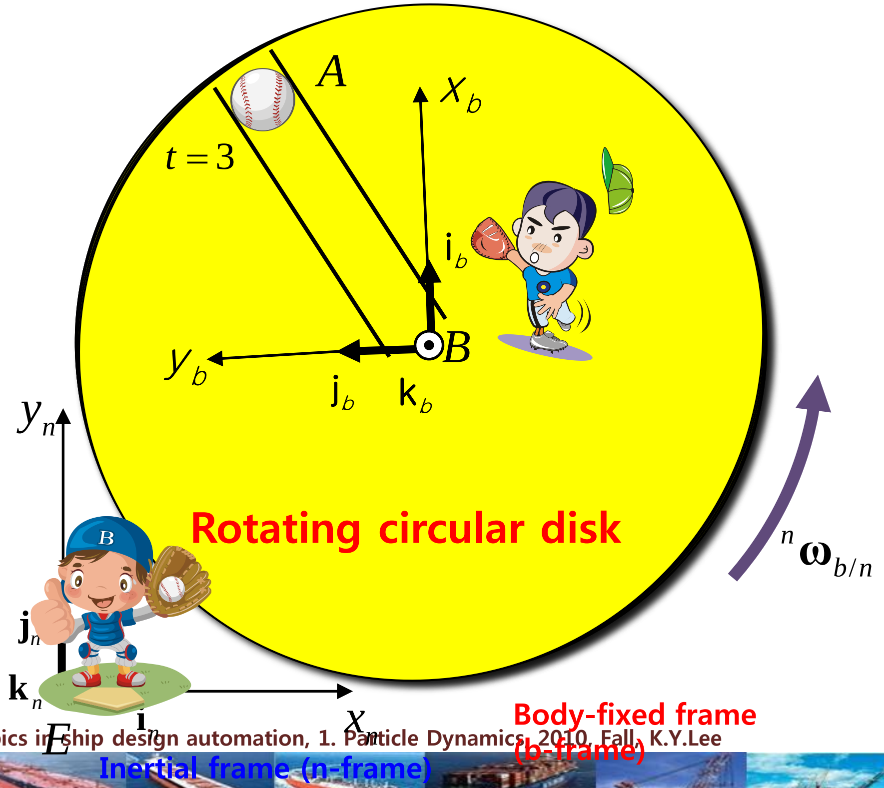
From the change of the tangential velocity, observer ① can recognize that there is certain force.

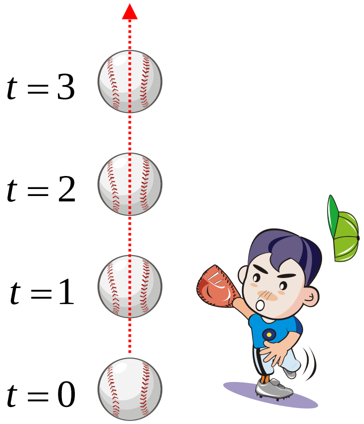
Example 3: Motion of a ball observed in the rotational frame and in the inertial frame

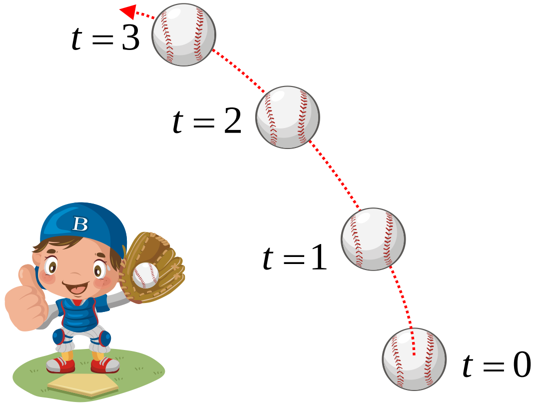
Person "B" is standing on the center of a large disk rotating with a constant angular velocity ${}^{n}\boldsymbol{\omega}_{b/n}$. He throws a ball "A" and the ball moves in a slot in the disk with a constant velocity.

Person "E" is standing still on the ground next to the disk. He also observes the ball "A".

Describe the motion of the ball from the person "B" and "E" respectively.







1.4 Coriolis effect

Curves in Mechanics. Velocity. Acceleration

$$s(t)=\int_a^t \sqrt{\mathbf{r}'\bullet\mathbf{r}'}\,d\tilde{t}$$

$$\frac{ds}{d\tilde{t}}=\sqrt{\mathbf{r}'\bullet\mathbf{r}'}$$

Let a curve C is represented by a parametric representation

$\mathbf{r}(t)$ with <u>time</u> t as parameter.

Velocity Vector $\mathbf{v}$: The tangent vector of C, $\mathbf{v}=\mathbf{r}'(t)$.

Speed |v| : $|\mathbf{v}|=|\mathbf{r}'|=\sqrt{\mathbf{r}'\bullet\mathbf{r}'}=\dfrac{ds}{dt}$ **ds : linear element.**

$$\mathbf{v}=\mathbf{r}'(t)=\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\frac{ds}{dt}=\mathbf{u}(s)\frac{ds}{dt}$$

($\mathbf{u}(s)$: unit tangent vector)

$$s(t)=\int_a^t \sqrt{\mathbf{r}'\bullet\mathbf{r}'}\,d\tilde{t}$$

$$\frac{ds}{d\tilde{t}}=\sqrt{\mathbf{r}'\bullet\mathbf{r}'}$$

Velocity Vector $\mathbf{v}$: The tangent vector of C, $\mathbf{v}=\mathbf{r}'(t)$

$$\mathbf{v}=\mathbf{r}'(t)=\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\frac{ds}{dt}=\mathbf{u}(s)\frac{ds}{dt}$$

($\mathbf{u}(s)$: unit tangent vector)

Acceleration Vector $\mathbf{a}$:

The second derivative of $\mathbf{r}(t)$, $\mathbf{a}(t)=\mathbf{v}'(t)=\mathbf{r}''(t)$.

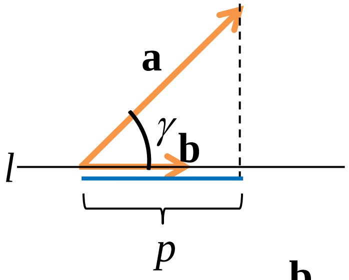
Acceleration $|\mathbf{a}|$: $|\mathbf{a}|=|\mathbf{r}''|=\sqrt{\mathbf{r}''\bullet\mathbf{r}''}=(d^2s/dt^2)$

$$\mathbf{a}(t)=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left(\mathbf{u}(s)\frac{ds}{dt}\right)=\frac{d\mathbf{u}}{ds}\left(\frac{ds}{dt}\right)^2+\mathbf{u}(s)\frac{d^2s}{dt^2}$$

$$|\mathbf{v}(t)| = c \qquad \mathbf{v}\bullet\mathbf{v} = |\mathbf{v}|^2 = c^2$$

Tangential and Normal Acceleration(접선가속도 & 법선가속도)

$$\mathbf{a}(t) = \frac{d\mathbf{u}}{ds}\left(\frac{ds}{dt}\right)^2 + \mathbf{u}(s)\frac{d^2 s}{dt^2} = \mathbf{a}_{\mathrm{norm}} + \mathbf{a}_{\mathrm{tan}}$$



$$p = |\mathbf{a}|\cos\gamma = \mathbf{a}\bullet\frac{\mathbf{b}}{|\mathbf{b}|}$$

(Sec. 9.2)

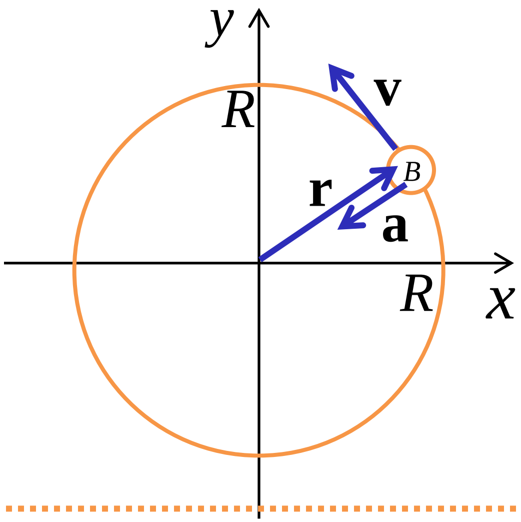
Normal Acceleration : $\mathbf{a}_{\mathrm{norm}} = \mathbf{a} - \mathbf{a}_{\mathrm{tan}}$

Tangential Acceleration :

$$|\mathbf{a}_{\mathrm{tan}}| = \mathbf{a}\bullet\frac{\mathbf{v}}{|\mathbf{v}|}$$

$$\mathbf{a}_{\mathrm{tan}} = |\mathbf{a}_{\mathrm{tan}}|\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\mathbf{a}\bullet\frac{\mathbf{v}}{|\mathbf{v}|}\right)\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{a}\bullet\mathbf{v}}{|\mathbf{v}||\mathbf{v}|}\cdot\mathbf{v} = \frac{\mathbf{a}\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}\cdot\mathbf{v}$$

Centripetal Acceleration. Centrifugal Force.



Circle C of radius R

$$\mathbf{r}(t) = [R\cos\omega t, R\sin\omega t]$$
$$= R\cos\omega t\mathbf{i} + R\sin\omega t\mathbf{j}$$

Velocity vector of small body B

$$\mathbf{v}(t) = \mathbf{r}'(t) = [-R\omega\sin\omega t, R\omega\cos\omega t]$$
$$= -R\omega\sin\omega\mathbf{i} + R\omega\cos\omega t\mathbf{j}$$

Speed of small body B

$$|\mathbf{v}(t)| = \|[-R\omega\sin\omega t, R\omega\cos\omega t]\|$$
$$= \sqrt{(R\omega)^2\sin^2\omega t + (R\omega)^2\cos^2\omega t} = R\omega$$

Acceleration vector of small body B

$$\mathbf{a}(t) = \mathbf{v}'(t) = [-R\omega^2\cos\omega t, -R\omega^2\sin\omega t]$$
$$= -R\omega^2\cos\omega\mathbf{i} - R\omega^2\sin\omega t\mathbf{j}$$

From $\mathbf{r}(t)$ and $\mathbf{a}(t)$

$$\mathbf{a} = -\omega^2\mathbf{r}$$

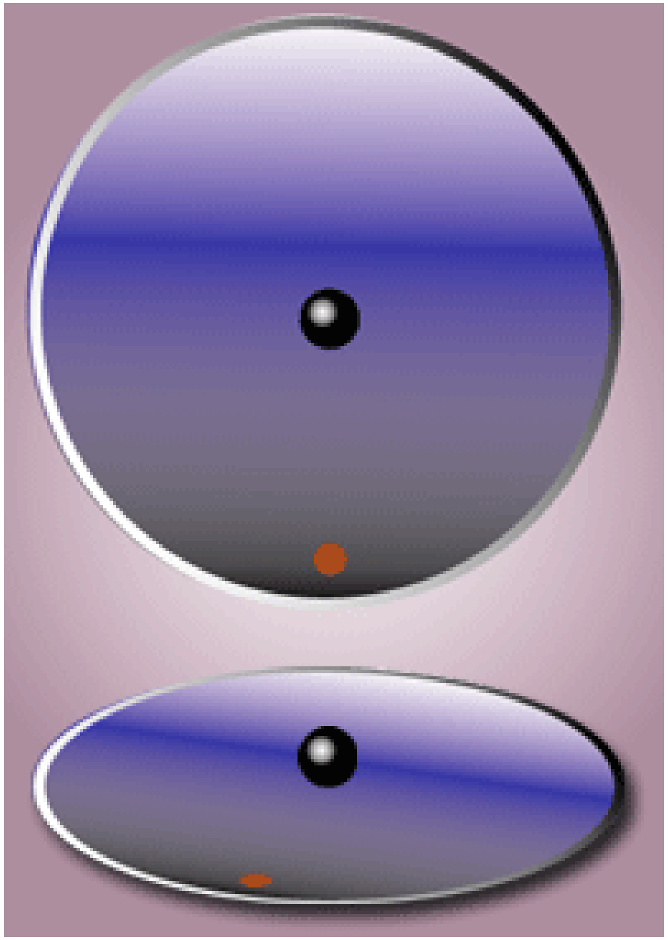
Direction of acceleration : center

Centripetal acceleration (구심가속도)

Coriolis effect

*http://en.wikipedia.org/wiki/Coriolis_effect

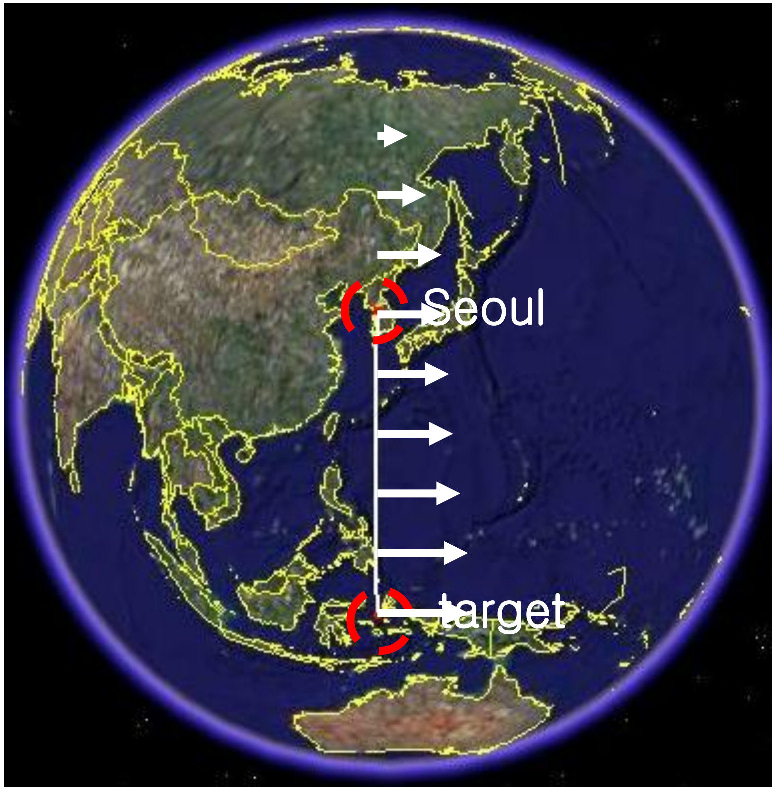
The Coriolis effect is the apparent deflection of moving objects from a straight path when they are viewed from a rotating frame of reference.



Coriolis effect (Northern Hemisphere)

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

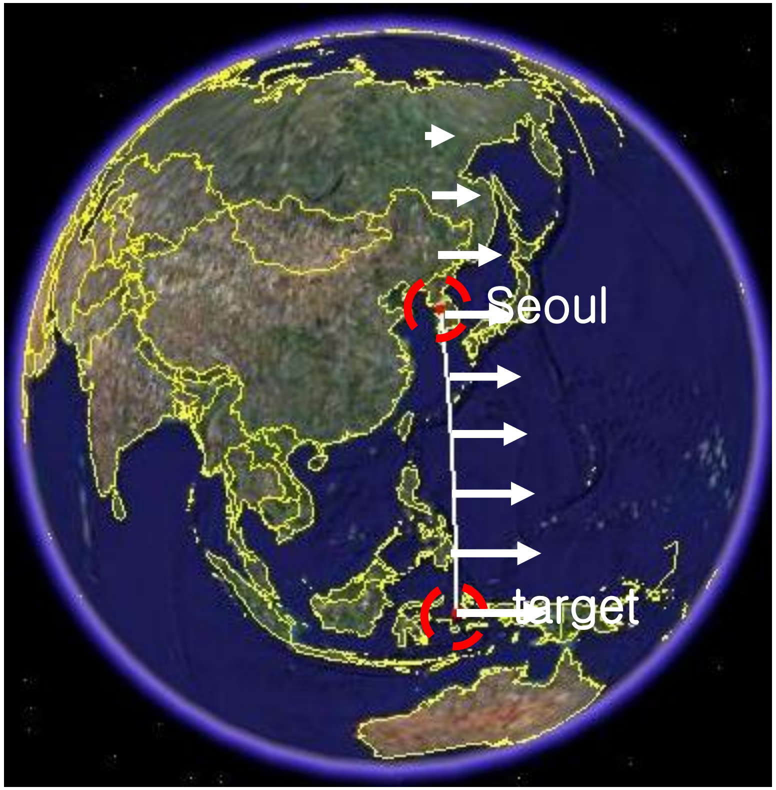
velocity caused by the rotation of Earth



Coriolis effect (Northern Hemisphere)

⇨: $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

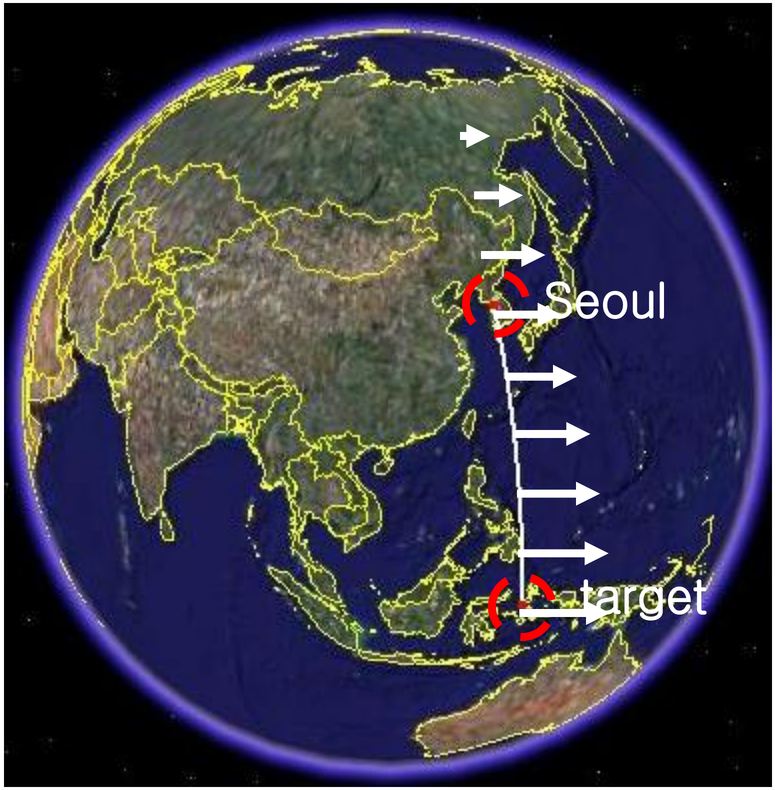
velocity caused by the rotation of Earth



Coriolis effect (Northern Hemisphere)

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

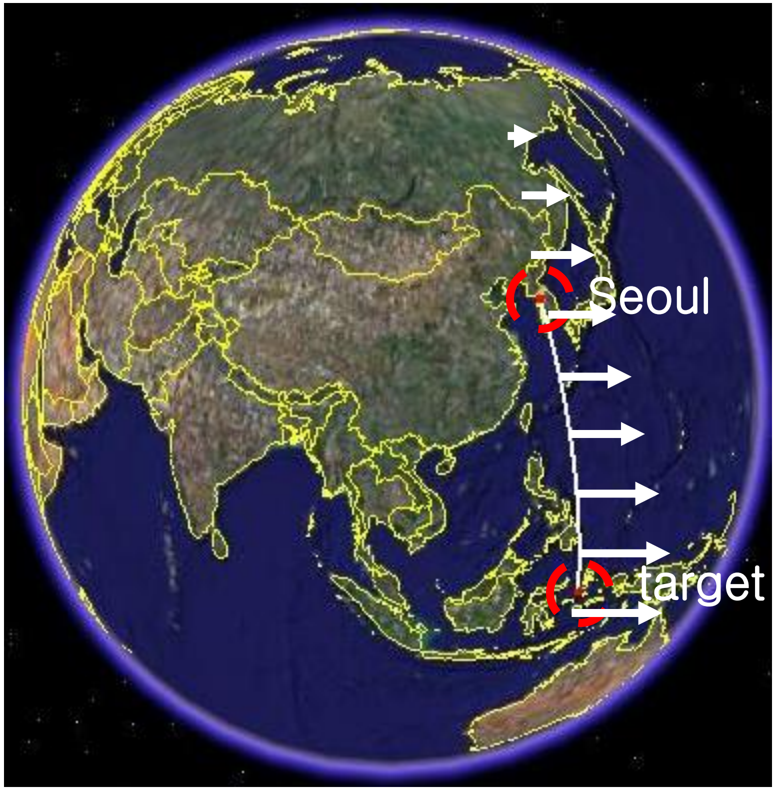
velocity caused by the rotation of Earth



Coriolis effect (Northern Hemisphere)

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

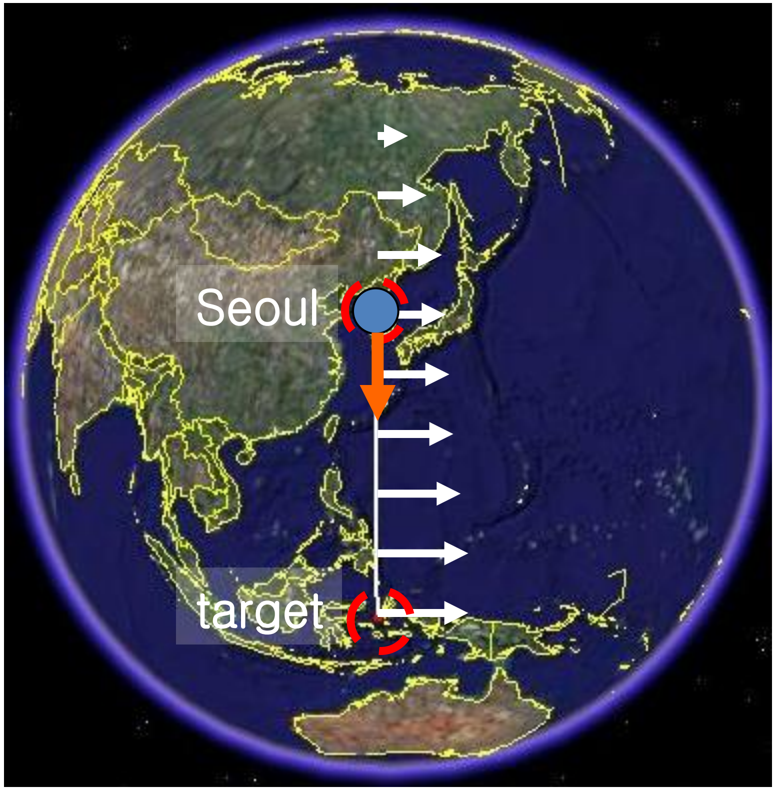
velocity caused by the rotation of Earth



Coriolis effect (Northern Hemisphere)

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

velocity caused by the rotation of Earth



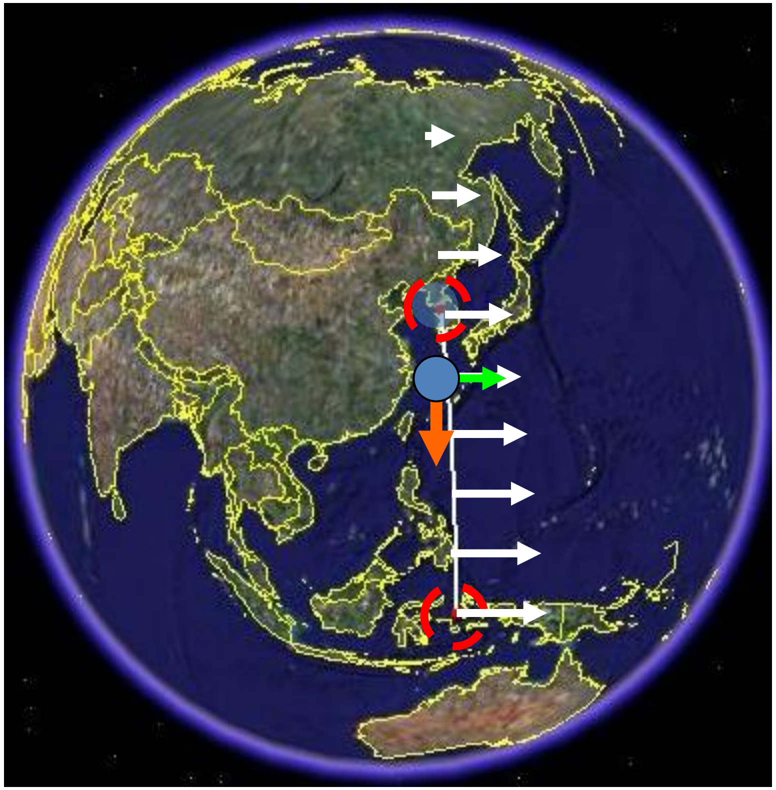
↓ initial velocity

→ initial velocity caused by the rotation of Earth

Coriolis effect (Northern Hemisphere)

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

velocity caused by the rotation of Earth



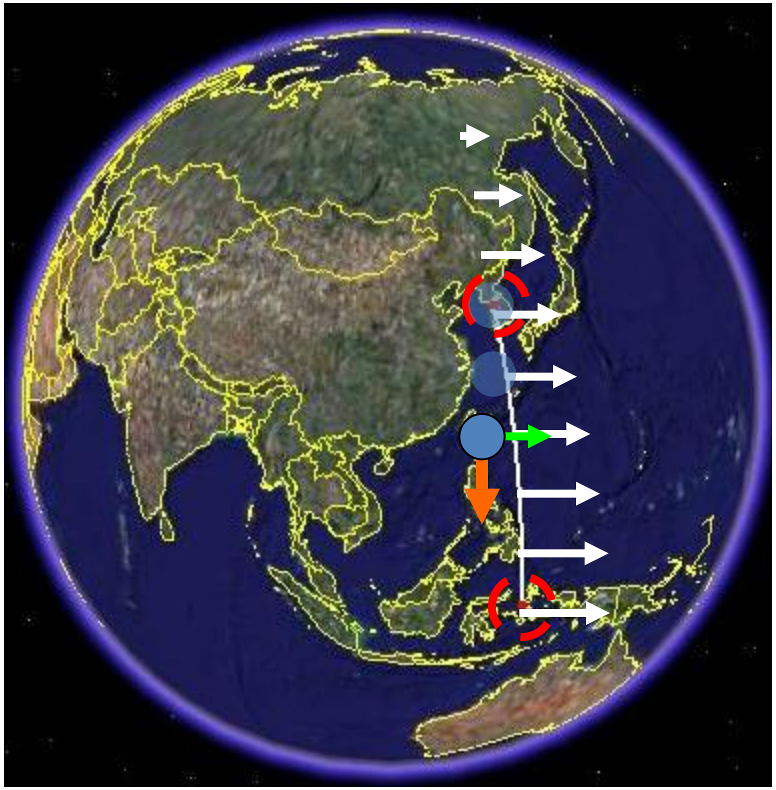
↓ initial velocity

→ initial velocity caused by the rotation of Earth

Coriolis effect (Northern Hemisphere)

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

velocity caused by the rotation of Earth



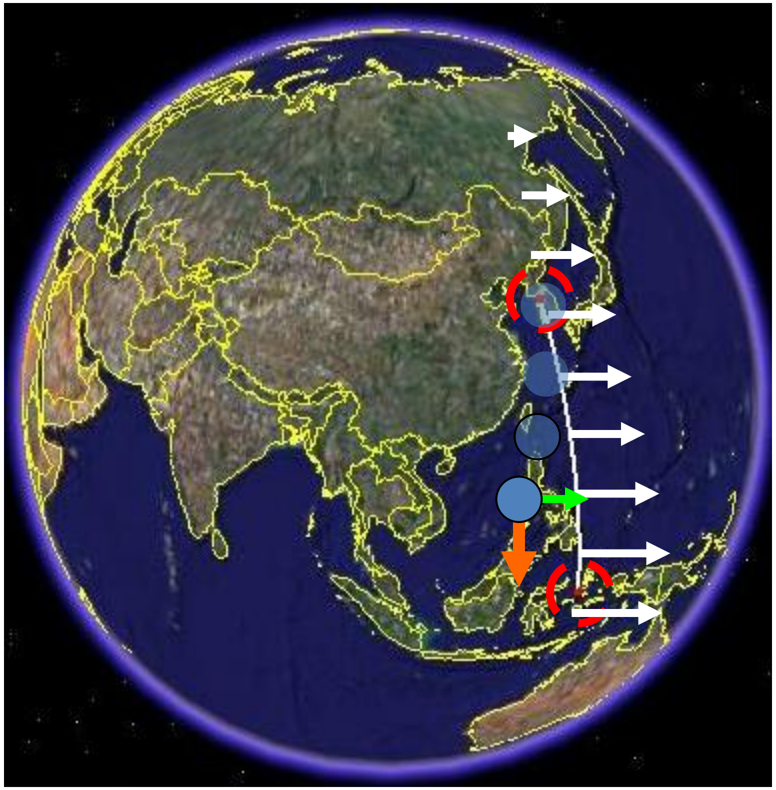
↓ initial velocity

→ initial velocity caused by the rotation of Earth

Coriolis effect (Northern Hemisphere)

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

velocity caused by the rotation of Earth



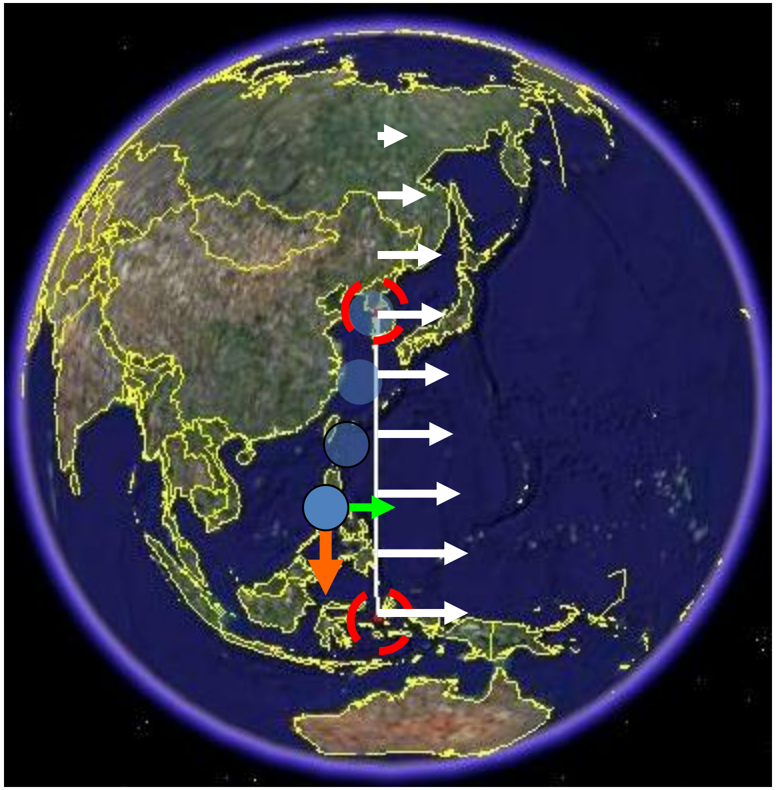
↓ initial velocity

→ initial velocity caused by the rotation of Earth

Coriolis effect (Northern Hemisphere)

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

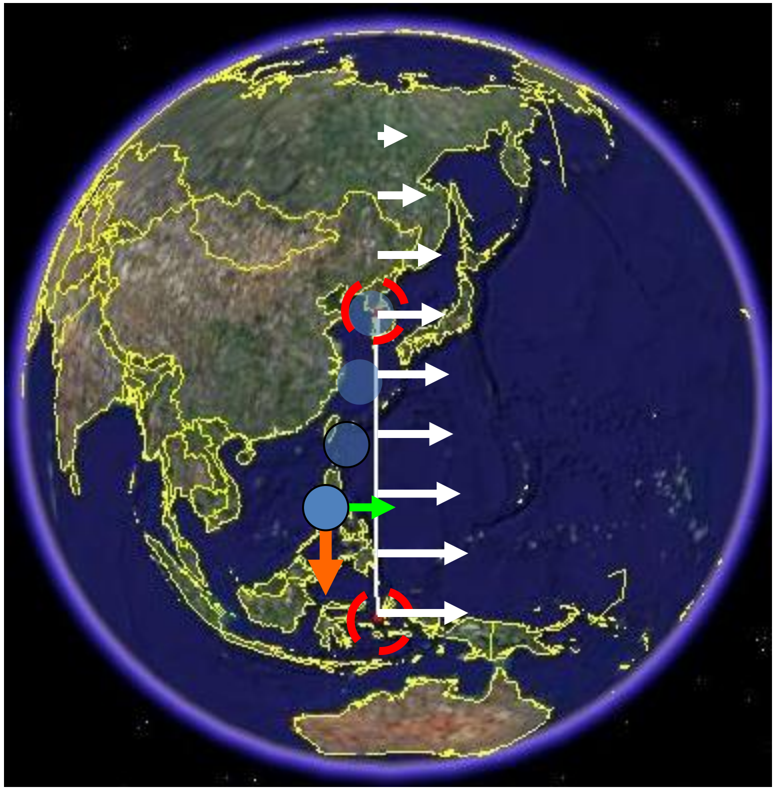
velocity caused by the rotation of Earth



↓ initial velocity

→ initial velocity caused by the rotation of Earth

Coriolis effect (Northern Hemisphere)



target에 정확히 물체를 떨어뜨리려면 Coriolis effect를 고려해야 한다.

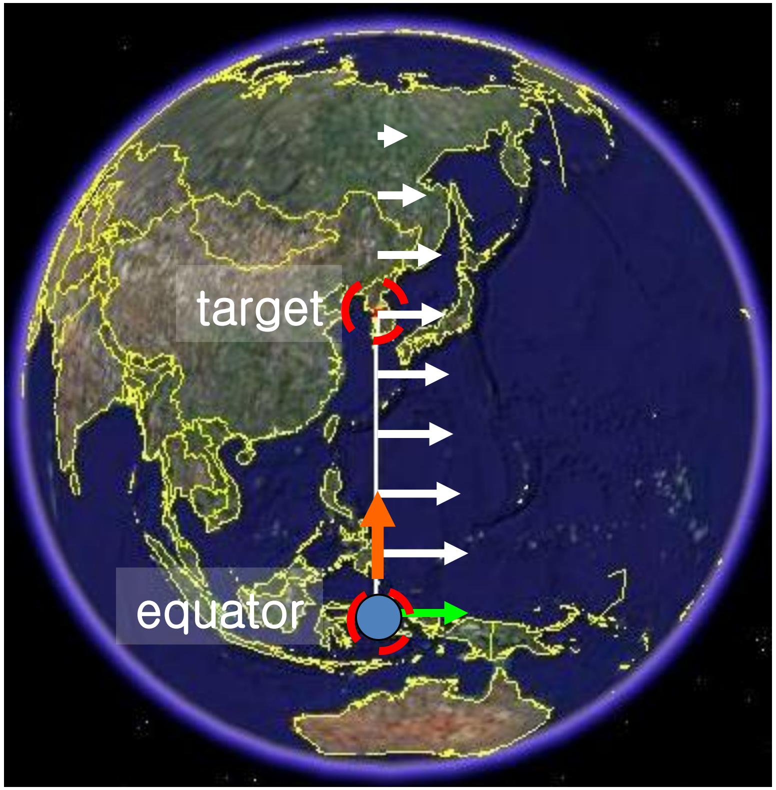
자오선을 따라서 target에 정확하게 물체를 떨어뜨리려면, 남쪽보다 오른쪽으로 더 기울여서 물체를 던져야 한다.

아니면, 계속해서 지구가 회전하는 방향(하얀 화살표)과 동일한 방향으로 가속해야 한다.

Coriolis effect

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

velocity caused by the rotation of Earth



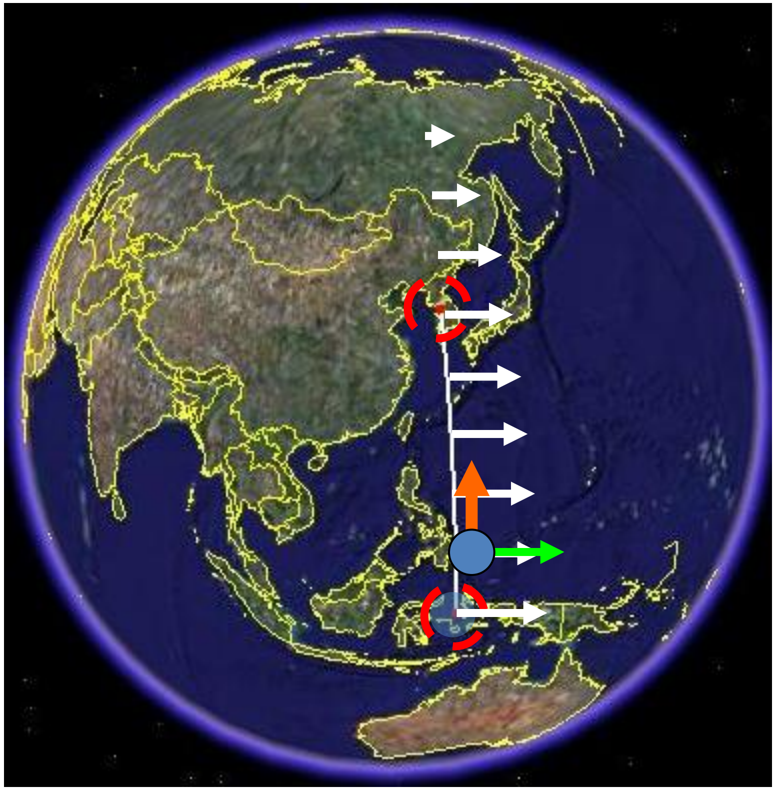
↑ initial velocity

→ initial velocity caused by the rotation of Earth

Coriolis effect

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

velocity caused by the rotation of Earth



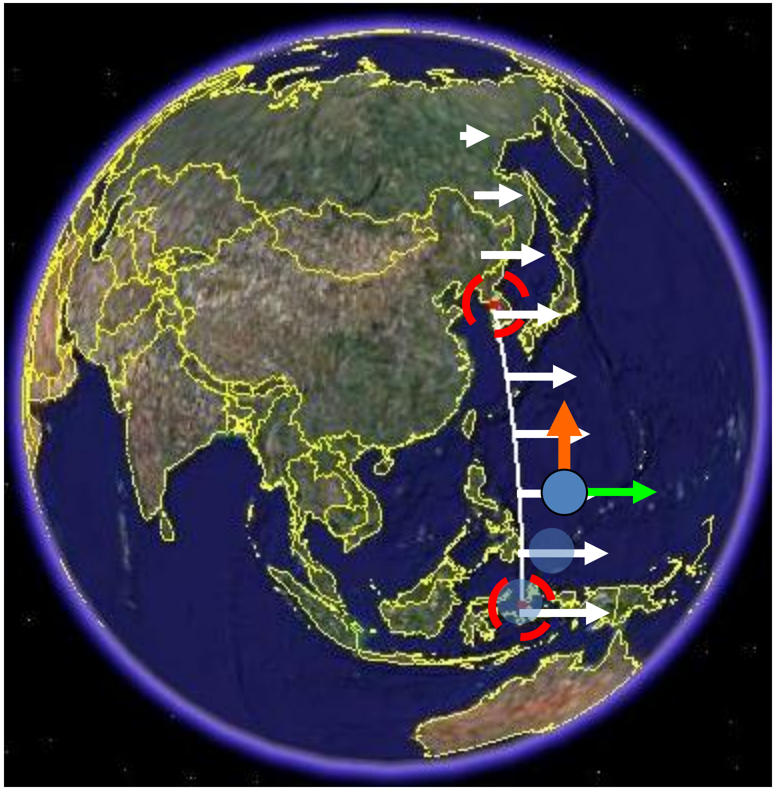
🡅 initial velocity

➔ initial velocity caused by the rotation of Earth

Coriolis effect

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

velocity caused by the rotation of Earth



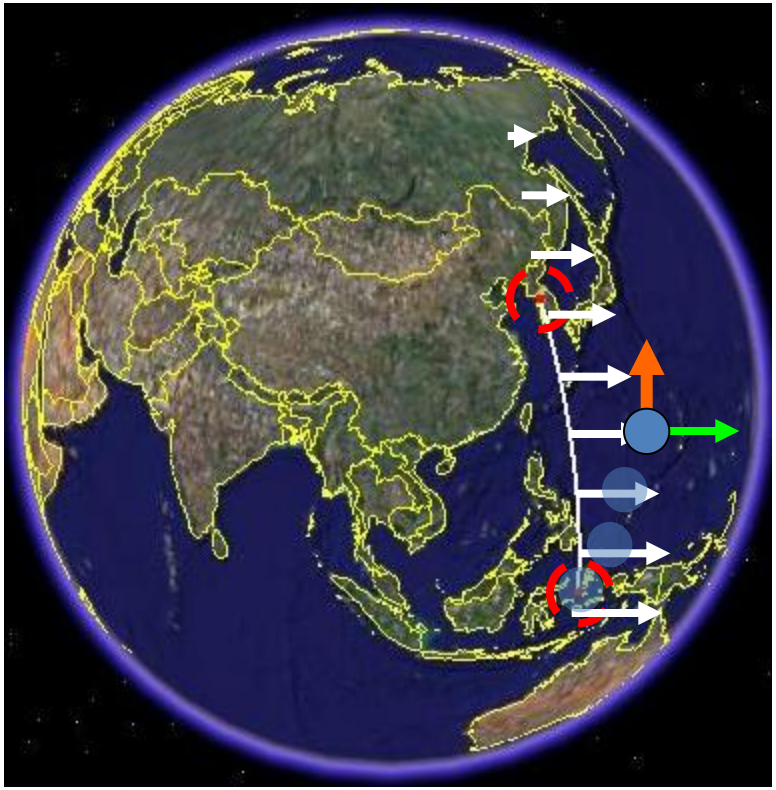
↑ initial velocity

→ initial velocity caused by the rotation of Earth

Coriolis effect

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

velocity caused by the rotation of Earth



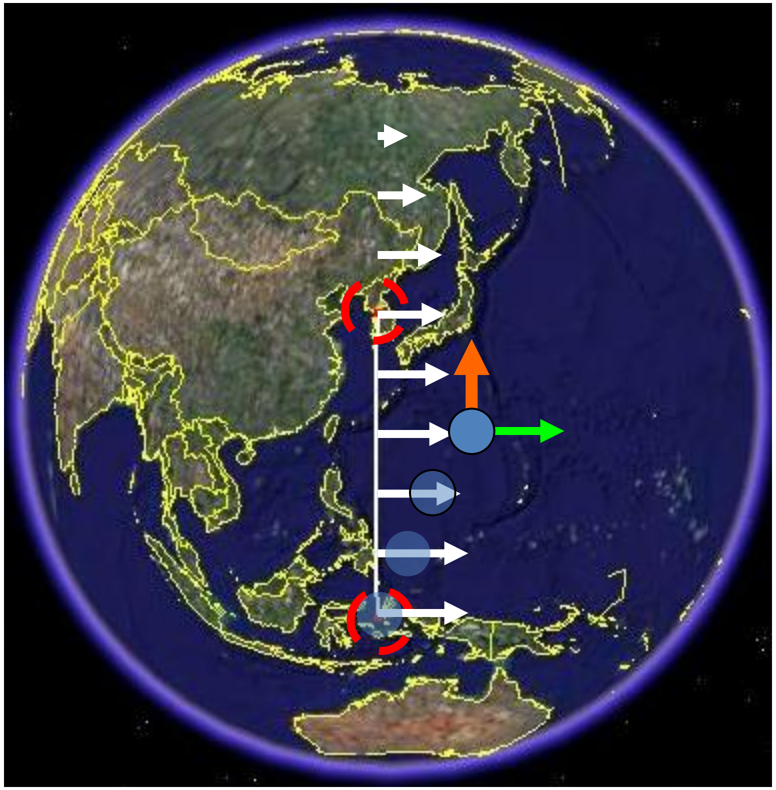
↑ initial velocity

→ initial velocity caused by the rotation of Earth

Coriolis effect

⇨ : $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

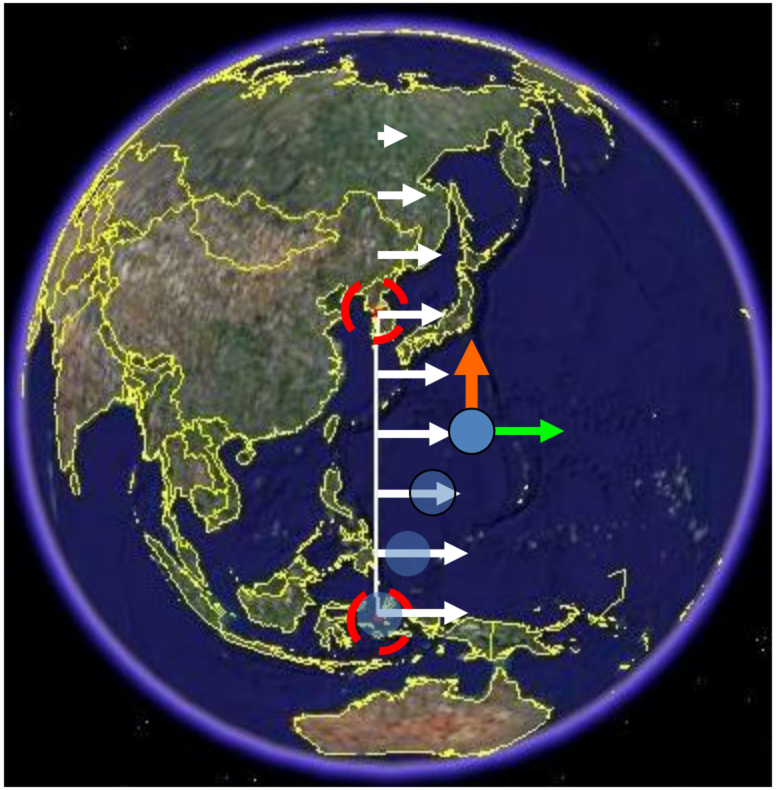
velocity caused by the rotation of Earth



↑ initial velocity

→ initial velocity caused by the rotation of Earth

Coriolis effect



자오선을 따라서 target에 정확하게 물체를 떨어뜨리려면, 북서쪽 기울여서 물체를 던져야 한다.

아니면, 계속해서 지구가 회전하는 방향(하얀 화살표)과 반대방향으로 가속해야 한다. (Example 9.5-8)

Superposition of Rotation. Coriolis Acceleration

$$\text{circle}: \mathbf{r}(t) = R\cos t \cdot \mathbf{i} + R\sin t \cdot \mathbf{j}$$

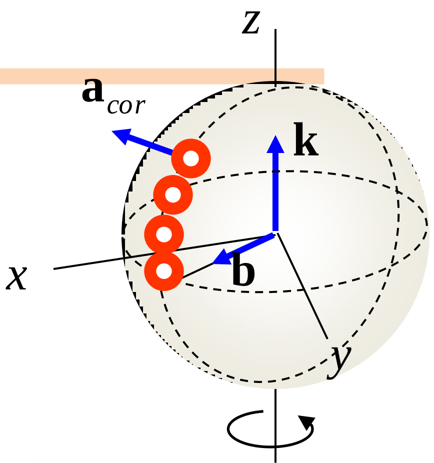


Fig. 209. Superposition of two rotations.

Question) A projectile is moving with constant speed (angular velocity γ) along a meridian (자오선, 경선) of the rotating earth in Fig. 209. Find its acceleration.

Solution) $\mathbf{r}(t) = R\cos\gamma t \cdot \mathbf{b}(t) + R\sin\gamma t\mathbf{k}$

$$\mathbf{b}(t) = \cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}$$

$$\mathbf{b}'(t) = -\omega\sin\omega t\mathbf{i} + \omega\cos\omega t\mathbf{j}$$

$$\mathbf{b}(t)\bullet\mathbf{b}'(t) = (\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j})\bullet(-\omega\sin\omega t\mathbf{i} + \omega\cos\omega t\mathbf{j}) = 0$$

$$\therefore \mathbf{b}(t) \perp \mathbf{b}'(t)$$

$$\mathbf{b}''(t) = -\omega^2\cos\omega t\mathbf{i} - \omega^2\sin\omega t\mathbf{j} = -\omega^2\mathbf{b}(t)$$

Coriolis Acceleration

$$\mathbf{b}(t) = \cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}$$

$$\mathbf{b}(t) \perp \mathbf{b}'(t),\ \ \mathbf{b}''(t) = -\omega^2\mathbf{b}(t)$$

$$\mathbf{r}(t) = R\cos\gamma t \cdot \mathbf{b}(t) + R\sin\gamma t\mathbf{k}$$

Velocity vector

$$\mathbf{v}(t) = \mathbf{r}'(t) = -\gamma R\sin\gamma t\mathbf{b} + R\cos\gamma t\mathbf{b}' + \gamma R\cos\gamma t\mathbf{k}$$

Acceleration vector

$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) &= -\left(\gamma^2 R\cos\gamma t\mathbf{b} + \gamma R\sin\gamma t\mathbf{b}'\right) \\ &\quad + \left(-\gamma R\sin\gamma t\mathbf{b}' + R\cos\gamma t\mathbf{b}''\right) - \gamma^2 R\sin\gamma t\mathbf{k} \\ &= R\cos\gamma t\mathbf{b}'' - 2\gamma R\sin\gamma t\mathbf{b}' - \gamma^2 R\cos\gamma t\mathbf{b} - \gamma^2 R\sin\gamma t\mathbf{k} \\ &= R\cos\gamma t\mathbf{b}'' - 2\gamma R\sin\gamma t\mathbf{b}' - \gamma^2 (R\cos\gamma t\mathbf{b} + R\sin\gamma t\mathbf{k}) \\ &= R\cos\gamma t\mathbf{b}'' - 2\gamma R\sin\gamma t\mathbf{b}' - \gamma^2\mathbf{r}\end{aligned}$$

Coriolis Acceleration

$$\mathbf{b}(t) = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}$$

$$\mathbf{b}(t) \perp \mathbf{b}'(t),\ \ \mathbf{b}''(t) = -\omega^2\mathbf{b}(t)$$

$$\mathbf{a}(t) = R\cos\gamma t\,\mathbf{b}'' - 2\gamma R\sin\gamma t\,\mathbf{b}' - \gamma^2\mathbf{r}$$

$R\cos\gamma t\,\mathbf{b}''$ **: Centripetal acceleration due to the rotation of the earth**
(지구가 회전을 함으로써 생기는 구심가속도**,** 지구 주위를 **projectile**이 도는 것을 고려하지 않음**.)**

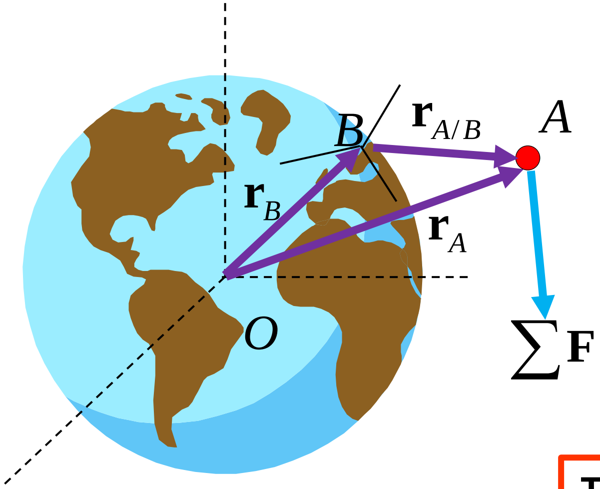
$-\gamma^2\mathbf{r}$ **: Centripetal acceleration due to the motion of the projectile on the meridian M of the rotating earth. (**지구 주위를 회전을 함으로써 생기는 구심가속도**,** 지구가 회전하는 것을 고려하지 않음**)**

$-2\gamma R\sin\gamma t\,\mathbf{b}'$ **: Coriolis acceleration due to the interaction of the two rotations.**

Earth rotates with the constant angular velocity

The earth rotates with the constant angular velocity ω.

Description of the acceleration of the point A form the inertial frame.



Newton's second law for the object A can be expressed in terms of the acceleration of A relative to the inertial reference frame:

$$\sum \mathbf{F} = m\mathbf{a}_A$$

The external force exerted on the point "A".
If the point "A" is in circular motion, the external force contains centripetal force.

The point "A" is accelerated in direction $\mathbf{a}_A$.

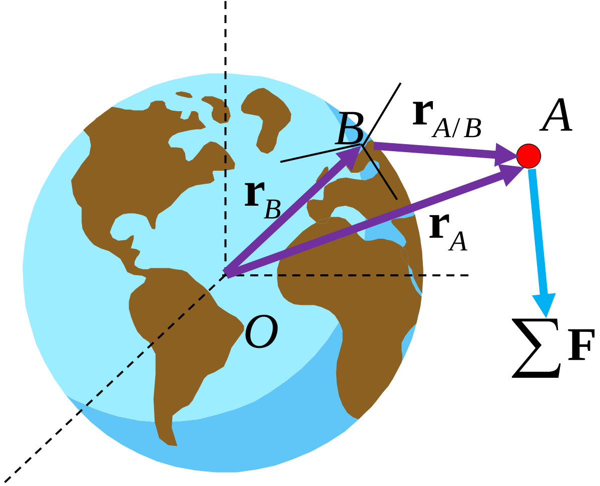
Example-The earth rotates with the constant angular velocity

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A,\,rel} + \boldsymbol{\omega}\times\mathbf{r}_{A/B}$$

$$\mathbf{a}_A = \mathbf{a}_B + \underbrace{\mathbf{a}_{A,\,rel} + 2\boldsymbol{\omega}\times\mathbf{v}_{A,\,rel} + \boldsymbol{\alpha}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/B})}_{\mathbf{a}_{A/B}}$$

The earth rotates with the constant angular velocity $\boldsymbol{\omega}$.

Description of the acceleration of the point A form the non-inertial frame, which is fixed on the earth.



Inertial frame $\sum\mathbf{F} = m\mathbf{a}_A$

Newton's second law also can be expressed in terms of the acceleration of A relative to the secondary reference frame whose origin is B:

$$\sum\mathbf{F} = m\left[\mathbf{a}_B + \mathbf{a}_{A,\,rel} + 2\boldsymbol{\omega}\times\mathbf{v}_{A,\,rel} + \boldsymbol{\alpha}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/B})\right]$$

$$\sum\mathbf{F} - m\cdot[\mathbf{a}_B + 2\boldsymbol{\omega}\times\mathbf{v}_{A,\,rel} + \cancel{\boldsymbol{\alpha}\times\mathbf{r}_{A/B}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/B})] = m\cdot\mathbf{a}_{A,\,rel}$$

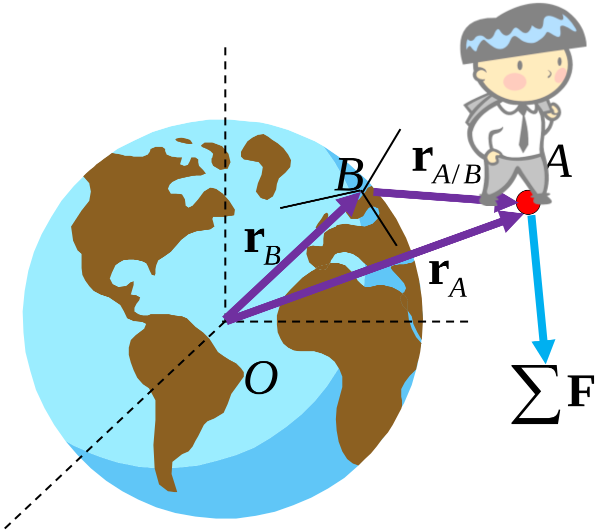
When Newton's second law is expressed in this way, "additional force" appear in the left side of the equation which are artifact arising from the motion of the secondary reference frame.

Example-The earth rotates with the constant angular velocity

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A,\,rel} + \boldsymbol{\omega}\times\mathbf{r}_{A/B}$$

$$\mathbf{a}_A = \mathbf{a}_B + \underbrace{\mathbf{a}_{A,\,rel} + 2\boldsymbol{\omega}\times\mathbf{v}_{A,\,rel} + \boldsymbol{\alpha}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/B})}_{\mathbf{a}_{A/B}}$$

The earth rotates with the constant angular velocity .



Inertial frame

$$\sum\mathbf{F} = m\mathbf{a}_A$$

Non-inertial frame(the origin is B)

$$\sum\mathbf{F} - m\cdot[\boxed{\mathbf{a}_B} + \boxed{2\boldsymbol{\omega}\times\mathbf{v}_{A,\,rel}} + \boxed{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/B})}] = m\cdot\mathbf{a}_{A,\,rel}$$

$-m\mathbf{a}_B$: If "A" is standing at the point "B" and observed at the point "B", the person "A" perceives additional force $-m\mathbf{a}_B$.

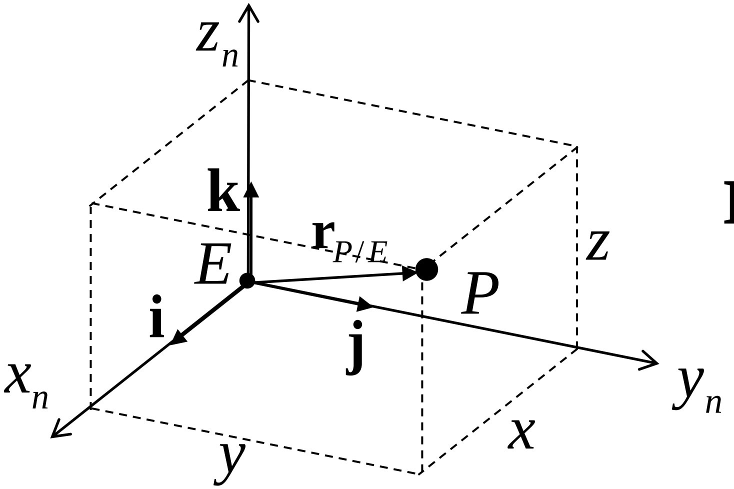
$-2m\boldsymbol{\omega}\times\mathbf{v}_{A,\,rel}$: If "A" in northern hemisphere that is moving tangent to the earth's surface travels north, certain force causes the person "A" to turn to the right. This force is Coriolis force due to the relative velocity and rotation of the frame.

$-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/B})$: If "A" lies on the earth's surface, point "B", and observed at the point "B", the person "A" rotates with the earth. The person "A" perceives the centripetal force and an additional force, "centrifugal force" $-m(r_{A/B}\omega^2)$.

1.5 Vector Decomposition

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, pp. 45

Vector Decomposition



$\mathbf{r}_{P/E}$

E → A start point of the position vector.

P → An end point of the position vector.

Vector is expressed in terms of unit vectors ~~of~~ directed along the coordinate~~s~~ axis

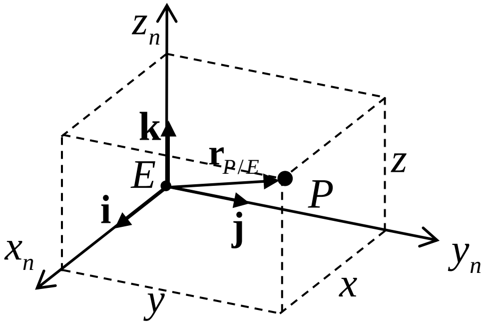
$$\mathbf{r}_{P/E} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \quad \Rightarrow \quad {}^{n}\mathbf{r}_{P/E} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

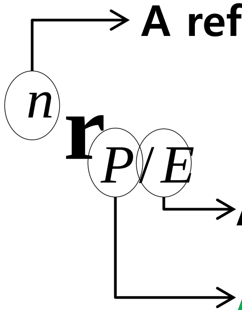
n → A reference frame where the vector is decomposed in.

Caution!!: The reference frame, where the vector is decomposed ~~in~~, does
(Check !!) not have any physical meaning. Only $\mathbf{r}_{P/E}$ has physical meaning

Jerry Ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, pp. 45

Vector Decomposition





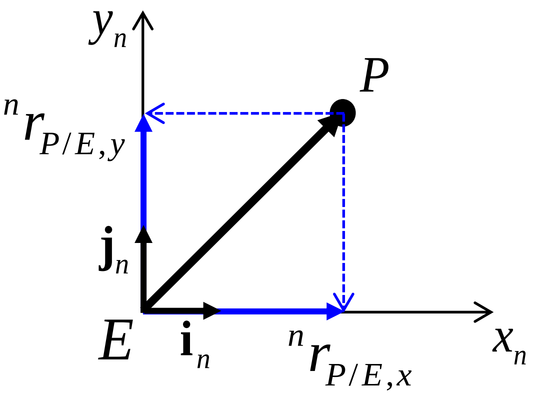
$${}^{n}\mathbf{r}_{P/E}$$

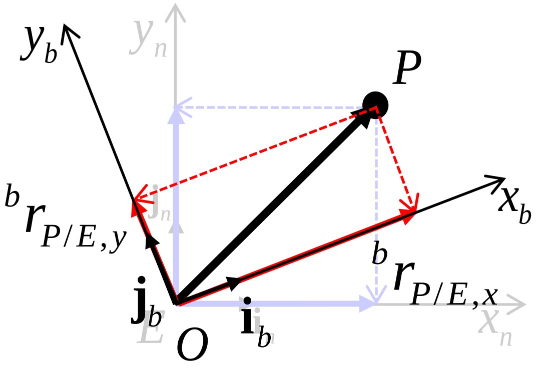
A reference frame where the vector is decomposed in.

A start point of the position vector.

An end point of the position vector.

Same vector may have different components with respect to the reference frame where the vector is decomposed in.





$${}^{\textcircled{n}}\mathbf{r}_{P/E} = {}^{n}r_{P/E,x}\mathbf{i}_n + {}^{n}r_{P/E,y}\mathbf{j}_n$$

$$= \begin{bmatrix} {}^{n}r_{P/E,x} \\ {}^{n}r_{P/E,y} \end{bmatrix}$$

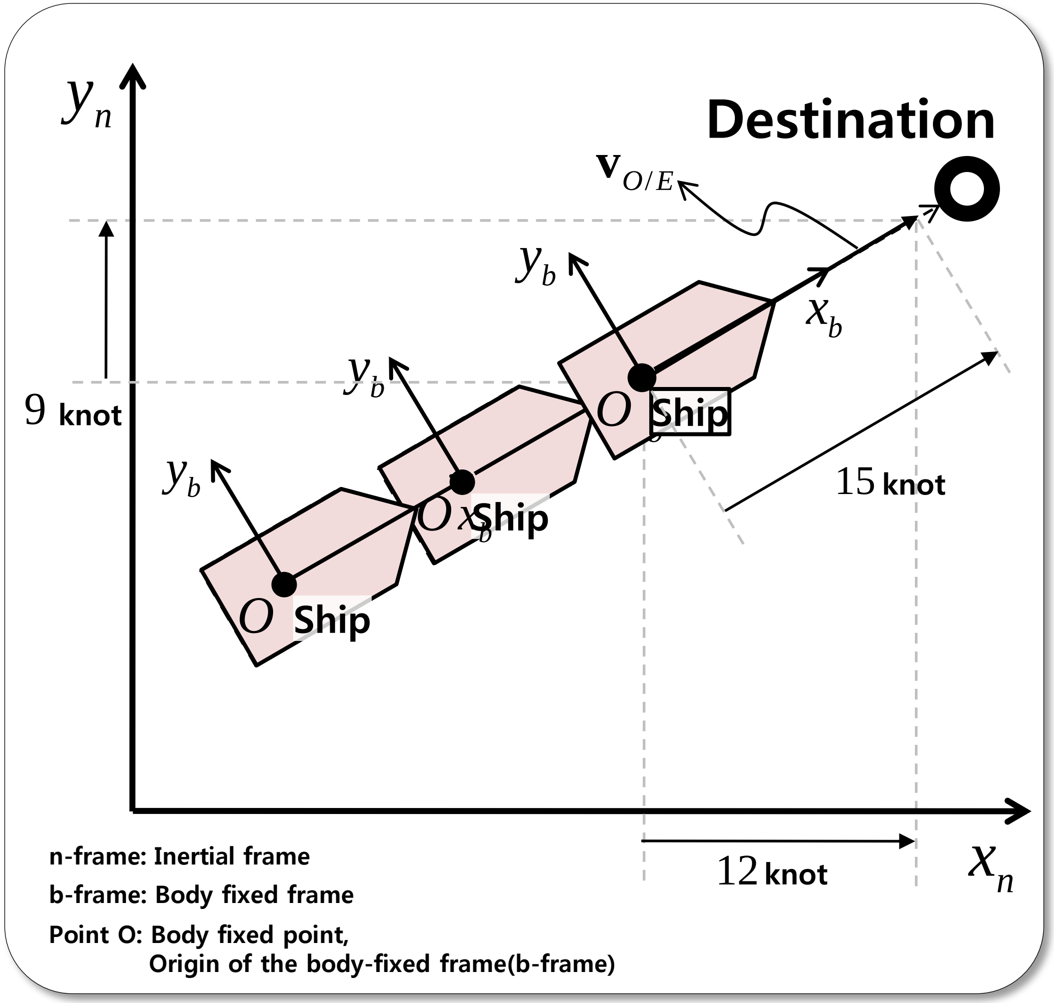
$${}^{\textcircled{b}}\mathbf{r}_{P/O} =^{b} r_{P/O,x}\mathbf{i}_b +^{b} r_{P/O,y}\mathbf{j}_b$$

$$= \begin{bmatrix} {}^{b}r_{P/O,x} \\ {}^{b}r_{P/O,y} \end{bmatrix}$$

Same vector

Different components

Vector Decomposition
- Example: Velocity vector decomposition



n-frame: Inertial frame

b-frame: Body fixed frame

Point O: Body fixed point,
Origin of the body-fixed frame(b-frame)

$\mathbf{v}_{O/E}$: Linear velocity of the **point O** with respect to **n-frame**$^{1)}$

Physical meaning

Component representation
(Vector decomposition)

$^{n}\mathbf{v}_{O/E}$: Linear velocity of the point O with respect to n-frame

decomposed in **n-frame**$^{2)}$

$$^{n}\mathbf{v}_{O/E} = [12 \quad 9]^T$$

$^{b}\mathbf{v}_{O/E}$: Linear velocity of the point O with respect to n-frame

decomposed in **b-frame**

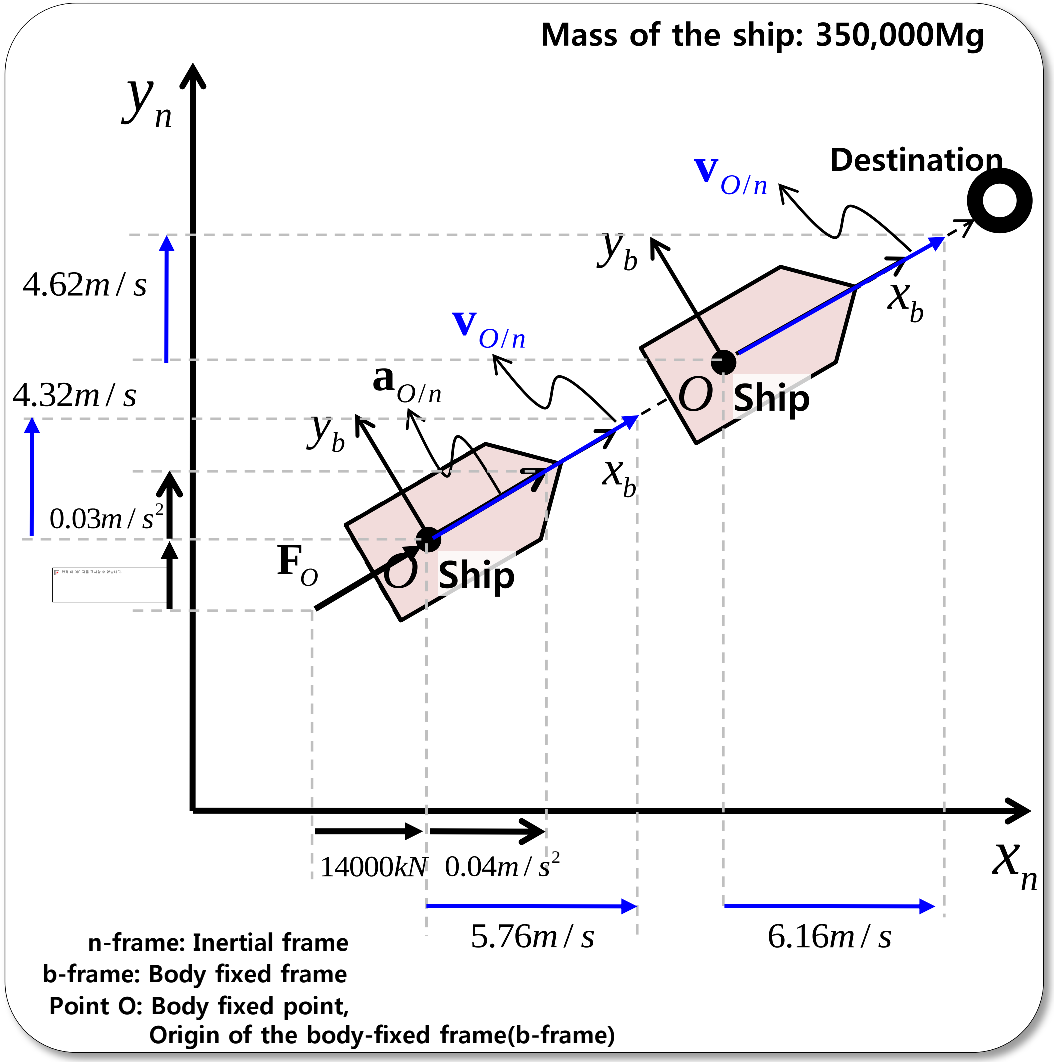
$$^{b}\mathbf{v}_{O/E} = [15 \quad 0]^T$$

1) c.f) Since an observer on the b-frame moves together with the ship, the linear velocity of the point O with respect to the b-frame $\mathbf{v}_{O/b}$ is always zero, which is trivial. That's why we consider the velocity with respect to the n-frame

2) The inertial frame is needed for global guidance, navigation and control e.g. to describe the motion and location of ship in transit between different continents(Fossen, p.19)
Since direction of hydrostatic force is invariant with respect to inertial frame, it is convenient to chose the inertial frame as a reference frame which the force vector is decomposed in.

Vector Decomposition

- Example: Velocity vector decomposition



n-frame: Inertial frame
b-frame: Body fixed frame
Point O: Body fixed point, Origin of the body-fixed frame(b-frame)

$\mathbf{F}_O = m\mathbf{a}_{O/n}$ **: Force Equation**

Component representation
(Vector decomposition)

$$^n\mathbf{F}_O = m\ {}^n\mathbf{a}_{O/n} \quad \cdots ①$$

: Force Equation
decomposed in n-frame

Given: $^n\mathbf{F}_O = \begin{bmatrix} 14000 \\ 10500 \end{bmatrix} kN, m = 350,000Mg, {}^n\mathbf{v}_{O/n}\Big|_{t=0} = \begin{bmatrix} 5.76 \\ 4.32 \end{bmatrix} m/s$

Find: $^n\mathbf{v}_{O/n}\Big|_{t=10}$

From equation ① $^n\mathbf{a}_{O/n} = {}^n\mathbf{F}_O / m$

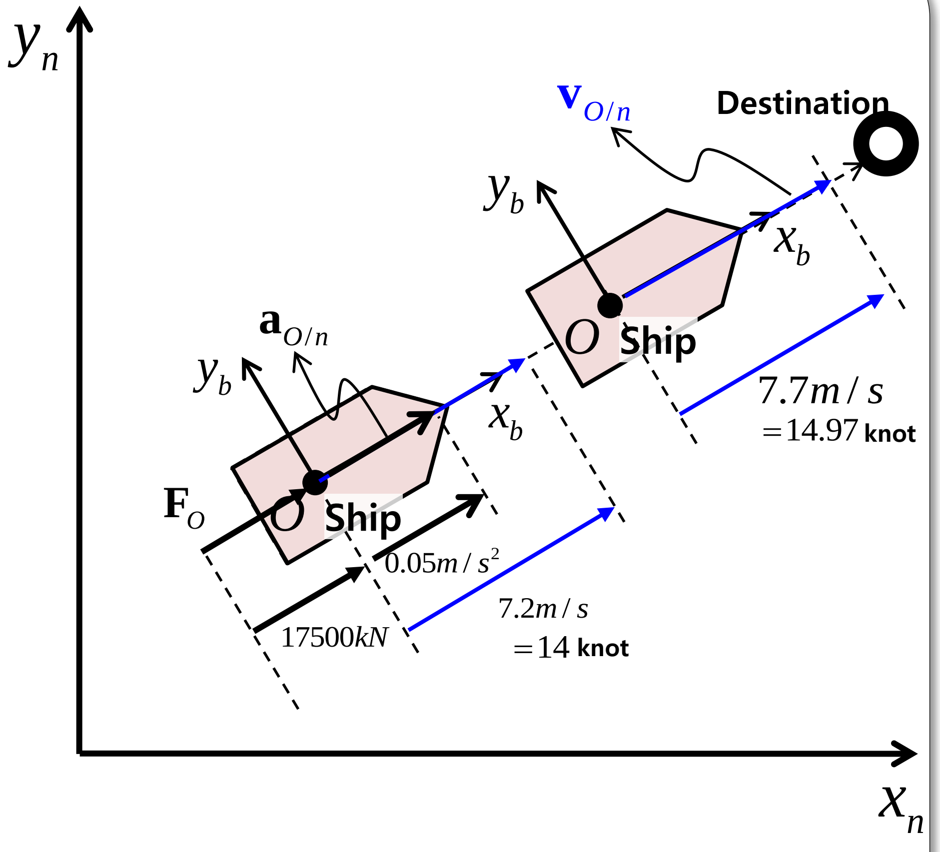
$$= \begin{bmatrix} 14000 \\ 10500 \end{bmatrix} \cdot \frac{1}{350000} = \begin{bmatrix} 0.04 \\ 0.03 \end{bmatrix}$$

$$^n\mathbf{v}_{O/n}\Big|_{t=10} = {}^n\mathbf{v}_{O/n}\Big|_{t=0} + 10 \times {}^n\mathbf{a}_{O/n}$$

$$= \begin{bmatrix} 5.76 \\ 4.32 \end{bmatrix} + 10 \begin{bmatrix} 0.04 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 6.16 \\ 4.62 \end{bmatrix}$$

Vector Decomposition
- Example: Velocity vector decomposition

Mass of the ship: 350,000Mg



n-frame: Inertial frame
b-frame: Body fixed frame
Point O: Body fixed point,
Origin of the body-fixed frame(b-frame)

$$\mathbf{F}_O = m\mathbf{a}_{O/n}$$ **: Force Equation**

Component representation
(Vector decomposition)

$$^{b}\mathbf{F}_O = m\,^{b}\mathbf{a}_{O/n} \quad \cdots ②$$

: Force Equation
decomposed in b-frame

Given: $^{b}\mathbf{F}_O = \begin{bmatrix} 17500 \\ 0 \end{bmatrix} kN\,, m = 350{,}000Mg\,, \,^{b}\mathbf{v}_{O/n}\Big|_{t=0} = \begin{bmatrix} 7.2 \\ 0 \end{bmatrix} m/s$

Find: $^{b}\mathbf{v}_{O/n}\Big|_{t=10}$

⇨ **From equation ②** $^{b}\mathbf{a}_{O/n} = \,^{b}\mathbf{F}_O / m$

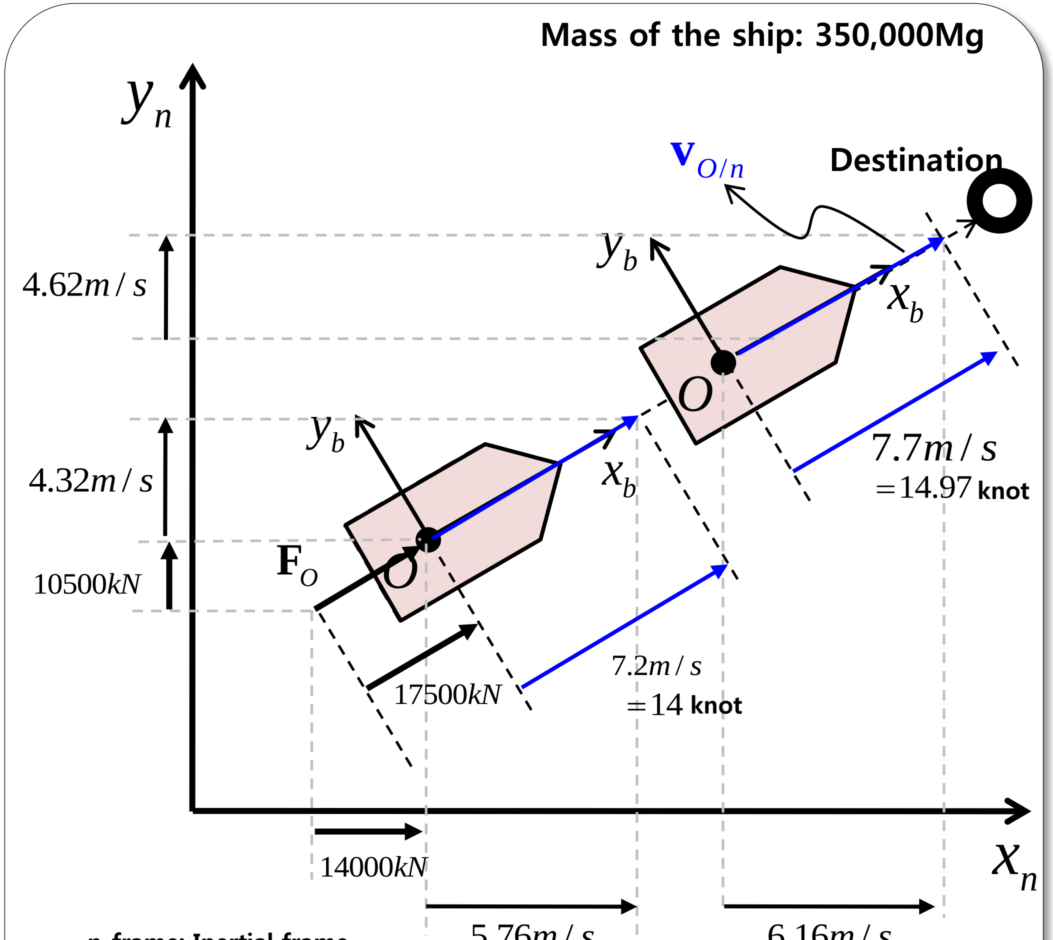
$$= \begin{bmatrix} 17500 \\ 0 \end{bmatrix} \cdot \frac{1}{350000} = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}$$

$$^{b}\mathbf{v}_{O/n}\Big|_{t=10} = \,^{b}\mathbf{v}_{O/n}\Big|_{t=0} + 10 \times \,^{b}\mathbf{a}_{O/n}$$

$$= \begin{bmatrix} 7.2 \\ 0 \end{bmatrix} + 10 \begin{bmatrix} 0.05 \\ 0 \end{bmatrix} = \begin{bmatrix} 7.7 \\ 0 \end{bmatrix}$$

Vector Decomposition

- Example: Velocity vector decomposition



n-frame: Inertial frame
b-frame: Body fixed frame
Point O: Body fixed point,
Origin of the body-fixed frame(b-frame)

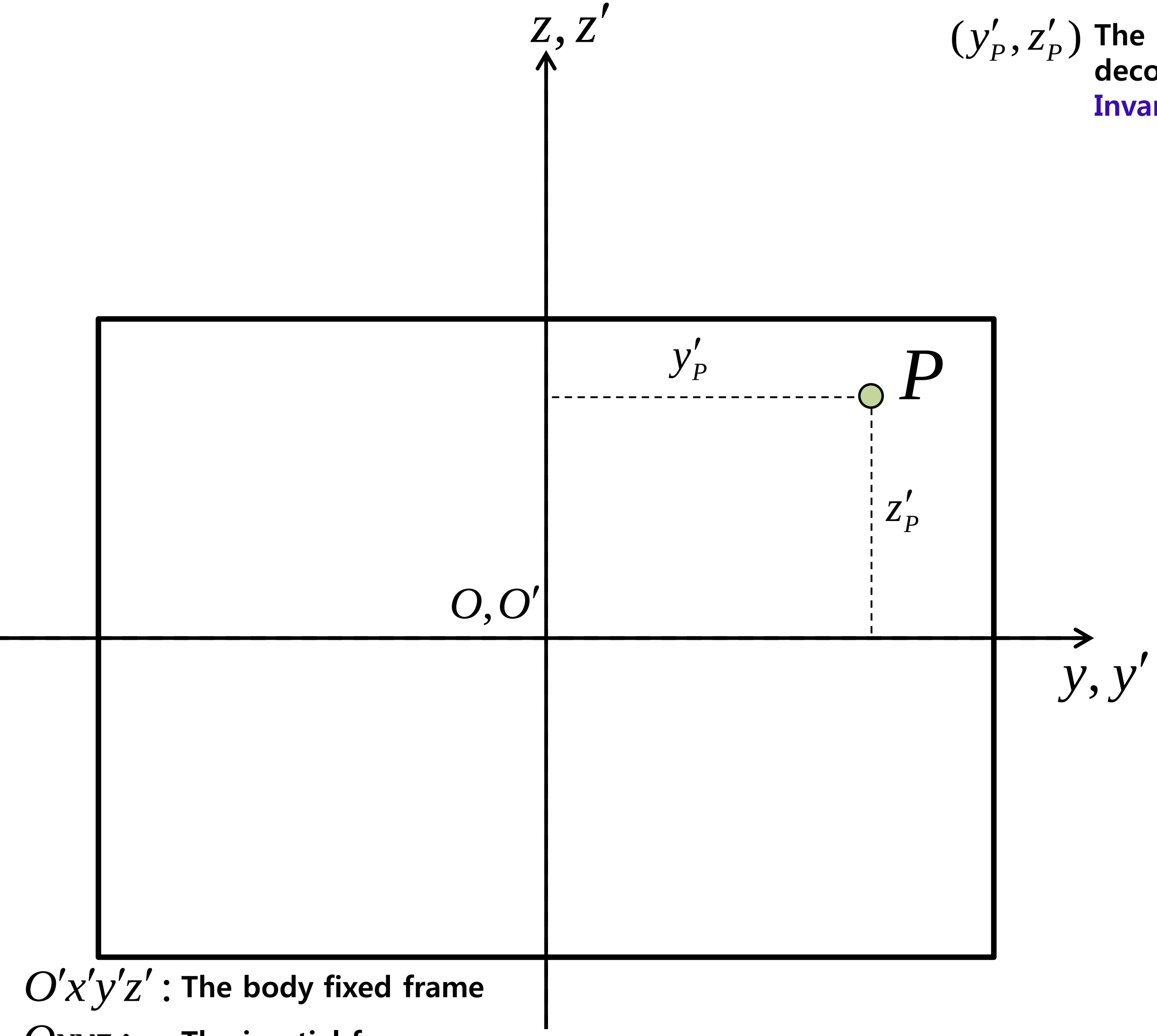
$\mathbf{F}_O = m\mathbf{a}_{O/n}$: Force Equation

Component representation
(Vector decomposition)

	${}^b\mathbf{F}_O = m\,{}^b\mathbf{a}_{O/n}$: Force Equation decomposed in b-frame	${}^n\mathbf{F}_O = m\,{}^n\mathbf{a}_{O/n}$: Force Equation decomposed in n-frame
Given	${}^b\mathbf{F}_O = \begin{bmatrix}17500\\0\end{bmatrix} kN$ ⇔ Same vector ${}^b\mathbf{v}_{O/n}\big\|_{t=0} = \begin{bmatrix}7.2\\0\end{bmatrix} m/$ ⇔ Same vector	${}^n\mathbf{F}_O = \begin{bmatrix}14000\\10500\end{bmatrix} kN$ ${}^n\mathbf{v}_{O/n}\big\|_{t=0} = \begin{bmatrix}5.76\\4.32\end{bmatrix} m/s$
Find	${}^b\mathbf{v}_{O/n}\big\|_{t=10} = \begin{bmatrix}7.7\\0\end{bmatrix}$ ⇔ Same vector	${}^n\mathbf{v}_{O/n}\big\|_{t=10} = \begin{bmatrix}6.16\\4.62\end{bmatrix}$

1.6 Coordinate Transformation

Representation of a Point "P" on an object with respect to the body fixed frame (decomposed in the body fixed frame)



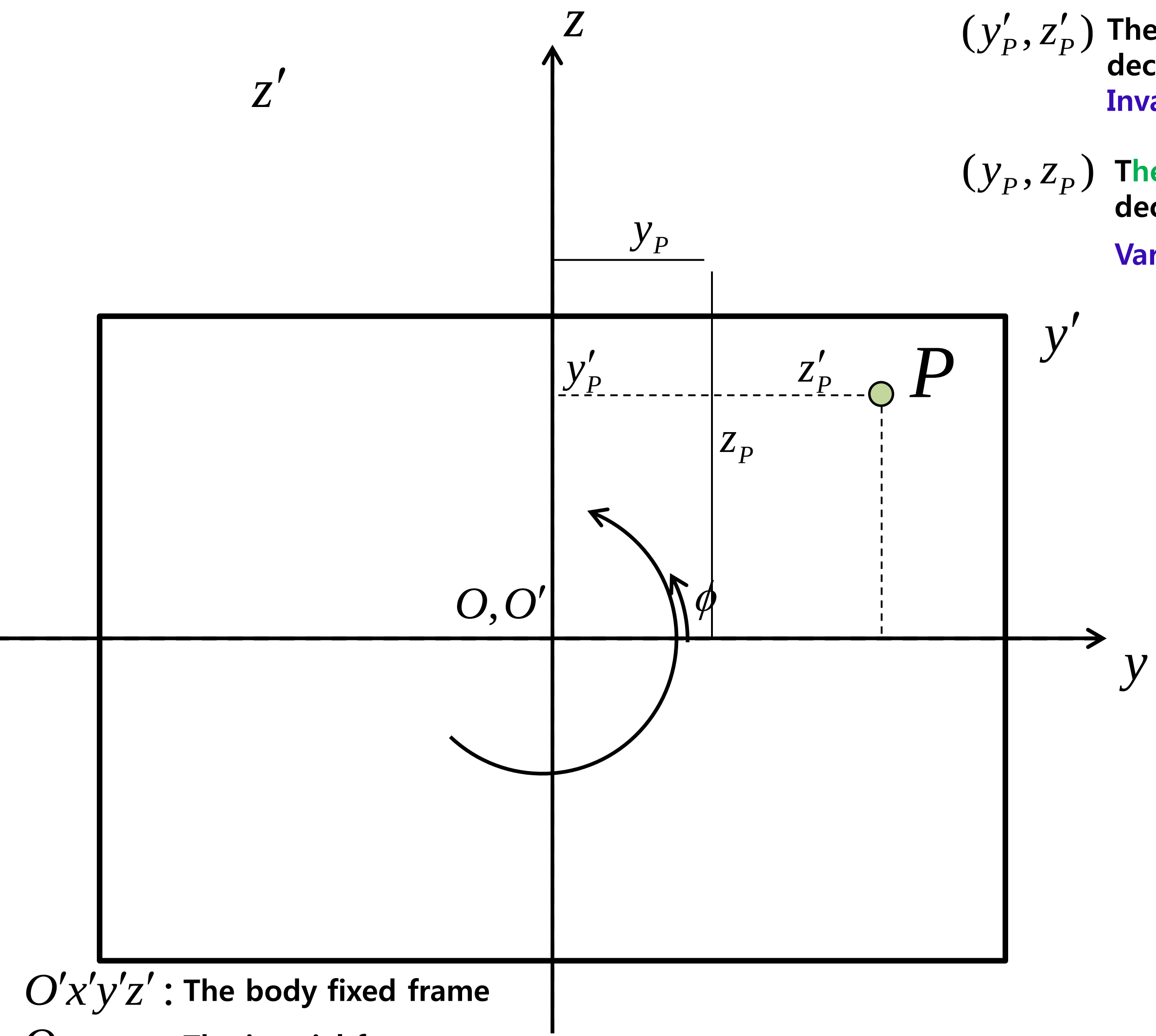
(y'_P, z'_P) **The Position vector of the point P decomposed in the body fixed frame**

Invariant with respect to the body fixed frame

$O'x'y'z'$: **The body fixed frame**

$Oxyz$: **The inertial frame**

Rotate the object with an angle of ϕ and then represent the point “P” on the object with respect to the inertial frame



(y'_P, z'_P) The Position vector of the point P decomposed in the body fixed frame
Invariant with respect to the body fixed frame

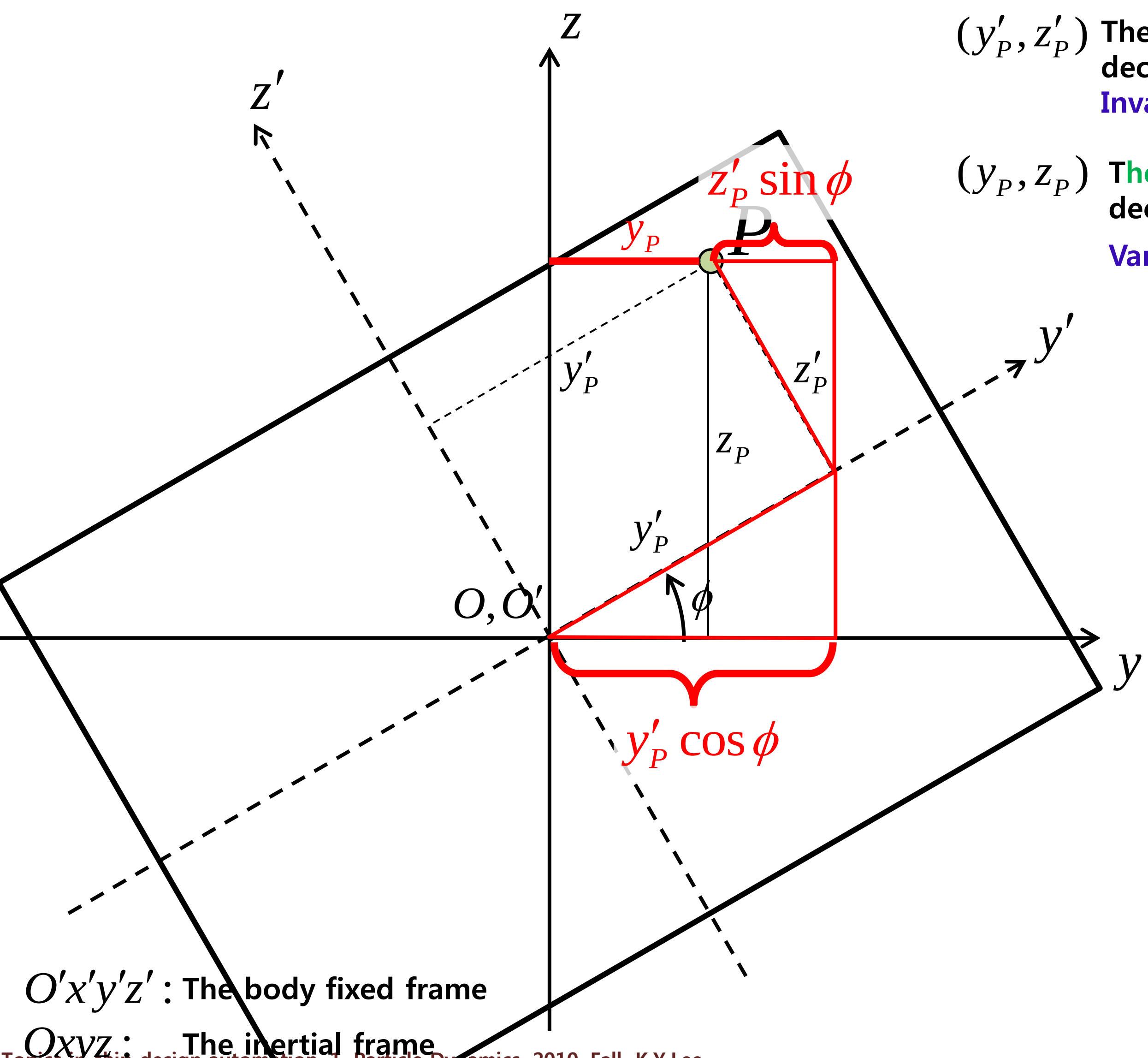
(y_P, z_P) The Position vector of the point P decomposed in the initial frame

Variant with respect to the inertial frame.

$O'x'y'z'$: The body fixed frame

$Oxyz$: The inertial frame

Coordinate Transformation of a Position Vector



(y'_P, z'_P) The Position vector of the point P decomposed in the body fixed frame

Invariant with respect to the body fixed frame

(y_P, z_P) The Position vector of the point P decomposed in the initial frame

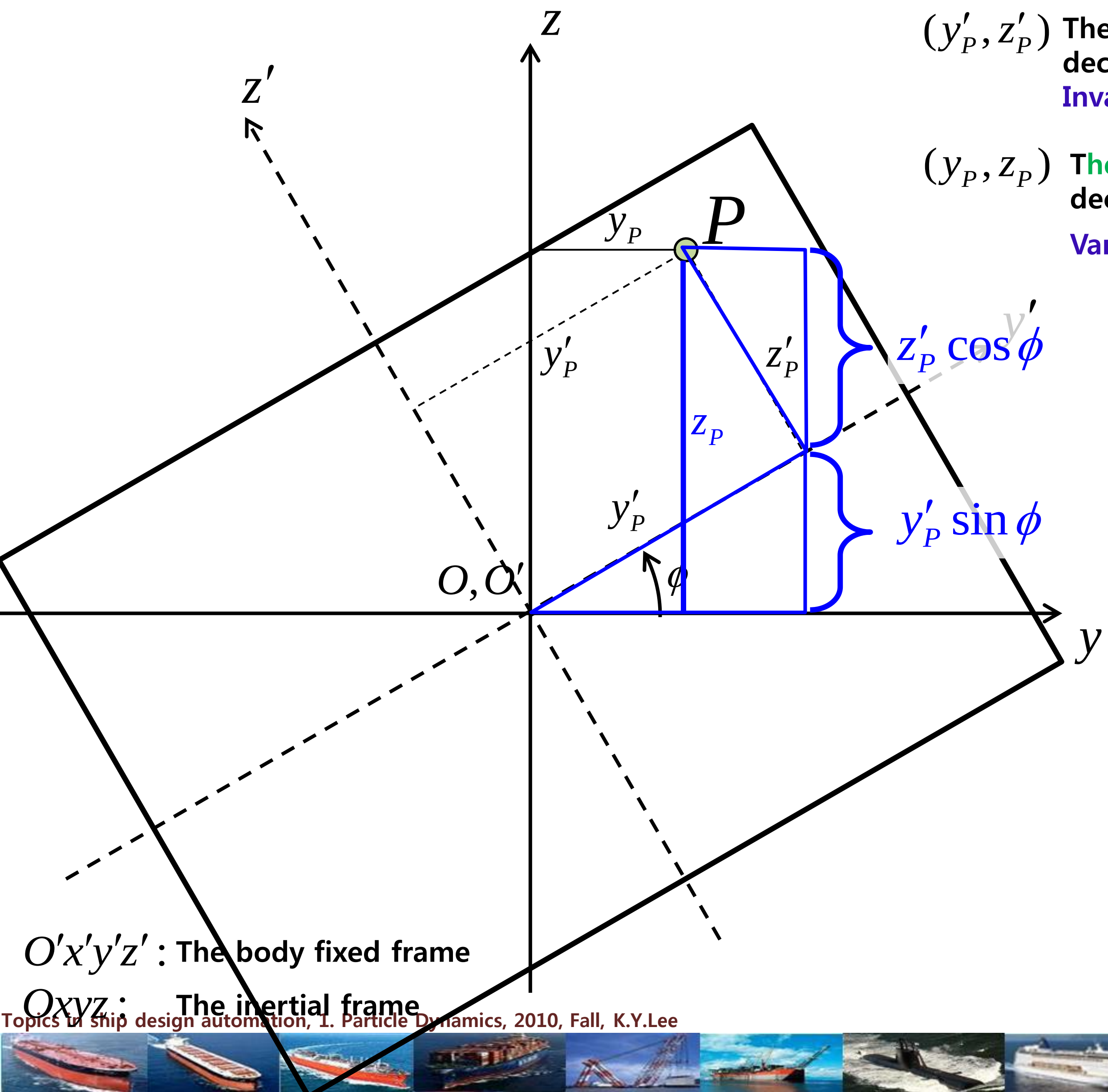
Variant with respect to the inertial frame.

$$y_P = y'_P \cos\phi - z'_P \sin\phi$$

$O'x'y'z'$: The body fixed frame

$Oxyz$: The inertial frame

Coordinate Transformation of a Position Vector



(y'_P, z'_P) The Position vector of the point P decomposed in the body fixed frame
Invariant with respect to the body fixed frame

(y_P, z_P) The Position vector of the point P decomposed in the initial frame
Variant with respect to the inertial frame.

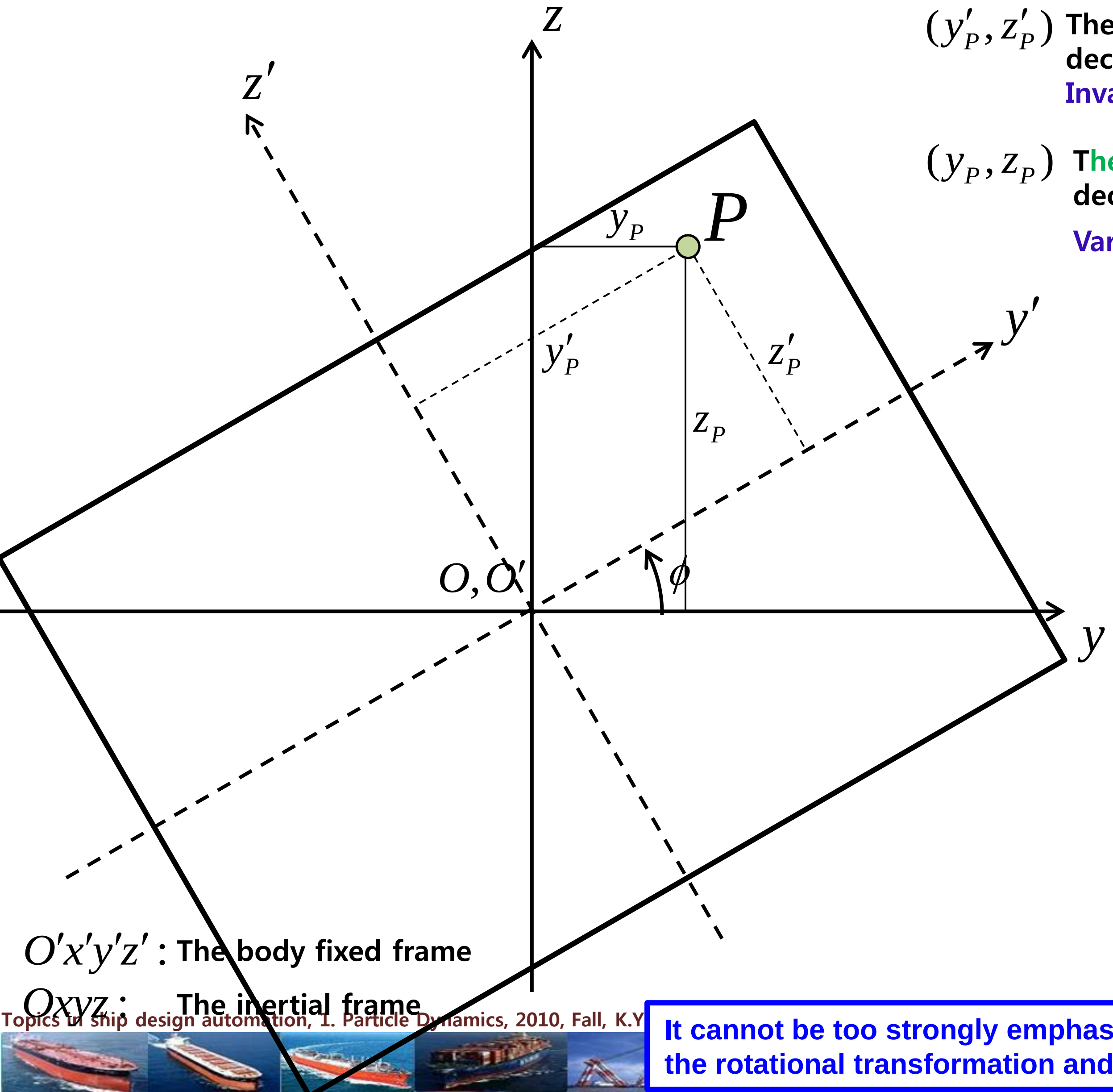
$$y_P = y'_P \cos\phi - z'_P \sin\phi$$

$$z_P = y'_P \sin\phi + z'_P \cos\phi$$

$O'x'y'z'$: The body fixed frame

$Oxyz$: The inertial frame

Coordinate Transformation of a Position Vector



$O'x'y'z'$ **: The body fixed frame**

$Oxyz$ **: The inertial frame**

(y'_P, z'_P) **The Position vector of the point P decomposed in the body fixed frame**

Invariant with respect to the body fixed frame

(y_P, z_P) **The Position vector of the point P decomposed in the initial frame**

Variant with respect to the inertial frame.

$$y_P = y'_P \cos\phi - z'_P \sin\phi$$

$$z_P = y'_P \sin\phi + z'_P \cos\phi$$

$\Downarrow$ **Matrix Form**

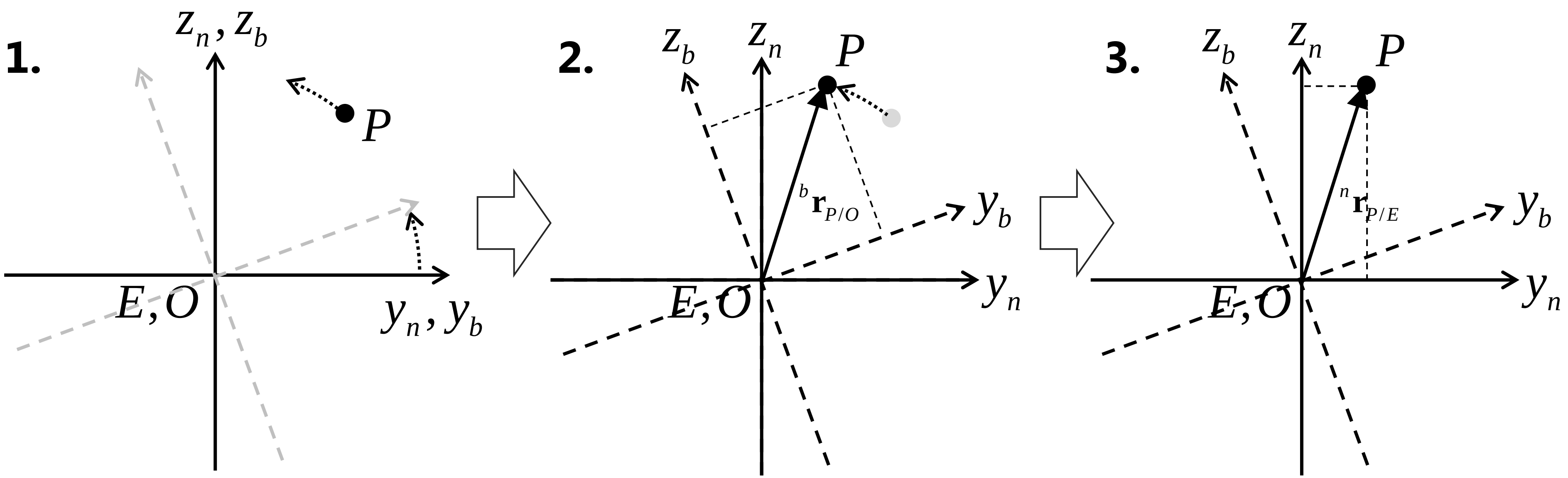
$$\begin{bmatrix} y_P \\ z_P \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y'_P \\ z'_P \end{bmatrix}$$

$${}^{n}\mathbf{r}_P = {}^{n}\mathbf{R}_b \, {}^{b}\mathbf{r}_P$$

It cannot be too strongly emphasized that the rotational transformation and the coordinate transformation are importan

Coordinate Transformation: Forward problem

$$ {}^{n}\mathbf{r}_{P/E} = {}^{n}\mathbf{R}_{b}\, {}^{b}\mathbf{r}_{P/O} $$



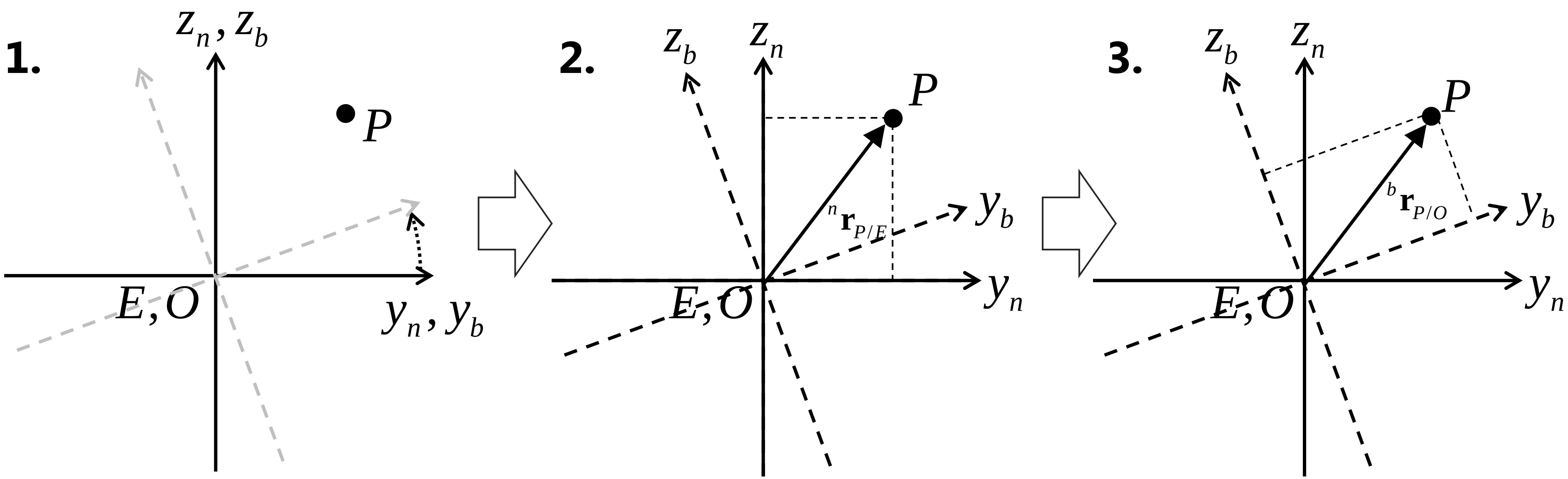
1. 문제정의: 점 **P**가 **b-frame**과 함께 회전하는 경우

2. 점 **P**가 **b-frame**과 함께 회전하였으므로, 알고 있는 벡터는
b-frame에서 기술한 점 **P**의 위치벡터 ${}^{b}\mathbf{r}_{P/O}$

3. 최종적으로 구하고자 하는 벡터는
n-frame에서 기술한 점 **P**의 위치벡터 ${}^{n}\mathbf{r}_{P/E}$

Coordinate Transformation: Inverse problem

$$^{n}\mathbf{r}_{P/E} = {}^{n}\mathbf{R}_{b}\,{}^{b}\mathbf{r}_{P/O}$$



1. **문제정의: 점 P는 n-frame과 함께 고정되어 있고 b-frame만 회전하는 경우**

2. 점 **P**가 **n-frame**과 함께 고정되어 있으므로, 알고 있는 벡터는
   n-frame에서 기술한 점 **P**의 위치벡터 $^{n}\mathbf{r}_{P/E}$

3. 최종적으로 구하고자 하는 벡터는
   b-frame에서 기술한 점 **P**의 위치벡터 $^{b}\mathbf{r}_{P/O}$

1.7 Rotating reference frame

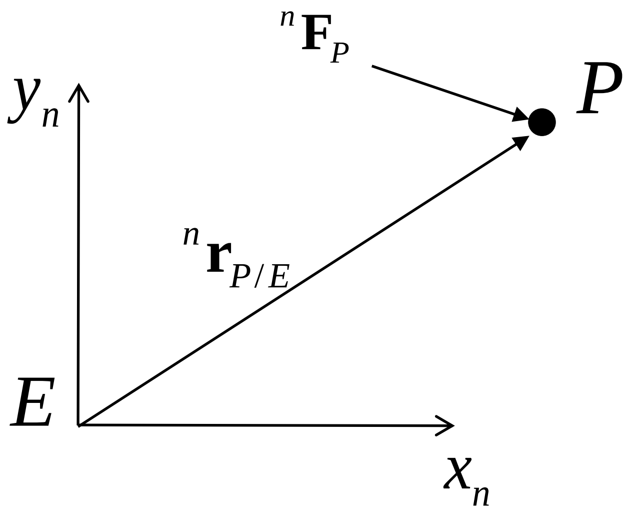
Jerry Ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.15

Relative Motion
- Rotating reference frame

Considering the point P, which is accelerated by a certain force. Description of the motion of the point P in n-frame is as follows.

E **: Origin of Inertial reference frame**

O **: Origin of Translating reference frame**



Newton's law is valid in any inertial reference frame

$$ {}^n\mathbf{F}_P = m_P \, {}^n\ddot{\mathbf{r}}_{P/E} $$

${}^n\mathbf{R}_b(\theta)$ **:Rotation matrix that transforms 3D vectors from b-frame to n-frame ~~coordinates~~.**

$$ {}^n\mathbf{R}_b = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} $$

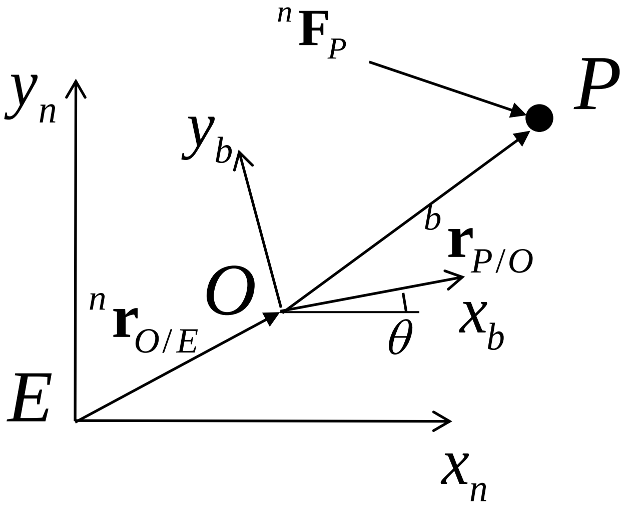
Jerry Ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.15

Relative Motion
- Rotating reference frame

Considering the point P, which is accelerated by a certain force. Description of the motion of the point P via b-frame is as follows.

E : Origin of Inertial reference frame

O : Origin of Translating reference frame



Newton's law is valid in any inertial reference frame

$${}^{n}\mathbf{F}_P = m_P \; {}^{n}\ddot{\mathbf{r}}_{P/E}$$

$${}^{n}\ddot{\mathbf{r}}_{P/E} = \frac{d^2}{dt^2} \; {}^{n}\mathbf{r}_{P/E}$$

$${}^{n}\mathbf{r}_{P/E} = {}^{n}\mathbf{r}_{O/E} + {}^{b}\mathbf{r}_{P/O}$$

➔These vectors can not be added because they are defined in ~~using~~ different frame ~~unit vector~~

$${}^{n}\mathbf{r}_{P/E} = {}^{n}\mathbf{r}_{O/E} + {}^{n}\mathbf{r}_{P/O}$$

➔ These vectors can be added because they are defined using the same unit vector.

$${}^{n}\mathbf{r}_{P/E} = {}^{n}\mathbf{r}_{O/E} + {}^{n}\mathbf{R}_b \cdot {}^{b}\mathbf{r}_{P/O}$$

${}^{n}\mathbf{R}_b(\theta)$:Rotation matrix that transforms 3D vectors from b-frame to n-frame ~~coordinates~~.

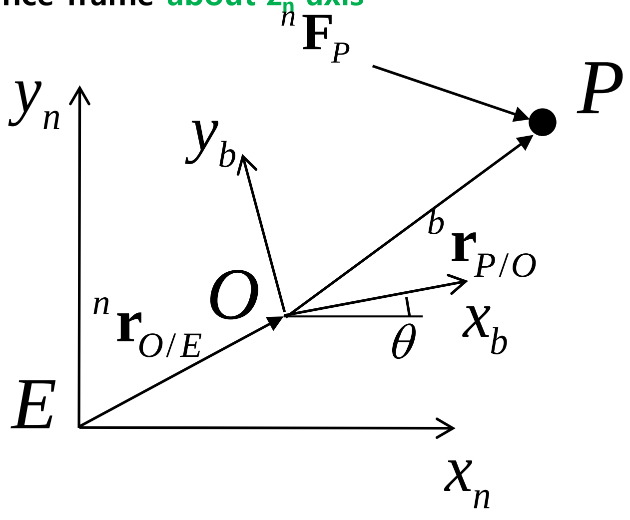
$${}^{n}\mathbf{R}_b = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Jerry Ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.15

Relative Motion
- Rotating reference frame

E : Origin of Inertial reference frame

O : Origin of Translating and Rotating reference frame about z_n axis



$${}^{n}\mathbf{r}_{P/E} = {}^{n}\mathbf{r}_{O/E} + {}^{n}\mathbf{R}_b \cdot {}^{b}\mathbf{r}_{P/O}$$

$\Downarrow$ 1st derivative w.r.t. the time

$$\frac{d}{dt}{}^{n}\mathbf{r}_{P/E} = \frac{d}{dt}{}^{n}\mathbf{r}_{O/E} + {}^{n}\mathbf{R}_b \cdot \frac{d}{dt}{}^{b}\mathbf{r}_{P/O} + {}^{n}\frac{d}{dt}{}^{n}\mathbf{R}_b \cdot {}^{b}\mathbf{r}_{P/O}$$

$\Downarrow$ $\frac{d}{dt}{}^{n}\mathbf{R}_b = {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{R}_b$ ▶

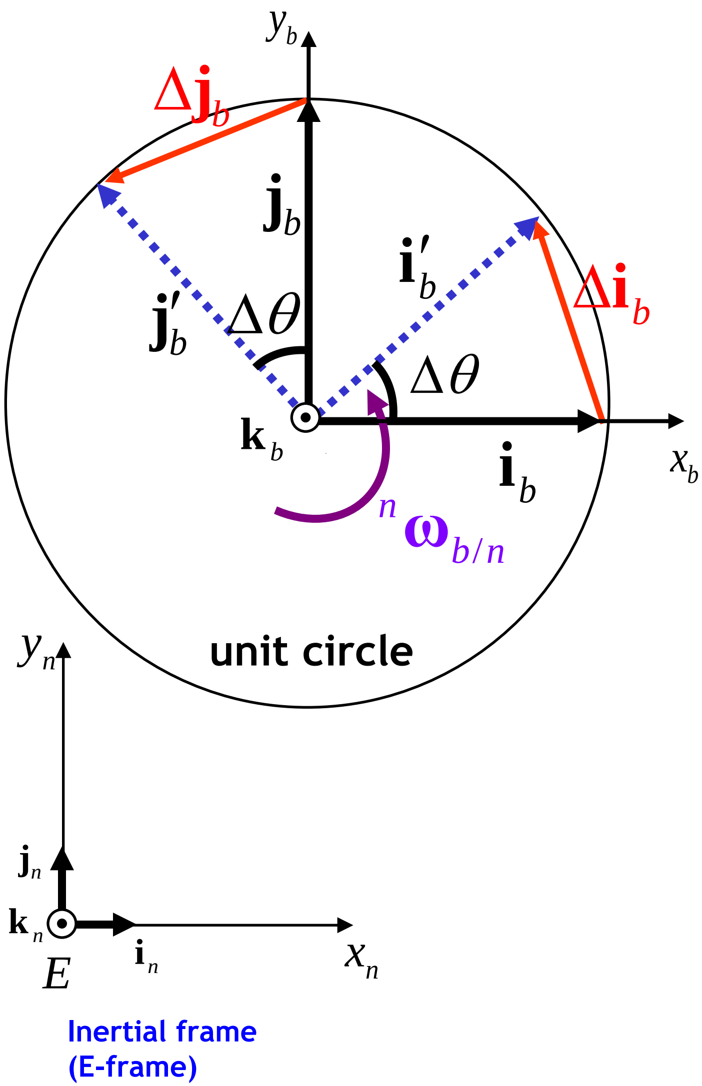
$$\frac{d}{dt}{}^{n}\mathbf{r}_{P/E} = \frac{d}{dt}{}^{n}\mathbf{r}_{O/E} + {}^{n}\mathbf{R}_b \cdot \frac{d}{dt}{}^{b}\mathbf{r}_{P/O} + {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{R}_b \cdot {}^{b}\mathbf{r}_{P/O}$$

${}^{E}\mathbf{R}_b(\theta)$:Rotation matrix that transforms 3D vectors from b to n coordinates.

$${}^{n}\mathbf{R}_b = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Time Derivative of a Rotating unit vector

$$\omega = \frac{d\theta}{dt} \qquad (\boldsymbol{\omega} = \dot{\theta}\mathbf{a} = \omega\mathbf{a})$$



Small Change in $\mathbf{i}_b$: $d\mathbf{i}_b$

$$\left\{ \begin{aligned} &\text{Magnitude: } \lim_{\Delta\theta\to0}|\Delta\mathbf{i}_b| = \lim_{\Delta\theta\to0}\left(|\mathbf{i}_b|\cdot\Delta\theta\right) = d\theta \\ &\text{Direction: } \Delta\theta\to0, \text{ direction of } \Delta\mathbf{i}_b \text{ converges to the direction of } \mathbf{j}_b. \end{aligned} \right.$$

$$d\mathbf{i}_{b(n)} = \lim_{\Delta\theta\to0}\Delta\mathbf{i}_{b(n)} = d\theta\,\mathbf{j}_{b(n)}$$

Small Change in $\mathbf{j}_b$: $d\mathbf{j}_b$

$$\left\{ \begin{aligned} &\text{Magnitude: } \lim_{\Delta\theta\to0}|\Delta\mathbf{j}_b| = \lim_{\Delta\theta\to0}\left(|\mathbf{j}_b|\cdot\Delta\theta\right) = d\theta \\ &\text{Direction: } \Delta\theta\to0, \text{ direction of } \Delta\mathbf{j}_b \text{ converges to the direction of } -\mathbf{i}_b. \end{aligned} \right.$$

$$d\mathbf{j}_b = \lim_{\Delta\theta\to0}\Delta\mathbf{j}_b = -d\theta\,\mathbf{i}_b$$

Time derivative of a rotating unit vector

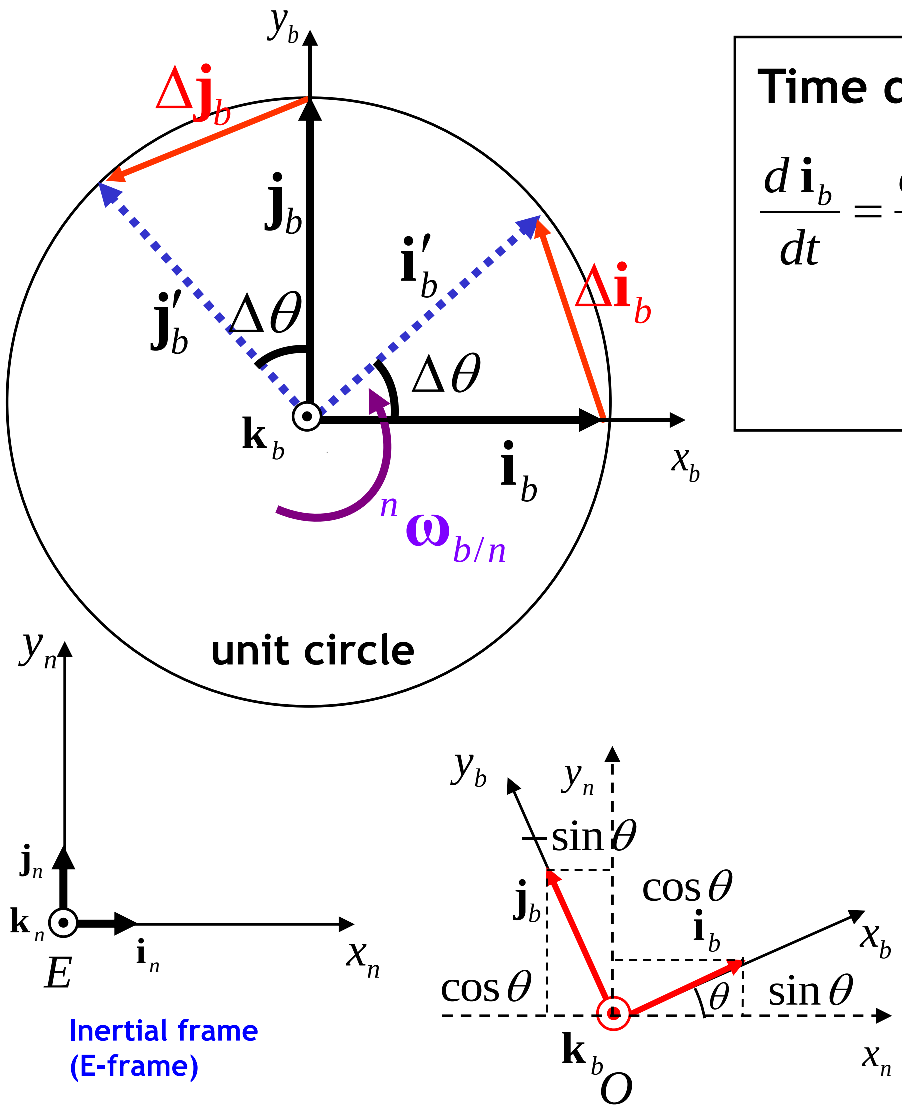
$$\frac{d\,\mathbf{i}_b}{dt} = \frac{d\theta\,\mathbf{j}_b}{dt} = \omega\,\mathbf{j}_b$$
$$= \omega_{b/n}(\mathbf{k}_b \times \mathbf{i}_b)$$
$$= {}^{n}\boldsymbol{\omega}_{b/n} \times \mathbf{i}_b$$

$$\frac{d\mathbf{j}_b}{dt} = -\frac{d\theta}{dt}\mathbf{i}_b = -\omega\,\mathbf{i}_b$$
$$= \omega_{b/n}(\mathbf{k}_b \times \mathbf{j}_b)$$
$$= {}^{n}\boldsymbol{\omega}_{b/n} \times \mathbf{j}_b$$

회전하는 단위 벡터의 시간에 대한 미분

$$\omega = \frac{d\theta}{dt}$$

$$(\boldsymbol{\omega} = \dot{\theta}\mathbf{a} = \omega\mathbf{a})$$



Inertial frame (E-frame)

Time derivative of a rotating unit vector

$$\frac{d\,\mathbf{i}_b}{dt} = \frac{d\theta\,\mathbf{j}_b}{dt} = \omega\,\mathbf{j}_b$$
$$= \omega_{b/n}(\mathbf{k}_b \times \mathbf{i}_b)$$
$$= {}^n\boldsymbol{\omega}_{b/n} \times \mathbf{i}_b$$

$$\frac{d\mathbf{j}_b}{dt} = -\frac{d\theta}{dt}\mathbf{i}_b = -\omega\,\mathbf{i}_b$$
$$= \omega_{b/n}(\mathbf{k}_b \times \mathbf{j}_b)$$
$$= {}^n\boldsymbol{\omega}_{b/n} \times \mathbf{j}_b$$

$$ {}^n\mathbf{R}_b(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = [\mathbf{i}_b \quad \mathbf{j}_b]$$

※ Derivative of a Rotation Matrix

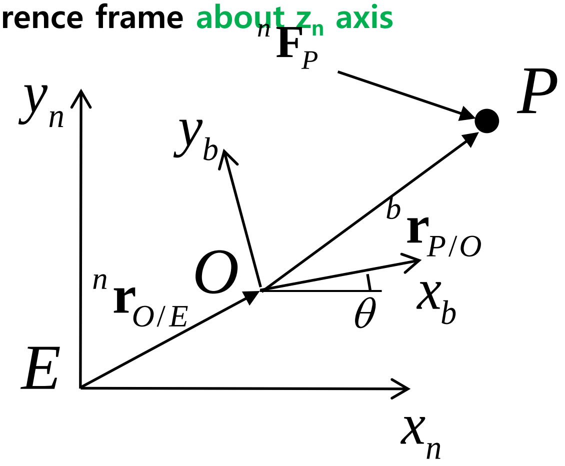
$$\frac{d\,{}^n\mathbf{R}_b(\theta)}{dt} = \begin{bmatrix} \frac{d\mathbf{i}_b}{dt} & \frac{d\mathbf{j}_b}{dt} \end{bmatrix}$$
$$= \begin{bmatrix} {}^n\boldsymbol{\omega}_{b/n} \times \mathbf{i}_b & {}^n\boldsymbol{\omega}_{b/n} \times \mathbf{j}_b \end{bmatrix}$$
$$= {}^n\boldsymbol{\omega}_{b/n} \times [\mathbf{i}_b \quad \mathbf{j}_b]$$
$$= {}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{R}_b$$

Jerry Ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.15

Relative Motion
- Rotating reference frame

E : Origin of Inertial reference frame

O : Origin of Translating and Rotating reference frame about z_n axis



${}^n\mathbf{R}_b$:Rotation matrix that transforms 3D vectors from b to n coordinates.

Newton's law is valid in any inertial reference frame

$${}^n\mathbf{F}_P = m_P\;{}^n\ddot{\mathbf{r}}_{P/E} \quad ,\; {}^n\ddot{\mathbf{r}}_{P/E} = \frac{d^2}{dt^2}\,{}^n\mathbf{r}_{P/E}$$

$$\frac{d}{dt}\,{}^n\mathbf{r}_{P/E} = \underline{\frac{d}{dt}\,{}^n\mathbf{r}_{O/E}} + \underline{{}^n\mathbf{R}_b\cdot\frac{d}{dt}\,{}^b\mathbf{r}_{P/O}} + \underline{{}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{R}_b\cdot{}^b\mathbf{r}_{P/O}}$$

2nd derivative w.r.t the time

$$\frac{d^2}{dt^2}\,{}^n\mathbf{r}_{P/E} = \underline{\frac{d}{dt}\,{}^b\mathbf{R}_b\cdot\frac{d}{dt}\,{}^b\mathbf{r}_{P/O}} + \underline{{}^n\mathbf{R}_b\cdot\frac{d^2\,{}^b\mathbf{r}_{P/O}}{dt^2}}$$
$$\underline{+\frac{d}{dt}\,{}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{R}_b\cdot{}^b\mathbf{r}_{P/O}} + \underline{{}^n\boldsymbol{\omega}_{b/n}\times\frac{d}{dt}\,{}^n\mathbf{R}_b\cdot{}^b\mathbf{r}_{P/O}}$$
$$\underline{+{}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{R}_b\cdot\frac{d}{dt}\,{}^b\mathbf{r}_{P/O}} + \underline{\frac{d^2}{dt^2}\,{}^n\mathbf{r}_{O/E}}$$

$$\frac{d}{dt}\,{}^n\mathbf{R}_b = {}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{R}_b$$

$$\frac{d^2}{dt^2}\,{}^n\mathbf{r}_{P/E} = \underline{{}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{R}_b\cdot\frac{d}{dt}\,{}^b\mathbf{r}_{P/O}} + \underline{{}^n\mathbf{R}_b\cdot\frac{d^2}{dt^2}\,{}^b\mathbf{r}_{P/O}} + \underline{\frac{d}{dt}\,{}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{R}_b\cdot{}^b\mathbf{r}_{P/O}}$$
$$\underline{+{}^n\boldsymbol{\omega}_{b/n}\times\left({}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{R}_b\cdot{}^b\mathbf{r}_{P/O}\right)} + \underline{{}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{R}_b\cdot\frac{d}{dt}\,{}^n\mathbf{r}_{P/O}} + \underline{\frac{d^2}{dt^2}\,{}^n\mathbf{r}_{O/E}}$$

$$\frac{d^2}{dt^2}{}^n\mathbf{r}_{P/E} = {}^n\mathbf{R}_b \cdot \frac{d^2}{dt^2}{}^b\mathbf{r}_{P/O} + \frac{d}{dt}{}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{R}_b \cdot {}^b\mathbf{r}_{P/O}$$
$$+{}^n\boldsymbol{\omega}_{b/n} \times \left({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{R}_b \cdot {}^b\mathbf{r}_{P/O}\right) + 2\left({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{R}_b \cdot \frac{d}{dt}{}^b\mathbf{r}_{P/O}\right) + \frac{d^2}{dt^2}{}^n\mathbf{r}_{O/E}$$

$$\Downarrow$$

$${}^n\ddot{\mathbf{r}}_{P/E} = {}^n\mathbf{R}_b \cdot {}^b\ddot{\mathbf{r}}_{P/O} + {}^n\boldsymbol{\alpha}_{b/n} \times {}^n\mathbf{r}_{P/O}$$
$$+{}^n\boldsymbol{\omega}_{b/n} \times \left({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{R}_b \cdot {}^b\mathbf{r}_{P/O}\right) + 2\left({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{R}_b \cdot {}^b\dot{\mathbf{r}}_{P/O}\right) + {}^n\ddot{\mathbf{r}}_{O/E}$$

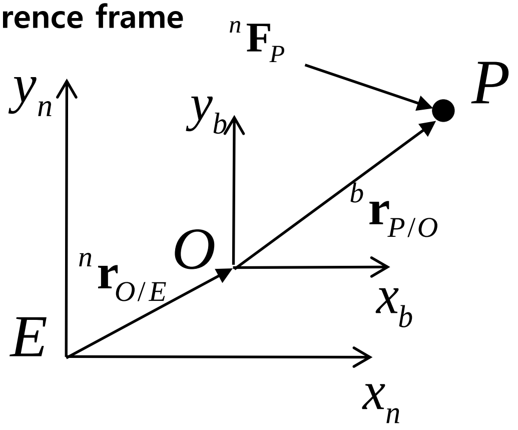
$$\Downarrow$$

$${}^n\ddot{\mathbf{r}}_{P/E} = {}^n\ddot{\mathbf{r}}_{P/O} + {}^n\boldsymbol{\alpha}_{b/n} \times {}^n\mathbf{r}_{P/O}$$
$$+{}^n\boldsymbol{\omega}_{b/n} \times \left({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{r}_{P/O}\right) + 2\left({}^n\boldsymbol{\omega}_{b/n} \times {}^n\dot{\mathbf{r}}_{P/O}\right) + {}^n\ddot{\mathbf{r}}_{O/E}$$

Relative Motion
- Rotating reference frame

E : Origin of Inertial reference frame

O : Origin of Translating and Rotating reference frame



If frame A is rotating reference frame

$$
\begin{aligned}
{}^{n}\ddot{\mathbf{r}}_{P/E} &= {}^{n}\ddot{\mathbf{r}}_{P/O} + {}^{n}\boldsymbol{\alpha}_{b/n} \times {}^{n}\mathbf{r}_{P/O} \\
&\quad + {}^{n}\boldsymbol{\omega}_{b/n} \times \left({}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{r}_{P/O} \right) + 2\left({}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\dot{\mathbf{r}}_{P/O} \right) + {}^{n}\ddot{\mathbf{r}}_{O/E}
\end{aligned}
$$

$$
{}^{n}\mathbf{F}_{P} = m_{P}\ {}^{n}\ddot{\mathbf{r}}_{P/E}
$$

⇩

$$
\begin{aligned}
{}^{n}\mathbf{F}_{P} &= m_{P}\left({}^{n}\ddot{\mathbf{r}}_{P/O} + {}^{n}\boldsymbol{\alpha}_{b/n} \times {}^{n}\mathbf{r}_{P/O} \right) \\
&\quad + m_{P}\left({}^{n}\boldsymbol{\omega}_{b/n} \times \left({}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{r}_{P/O} \right) + 2\left({}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\dot{\mathbf{r}}_{P/O} \right) + {}^{n}\ddot{\mathbf{r}}_{O/E} \right)
\end{aligned}
$$

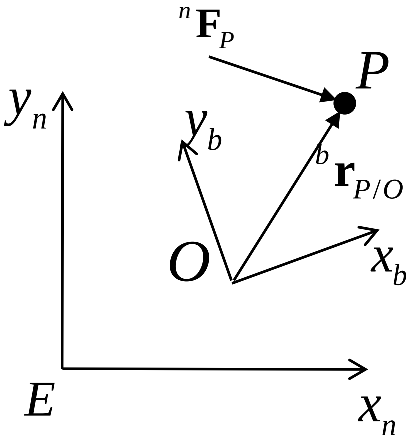
⇩

$$
\begin{aligned}
&{}^{n}\mathbf{F}_{P} - m_{P}\left({}^{n}\boldsymbol{\alpha}_{b/n} \times {}^{n}\mathbf{r}_{P/O} \right) - m_{P}\left({}^{n}\boldsymbol{\omega}_{b/n} \times \left({}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{r}_{P/O} \right) \right) \\
&\quad - 2m_{P}\left({}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\dot{\mathbf{r}}_{P/O} \right) - m_{P}\ {}^{n}\ddot{\mathbf{r}}_{O/E} = m_{P}\ {}^{n}\ddot{\mathbf{r}}_{P/O}
\end{aligned}
$$

Relative Motion
- Rotating reference frame

✓ Meaning of each term

$$ {}^n\mathbf{F}_P - m_P\left({}^n\boldsymbol{\alpha}_{b/n}\times{}^n\mathbf{r}_{P/O}\right) - m_P\left({}^n\boldsymbol{\omega}_{b/n}\times\left({}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{r}_{P/O}\right)\right) - 2m_P\left({}^n\boldsymbol{\omega}_{b/n}\times{}^n\dot{\mathbf{r}}_{P/O}\right) - m_P\,{}^n\ddot{\mathbf{r}}_{O/E} = m_P\,{}^n\ddot{\mathbf{r}}_{P/O} $$



LHS

${}^n\mathbf{F}_P$: **External resultant force** exerted on point mass P

$-m_P\left({}^n\boldsymbol{\alpha}_{b/n}\times{}^n\mathbf{r}_{P/O}\right)$: **Inertial force** caused by angular acceleration of b-frame

$-m_P\left({}^n\boldsymbol{\omega}_{b/n}\times\left({}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{r}_{P/O}\right)\right)$: **Inertial force** caused by angular velocity of b-frame
→ **Centrifugal Force**

$-2m_P\left({}^n\boldsymbol{\omega}_{b/n}\times{}^n\dot{\mathbf{r}}_{P/O}\right)$: **Inertial force** caused by linear velocity of P with respect to b-frame and angular velocity of b-frame
→ **Coriolis Force**

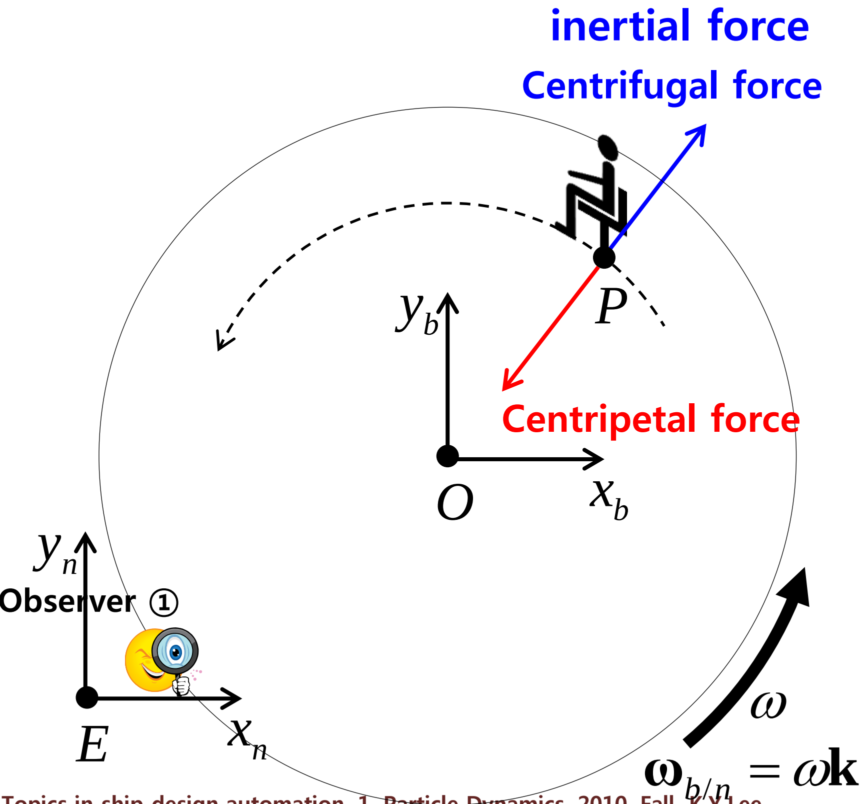
RHS

$m_P\,{}^n\ddot{\mathbf{r}}_{P/O}$: **Relative Acceleration** with respect to b-frame and **mass** of point mass P

Relative Motion
- Examples of rotating reference frame

Case #1

- A chair is fixed on a circular disk which is rotating with an angular velocity ω.
- What kind of forces does a person sitting on the chair feel?



Description from the observer ①

The person sitting on the chair revolves around the center of the disk.

It shows that the centripetal force is exerted on the person

Description from the person sitting on the chair.

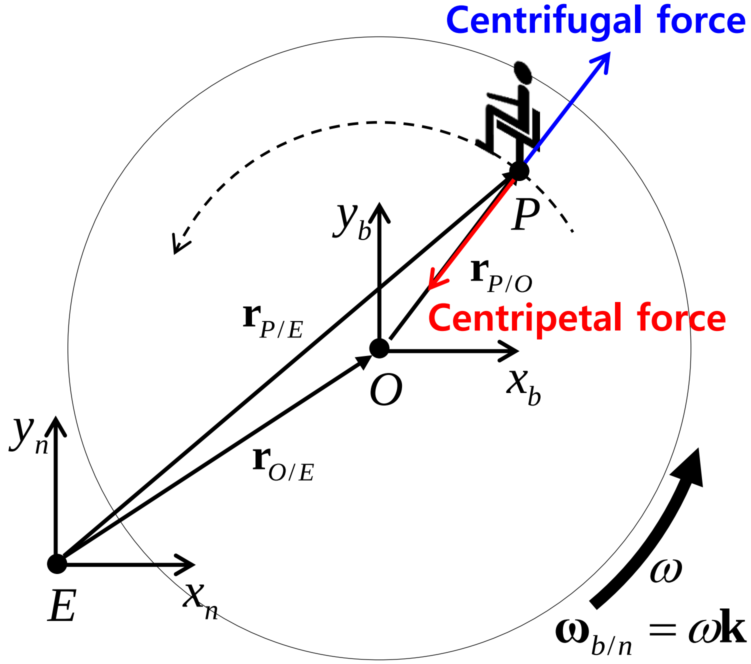
The person sitting on the chair feels centrifugal force. ← inertial force

[sen'trɪpɪtl] [sen'trɪfjʊgl]

Relative Motion
- Examples of rotating reference frame

Case #1

- A chair is fixed on a circular disk which is rotating with an angular velocity ω.
- What kind of forces does a person sitting on the chair feel?



- We apply Newton's 2nd law to the person on the chair

$$m_P\,{}^n\ddot{\mathbf{r}}_{P/E} = \mathbf{F}_P$$

$$m_P\{{}^n\ddot{\mathbf{r}}_{O/E} + {}^n\ddot{\mathbf{r}}_{P/O} + ({}^n\dot{\boldsymbol{\omega}}_{b/n} \times {}^n\mathbf{r}_{P/O}) + 2({}^n\boldsymbol{\omega}_{b/n} \times {}^n\dot{\mathbf{r}}_{P/O}) + ({}^n\boldsymbol{\omega}_{b/n} \times ({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{r}_{P/O}))\} = \mathbf{F}_P$$

$$m_P\,{}^n\ddot{\mathbf{r}}_{O/E} + \boxed{m_P\,{}^n\ddot{\mathbf{r}}_{P/O}} + m_P({}^n\dot{\boldsymbol{\omega}}_{b/n} \times {}^n\mathbf{r}_{P/O}) + 2m_P({}^n\boldsymbol{\omega}_{b/n} \times {}^n\dot{\mathbf{r}}_{P/O}) + m_P({}^n\boldsymbol{\omega}_{b/n} \times ({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{r}_{P/O})) = \mathbf{F}_P$$

$$\boxed{m_P\,{}^n\ddot{\mathbf{r}}_{P/O}} = \mathbf{F}_P - m_P\,\cancel{{}^n\ddot{\mathbf{r}}_{O/E}} - m_P(\cancel{{}^n\dot{\boldsymbol{\omega}}_{b/n}} \times {}^n\mathbf{r}_{P/O}) - 2m_P({}^n\boldsymbol{\omega}_{b/n} \times \cancel{{}^n\dot{\mathbf{r}}_{P/O}}) \boxed{- m_P({}^n\boldsymbol{\omega}_{b/n} \times ({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{r}_{P/O}))}$$

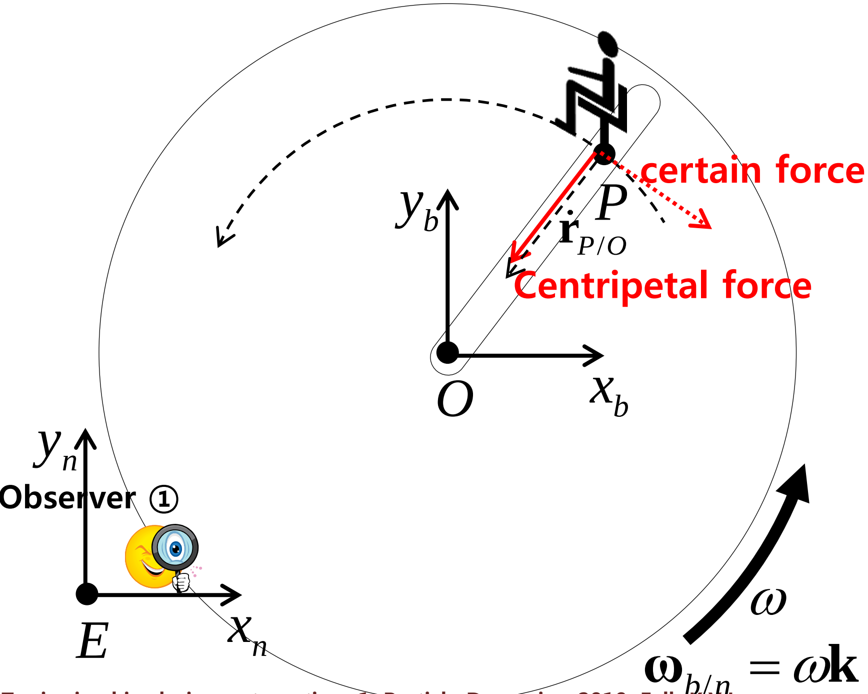
inertial force

The person feels centrifugal force

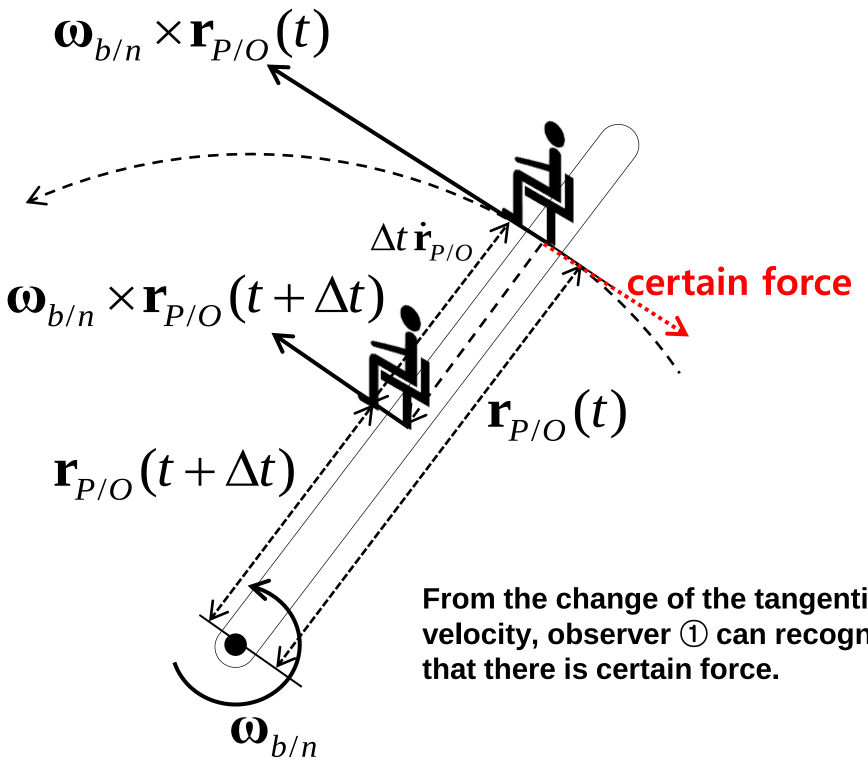
Relative Motion
- Examples of rotating reference frame

Case #2

- A chair moves with velocity v along the line on a circular disk which is rotating with an angular velocity ω.
- What kind of forces does a person sitting on the chair feel?



Description from the observer ①

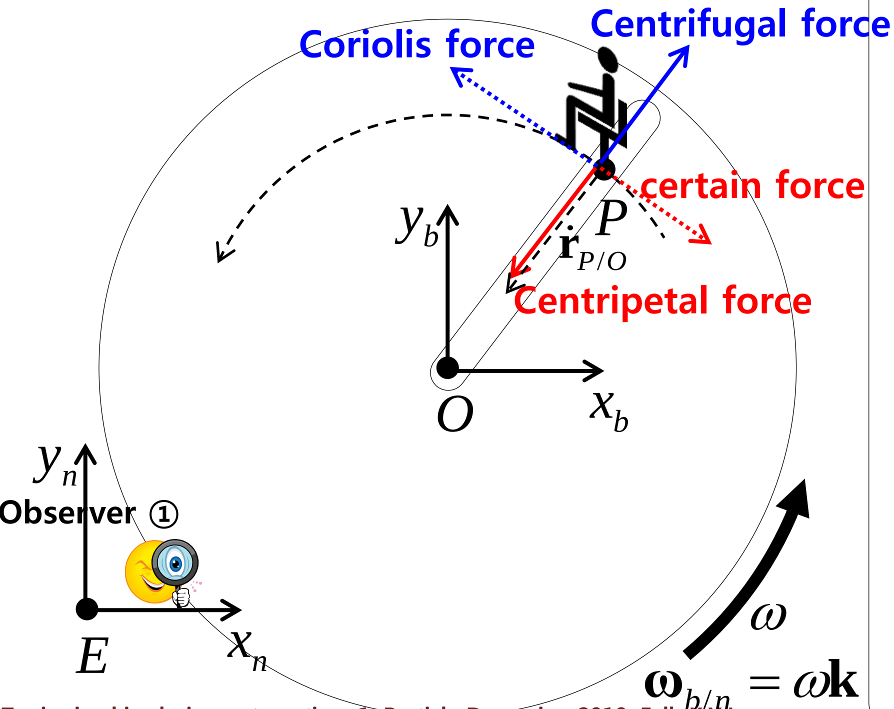


From the change of the tangential velocity, observer ① can recognize that there is certain force.

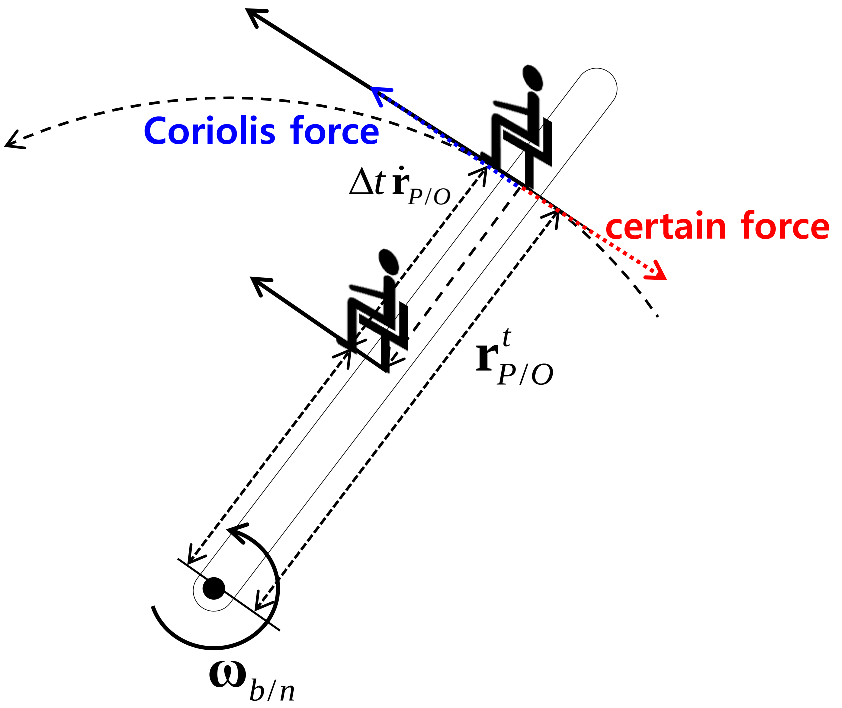
Relative Motion
- Examples of rotating reference frame

Case #2

- A chair moves with velocity v along the line on a circular disk which is rotating with an angular velocity ω.
- What kind of forces does a person sitting on the chair feel?



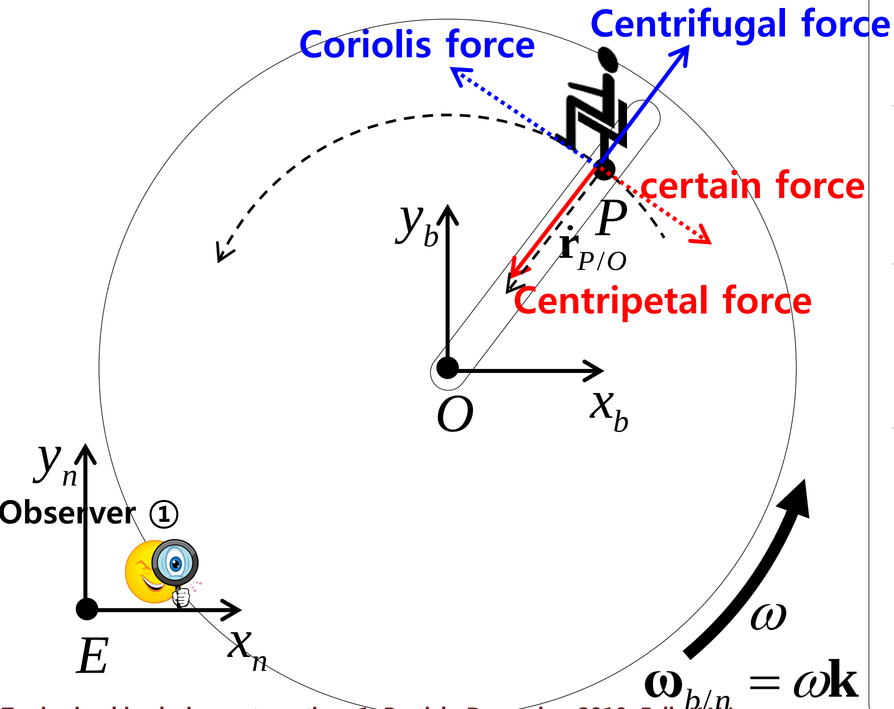
The person sitting on the chair feels **Coriolis force**.



Relative Motion
- Examples of rotating reference frame

Case #2

- A chair moves with velocity v along the line on a circular disk which is rotating with an angular velocity ω.
- What kind of forces does a person sitting on the chair feel?



- We apply Newton's 2nd law to the person on the chair

$$m_P\,{}^n\ddot{\mathbf{r}}_{P/E} = \mathbf{F}_P$$

$$\begin{aligned} & m_P\{{}^n\ddot{\mathbf{r}}_{O/E} + {}^n\ddot{\mathbf{r}}_{P/O} + ({}^n\dot{\boldsymbol{\omega}}_{b/n} \times {}^n\mathbf{r}_{P/O}) \\ & +2({}^n\boldsymbol{\omega}_{b/n} \times {}^n\dot{\mathbf{r}}_{P/O}) + ({}^n\boldsymbol{\omega}_{b/n} \times ({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{r}_{P/O}))\} = \mathbf{F}_P \end{aligned}$$

$$\begin{aligned} & m_P\,{}^n\ddot{\mathbf{r}}_{O/E} + \boxed{m_P\,{}^n\ddot{\mathbf{r}}_{P/O}} + m_P({}^n\dot{\boldsymbol{\omega}}_{b/n} \times {}^n\mathbf{r}_{P/O}) \\ & +2m_P({}^n\boldsymbol{\omega}_{b/n} \times {}^n\dot{\mathbf{r}}_{P/O}) + m_P({}^n\boldsymbol{\omega}_{b/n} \times ({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{r}_{P/O})) = \mathbf{F}_P \end{aligned}$$

$$\begin{aligned} \boxed{m_P\,{}^n\ddot{\mathbf{r}}_{P/O}} = \mathbf{F}_P - m_P\,\cancel{{}^n\ddot{\mathbf{r}}_{O/E}} - m_P(\cancel{{}^n\dot{\boldsymbol{\omega}}_{b/n}} \times {}^n\mathbf{r}_{P/O}) \\ \boxed{-2m_P({}^n\boldsymbol{\omega}_{b/n} \times {}^n\dot{\mathbf{r}}_{P/O})} \boxed{-m_P({}^n\boldsymbol{\omega}_{b/n} \times ({}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{r}_{P/O}))} \end{aligned}$$

inertial force

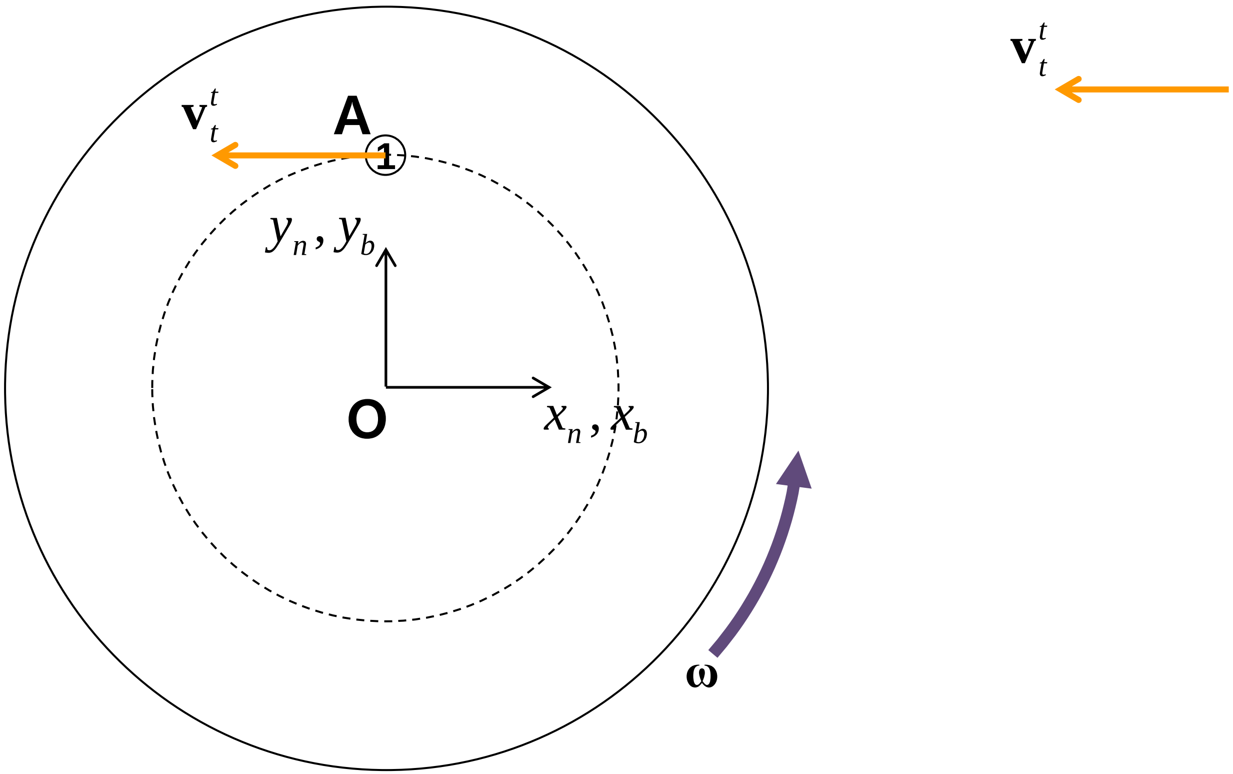
The person feels Coriolis and centrifugal force

1.8 Centrifugal and Coriolis Acceleration

Example) Rotating Disk
- observed in n-frame (1/5)

A point “A” is fixed on a ~~rotating~~ disk rotating with a constant angular velocity.

Velocity of the point A observed in n-frame.

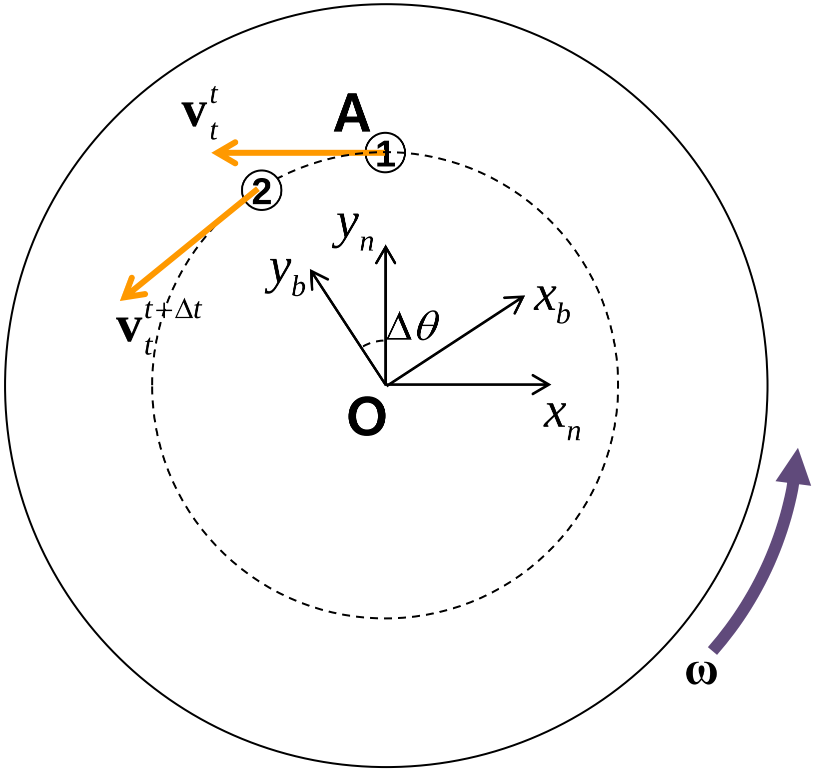


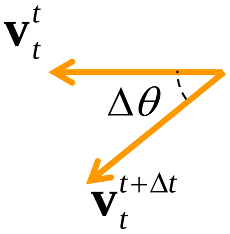
n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

$\mathbf{v}$ ⓣ t: time “t”
ⓣ t: Tangential vector, n: normal vector

Example) Rotating Disk (2/5)

Velocity of the point A observed in n-frame.

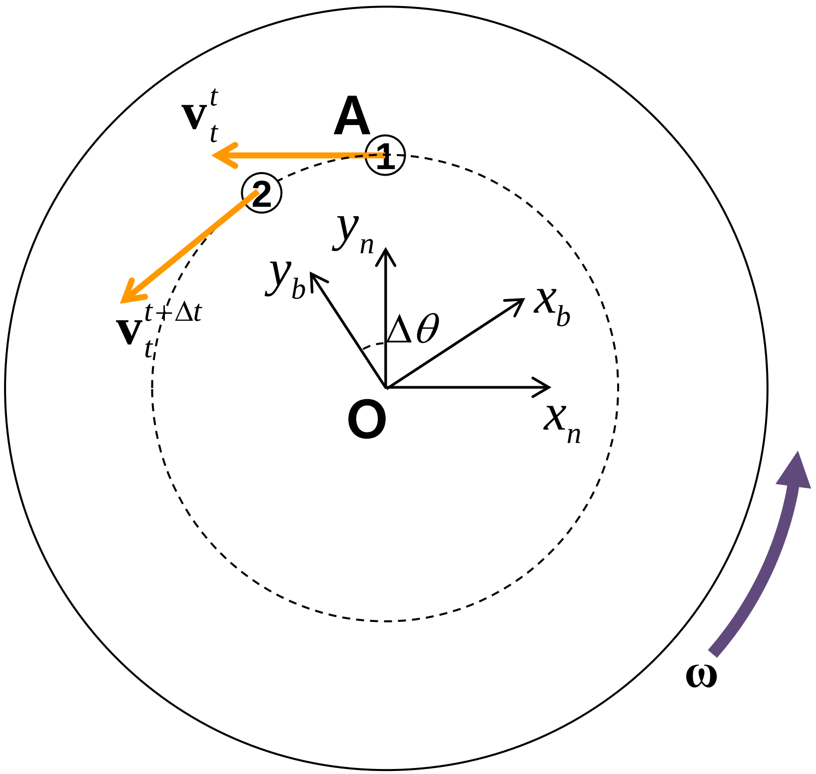


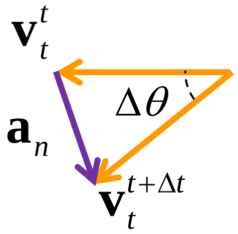


n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

Example) Rotating Disk (3/5)

Velocity of the point A observed in n-frame.

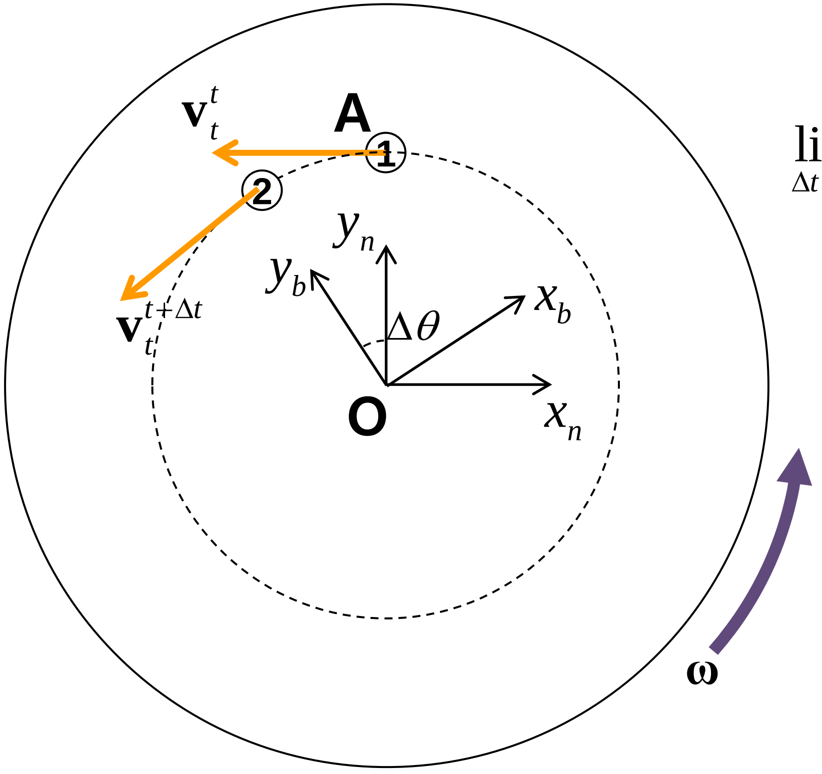




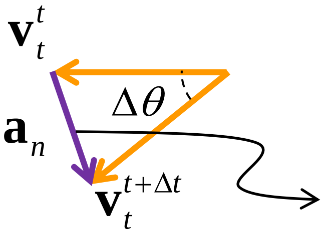
n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

Example) Rotating Disk – Centripetal Acceleration (4/5)

Velocity of the point A observed in n-frame.



$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \mathbf{a}_n$$



Centripetal Acceleration

$$\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{r}_{A/O} \right)$$

Magnitude of $\mathbf{a}_n$

$$\mathbf{v}_t^t = \boldsymbol{\omega} \times \mathbf{r}_{A/O}$$

$$\left| \mathbf{v}_t^t \right| = \omega r$$

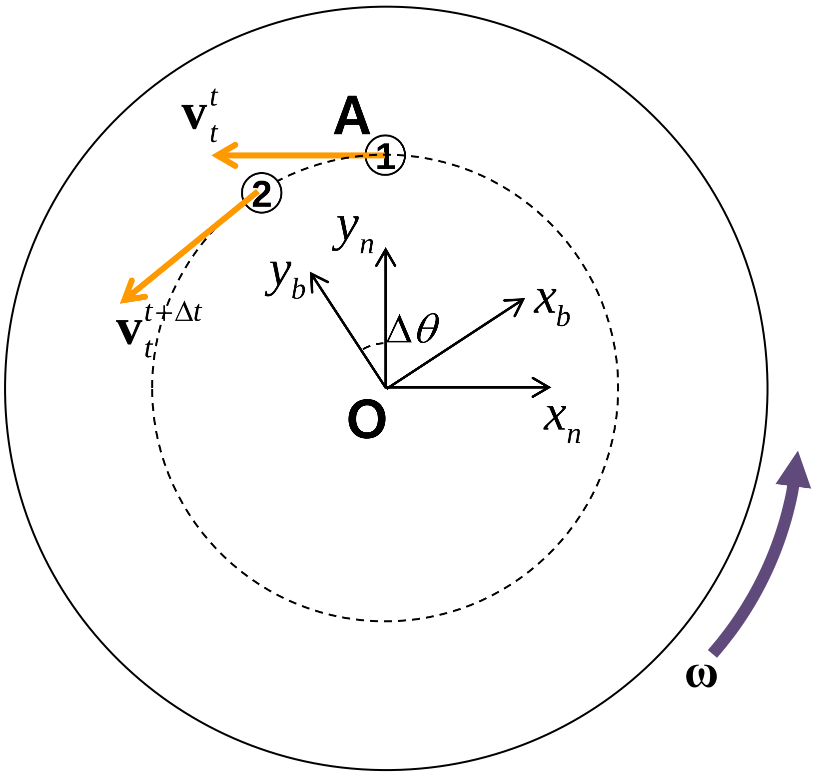
$$\lim_{\Delta\theta \to 0} \left| \Delta \mathbf{v} \right| \approx \left| \mathbf{v}_t^t \right| \cdot \Delta\theta = \omega r \cdot \Delta\theta$$

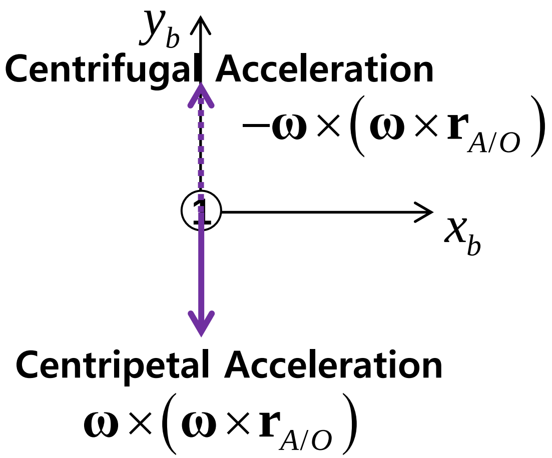
$$\left| \mathbf{a}_n \right| = \lim_{\Delta t \to 0} \left| \frac{\Delta \mathbf{v}}{\Delta t} \right| \approx \left| \mathbf{v}_t^t \right| \cdot \frac{\Delta\theta}{\Delta t} = \omega r \cdot \omega = \omega^2 r$$

n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

Example) Rotating Disk : observed in b-frame
- Centrifugal Acceleration (5/5)

Velocity of the point A observed in b-frame.





Centrifugal Acceleration

$-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/O})$

Centripetal Acceleration

$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/O})$

Since, the point A observed in b-frame is not accelerated, there should be an additional force exerted on the point A except the centripetal force.

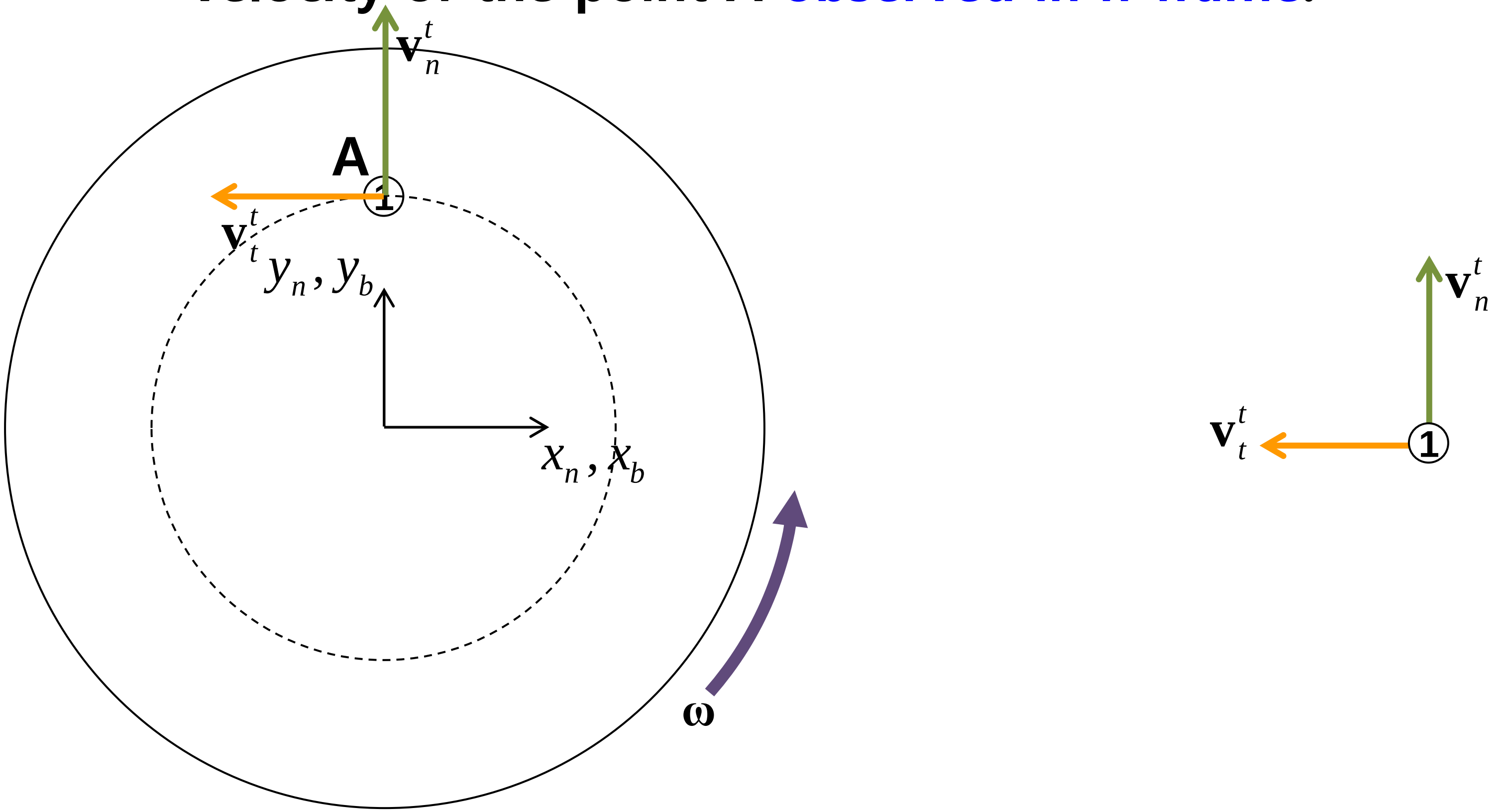
The additional force is a centrifugal force.

n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

Example) Rotating Disk - Coriolis Acceleration(1/9)

A point "A" is moving along a slot with a constant velocity, and the slot is on a disk rotating with a constant angular velocity.

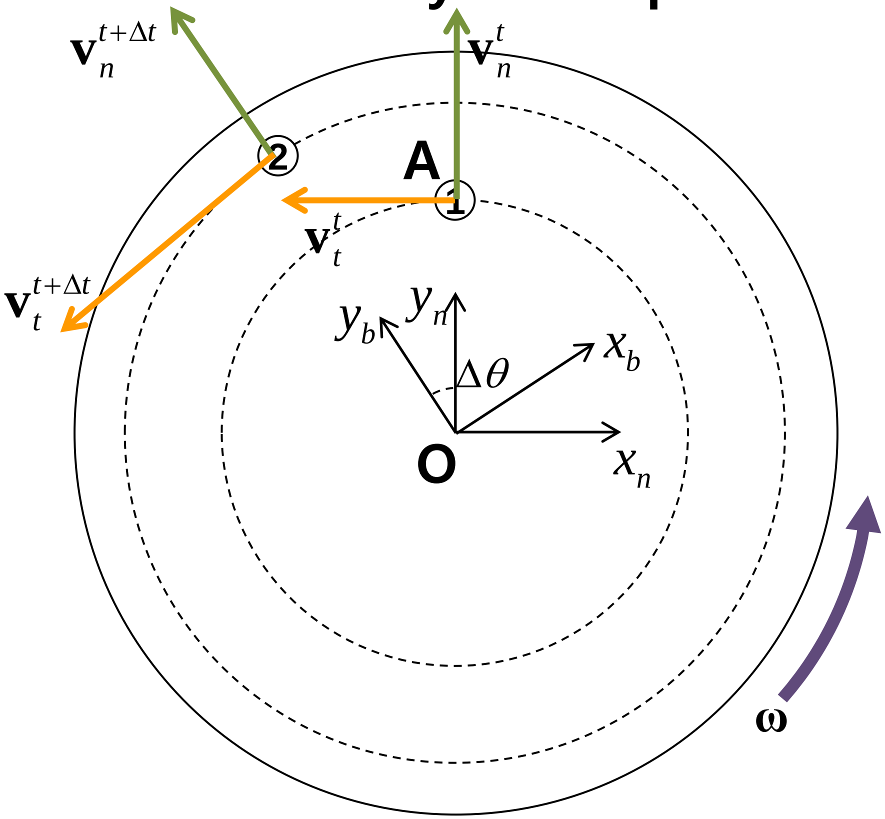
Velocity of the point A observed in n-frame.

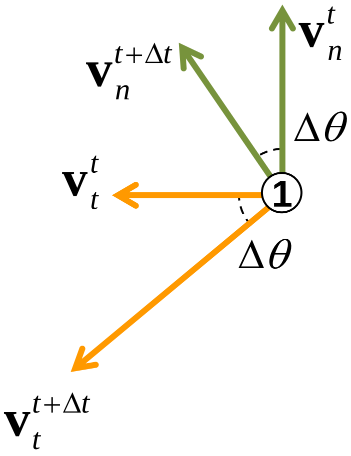


n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

Example) Rotating Disk - Coriolis Acceleration(2/9)

Velocity of the point A observed in n-frame.

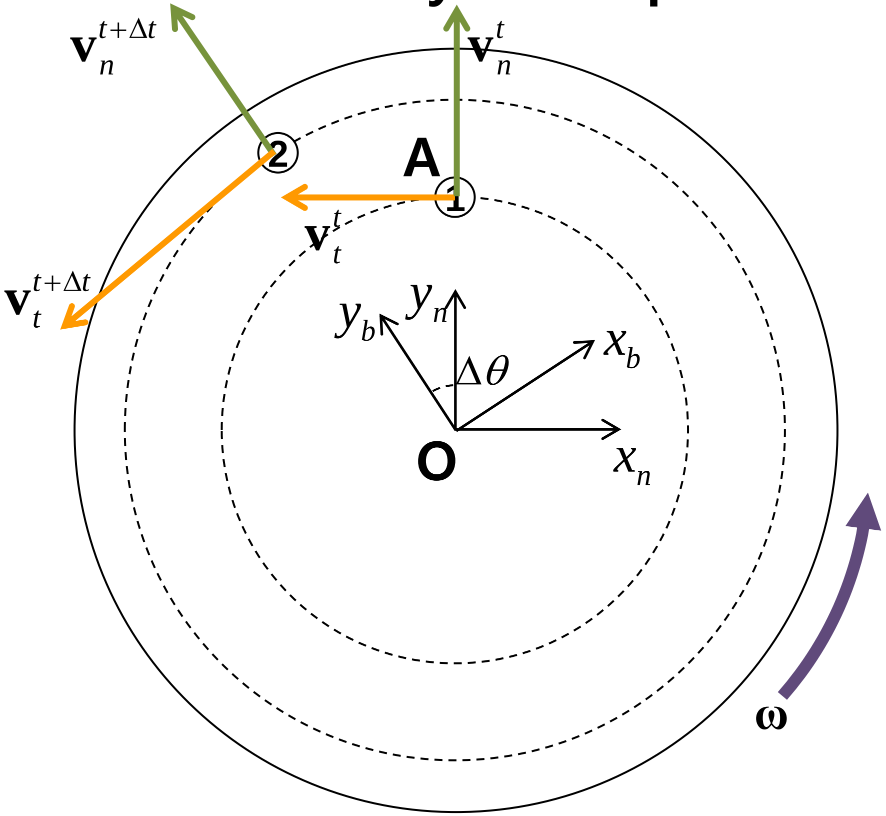


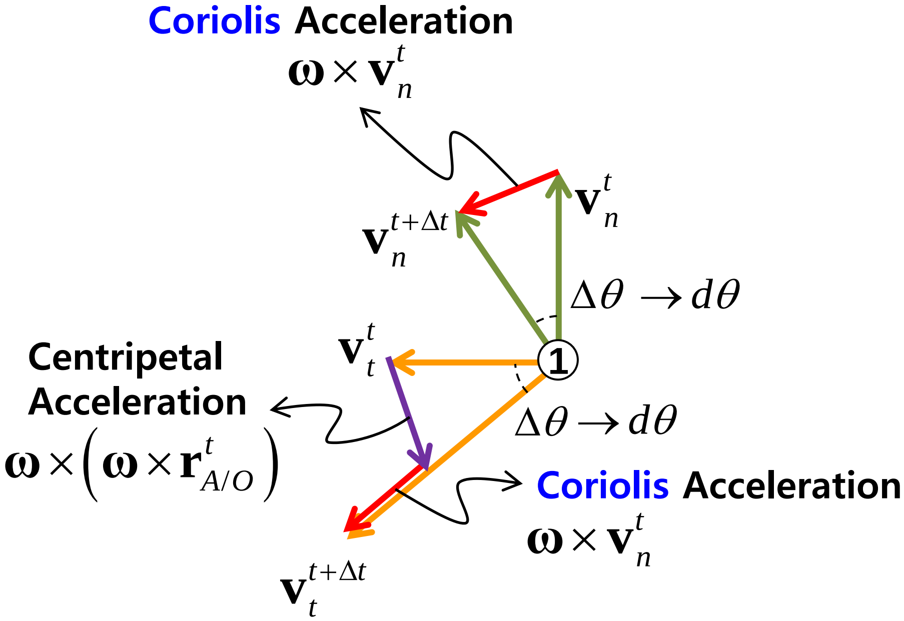


n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

Example) Rotating Disk - Coriolis Acceleration(3/9)

Velocity of the point A observed in n-frame.





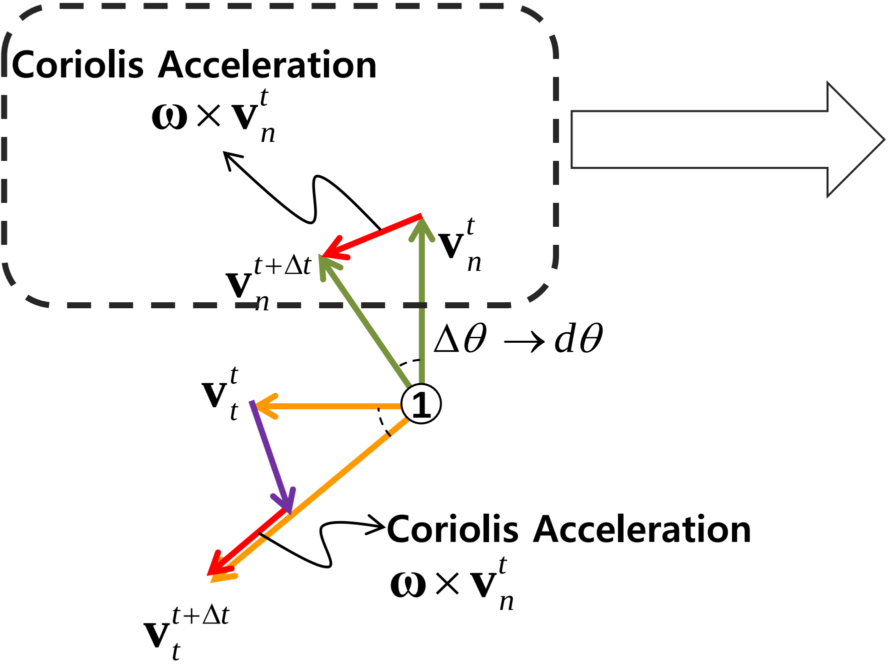
Coriolis Acceleration
$\boldsymbol{\omega}\times\mathbf{v}_n^t$

Centripetal Acceleration
$\boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}_{A/O}^t\right)$

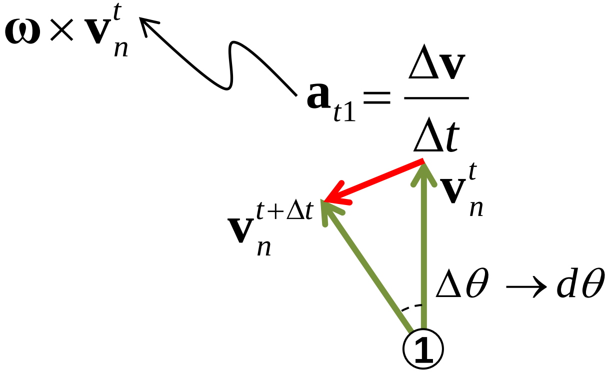
Coriolis Acceleration
$\boldsymbol{\omega}\times\mathbf{v}_n^t$

n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

Example) Rotating Disk - Coriolis Acceleration(4/9)



Coriolis Acceleration



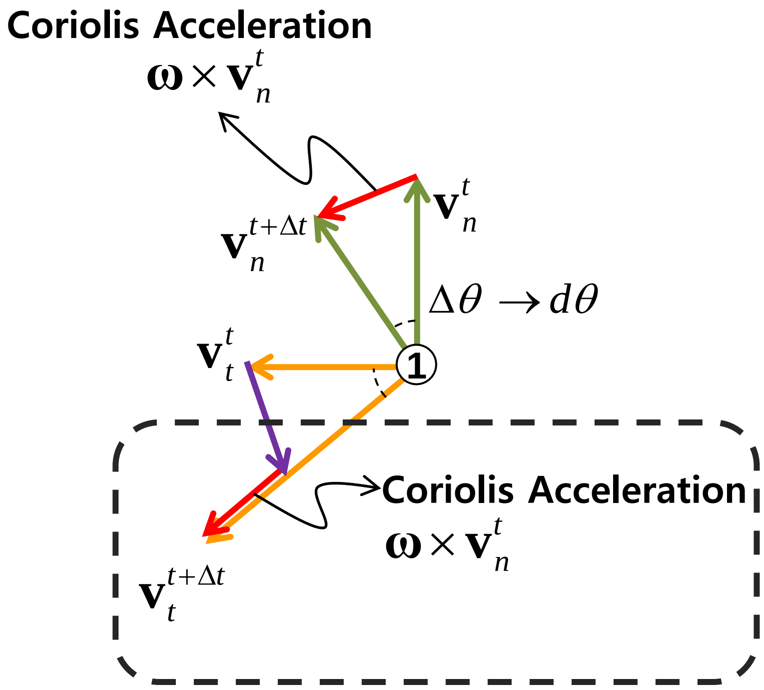
$$\mathbf{a}_{t1} = \frac{\Delta\mathbf{v}}{\Delta t}$$

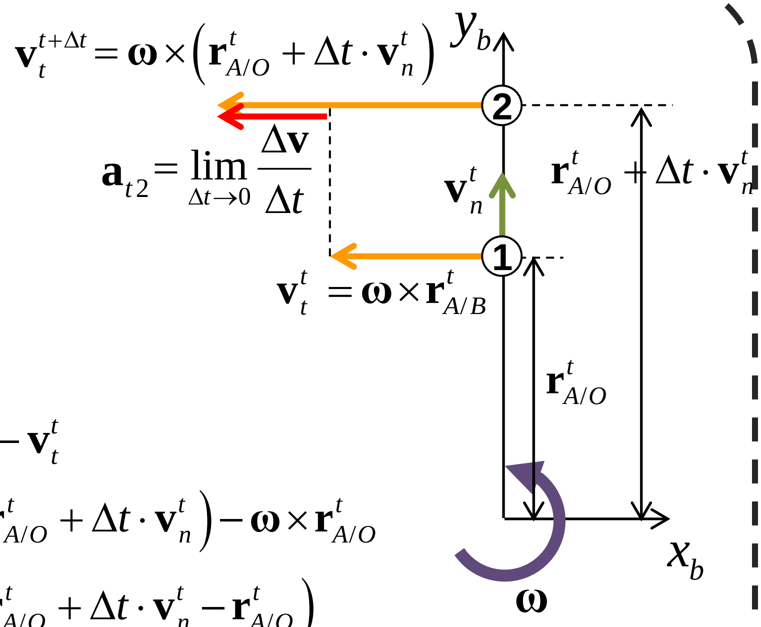
Magnitude of $\mathbf{a}_t$

$$\lim_{\Delta\theta\to 0}|\Delta\mathbf{v}| \approx |\mathbf{v}_n^t|\cdot\Delta\theta \quad (=\mathbf{v}_n^t\cdot\Delta\theta)$$

$$|\mathbf{a}_{t1}| = \lim_{\Delta t\to 0}\left|\frac{\Delta\mathbf{v}}{\Delta t}\right| \approx |\mathbf{v}_n^t|\cdot\frac{\Delta\theta}{\Delta t} = |\mathbf{v}_n^t|\omega$$

Rotating Disk: Coriolis Acceleration(5/9)





$$\mathbf{v}_t^{t+\Delta t} = \boldsymbol{\omega}\times\left(\mathbf{r}_{A/O}^t + \Delta t\cdot\mathbf{v}_n^t\right)$$

$$\mathbf{a}_{t2} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t}$$

$$\mathbf{v}_t^t = \boldsymbol{\omega}\times\mathbf{r}_{A/B}^t$$

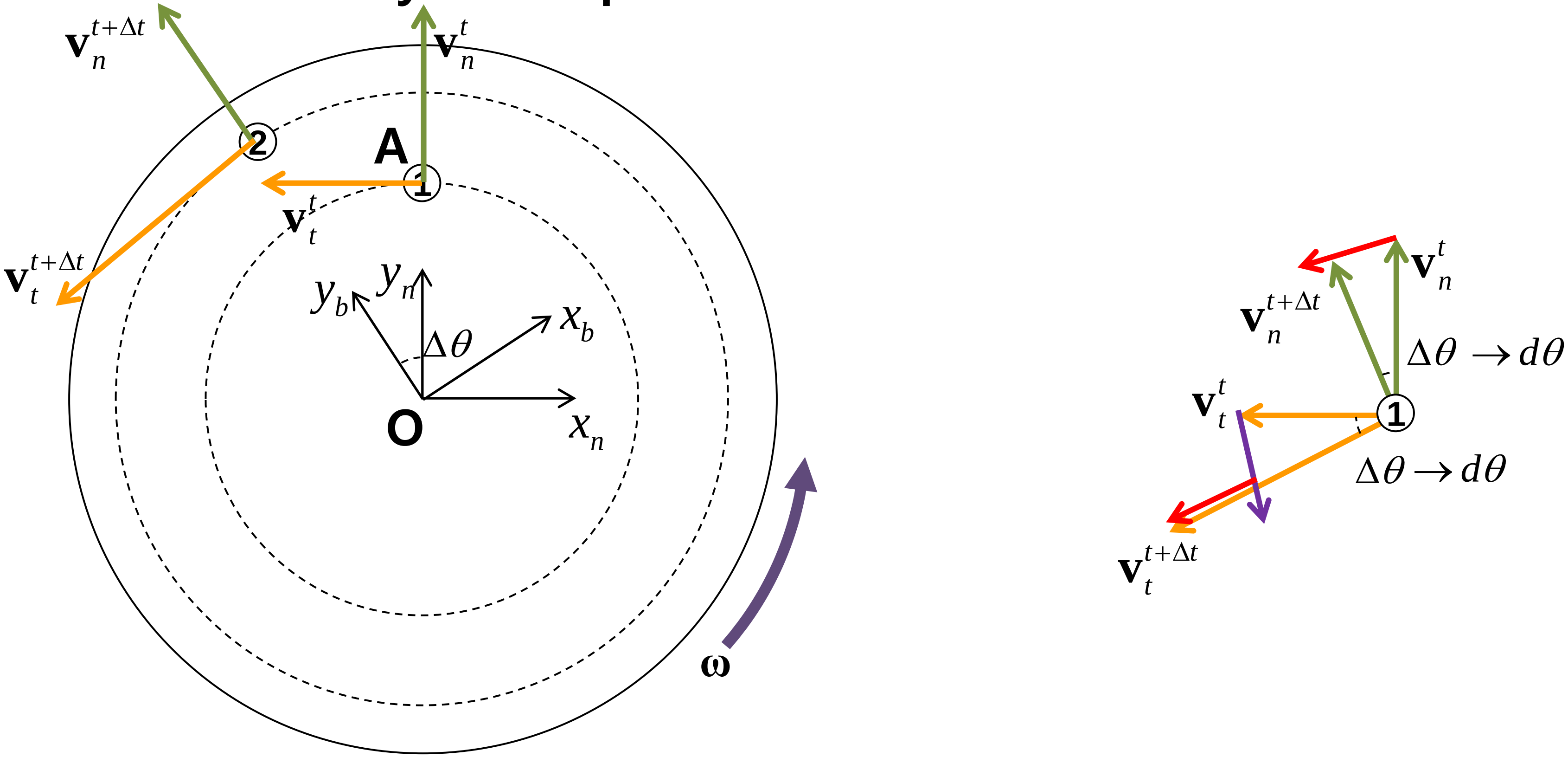
$$\begin{aligned}\Delta\mathbf{v} &= \mathbf{v}_t^{t+\Delta t} - \mathbf{v}_t^t\\ &= \boldsymbol{\omega}\times\left(\mathbf{r}_{A/O}^t + \Delta t\cdot\mathbf{v}_n^t\right) - \boldsymbol{\omega}\times\mathbf{r}_{A/O}^t\\ &= \boldsymbol{\omega}\times\left(\mathbf{r}_{A/O}^t + \Delta t\cdot\mathbf{v}_n^t - \mathbf{r}_{A/O}^t\right)\\ &= \boldsymbol{\omega}\times\Delta t\cdot\mathbf{v}_n^t\end{aligned}$$

$$\mathbf{a}_{t2} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\boldsymbol{\omega}\times\Delta t\cdot\mathbf{v}_n^t}{\Delta t} = \boldsymbol{\omega}\times\mathbf{v}_n^t$$

Rotating Disk: Coriolis Acceleration(6/9)

A point "A" is moving along a slot with a constant velocity, and the slot is on a disk rotating with a constant angular velocity.

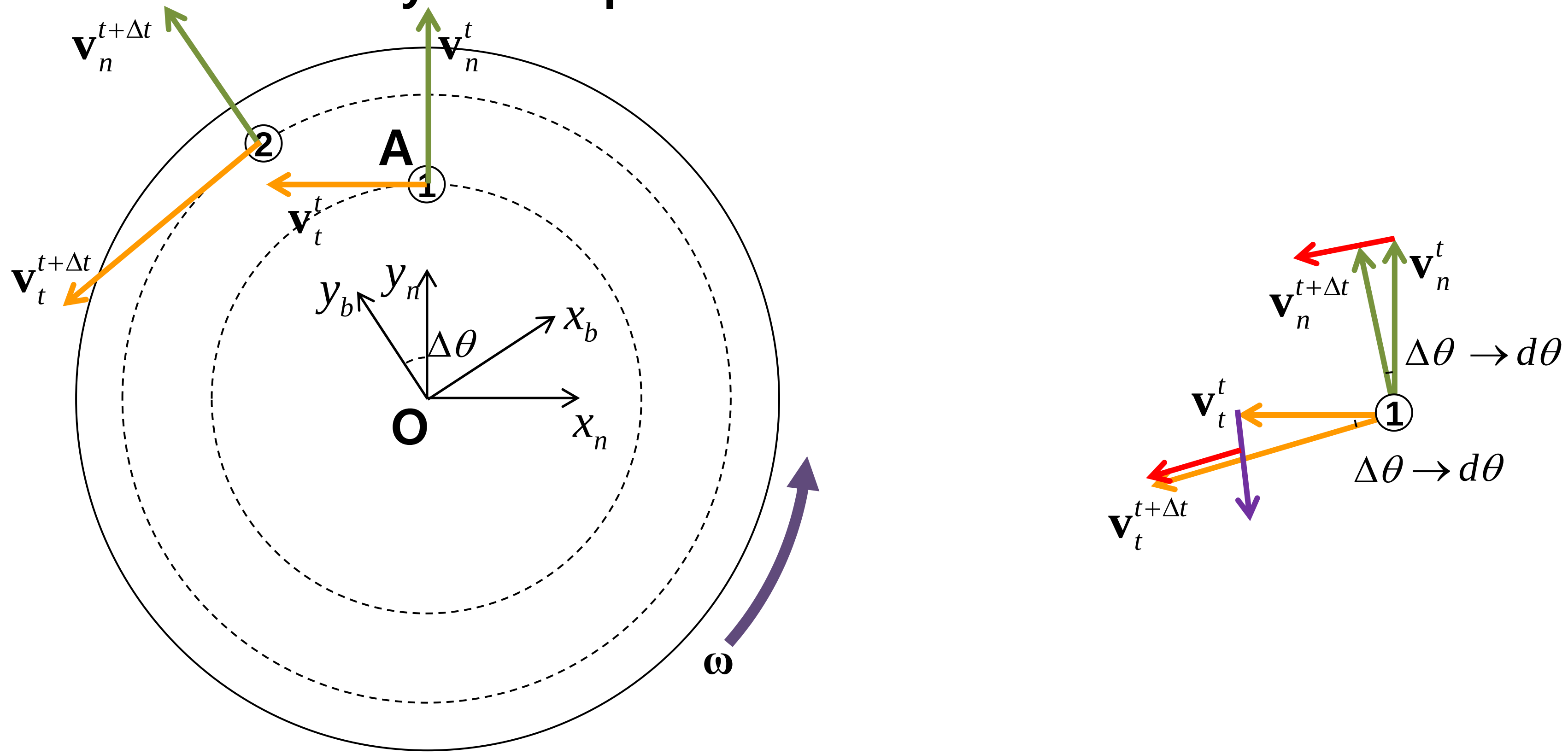
Velocity of the point A observed in n-frame.



n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

Rotating Disk - Coriolis Acceleration(7/9)

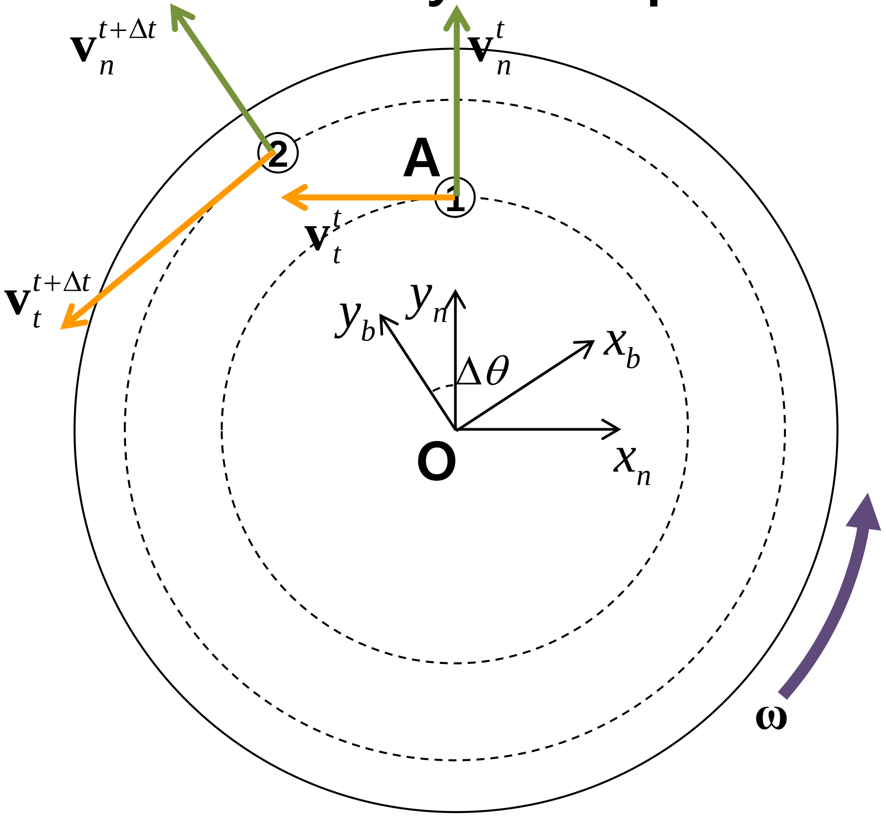
Velocity of the point A observed in n-frame.

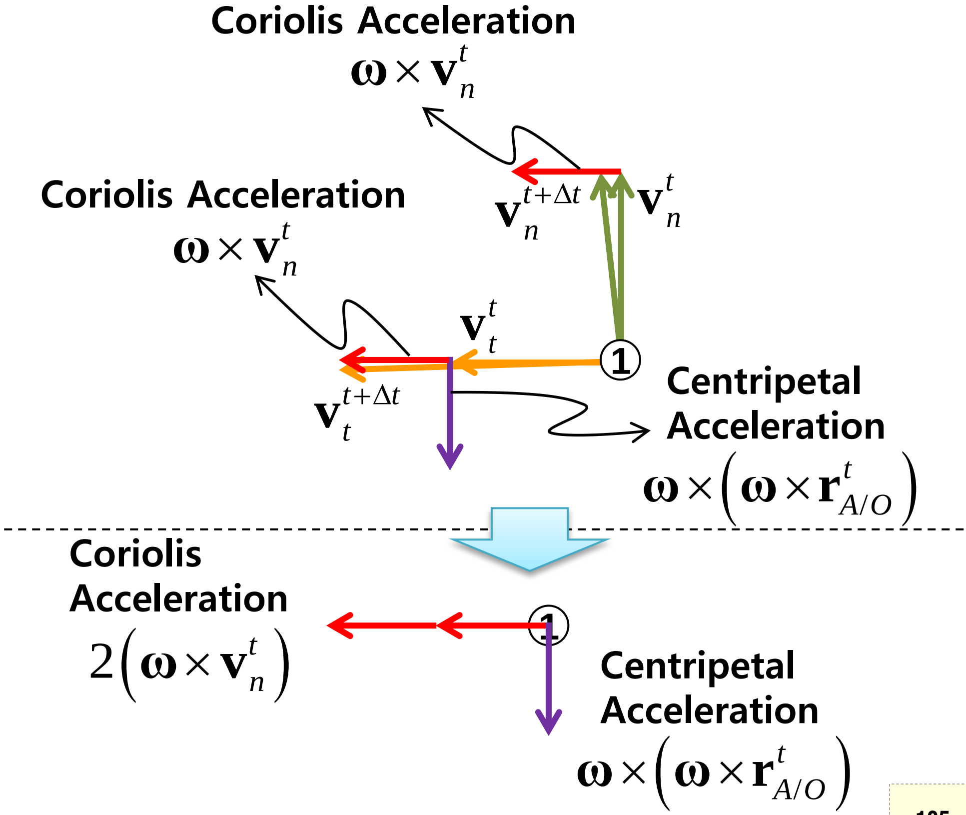


n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

Rotating Disk: Coriolis Acceleration
- observed in n-frame (8/9)

Velocity of the point A observed in n-frame.





Coriolis Acceleration

$\boldsymbol{\omega}\times\mathbf{v}_n^t$

Coriolis Acceleration

$\boldsymbol{\omega}\times\mathbf{v}_n^t$

Centripetal Acceleration

$\boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}_{A/O}^t\right)$

Coriolis Acceleration

$2\left(\boldsymbol{\omega}\times\mathbf{v}_n^t\right)$

Centripetal Acceleration

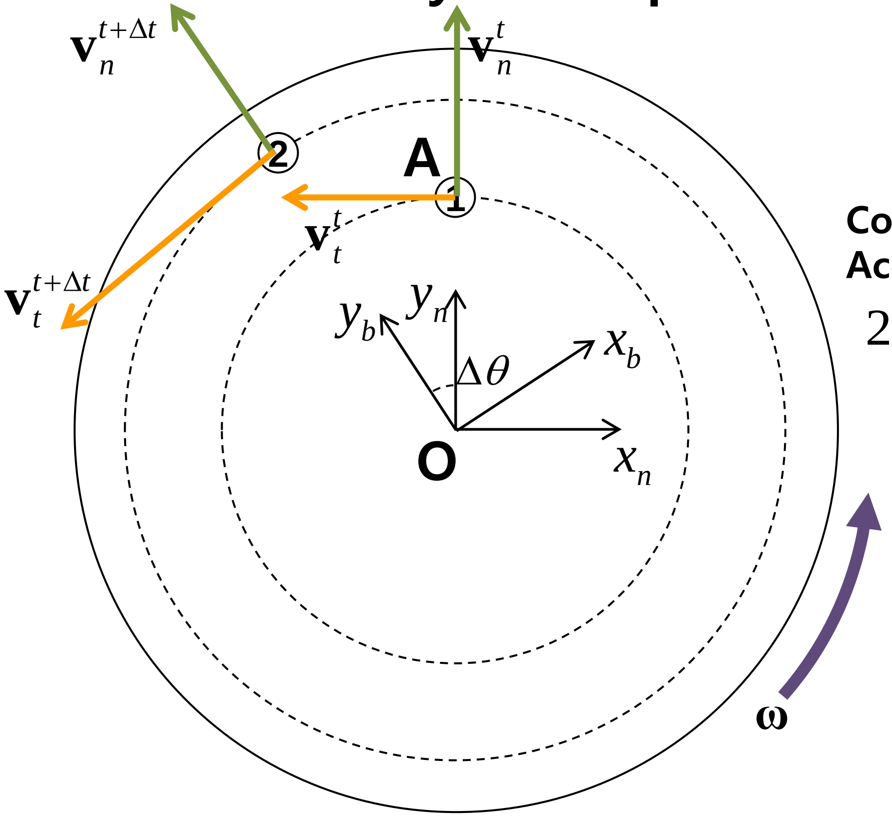
$\boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}_{A/O}^t\right)$

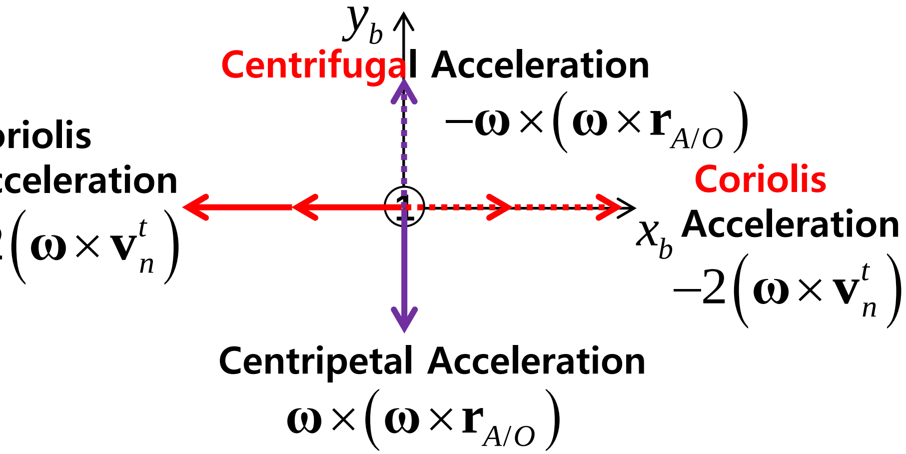
n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

Rotating Disk : Coriolis Acceleration
- observed in b-frame (9/9)

A point "A" is moving along a slot with a constant velocity, and the slot is on a disk rotating with a constant angular velocity.

Velocity of the point A observed in b-frame.





Centrifugal Acceleration

$-\omega\times(\omega\times\mathbf{r}_{A/O})$

Coriolis Acceleration

$2(\omega\times\mathbf{v}_n^t)$

Coriolis Acceleration

$-2(\omega\times\mathbf{v}_n^t)$

Centripetal Acceleration

$\omega\times(\omega\times\mathbf{r}_{A/O})$

Since, the point A observed in b-frame is not accelerated, there should be an additional force exerted on the point A except the centripetal force.

The additional force is a centrifugal force and Coriolis force

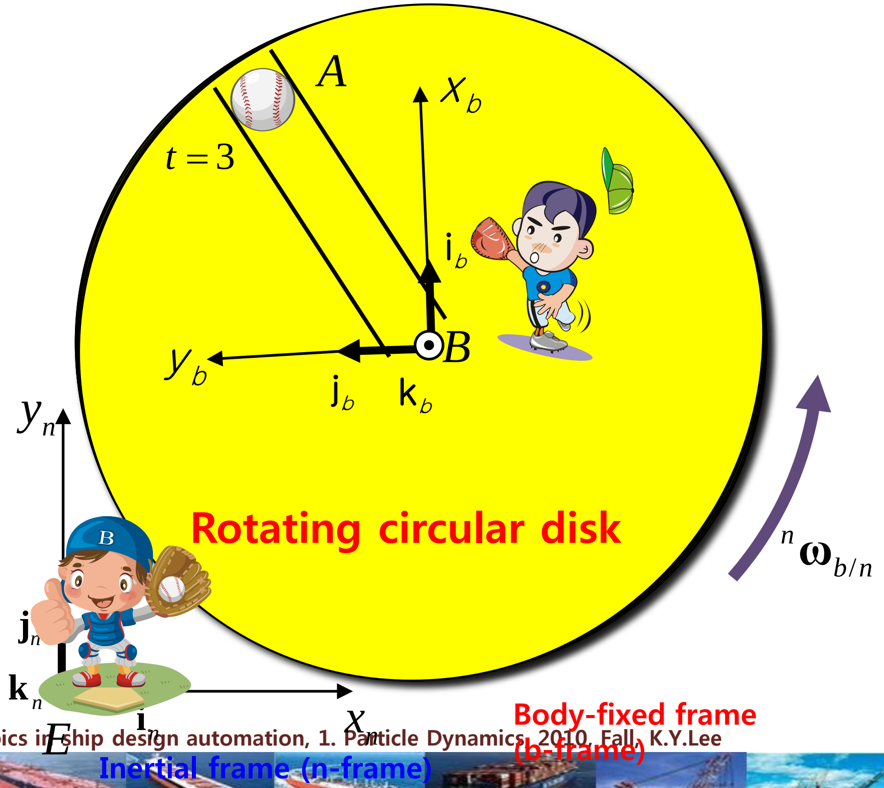
n-frame: an inertial frame.
b-frame: a frame fixed on the center of the disk.

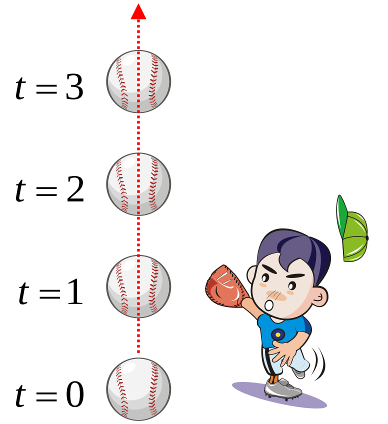
1.9 Motion of a ball
1) observed in rotational frame
2) observed in inertial frame

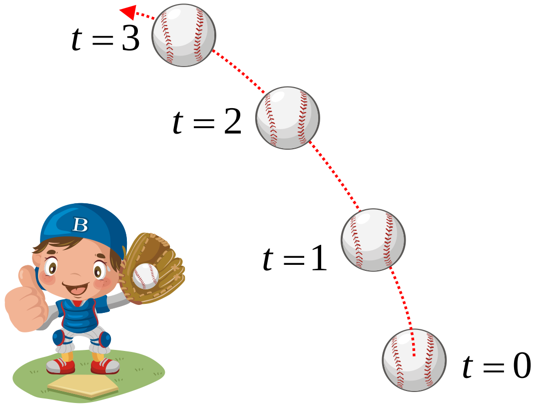
Example 3: Motion of a ball observed in the rotational frame and in the inertial frame

Person "B" is standing on the center of a large disk rotating with a constant angular velocity ${}^{n}\boldsymbol{\omega}_{b/n}$. He throws a ball "A" and the ball moves in a slot in the disk with a constant velocity.
Person "E" is standing still on the ground next to the disk. He also observes the ball "A".

Describe the motion of the ball from the person "B" and "E" respectively.







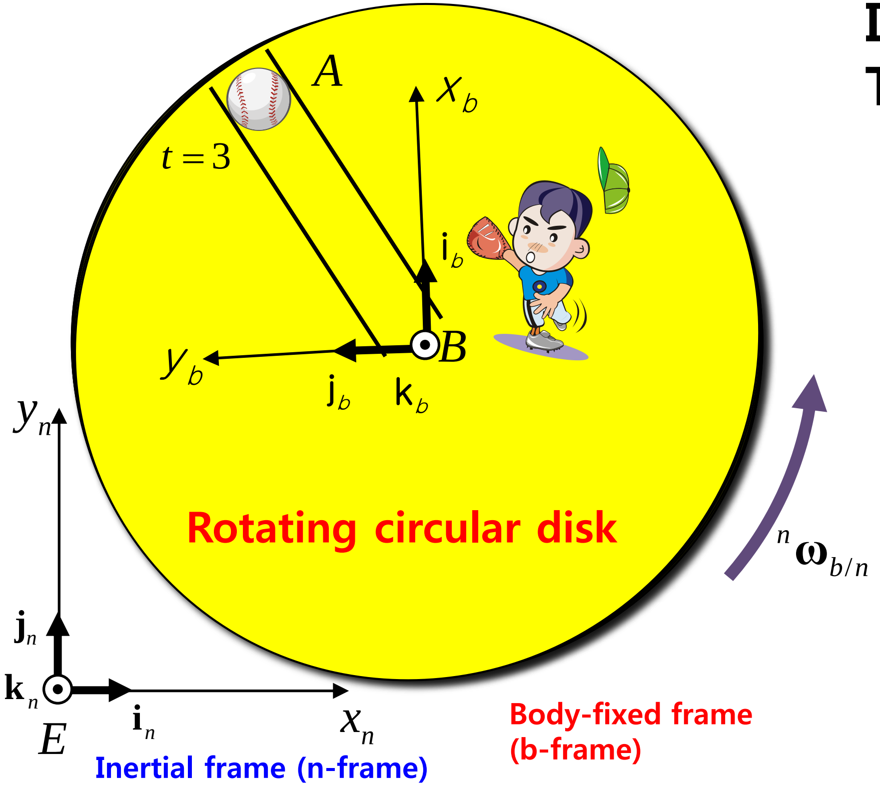
Velocity vector of the point with respect to rotating reference frame – example) rotating disk

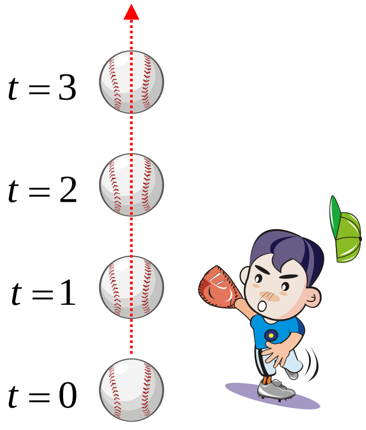
Given: ☐ **Find:** ☐

Person "B" is standing on the center of a large disk rotating with a constant angular velocity ${}^{n}\boldsymbol{\omega}_{b/n}$. He throws a ball "A" and the ball moves in a slot in the disk with a constant velocity.

$$\mathbf{v}_A = \boxed{\mathbf{v}_B} + \boxed{\mathbf{v}_{A,\,rel}} + \boxed{\boldsymbol{\omega}} \times \boxed{\mathbf{r}_{A/B}}$$

In this equation,
To find $\mathbf{v}_A$, $\mathbf{v}_{A,\,rel}$ is "given" variable.





Given

$\mathbf{v}_{A,rel}$

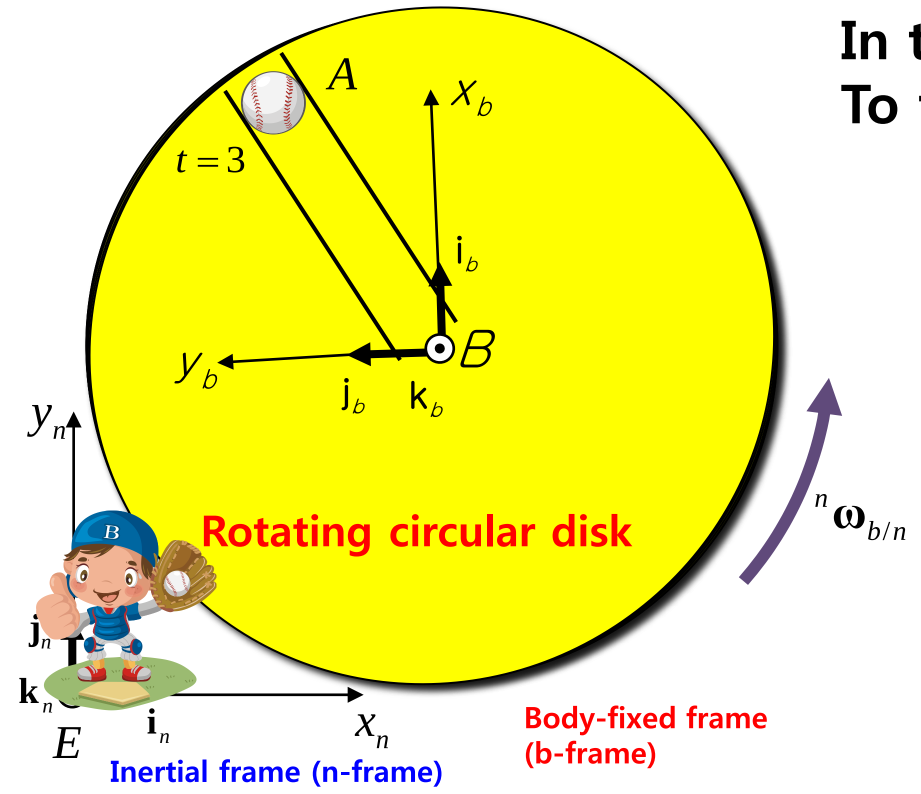
Velocity vector of the point with respect to rotating reference frame – example) rotating disk

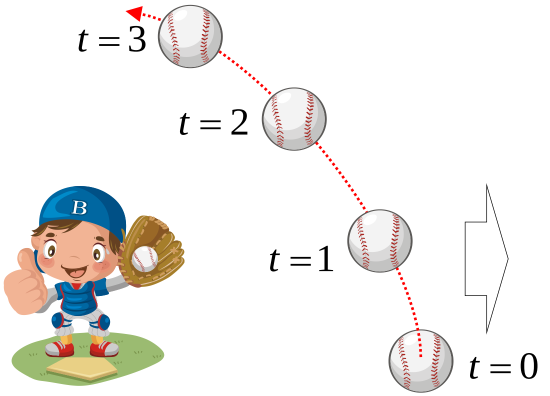
Given: ☐ Find: ☐

Person "E" is standing still on the ground next to the disk. He observes the ball "A" and the ball moves in a slot in the disk with a constant velocity.

$$\boxed{\mathbf{v}_A} = \boxed{\mathbf{v}_B} + \boxed{\mathbf{v}_{A,\,rel}} + \boxed{\boldsymbol{\omega}} \times \boxed{\mathbf{r}_{A/B}}$$

In this equation,
To find $\mathbf{v}_A$,$\mathbf{v}_{A,\,rel}$ is "given" variable.



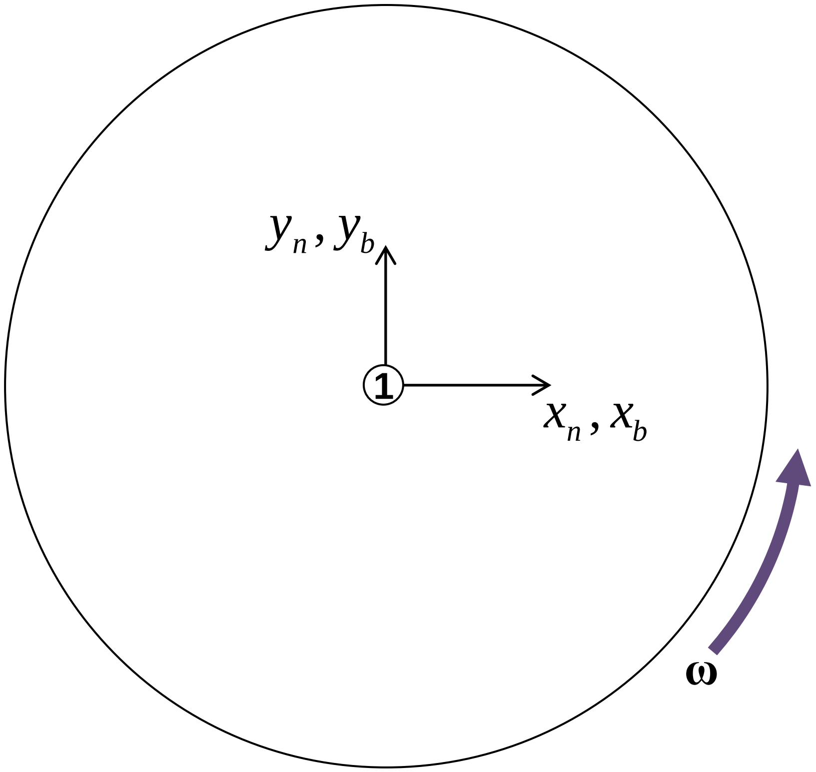


Find

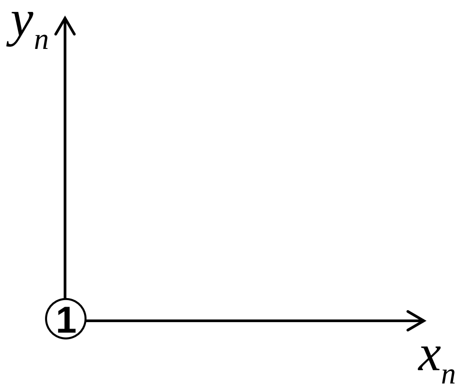
$\mathbf{v}_A$

Position vector of the point with respect to rotating reference frame – example) rotating disk

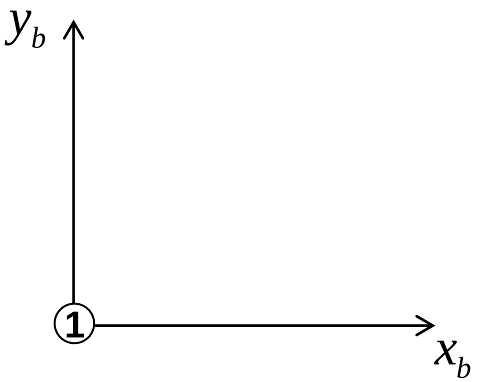
The position of point A



Description of the motion of the ball from the person "B".
→The position vector of point A
expressed in terms of unit vectors of n-frame



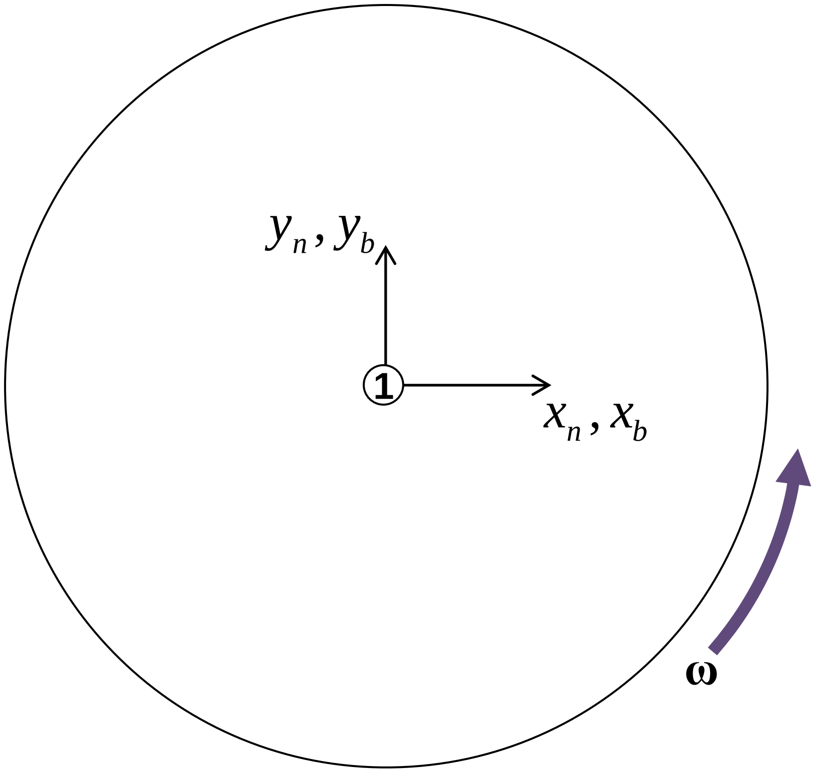
Description of the motion of the ball from the person "E".
→The position vector of point A
expressed in terms of unit vectors of b-frame

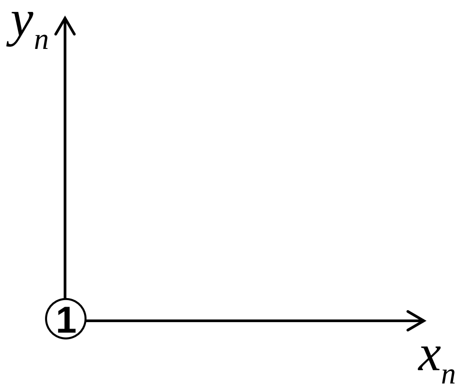


Position vector of the point with respect to rotating reference frame – example) rotating disk

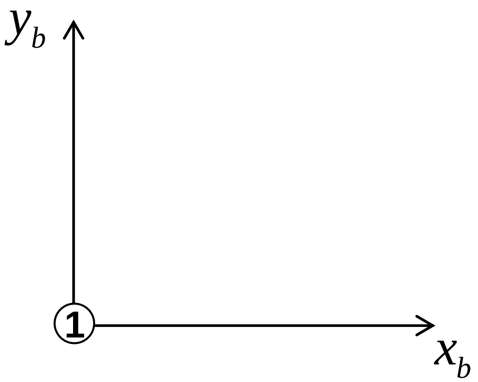
Description of the motion of the ball from the person "B".
→The position vector of point A
expressed in terms of unit vectors of n-frame

The position of point A





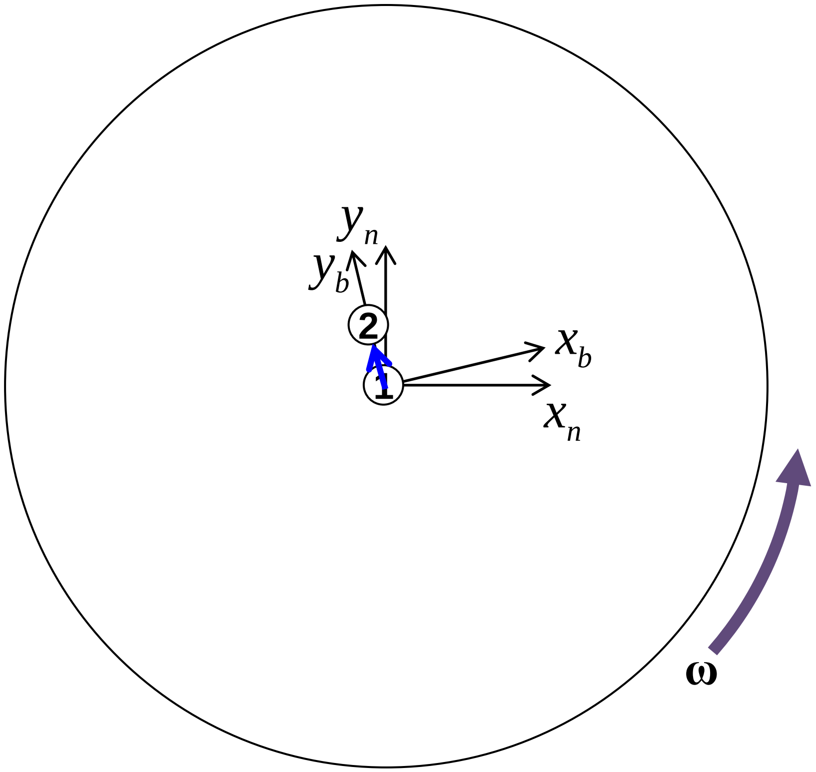
Description of the motion of the ball from the person "E".
→The position vector of point A
expressed in terms of unit vectors of b-frame

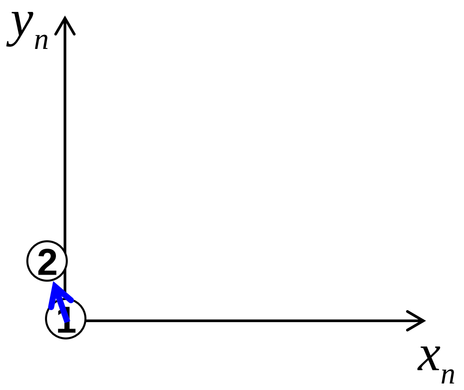


Position vector of the point with respect to rotating reference frame – example) rotating disk

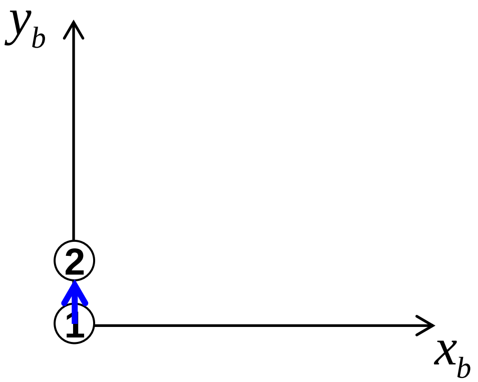
Description of the motion of the ball from the person “B”.
→The position vector of point A
expressed in terms of unit vectors of n-frame

The position of point A



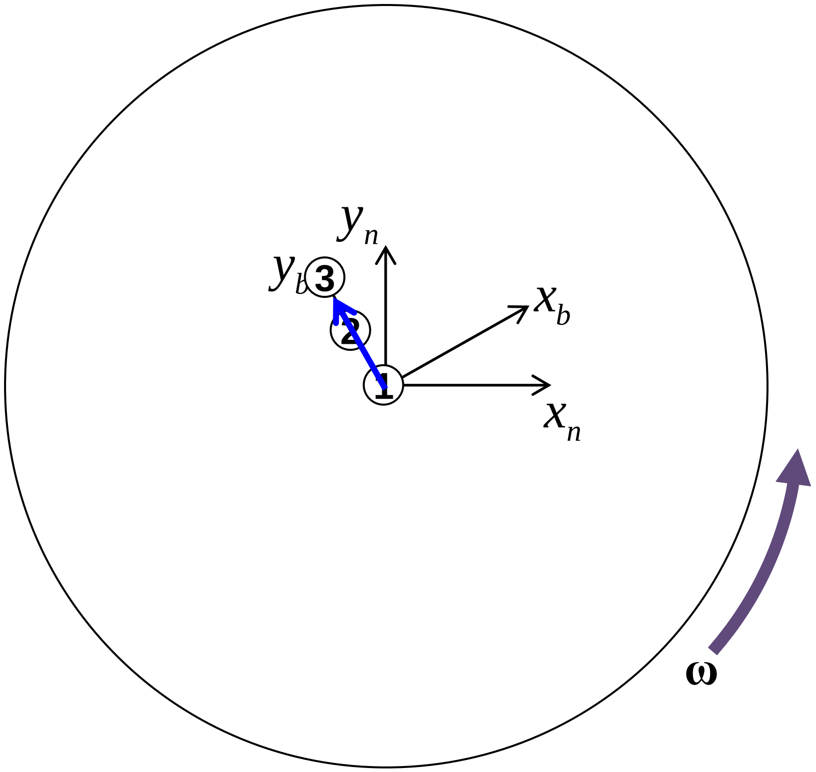


Description of the motion of the ball from the person “E”.
→The position vector of point A
expressed in terms of unit vectors of b-frame

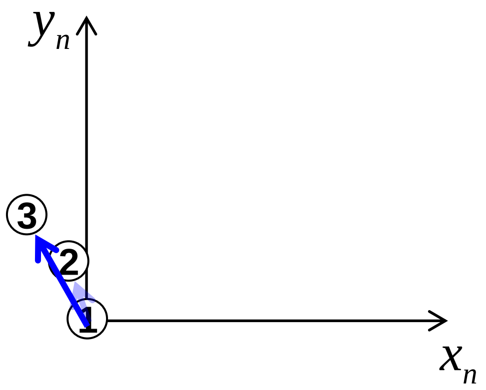


Position vector of the point with respect to rotating reference frame – example) rotating disk

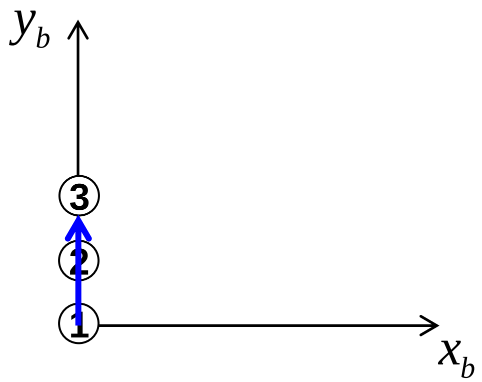
The position of point A



Description of the motion of the ball from the person "B".
→The position vector of point A
expressed in terms of unit vectors of n-frame



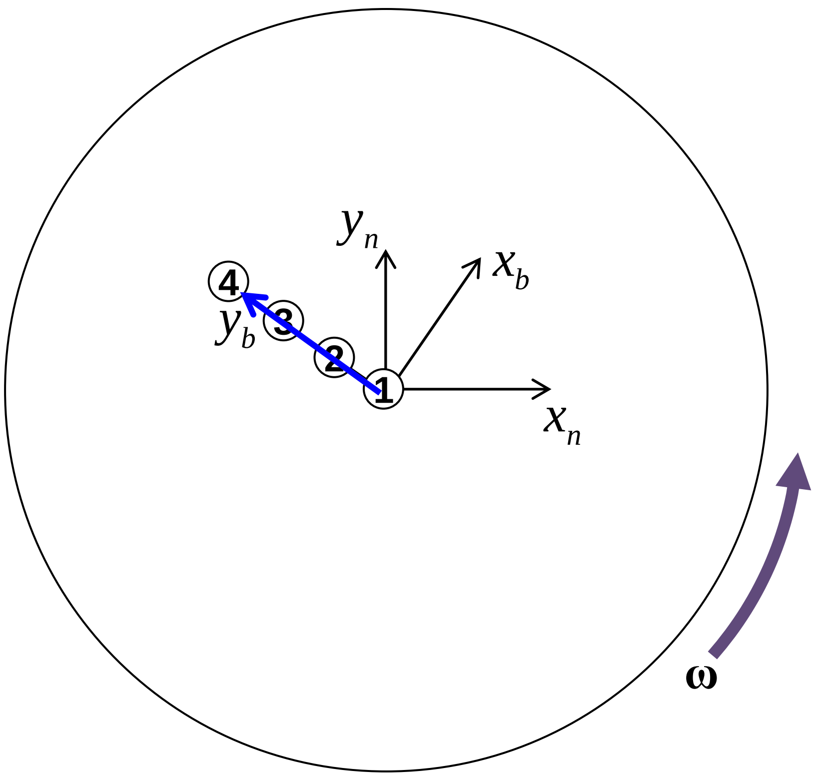
Description of the motion of the ball from the person "E".
→The position vector of point A
expressed in terms of unit vectors of b-frame

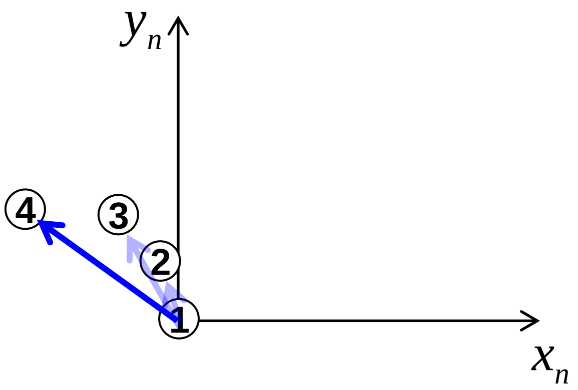


Position vector of the point with respect to rotating reference frame – example) rotating disk

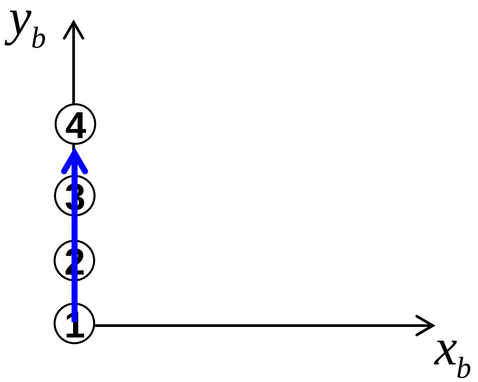
Description of the motion of the ball from the person "B".
→The position vector of point A
expressed in terms of unit vectors of n-frame

The position of point A



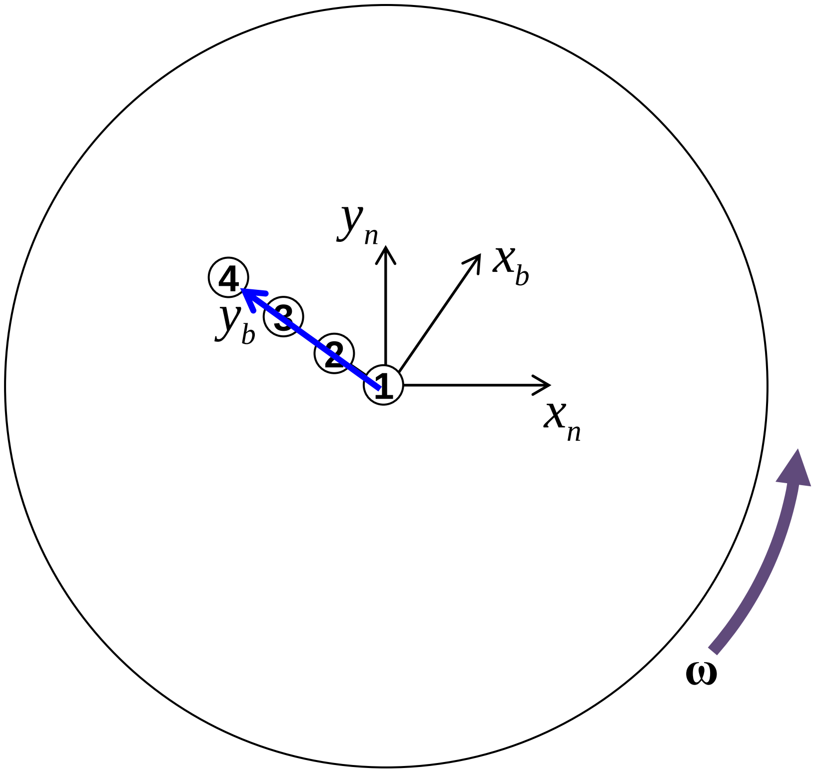


Description of the motion of the ball from the person "E".
→The position vector of point A
expressed in terms of unit vectors of b-frame



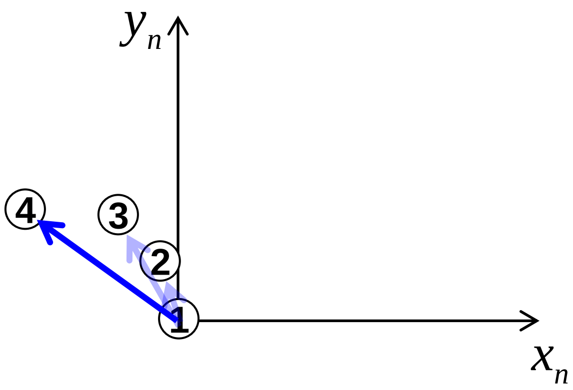
Position vector of the point with respect to rotating reference frame – example) rotating disk

The position of point A



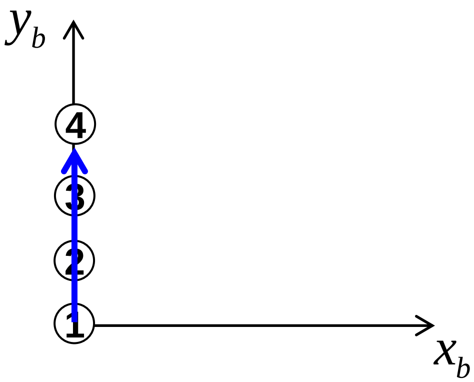
Description of the motion of the ball from the person "B".

→The position vector of point A
expressed in terms of unit vectors of n-frame



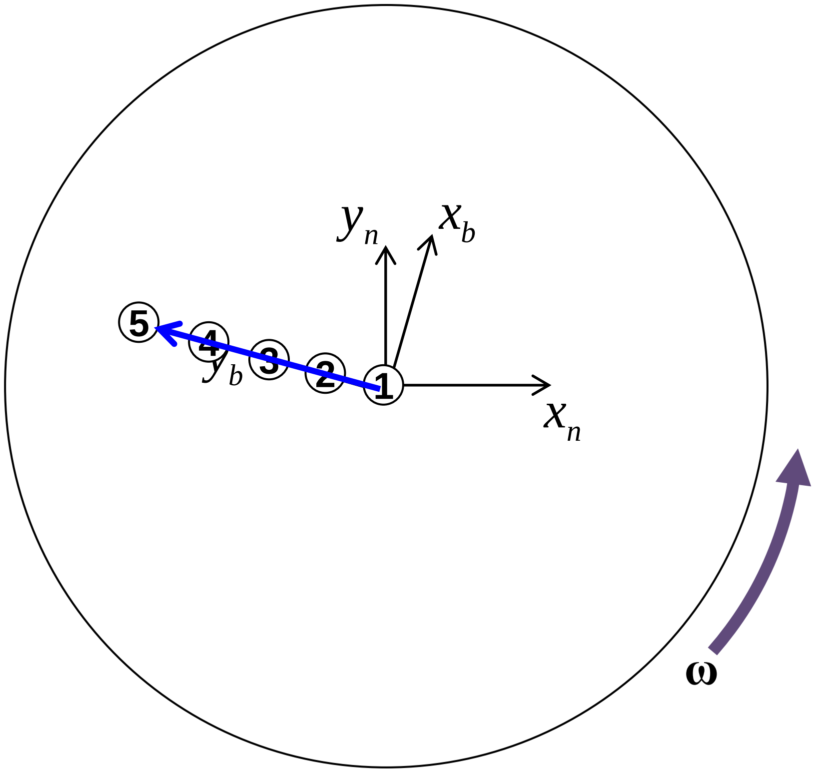
Description of the motion of the ball from the person "E".

→The position vector of point A
expressed in terms of unit vectors of b-frame

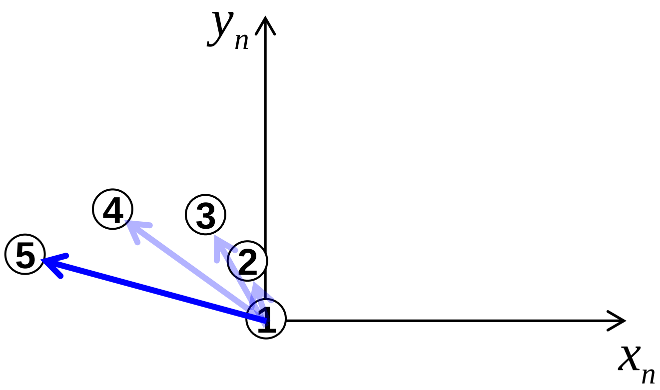


Position vector of the point with respect to rotating reference frame – example) rotating disk

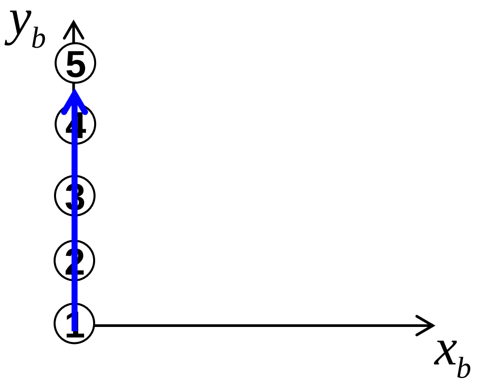
The position of point A



Description of the motion of the ball from the person "B".
→The position vector of point A
expressed in terms of unit vectors of n-frame

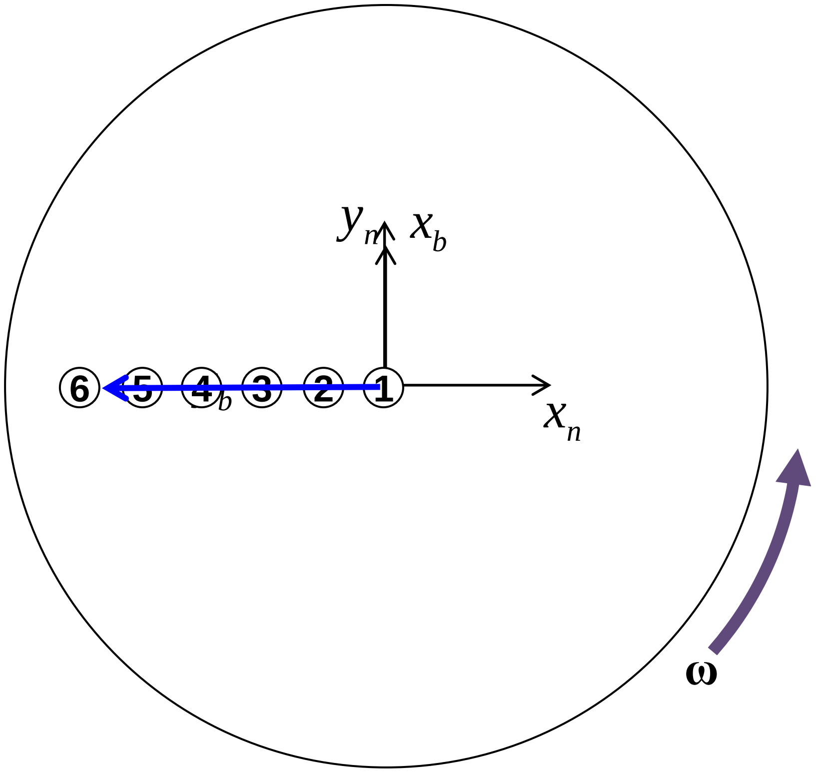


Description of the motion of the ball from the person "E".
→The position vector of point A
expressed in terms of unit vectors of b-frame

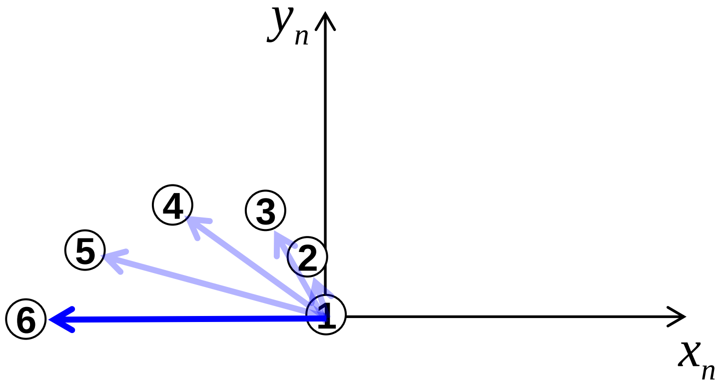


Position vector of the point with respect to rotating reference frame – example) rotating disk

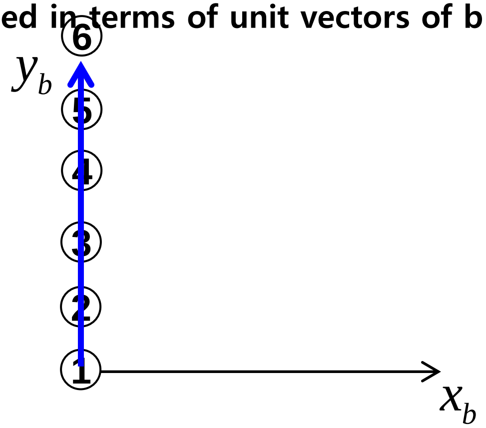
The position of point A



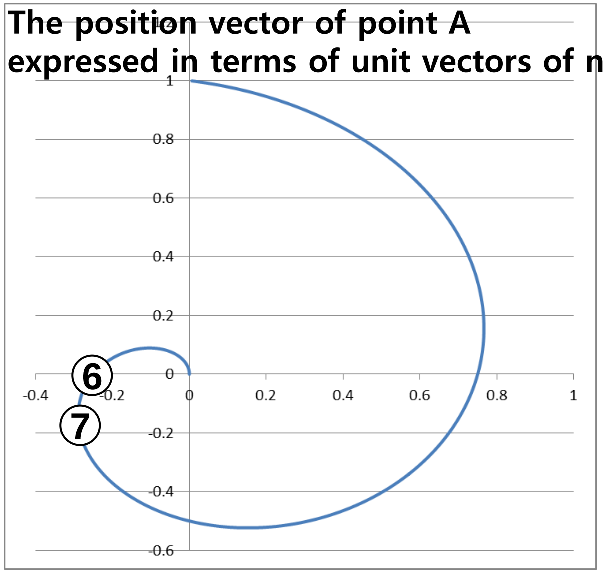
Describe the motion of the ball from the person "B".
→The position vector of point A expressed in terms of unit vectors of n-frame



Describe the motion of the ball from the person "E".
→The position vector of point A expressed in terms of unit vectors of b-frame

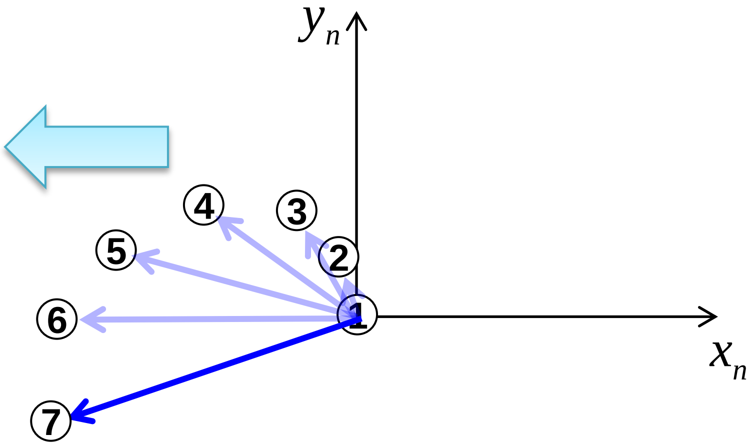


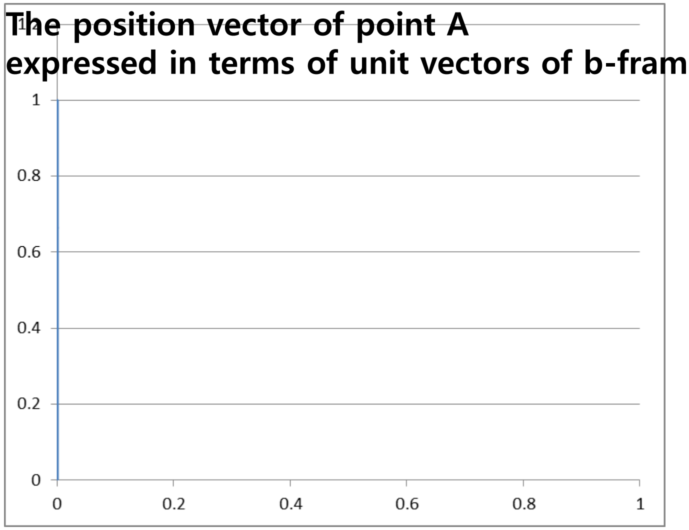
Position vector of the point with respect to rotating reference frame – example) rotating disk



The position vector of point A
expressed in terms of unit vectors of n-frame

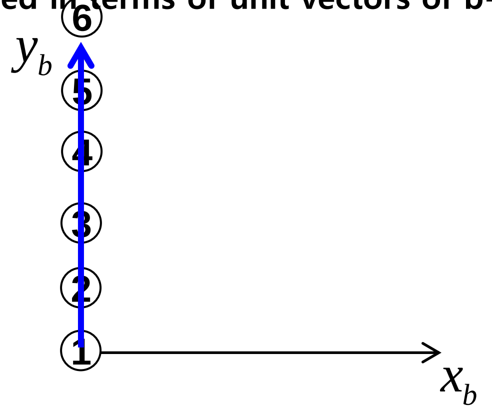
→The position vector of point A
expressed in terms of unit vectors of n-frame





The position vector of point A
expressed in terms of unit vectors of b-frame

→The position vector of point A
expressed in terms of unit vectors of b-frame



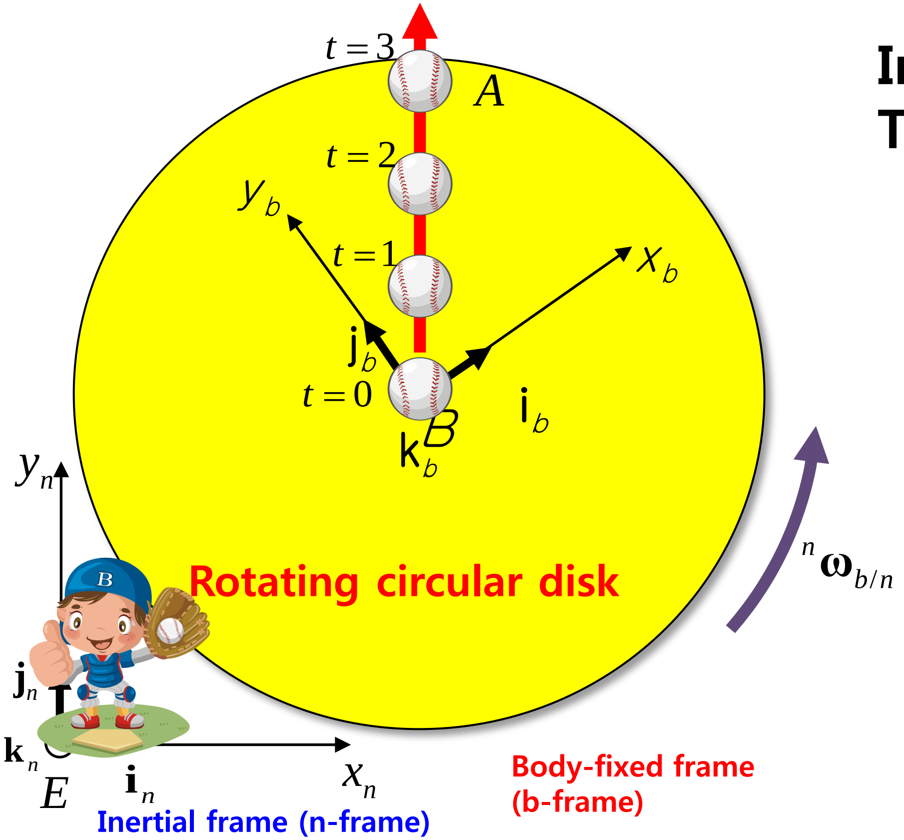
Velocity vector of the point with respect to rotating reference frame – example) rotating disk

Given: ☐ Find: ☐

Person "E" is standing on the ground next to the disk. He observes a ball "A" and the ball moves horizontally with a constant velocity $\mathbf{v}_A$.

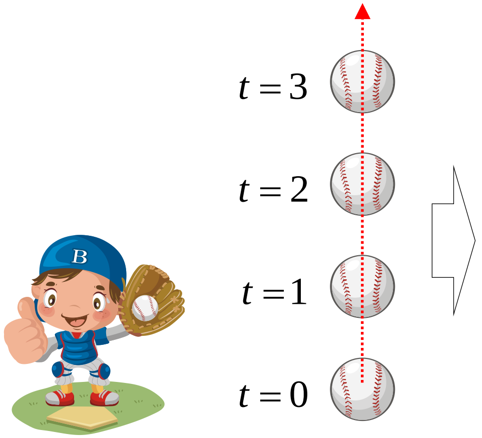
$$\boxed{\mathbf{v}_A} = \boxed{\mathbf{v}_B} + \mathbf{v}_{A,\,rel} + \boxed{\boldsymbol{\omega}} \times \boxed{\mathbf{r}_{A/B}}$$

In this equation,
To find $\mathbf{v}_{A,\,rel}$, $\mathbf{v}_A$ is "given" variable.



Body-fixed frame (b-frame)

Inertial frame (n-frame)



Given

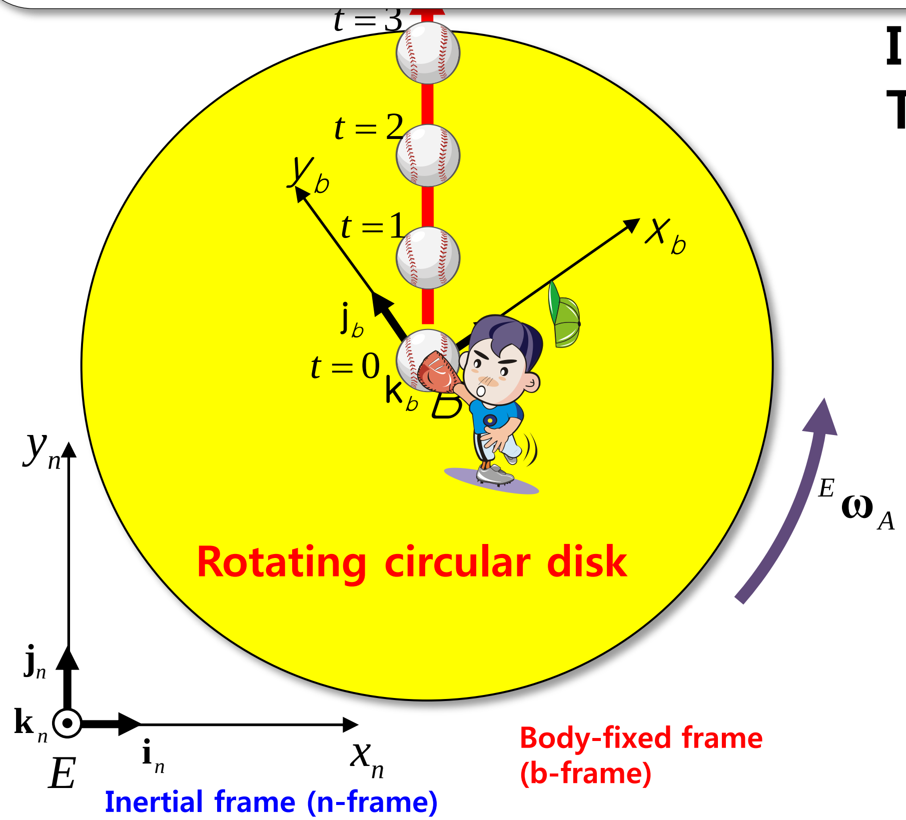
$\mathbf{v}_A$

Velocity vector of the point with respect to rotating reference frame – example) rotating disk

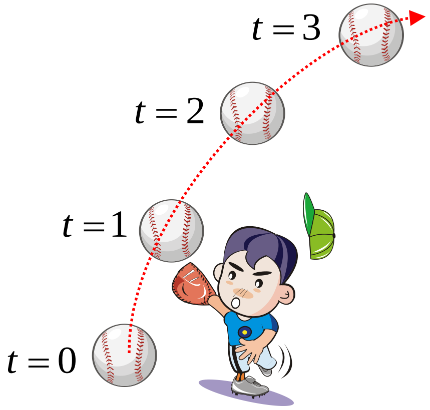
Given: ☐ Find: ☐

Person "B" is standing on the center of a large disk rotating with a constant angular velocity ${}^{n}\boldsymbol{\omega}_{b/n}$. He throws a ball "A" horizontally with a constant velocity $\mathbf{v}_A$

$$\boxed{\mathbf{v}_A}=\boxed{\mathbf{v}_B}+\boxed{\mathbf{v}_{A,\ rel}}+\boxed{\boldsymbol{\omega}}\times\boxed{\mathbf{r}_{A/B}}$$



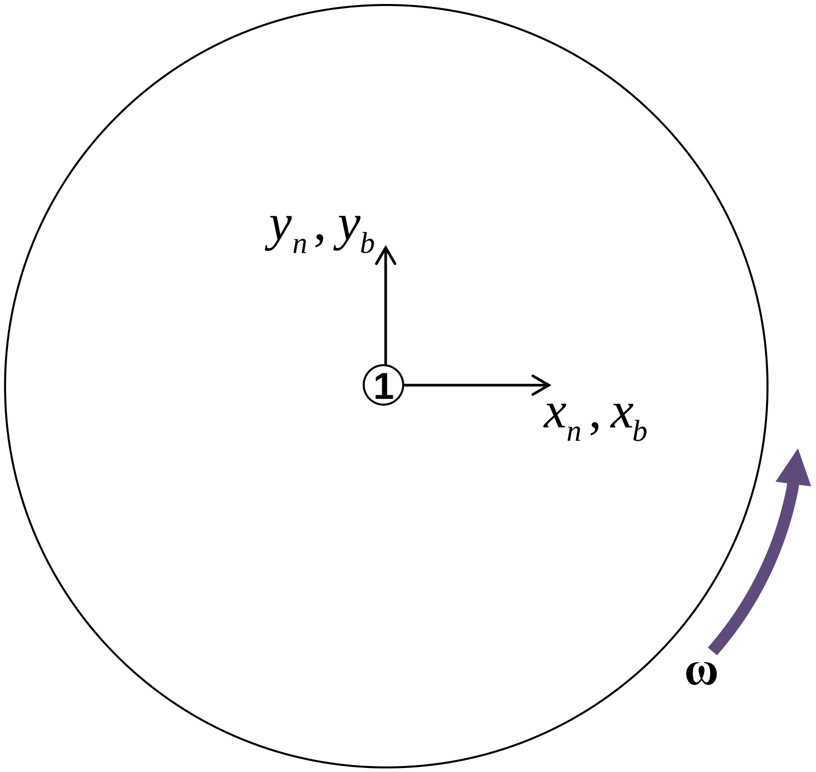
In this equation,
To find $\mathbf{v}_{A,\ rel}$, $\mathbf{v}_A$ is "given" variable.



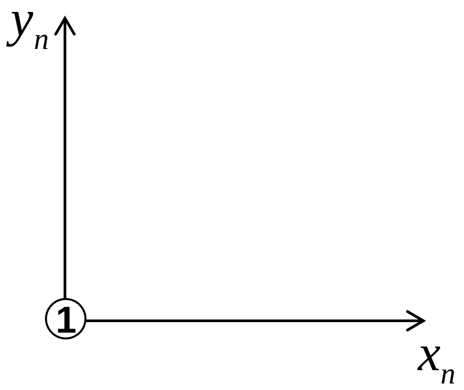
Find

$\mathbf{v}_{A,rel}$

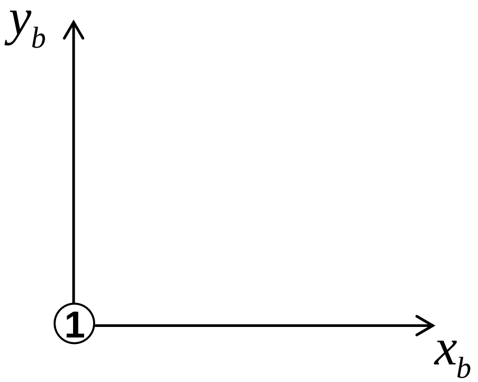
Position vector of the point with respect to rotating reference frame – example) rotating disk



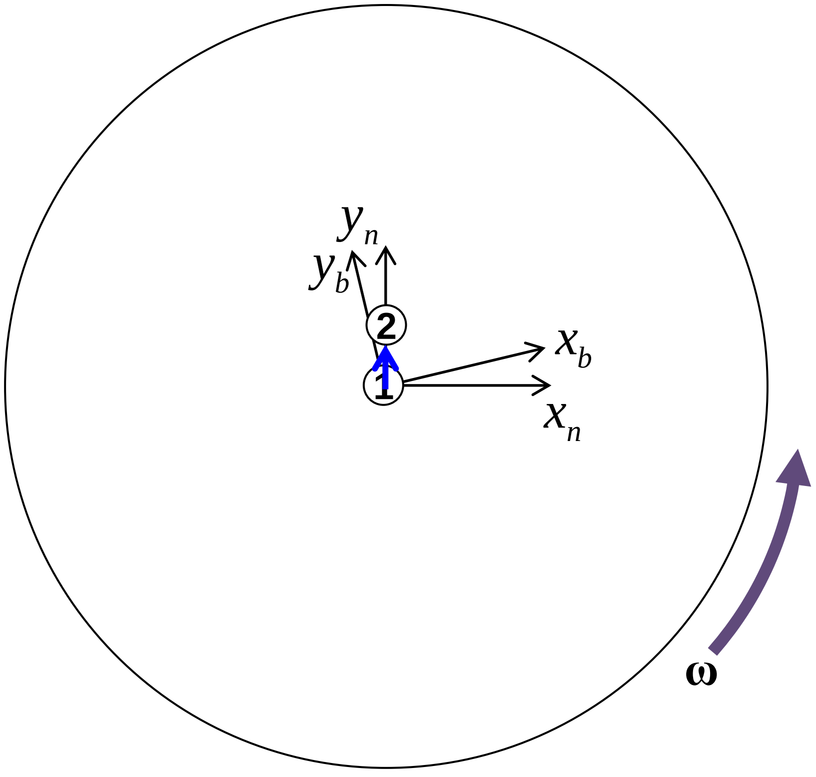
The position vector of the point A from the point B expressed in terms of unit vectors of n-frame



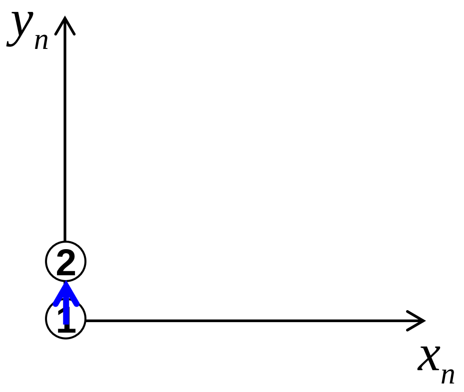
The position vector of the point A from the point B expressed in terms of unit vectors of b-frame



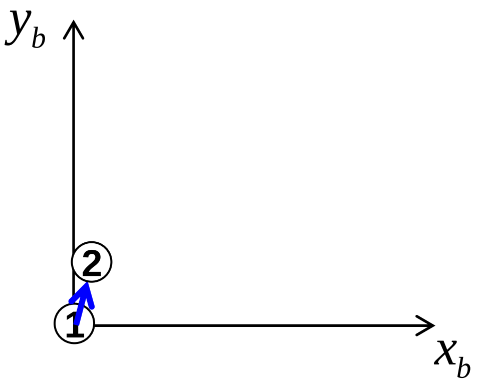
Position vector of the point with respect to rotating reference frame – example) rotating disk



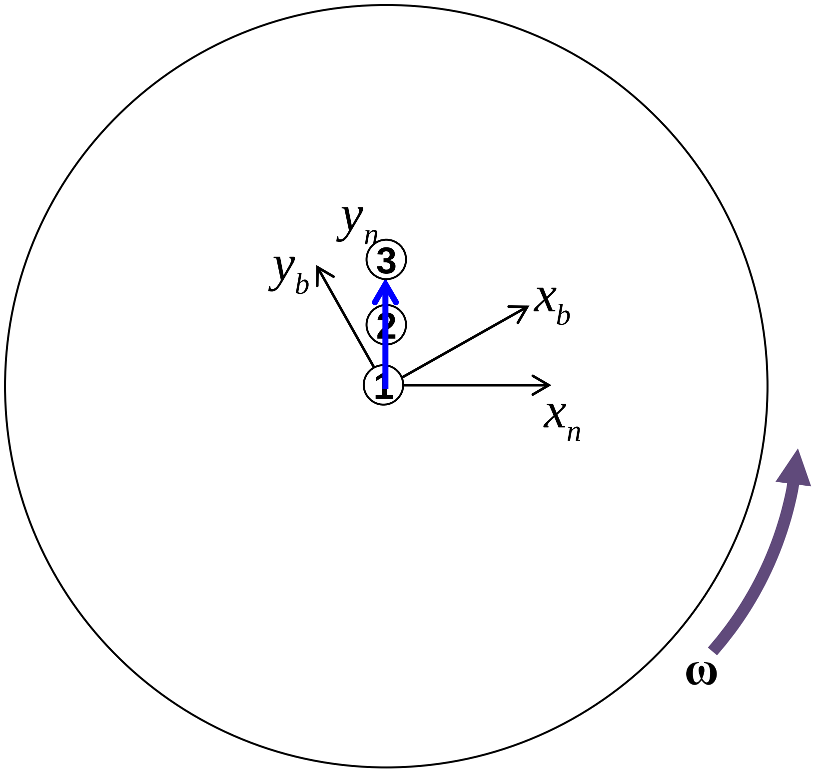
The position vector of the point A from the point B expressed in terms of unit vectors of n-frame



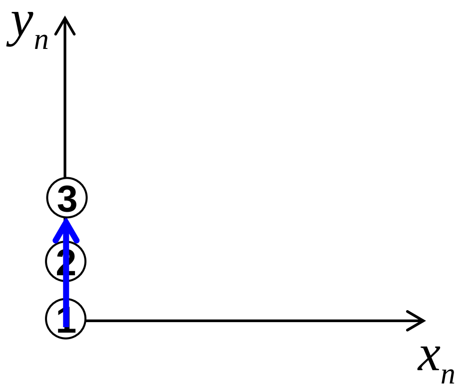
The position vector of the point A from the point B expressed in terms of unit vectors of b-frame



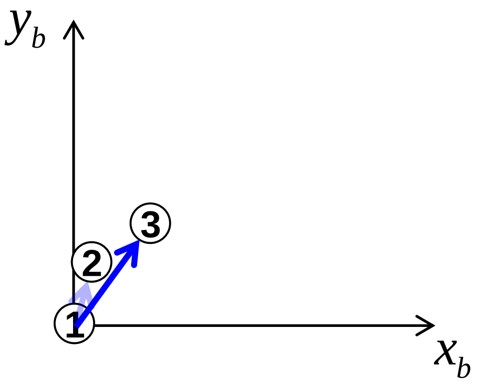
Position vector of the point with respect to rotating reference frame – example) rotating disk



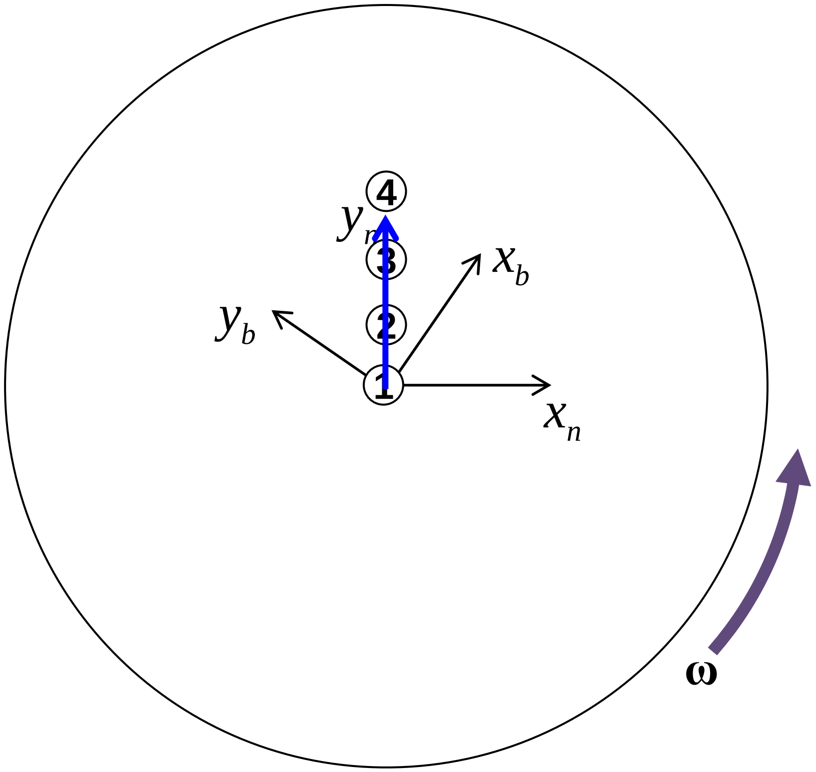
The position vector of the point A from the point B expressed in terms of unit vectors of n-frame



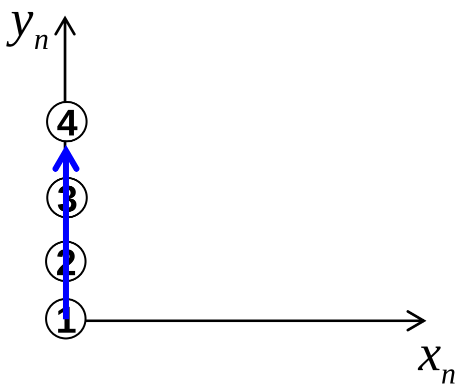
The position vector of the point A from the point B expressed in terms of unit vectors of b-frame



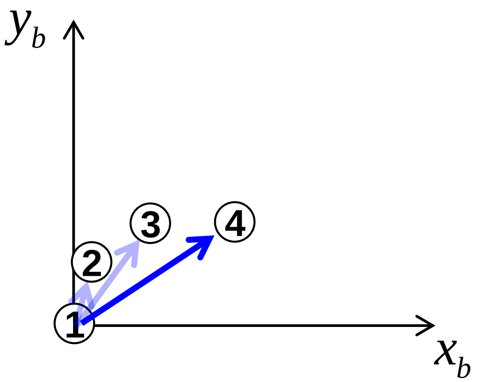
Position vector of the point with respect to rotating reference frame – example) rotating disk



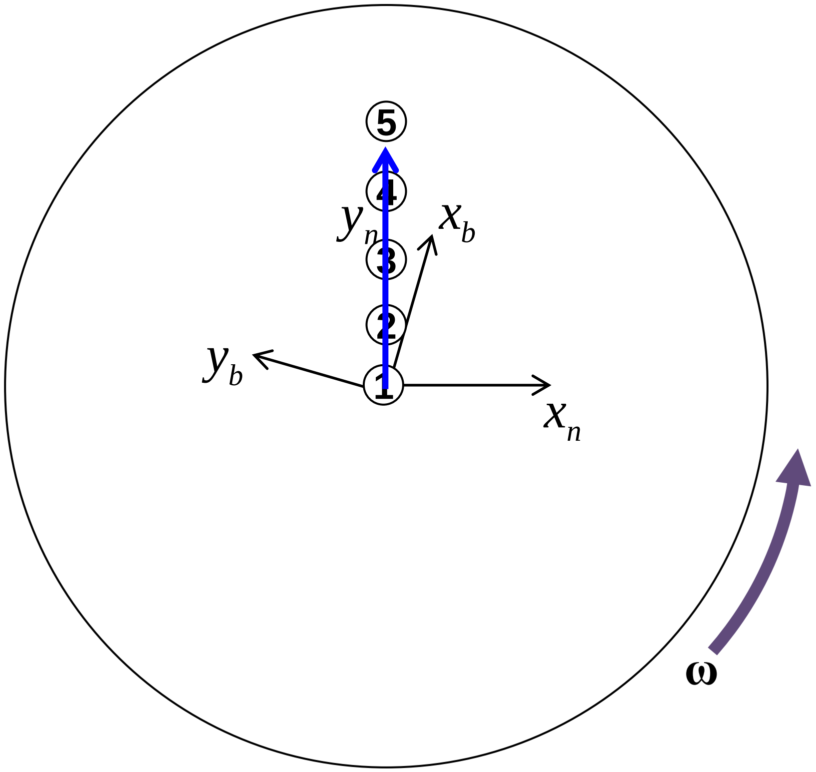
The position vector of the point A from the point B expressed in terms of unit vectors of n-frame



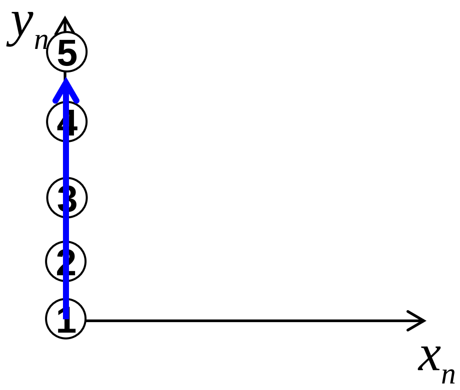
The position vector of the point A from the point B expressed in terms of unit vectors of b-frame



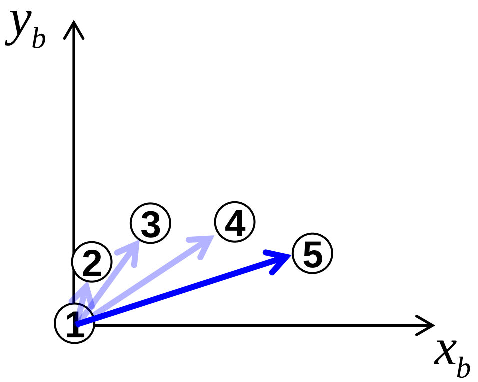
Position vector of the point with respect to rotating reference frame – example) rotating disk



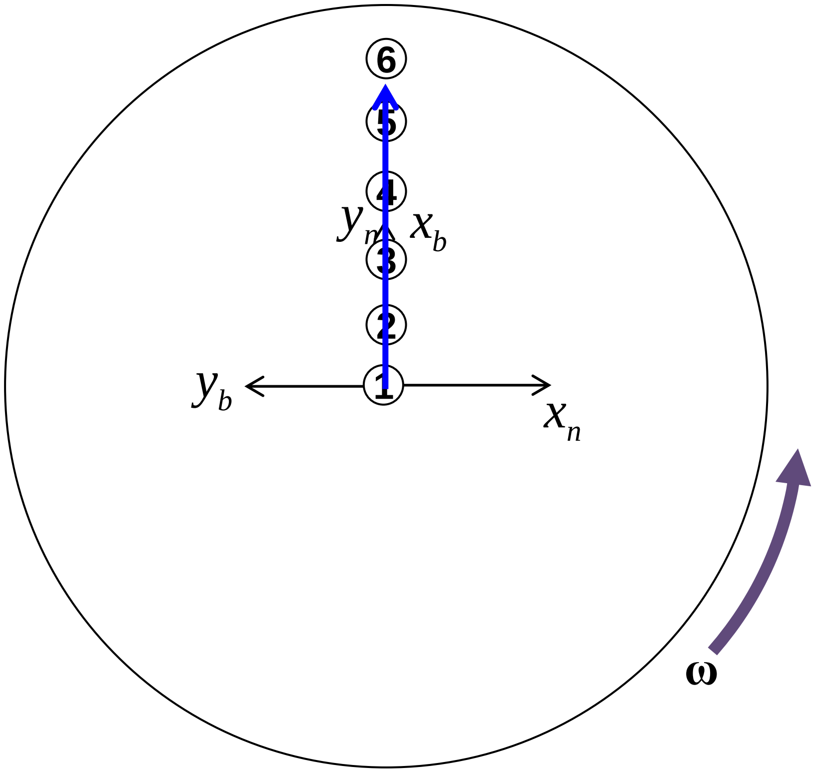
The position vector of the point A from the point B expressed in terms of unit vectors of n-frame



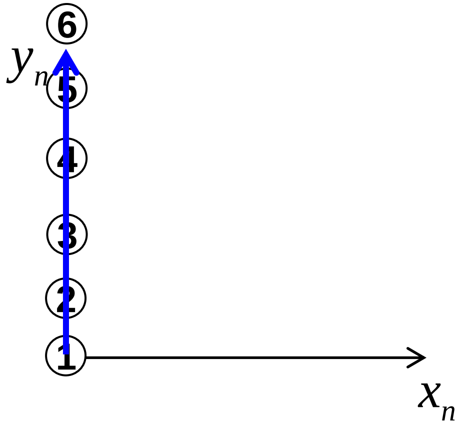
The position vector of the point A from the point B expressed in terms of unit vectors of b-frame



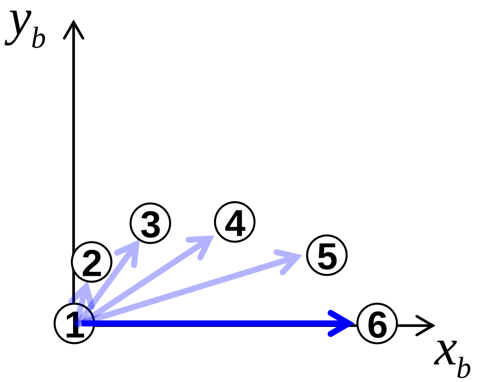
Position vector of the point with respect to rotating reference frame – example) rotating disk



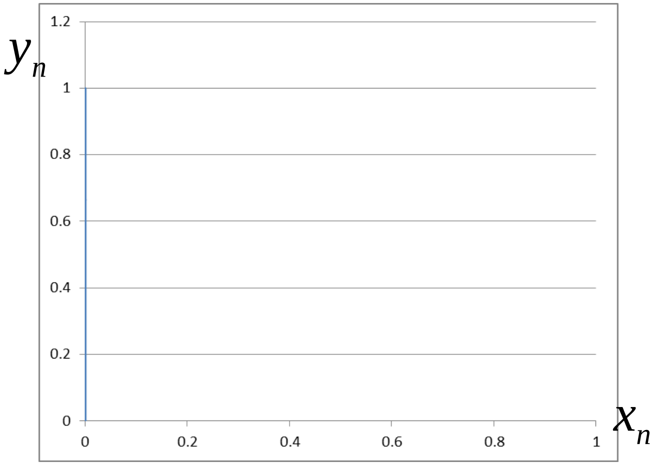
The position vector of the point A from the point B expressed in terms of unit vectors of n-frame



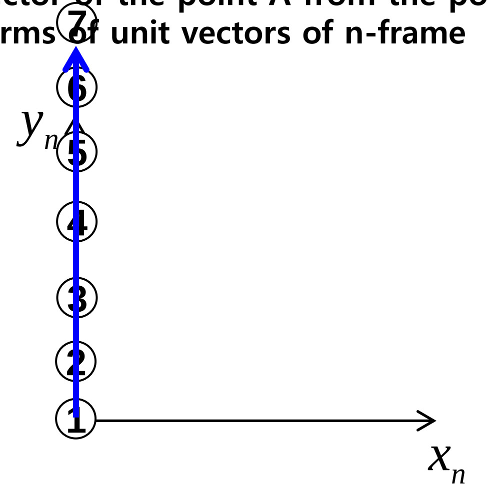
The position vector of the point A from the point B expressed in terms of unit vectors of b-frame

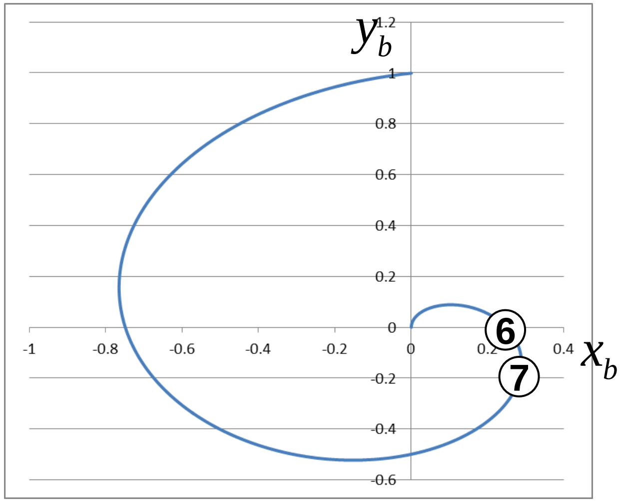


Position vector of the point with respect to rotating reference frame – example) rotating disk

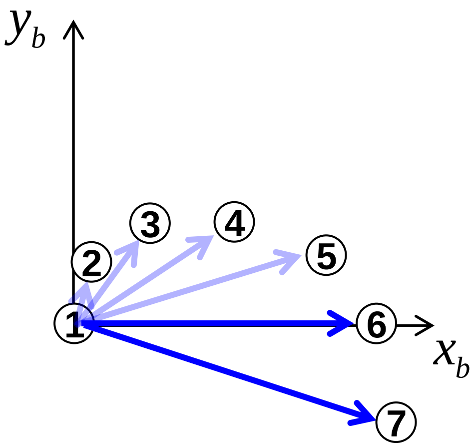


The position vector of the point A from the point B expressed in terms of unit vectors of n-frame





The position vector of the point A from the point B expressed in terms of unit vectors of b-frame



Naval Architecture & Ocean Engineering

Topics in ship design automation

2. Single Rigidbody Dynamics

Prof. Kyu-Yeul Lee

Fall, 2010

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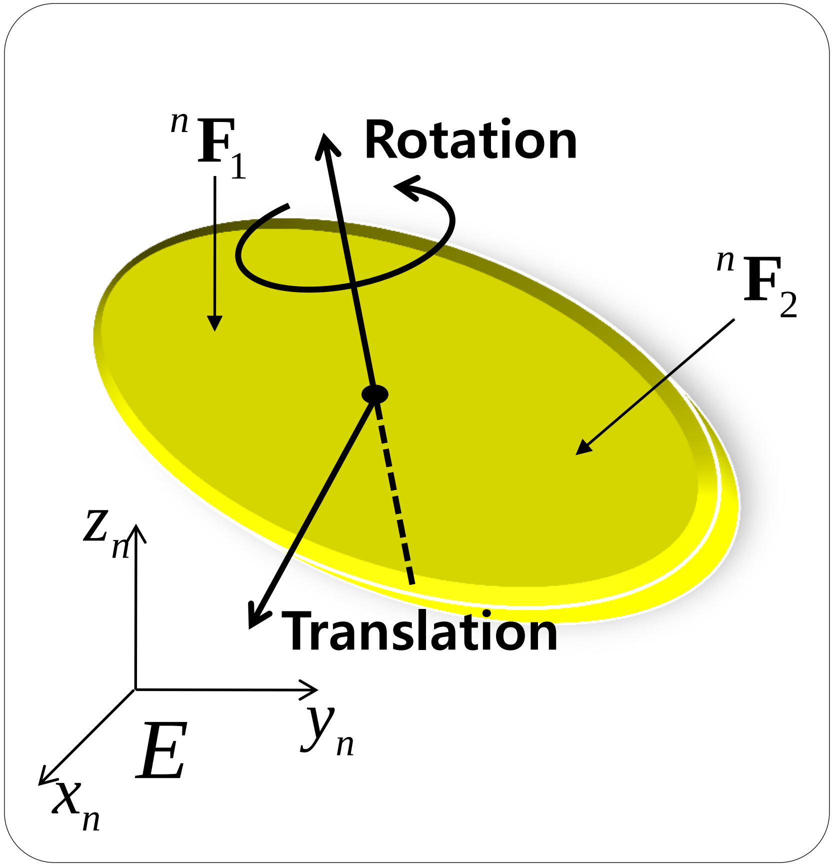
2.1 Derivation of Equations of rigid body motion

Spatial motion of the rigid body - Problem definition



1. Forces are acting on a rigid body.

2. Then the rigid body will translate and rotate.

3. We expect that the resultant force and moment are related to the translational and rotational motion.

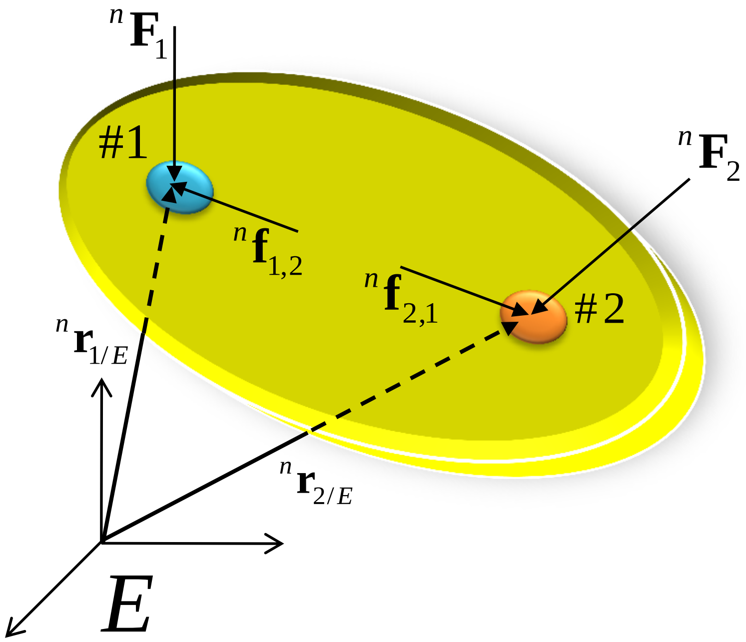
4. We shall confirm and quantify these expectations!!

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Translation of the rigid body in spatial motion - External and internal forces acting on the particles

We begin by considering a rigid-body composed of two particles.

The rigid body composed of two particles



$n-frame$: **The inertial frame**

m_1, m_2: **The mass of the each particle**

$$ {}^n\mathbf{F}_1,\ {}^n\mathbf{F}_2 $$

: The external forces acting on the each particle

$$ {}^n\mathbf{f}_{1,2},\ {}^n\mathbf{f}_{2,1} $$

: The interaction forces between the particles

$$ {}^n\mathbf{f}_{1,2} = -{}^n\mathbf{f}_{2,1} $$

: According to Newton's Third Law, a pair of interaction forces such as these are equal in magnitude and oppositely directed.

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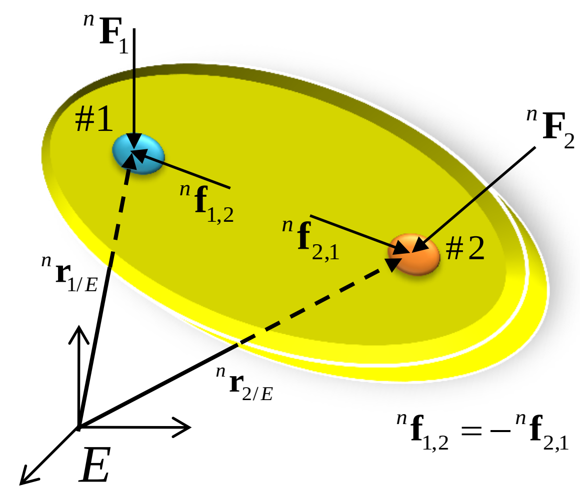
Translation of the rigid body in spatial(3D) motion - Newton equation for the rigid body(1/4)

According to Newton's Second Law

$$ {}^{n}\mathbf{F}_1 + {}^{n}\mathbf{f}_{1,2} = m_1 \frac{d^2\,{}^{n}\mathbf{r}_{1/E}}{dt^2}, \quad {}^{n}\mathbf{F}_2 + {}^{n}\mathbf{f}_{2,1} = m_2 \frac{d^2\,{}^{n}\mathbf{r}_{2/E}}{dt^2} $$



n-Frame: Inertial Frame

${}^{n}\mathbf{F}_j, {}^{n}\mathbf{F}_k$: The external forces exerted on the respective particles

${}^{n}\mathbf{f}_{k,j}, {}^{n}\mathbf{f}_{j,k}$: The interaction forces between the particles

m_1, m_2 : mass of the respective particles

Summation of the two equations of motions

$$ {}^{n}\mathbf{F}_1 + \cancel{{}^{n}\mathbf{f}_{1,2}} + {}^{n}\mathbf{F}_2 + \cancel{{}^{n}\mathbf{f}_{2,1}} = m_1 \frac{d^2\,{}^{n}\mathbf{r}_{1/E}}{dt^2} + m_2 \frac{d^2\,{}^{n}\mathbf{a}_{2/E}}{dt^2} $$

Internal(interaction) forces cancel in this sum.

$$ \boxed{{}^{n}\mathbf{F}_1 + {}^{n}\mathbf{F}_2} = m_1 \frac{d^2\,{}^{n}\mathbf{r}_{1/E}}{dt^2} + m_2 \frac{d^2\,{}^{n}\mathbf{r}_{2/E}}{dt^2} $$

The internal forces give no contribution to the resultant force.

$$ {}^{n}\mathbf{F}_1 + {}^{n}\mathbf{F}_2 = \frac{d^2}{dt^2}\left(m_1\,{}^{n}\mathbf{r}_{1/E} + m_2\,{}^{n}\mathbf{r}_{2/E} \right) $$

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Translation of the rigid body in spatial(3D) motion - Newton equation for the rigid body(2/4)

$$ {}^{n}\mathbf{F}_1 + {}^{n}\mathbf{F}_2 = \frac{d^2}{dt^2}\left(m_1\,{}^{n}\mathbf{r}_{1/E} + m_2\,{}^{n}\mathbf{r}_{2/E}\right) \cdots (1) $$

According to the definition of center of mass G.

$$ {}^{n}\mathbf{r}_{G/E} = \frac{m_1\,{}^{n}\mathbf{r}_{1/E} + m_2\,{}^{n}\mathbf{r}_{2/E}}{m_{system}} \quad , where\ m_{system} = m_1 + m_2 $$

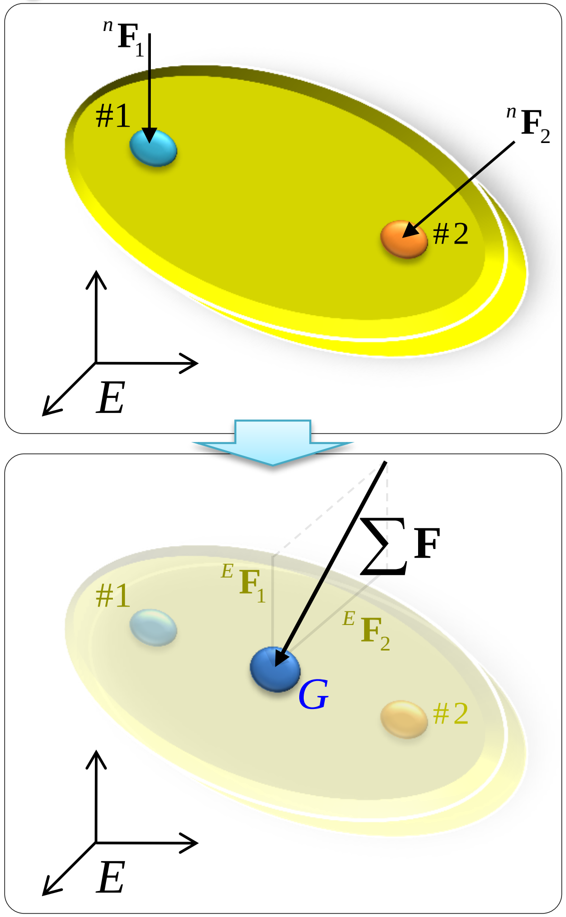
$$ m_{system}\,{}^{n}\mathbf{r}_{G/E} = m_1\,{}^{n}\mathbf{r}_{1/E} + m_2\,{}^{n}\mathbf{r}_{2/E} \cdots (2) $$

Substituting (2) into (1)

$$ {}^{n}\mathbf{F}_1 + {}^{n}\mathbf{F}_2 = \frac{d^2}{dt^2}\left(m_{system}\,{}^{n}\mathbf{r}_{G/E}\right) $$

$\sum \mathbf{F} = {}^{n}\mathbf{F}_1 + {}^{n}\mathbf{F}_2$, m_{system} is time invariant.

$$ \boxed{\sum \mathbf{F} = m_{system} \frac{d^2}{dt^2}\,{}^{n}\mathbf{r}_{G/E}} $$



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Translation of the rigid body in spatial(3D) motion - Newton equation for the rigid body(3/4)

$$^{n}\mathbf{F}_1 + {}^{n}\mathbf{F}_2 = \frac{d^2}{dt^2}\left(m_1\,{}^{n}\mathbf{r}_{1/E} + m_2\,{}^{n}\mathbf{r}_{2/E}\right) \cdots\text{(1)}$$

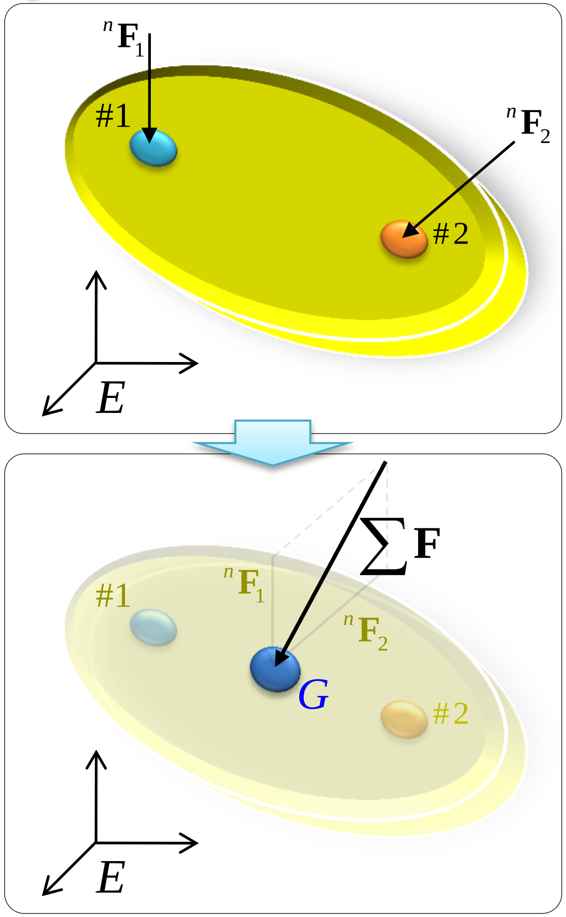
According to the definition of center of mass G.

From this expression, we recognize that, although he posed the Second Law for a particle, Newton actually captured the behavior of the center of mass of any system of particles.

$$^{n}\mathbf{F}_1 + {}^{n}\mathbf{F}_2 = \frac{d^2}{dt^2}\left(m_{system}\,{}^{n}\mathbf{r}_{G/E}\right)$$

$\sum\mathbf{F} = {}^{n}\mathbf{F}_1 + {}^{n}\mathbf{F}_2$, m_{system} is time invariant.

$$\boxed{\sum\mathbf{F} = m_{system}\frac{d^2}{dt^2}\,{}^{n}\mathbf{r}_{G/E}}$$



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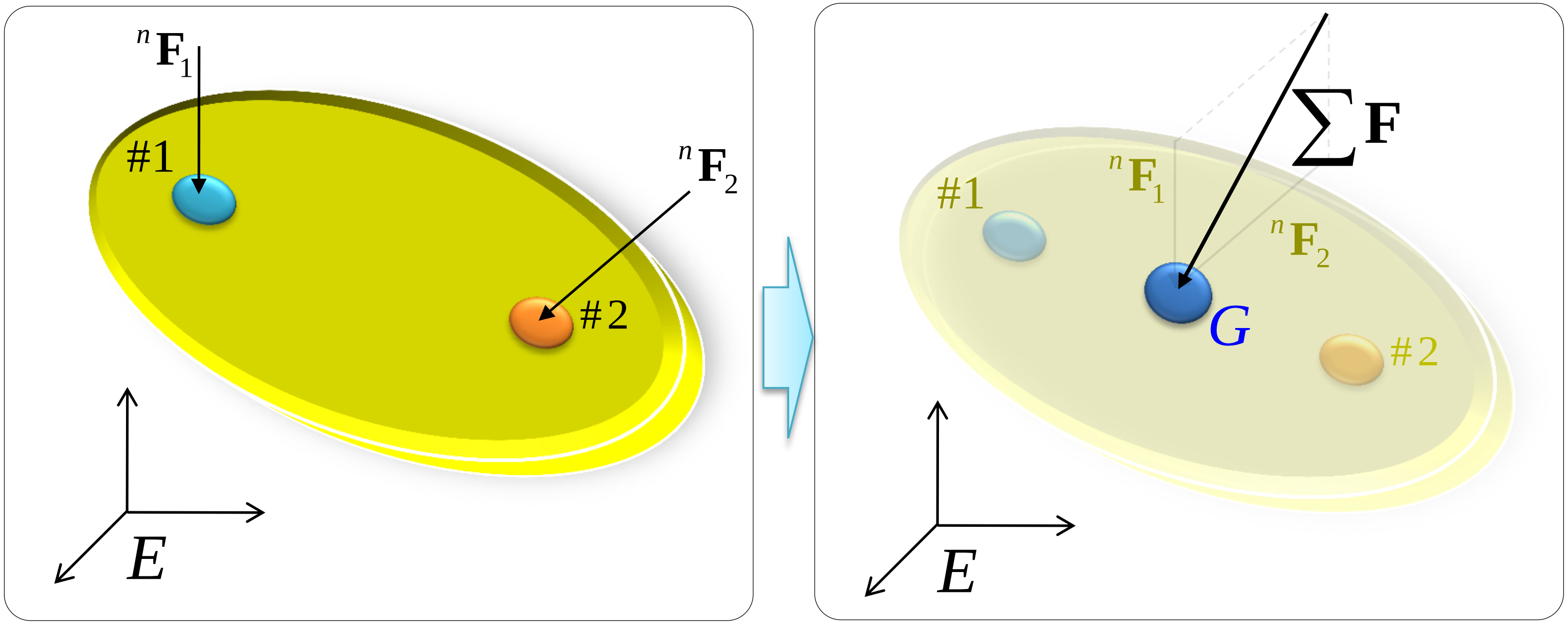
Translation of the rigid body in spatial(3D) moti
- Newton equation for the rigid body(4/4)

$$\sum \mathbf{F} = m_{system} \frac{d^2}{dt^2} {}^n\mathbf{r}_{G/E}$$

$$\sum \mathbf{F} = {}^n\mathbf{F}_1 + {}^n\mathbf{F}_2$$

m_{system} : Total mass of the system

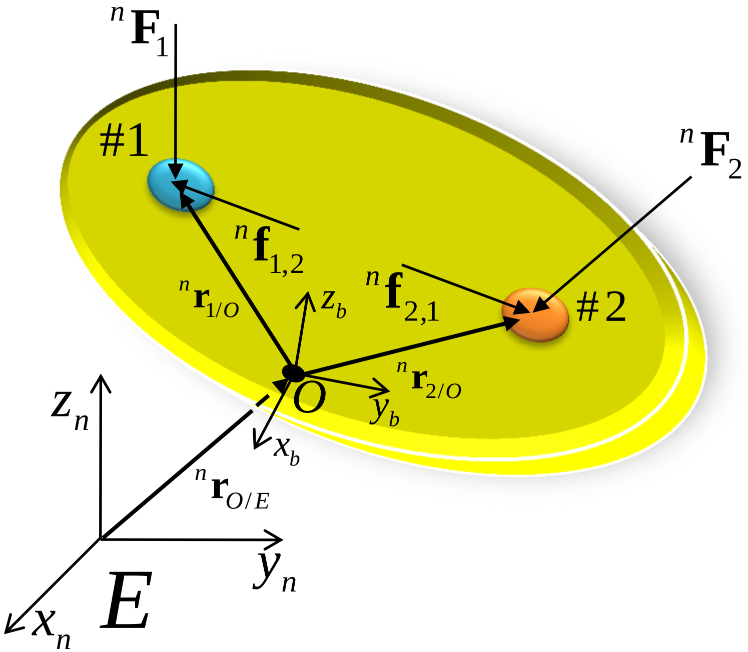


The resultant force can be considered to act on the center of mass of the system of particles.

The forces which acts on any point of the system of particles can be considered to act on the center of mass.

Rotation of the rigid body in spatial(3D) motion
- Moment equation

The rigid body composed of two particles



n-frame: Inertial Frame

$^{n}\mathbf{F}_1$, $^{n}\mathbf{F}_2$: The external forces exerted on the respective particles

$^{n}\mathbf{f}_{1,2}$, $^{n}\mathbf{f}_{2,1}$: The interaction forces between the particles

Point O: Arbitrary point O in the body

b-frame: Body fixed frame

$^{n}\mathbf{r}_{1/O}$, $^{n}\mathbf{r}_{2/O}$: The position vector of the respective particles with respect to the point O

$\sum {}^{n}\mathbf{M}_O$: Total moment about the point O

Moment equation

$$\sum {}^{n}\mathbf{M}_O = {}^{n}\mathbf{r}_{1/O} \times \left({}^{n}\mathbf{F}_1 + {}^{n}\mathbf{f}_{1,2} \right) + {}^{n}\mathbf{r}_{2/O} \times \left({}^{n}\mathbf{F}_2 + {}^{n}\mathbf{f}_{2,1} \right)$$

$$= {}^{n}\mathbf{r}_{1/O} \times m_1 {}^{n}\ddot{\mathbf{r}}_{1/E} + {}^{n}\mathbf{r}_{2/O} \times m_2 {}^{n}\ddot{\mathbf{r}}_{2/E}$$

According to Newton's Second Law

$$^{n}\mathbf{F}_1 + {}^{n}\mathbf{f}_{1,2} = m_1 {}^{n}\ddot{\mathbf{r}}_{1/E}, \qquad {}^{n}\mathbf{F}_2 + {}^{n}\mathbf{f}_{2,1} = m_2 {}^{n}\ddot{\mathbf{r}}_{2/E}, \quad where\ \ddot{\mathbf{r}} = \frac{d^2}{dt^2}\mathbf{r}$$

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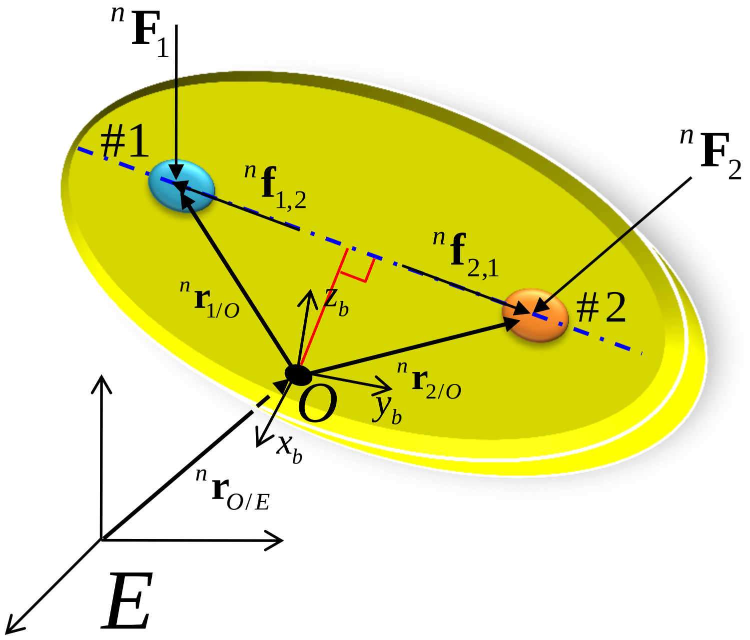
Rotation of the rigid body in spatial(3D) motion

- Total moment about axis-z through the point arbitrary point O

The rigid body composed of two particles



$$\boxed{\sum {}^n\mathbf{M}_O} = {}^n\mathbf{r}_{1/O} \times m_1 \, {}^n\ddot{\mathbf{r}}_{1/E} + {}^n\mathbf{r}_{2/O} \times m_2 \, {}^n\ddot{\mathbf{r}}_{2/E}$$

LHS: Total moment about point O

The interaction forces are colinear, meaning that they have the same line of action.

The perpendicular distance from point O to their line of action is identical.

The moments about point O exerted by each interaction force are equal in magnitude, but directed oppositely

$$\boxed{{}^n\mathbf{r}_{1/O} \times {}^n\mathbf{f}_{1,2}} = \boxed{-{}^n\mathbf{r}_{2/O} \times {}^n\mathbf{f}_{2,1}}$$

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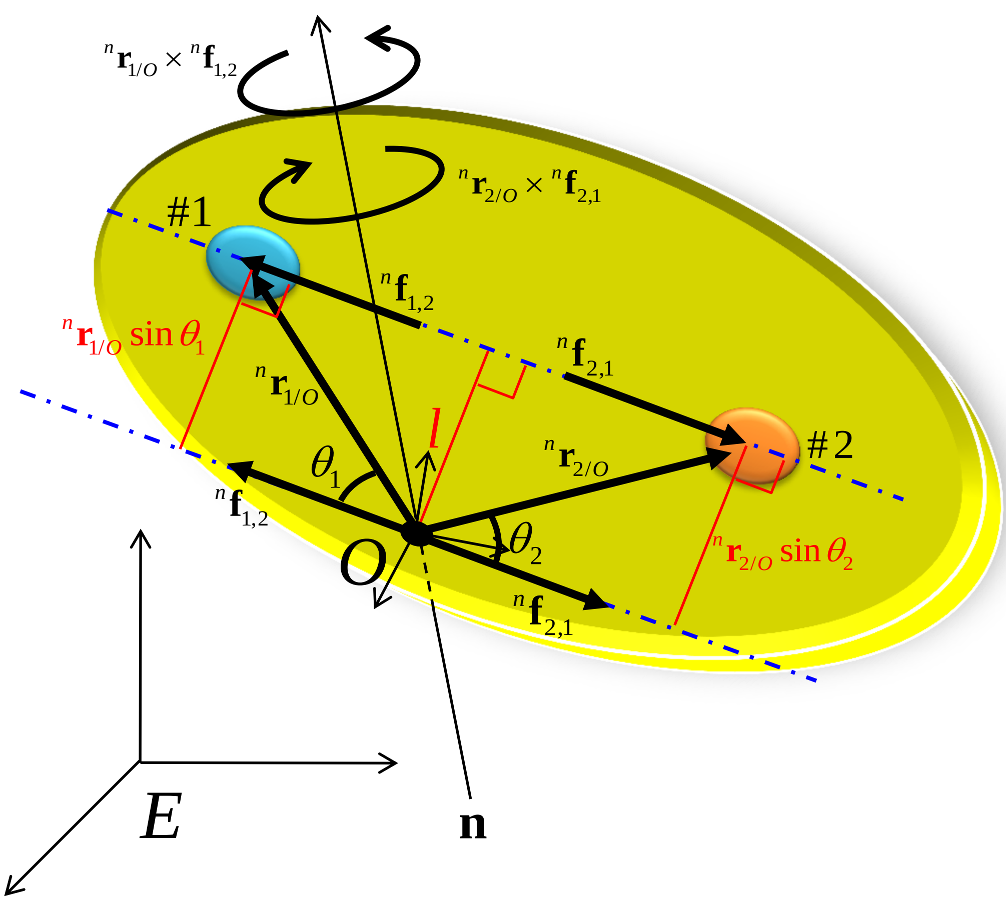
Rotation of the rigid body in spatial(3D) motion
- Moment exerted by interaction forces



The moments about point O
exerted by each interaction force are
equal in magnitude, but directed oppositely

$${}^n\mathbf{r}_{1/O}\times{}^n\mathbf{f}_{1,2}$$

- Direction: Counter clockwise about axis-n
- Magnitude:

$$\left|{}^n\mathbf{r}_{1/O}\times{}^n\mathbf{f}_{1,2}\right| = \left|{}^n\mathbf{f}_{1,2}\right|\left|{}^n\mathbf{r}_{1/O}\right|\sin\theta_1$$
$$=\left|{}^n\mathbf{f}_{1,2}\right| l$$

$${}^n\mathbf{r}_{2/A}\times{}^n\mathbf{f}_{2,1}$$

- Direction: Clockwise about axis-n
- Magnitude:

$$\left|{}^n\mathbf{r}_{2/O}\times{}^n\mathbf{f}_{2,1}\right| = \left|{}^n\mathbf{f}_{2,1}\right|\left|{}^n\mathbf{r}_{2/O}\right|\sin\theta_2$$
$$=\left|{}^n\mathbf{f}_{2,1}\right| l$$

$$\boxed{{}^n\mathbf{r}_{1/O}\times{}^n\mathbf{f}_{1,2}} = \boxed{-{}^n\mathbf{r}_{2/O}\times{}^n\mathbf{f}_{2,1}}$$

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Rotation of the rigid body in spatial(3D) motion
- Total moment about axis-z through the point arbitrary point O

$$\underline{\sum {}^{n}\mathbf{M}_{O}} = {}^{n}\mathbf{r}_{1/O} \times m_1 \, {}^{n}\ddot{\mathbf{r}}_{1/E} + {}^{n}\mathbf{r}_{2/O} \times m_2 \, {}^{n}\ddot{\mathbf{r}}_{2/E}$$

- LHS: The moment about O

$$\underline{\sum {}^{n}\mathbf{M}_{O}} = \underline{{}^{n}\mathbf{r}_{1/O} \times {}^{n}\mathbf{F}_{1} + {}^{n}\mathbf{r}_{2/O} \times {}^{n}\mathbf{F}_{2}}$$

This show that only the external forces contribute to the resultant moment about axis-z through point O

The internal forces does not contribute to the resultant moment.

The moment sum of this system

$$\begin{aligned}
\sum {}^{n}\mathbf{M}_{O} &= {}^{n}\mathbf{r}_{1/O} \times \left({}^{n}\mathbf{F}_{1} + {}^{n}\mathbf{f}_{1,2}\right) + {}^{n}\mathbf{r}_{2/O} \times \left({}^{n}\mathbf{F}_{2} + {}^{n}\mathbf{f}_{2,1}\right) \\
&= {}^{n}\mathbf{r}_{1/O} \times {}^{n}\mathbf{F}_{1} + \boxed{{}^{n}\mathbf{r}_{1/O} \times {}^{n}\mathbf{f}_{1,2}} + {}^{n}\mathbf{r}_{2/O} \times {}^{n}\mathbf{F}_{2} + {}^{n}\mathbf{r}_{2/O} \times {}^{n}\mathbf{f}_{2,1} \\
&= {}^{n}\mathbf{r}_{1/O} \times {}^{n}\mathbf{F}_{1} \boxed{\cancel{- {}^{n}\mathbf{r}_{2/O} \times {}^{n}\mathbf{f}_{2,1}}} + {}^{n}\mathbf{r}_{2/O} \times {}^{n}\mathbf{F}_{2} + \cancel{{}^{n}\mathbf{r}_{2/O} \times {}^{n}\mathbf{f}_{2,1}} \\
&= {}^{n}\mathbf{r}_{1/O} \times {}^{n}\mathbf{F}_{1} + {}^{n}\mathbf{r}_{2/O} \times {}^{n}\mathbf{F}_{2}
\end{aligned}$$

$$\boxed{{}^{n}\mathbf{r}_{1/O} \times {}^{n}\mathbf{f}_{1,2}} = \boxed{-{}^{n}\mathbf{r}_{2/O} \times {}^{n}\mathbf{f}_{2,1}}$$

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Rotation of the rigid body in spatial(3D) motion

- Derivation of the moment equation

$$\sum {}^{n}\mathbf{M}_O = \underline{{}^{n}\mathbf{r}_{1/O} \times m_1 {}^{n}\ddot{\mathbf{r}}_{1/E}} + {}^{n}\mathbf{r}_{2/O} \times m_2 {}^{n}\ddot{\mathbf{r}}_{2/E}$$

$$\Downarrow$$

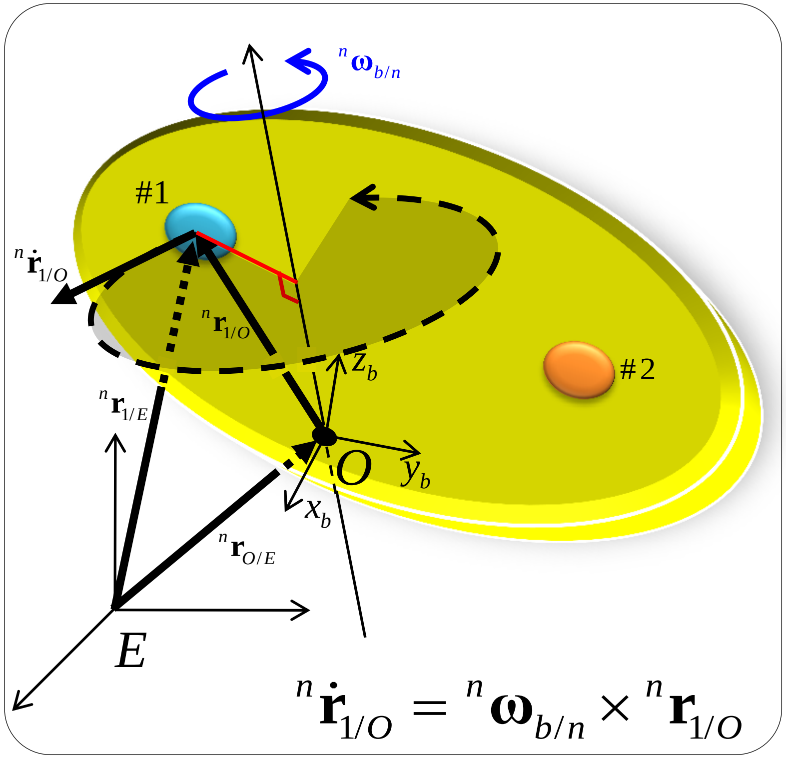
$${}^{n}\mathbf{r}_{1/O} \times m_1 \left({}^{n}\ddot{\mathbf{r}}_{O/E} + {}^{n}\ddot{\mathbf{r}}_{1/O} \right)$$

$$= m_1 {}^{n}\mathbf{r}_{1/O} \times {}^{n}\ddot{\mathbf{r}}_{O/E} + m_1 {}^{n}\mathbf{r}_{1/O} \times {}^{n}\ddot{\mathbf{r}}_{1/O}$$

$$= m_1 {}^{n}\mathbf{r}_{1/O} \times {}^{n}\ddot{\mathbf{r}}_{O/E} + m_1 {}^{n}\mathbf{r}_{1/O} \times \frac{d}{dt} {}^{n}\dot{\mathbf{r}}_{1/O}$$

$$= m_1 {}^{n}\mathbf{r}_{1/O} \times {}^{n}\ddot{\mathbf{r}}_{O/E} + \frac{d}{dt}\left(m_1 {}^{n}\mathbf{r}_{1/O} \times {}^{n}\dot{\mathbf{r}}_{1/O} \right)$$

$$\underline{= m_1 {}^{n}\mathbf{r}_{1/O} \times {}^{n}\ddot{\mathbf{r}}_{O/E} + \frac{d}{dt}\left(m_1 {}^{n}\mathbf{r}_{1/O} \times {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{r}_{1/O} \right)}$$



$${}^{n}\dot{\mathbf{r}}_{1/O} = {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{r}_{1/O}$$

$$\frac{d}{dt}\left(m_1 {}^{n}\mathbf{r}_{1/O} \times {}^{n}\dot{\mathbf{r}}_{1/O} \right) = m_1 \frac{d}{dt} {}^{n}\mathbf{r}_{1/O} \times {}^{n}\dot{\mathbf{r}}_{1/O} + m_1 {}^{n}\mathbf{r}_{1/O} \times \frac{d}{dt} {}^{n}\dot{\mathbf{r}}_{1/O}$$

$$= \cancel{m_1 {}^{n}\dot{\mathbf{r}}_{1/O} \times {}^{n}\dot{\mathbf{r}}_{1/O}} + m_1 {}^{n}\mathbf{r}_{1/O} \times \frac{d}{dt} {}^{n}\dot{\mathbf{r}}_{1/O}$$

$$= m_1 {}^{n}\mathbf{r}_{1/O} \times \frac{d}{dt} {}^{n}\dot{\mathbf{r}}_{1/O}$$

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Rotation of the rigid body in spatial(3D) motion
- Definition of the mass moment of inertia

$$\sum {}^n\mathbf{M}_O = \underline{{}^n\mathbf{r}_{1/O}\times m_1\,{}^n\ddot{\mathbf{r}}_{1/E}} + {}^n\mathbf{r}_{2/O}\times m_2\,{}^n\ddot{\mathbf{r}}_{2/E}$$

$$\underline{m_1\,{}^n\mathbf{r}_{1/O}\times{}^n\ddot{\mathbf{r}}_{O/E} + \frac{d}{dt}\left(\boxed{m_1\,{}^n\mathbf{r}_{1/O}\times{}^n\boldsymbol{\omega}_{b/n}\times{}^n\mathbf{r}_{1/O}}\right)}$$

$$\underline{m_1\,{}^n\mathbf{r}_{1/O}}\times\underline{{}^n\boldsymbol{\omega}_{b/n}}\times\underline{{}^n\mathbf{r}_{1/O}} = \underline{{}^n\mathbf{I}_{O,1}}\;\underline{{}^n\boldsymbol{\omega}_{b/n}}$$

$$= m_1\begin{bmatrix} {}^n r_{1/O,x} \\ {}^n r_{1/O,y} \\ {}^n r_{1/O,z} \end{bmatrix}\times\begin{bmatrix} {}^n\omega_{b/n,x} \\ {}^n\omega_{b/n,y} \\ {}^n\omega_{b/n,z} \end{bmatrix}\times\begin{bmatrix} {}^n r_{1/O,x} \\ {}^n r_{1/O,y} \\ {}^n r_{1/O,z} \end{bmatrix} = \begin{bmatrix} m_1\left({}^n r_{1/O,y}{}^2 + {}^n r_{1/O,z}{}^2\right) & -m_1\,{}^n r_{1/O,x}\,{}^n r_{1/O,y} & -m_1\,{}^n r_{1/O,x}\,{}^n r_{1/O,z} \\ -m_1\,{}^n r_{1/O,x}\,{}^n r_{1/O,y} & m_1\left({}^n r_{1/O,x}{}^2 + {}^n r_{1/O,z}{}^2\right) & -m_1\,{}^n r_{1/O,y}\,{}^n r_{1/O,z} \\ -m_1\,{}^n r_{1/O,x}\,{}^n r_{1/O,z} & -m_1\,{}^n r_{1/O,y}\,{}^n r_{1/O,z} & m_1\left({}^n r_{1/O,x}{}^2 + {}^n r_{1/O,y}{}^2\right) \end{bmatrix}\begin{bmatrix} {}^n\omega_{b/n,x} \\ {}^n\omega_{b/n,y} \\ {}^n\omega_{b/n,z} \end{bmatrix}$$

${}^n\mathbf{I}_{O,1}$: **Mass moment of inertia of the particle #1 about point O**

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Rotation of the rigid body in spatial(3D) motion
- Moment equation

$$\sum {}^n\mathbf{M}_O = \underline{{}^n\mathbf{r}_{1/O}\times m_1\,{}^n\ddot{\mathbf{r}}_{1/E}} + {}^n\mathbf{r}_{2/O}\times m_2\,{}^n\ddot{\mathbf{r}}_{2/E}$$

$$\Downarrow$$

$$\underline{m_1\,{}^n\mathbf{r}_{1/O}\times{}^n\ddot{\mathbf{r}}_{O/E} + \frac{d}{dt}\left({}^n\mathbf{I}_O\,{}^n\boldsymbol{\omega}_{b/n}\right)}$$

$$\sum {}^n\mathbf{M}_O = \underline{{}^n\mathbf{r}_{1/O}\times m_1\,{}^n\ddot{\mathbf{r}}_{1/E}} + \underline{{}^n\mathbf{r}_{2/O}\times m_2\,{}^n\ddot{\mathbf{r}}_{2/E}}$$

In the same way for deriving the first term

$$= \underline{m_1\,{}^n\mathbf{r}_{1/O}\times{}^n\ddot{\mathbf{r}}_{O/E} + \boxed{\frac{d}{dt}\left({}^n\mathbf{I}_{O,1}\,{}^n\boldsymbol{\omega}_{b/n}\right)}} + \underline{\boxed{m_2\,{}^n\mathbf{r}_{2/O}\times{}^n\ddot{\mathbf{r}}_{O/E}} + \frac{d}{dt}\left({}^n\mathbf{I}_{O,2}\,{}^n\boldsymbol{\omega}_{b/n}\right)}$$

$$= m_1\,{}^n\mathbf{r}_{1/O}\times{}^n\underline{\ddot{\mathbf{r}}_{O/E}} + \boxed{m_2\,{}^n\mathbf{r}_{2/O}\times{}^n\underline{\ddot{\mathbf{r}}_{O/E}}} + \boxed{\frac{d}{dt}\left({}^n\mathbf{I}_{O,1}\,\underline{{}^n\boldsymbol{\omega}_{b/n}}\right)} + \frac{d}{dt}\left({}^n\mathbf{I}_{O,2}\,\underline{{}^n\boldsymbol{\omega}_{b/n}}\right)$$

According to the distributive law

$$= \left(\boxed{m_1\,{}^n\mathbf{r}_{1/O} + m_2\,{}^n\mathbf{r}_{2/O}}\right)\times\underline{{}^n\ddot{\mathbf{r}}_{O/E}} + \frac{d}{dt}\left(\left(\boxed{{}^n\mathbf{I}_{O,1} + {}^n\mathbf{I}_{O,2}}\right)\underline{{}^n\boldsymbol{\omega}_{b/n}}\right)$$

Applying definition of the center of mass

$${}^n\mathbf{r}_{G/O} = \frac{m_1\,{}^n\mathbf{r}_{1/O} + m_2\,{}^n\mathbf{r}_{2/O}}{m_{system}}\ ,where\ m_{system} = m_1 + m_2 \Rightarrow m_{system}\,{}^n\mathbf{r}_{G/O} = m_1\,{}^n\mathbf{r}_{1/O} + m_2\,{}^n\mathbf{r}_{2/O}$$

$$= \boxed{m_{system}\,{}^n\mathbf{r}_{G/O}}\times{}^n\ddot{\mathbf{r}}_{O/E} + \frac{d}{dt}\left(\boxed{{}^n\mathbf{I}_O}\,{}^n\boldsymbol{\omega}_{b/n}\right)\quad ,where\ {}^n\mathbf{I}_O = {}^n\mathbf{I}_{O,1} + {}^n\mathbf{I}_{O,2}$$

How can we differentiate ${}^E\mathbf{I}_A$ with respect to time?

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Rotation of the rigid body in spatial(3D) motion - Moment equation

$$\sum {}^n\mathbf{M}_O = {}^n\mathbf{r}_{1/O} \times m_1\,{}^n\ddot{\mathbf{r}}_{1/E} + {}^n\mathbf{r}_{2/O} \times m_2\,{}^n\ddot{\mathbf{r}}_{2/E}$$

$$\Downarrow$$

$$m_1\,{}^n\mathbf{r}_{1/O} \times {}^n\ddot{\mathbf{r}}_{O/E} + \frac{d}{dt}\left({}^n\mathbf{I}_O\,{}^n\boldsymbol{\omega}_{b/n}\right)$$

$$\sum {}^n\mathbf{M}_O = \underbrace{{}^n\mathbf{r}_{1/O} \times m_1\,{}^n\ddot{\mathbf{r}}_{1/E}} + \underbrace{{}^n\mathbf{r}_{2/O} \times m_2\,{}^n\ddot{\mathbf{r}}_{2/E}}$$

In the same way for deriving the first term

$$= \underbrace{m_1\,{}^n\mathbf{r}_{1/O} \times {}^n\ddot{\mathbf{r}}_{O/E} + \boxed{\frac{d}{dt}\left({}^n\mathbf{I}_{O,1}\,{}^n\boldsymbol{\omega}_{b/n}\right)}} + \underbrace{\boxed{m_2\,{}^n\mathbf{r}_{2/O} \times {}^n\ddot{\mathbf{r}}_{O/E}} + \frac{d}{dt}\left({}^n\mathbf{I}_{O,2}\,{}^n\boldsymbol{\omega}_{b/n}\right)}$$

We want to transform the inertia matrix corresponding to the n-frame ${}^n\mathbf{I}_O$, which is not constant, to the inertia matrix corresponding to the b-frame ${}^b\mathbf{I}_O$, which is constant.

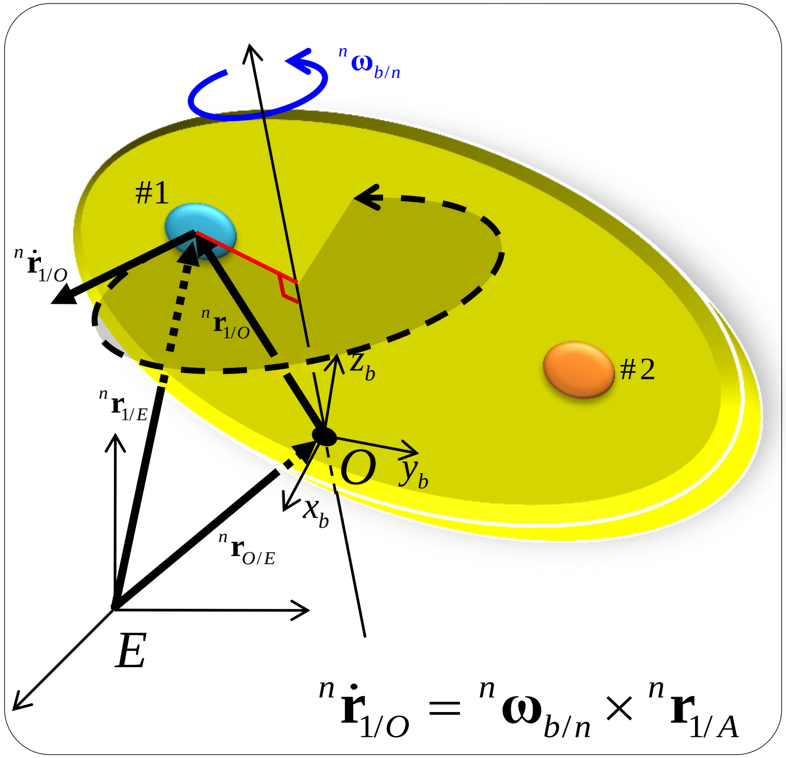
$${}^n\mathbf{r}_{G/O} = \frac{m_1\,{}^n\mathbf{r}_{1/O} + m_2\,{}^n\mathbf{r}_{2/O}}{m_{system}}$$, where m_{system} $_1 + m_2$ ⇨ $m_{system}\,{}^n\mathbf{r}_{G/O} = m_1\,{}^n\mathbf{r}_{1/O} + m_2\,{}^n\mathbf{r}_{2/O}$

$$= \boxed{m_{system}\,{}^n\mathbf{r}_{G/O}} \times {}^n\ddot{\mathbf{r}}_{O/E} + \frac{d}{dt}\left(\boxed{{}^n\mathbf{I}_O}\,{}^n\boldsymbol{\omega}_{b/n}\right)$$, where ${}^n\mathbf{I}_O = {}^n\mathbf{I}_{O,1} + {}^n\mathbf{I}_{O,2}$ ▶

How can we differentiate ${}^E\mathbf{I}_A$ with respect to time?

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Rotational transformation of the inertia matrix - Kinetic energy of the rotating body



$$^{n}\dot{\mathbf{r}}_{1/O} = {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{r}_{1/A}$$

- The point O is stationary point.
- The angular velocity of the body is $^{n}\boldsymbol{\omega}_{b/n}$

$$^{E}\mathbf{r}_{i/A} = \begin{bmatrix} ^{E}x_i \\ ^{E}y_i \\ ^{E}z_i \end{bmatrix}, {}^{n}\mathbf{I}_{O,i} = \begin{bmatrix} m_i\left(^{n}y_i^{2} + {}^{n}z_i^{2}\right) & -m_i\, ^{n}x_i\, ^{n}y_i & -m_i\, ^{n}x_i\, ^{n}z_i \\ -m_i\, ^{n}x_i\, ^{n}y_i & m_i\left(^{n}x_i^{2} + {}^{n}z_i^{2}\right) & -m_i\, ^{n}y_i\, ^{n}z_i \\ -m_i\, ^{n}x_i\, ^{n}z_i & -m_i\, ^{n}y_i\, ^{n}z_i & m_i\left(^{n}x_i^{2} + {}^{n}y_i^{2}\right) \end{bmatrix}$$

$$^{n}\mathbf{I}_O = \sum_{i=1}^{2} {}^{n}\mathbf{I}_{O,i}$$

Kinetic Energy T

$$T_i = \frac{1}{2} m_i \cdot {}^{n}\dot{\mathbf{r}}_{i/O} \cdot {}^{n}\dot{\mathbf{r}}_{i/O}, \qquad T = \sum_{i=1}^{2} T_i$$

$$T_i = \frac{1}{2} m_i \cdot \left({}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{r}_{i/O}\right) \cdot \left({}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{r}_{i/O}\right)$$

$$T = \frac{1}{2}\, {}^{n}\boldsymbol{\omega}_{b/n}^{T}\, {}^{n}\mathbf{I}_O\, {}^{n}\boldsymbol{\omega}_{b/n}$$

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, pp.235~pp.236, pp.258~pp.260

Rotational transformation of the inertia matrix

$$T = \frac{1}{2}\, {}^{n}\boldsymbol{\omega}_{b/n}^{\;T}\, \boxed{{}^{n}\mathbf{I}_{O}}\, {}^{n}\boldsymbol{\omega}_{b/n} \qquad \cdots\ (1)$$

- The same value of the kinetic energy result if angular velocity and the inertia properties are referred to the body fixed frame(A-frame), so

$$\begin{aligned}
T &= \frac{1}{2}\, {}^{b}\boldsymbol{\omega}_{b/n}^{\;T}\, {}^{b}\mathbf{I}_{O}\, {}^{b}\boldsymbol{\omega}_{b/n} \\
&= \frac{1}{2}\left({}^{b}\mathbf{R}_{n}\, {}^{n}\boldsymbol{\omega}_{b/n}\right)^{T} {}^{b}\mathbf{I}_{O}\left({}^{b}\mathbf{R}_{n}\, {}^{n}\boldsymbol{\omega}_{b/n}\right) \\
&= \frac{1}{2}\, {}^{n}\boldsymbol{\omega}_{b/n}^{\;T}\, {}^{b}\mathbf{R}_{n}^{\;T}\, {}^{b}\mathbf{I}_{O}\, {}^{b}\mathbf{R}_{n}\, {}^{n}\boldsymbol{\omega}_{b/n} \\
&= \frac{1}{2}\, {}^{n}\boldsymbol{\omega}_{b/n}^{\;T}\, \boxed{{}^{n}\mathbf{R}_{b}\, {}^{b}\mathbf{I}_{O}\, {}^{b}\mathbf{R}_{n}}\, {}^{n}\boldsymbol{\omega}_{b/n} \qquad \cdots\ (2)
\end{aligned}$$

(1) and (2) should be same.

Rotation transformation

$$\Rightarrow \quad {}^{n}\mathbf{I}_{O} = {}^{n}\mathbf{R}_{b}\, {}^{b}\mathbf{I}_{O}\, {}^{b}\mathbf{R}_{n}$$

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, pp.235~pp.236, pp.258~pp.260

Rotation of the rigid body in spatial(3D) motion - Moment equation

$${}^n\mathbf{I}_O = {}^n\mathbf{R}_b\,{}^b\mathbf{I}_O\,{}^b\mathbf{R}_n$$

$$\sum {}^n\mathbf{M}_O = {}^n\mathbf{r}_{1/O} \times m_1\,{}^n\ddot{\mathbf{r}}_{1/E} + {}^n\mathbf{r}_{2/O} \times m_2\,{}^n\ddot{\mathbf{r}}_{2/E}$$

$$= m_{system}\,{}^n\mathbf{r}_{G/O} \times {}^n\ddot{\mathbf{r}}_{O/E} + \underline{\frac{d}{dt}\left({}^n\mathbf{I}_O\,{}^n\boldsymbol{\omega}_{b/n}\right)}$$

${}^n\mathbf{I}_O$ **is not constant, but** ${}^b\mathbf{I}_O$ **is constant.**

$$\underline{\frac{d}{dt}\left({}^n\mathbf{I}_O\,{}^n\boldsymbol{\omega}_{b/n}\right)} = \frac{d}{dt}\left({}^n\mathbf{R}_b\,{}^b\mathbf{I}_O\,{}^b\mathbf{R}_n\,{}^n\boldsymbol{\omega}_{b/n}\right)$$

${}^b\mathbf{I}_O$ **is constant.**

$$= \frac{d}{dt}\left({}^n\mathbf{R}_b\right){}^b\mathbf{I}_O\,{}^b\mathbf{R}_n\,{}^n\boldsymbol{\omega}_{b/n} + {}^n\mathbf{R}_b\,\cancel{\frac{d}{dt}\left({}^b\mathbf{I}_O\right)}\,{}^b\mathbf{R}_n\,{}^n\boldsymbol{\omega}_{b/n}$$

$$+ {}^n\mathbf{R}_b\,{}^b\mathbf{I}_O\,\frac{d}{dt}\left({}^b\mathbf{R}_n\right){}^n\boldsymbol{\omega}_{b/n} + {}^n\mathbf{R}_b\,{}^b\mathbf{I}_O\,{}^b\mathbf{R}_n\,\frac{d}{dt}\left({}^n\boldsymbol{\omega}_{b/n}\right)$$

$$= \underline{{}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{R}_b\,{}^b\mathbf{I}_O\,{}^b\mathbf{R}_n\,{}^n\boldsymbol{\omega}_{b/n}}$$

$$\underline{+ \cancel{{}^n\mathbf{R}_b\,{}^b\mathbf{I}_O\,{}^b\boldsymbol{\omega}_{b/n} \times \underbrace{\boxed{{}^b\mathbf{R}_n\,{}^n\boldsymbol{\omega}_{b/n}}}_{{}^b\boldsymbol{\omega}_{b/n}}} + {}^n\mathbf{R}_b\,{}^b\mathbf{I}_O\,{}^b\mathbf{R}_n\,{}^n\dot{\boldsymbol{\omega}}_{b/n}}$$

Topics in ship design automation, 2. Single Rigidbody Dynamics, 2010, Fall, K.Y.Lee

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Rotation of the rigid body in spatial(3D) motion - Moment equation

$${}^n\mathbf{I}_O = {}^n\mathbf{R}_b\,{}^b\mathbf{I}_O\,{}^b\mathbf{R}_n$$

$$\sum {}^n\mathbf{M}_O = {}^n\mathbf{r}_{1/O} \times m_1\,{}^n\ddot{\mathbf{r}}_{1/E} + {}^n\mathbf{r}_{2/O} \times m_2\,{}^n\ddot{\mathbf{r}}_{2/E}$$

$$= \underline{m_{system}\,{}^n\mathbf{r}_{G/O} \times {}^n\ddot{\mathbf{r}}_{O/E}} + \underline{\frac{d}{dt}\left({}^n\mathbf{I}_O\,{}^n\boldsymbol{\omega}_{b/n}\right)}$$

$$\frac{d}{dt}\left({}^n\mathbf{I}_O\,{}^n\boldsymbol{\omega}_{b/n}\right) = {}^n\boldsymbol{\omega}_{b/n} \times \boxed{{}^n\mathbf{R}_b\,{}^b\mathbf{I}_O\,{}^b\mathbf{R}_n}\,{}^n\boldsymbol{\omega}_{b/n} + \boxed{{}^n\mathbf{R}_b\,{}^b\mathbf{I}_O\,{}^b\mathbf{R}_n}\,{}^n\dot{\boldsymbol{\omega}}_{b/n}$$

$$= \underline{{}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{I}_O\,{}^n\boldsymbol{\omega}_{b/n} + {}^n\mathbf{I}_O\,{}^n\dot{\boldsymbol{\omega}}_{b/n}}$$

Moment equation

$$\sum {}^n\mathbf{M}_O = \underline{m_{system}\,{}^n\mathbf{r}_{G/O} \times {}^n\ddot{\mathbf{r}}_{O/E}} + \underline{{}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{I}_O\,{}^n\boldsymbol{\omega}_{b/n} + {}^n\mathbf{I}_O\,{}^n\dot{\boldsymbol{\omega}}_{b/n}}$$

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Rotation of the rigid body in spatial(3D) motion - Simplified version of Moment equation

Moment equation $\sum {}^{n}\mathbf{M}_O = m_{system}\, {}^{n}\mathbf{r}_{G/O} \times {}^{n}\ddot{\mathbf{r}}_{O/E} + {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{I}_O\, {}^{n}\boldsymbol{\omega}_{b/n} + {}^{n}\mathbf{I}_O\, {}^{n}\dot{\boldsymbol{\omega}}_{b/n}$

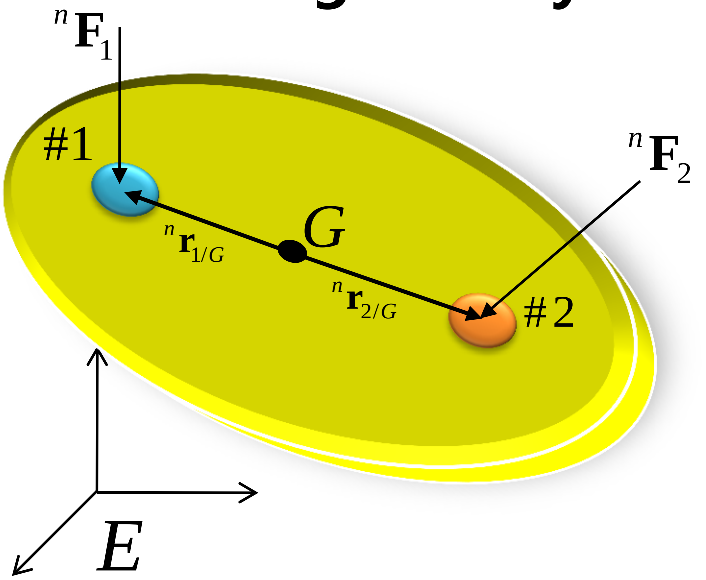
If the point O coincides with the center of mass G

$$\sum {}^{n}\mathbf{M}_G = m_{system}\, \cancel{{}^{n}\mathbf{r}_{G/G}} \times {}^{n}\ddot{\mathbf{r}}_{G/E} + {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{I}_G\, {}^{n}\boldsymbol{\omega}_{b/n} + {}^{n}\mathbf{I}_G\, {}^{n}\dot{\boldsymbol{\omega}}_{b/n}$$

$$\sum {}^{n}\mathbf{M}_G = {}^{n}\mathbf{I}_G\, {}^{n}\dot{\boldsymbol{\omega}}_{b/n} + {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{I}_G\, {}^{n}\boldsymbol{\omega}_{b/n}$$

⇦ The simplified version of the moment equation

The rigid body composed of the two particles



$$\sum {}^{n}\mathbf{M}_G = {}^{n}\mathbf{I}_G\, {}^{n}\dot{\boldsymbol{\omega}}_{b/n} + {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{I}_G\, {}^{n}\boldsymbol{\omega}_{b/n}$$

The point, which the moment is calculated about, should be the center of mass G

$$\sum {}^{n}\mathbf{M}_G = {}^{n}\mathbf{r}_{1/G} \times {}^{n}\mathbf{F}_1 + {}^{n}\mathbf{r}_{2/G} \times {}^{n}\mathbf{F}_2$$

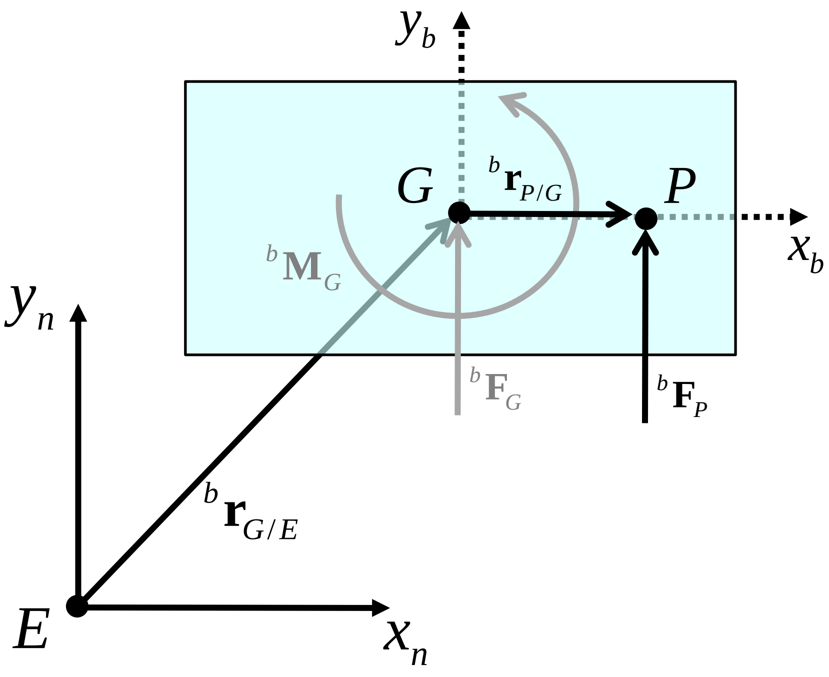
Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

2.2 Equations of a rigid body motion

Equations of 2 dimensional rigid body motion

1) Rigid body is in 2 dimensional motion
2) Kinematical reference point[1)]: Center of mass G

What is equations of 2 dimensional rigid body motion?



m : The mass of the rigid body

n-frame: Inertial frame $x_n\, y_n\, z_n$

Point E: Origin of the inertial frame(n-frame)

b-frame: Body fixed frame $x_b\, y_b\, z_b$

Point G: Center of mass, Origin of the body-fixed frame(b-frame)

Force equation[2)]

$$m\ {}^b\ddot{\mathbf{r}}_{G/E} = {}^b\mathbf{F}_G$$

Derivation ▶

m : The mass of the rigid body

${}^b\mathbf{r}_{G/E}$: Position vector of the center of mass G with respect to E decomposed in b-frame[3)]

${}^b\mathbf{F}_G$: Force exerted on the point G decomposed in b-frame, equivalent to ${}^b\mathbf{F}_P$, $({}^b\mathbf{F}_G = {}^b\mathbf{F}_P)$

Moment(Euler) equation[4)]

$${}^b\mathbf{I}_G\ {}^b\dot{\boldsymbol{\omega}}_{b/n} = {}^b\mathbf{M}_G$$

Derivation ▶

${}^b\mathbf{I}_G$: Mass moment of inertia of the rigid body about z_b-axis

${}^b\boldsymbol{\omega}_{b/n}$: Angular velocity of the b-frame with respect to n-frame decomposed in b-frame

${}^b\mathbf{M}_G$: Moment about z_b-axis decomposed in b-frame,

$${}^b\mathbf{M}_G = {}^b\mathbf{r}_{P/G} \times {}^b\mathbf{F}_P$$

c.f.) Moment equation of 3 dimensional rigid body motion

$${}^b\mathbf{M}_G = {}^b\mathbf{I}_G\ {}^b\dot{\boldsymbol{\omega}}_{b/n} + \boxed{{}^b\boldsymbol{\omega}_{b/n} \times {}^b\mathbf{I}_G\ {}^b\boldsymbol{\omega}_{b/n}}$$

What if the kinematical reference point is arbitrary point O? ▶

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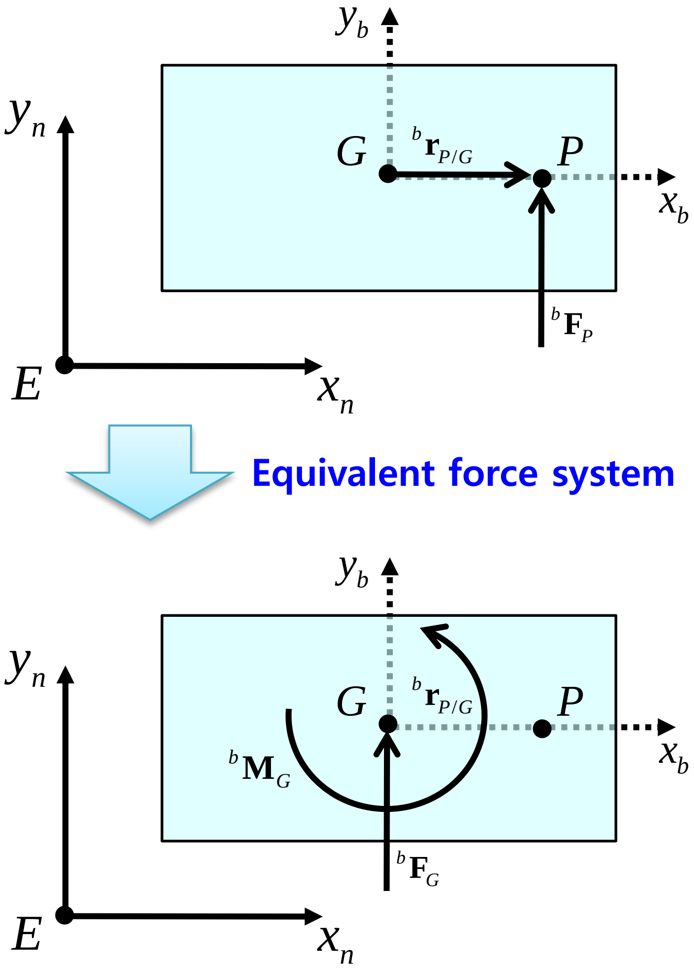
1), 2), 4) Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.297, p.234, p.234
3) Fossen, T. I., Marine Control Systesm, Marine Cybernetics, 2002, p.20

Force and moment exerted on a rigid body



n-frame: Inertial frame

Point E: Origin of the inertial frame(n-frame)

b-frame: Body fixed frame

Point G: Center of mass,
Origin of the body-fixed frame(b-frame)

$$ {}^b\mathbf{I}_G \, {}^b\dot{\boldsymbol{\omega}}_{b/n} = \boxed{{}^b\mathbf{M}_G} $$

${}^b\mathbf{F}_P$: Force acting on the point P decomposed in the b-frame

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${}^b\mathbf{F}_G$: Force acting on the point G decomposed in the b-frame

$$ {}^b\mathbf{F}_G = {}^b\mathbf{F}_P $$

- The translational motion is independent of the point where the external force is exerted. (Fossen, 2002, pp. 54)

${}^b\mathbf{M}_G$: Moment about z_b-axis decomposed in the b-frame

$$ {}^b\mathbf{M}_G = {}^b\mathbf{r}_{P/G} \times {}^b\mathbf{F}_P $$

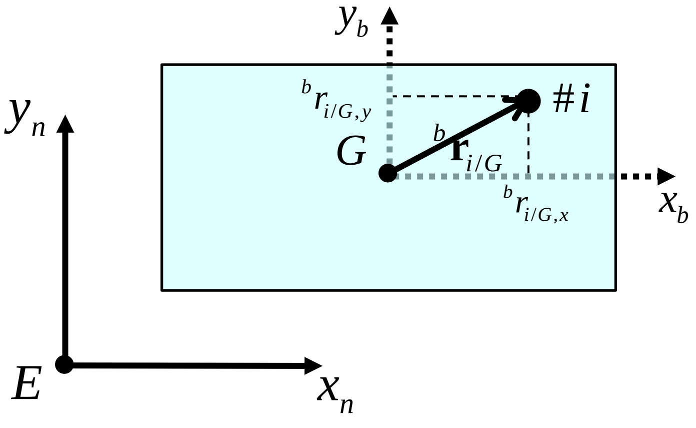
The moment is generated by the force exerted on the point P

- we consider the moment exerted by each interaction force.
- it is reasonable to expect that the resultant moment of a set of forces represents the rotational influence

Mass Moment of Inertia

$$\boxed{{}^{b}\mathbf{I}_{G}}\,{}^{b}\dot{\boldsymbol{\omega}}_{b/n} = {}^{b}\mathbf{M}_{G}$$

The rotational motion depends not only on the mass of the body, but also on how its mass is distributed[1].



The mass moment of inertia is a measure of the resistance that a body offers to changes in its rotational motion, just as mass is a measure of the resistance that a body offers to changes in its translational motion[2]

Mass moment of inertia of the rigid body in 2D motion

$${}^{b}\mathbf{I}_{G} = \sum_{i=1}^{n} m_i\, {}^{b}\mathbf{r}_{i/G}^{2}$$

Mass moment of inertia of the rigid body in 3D motion

$${}^{b}\mathbf{I}_{G} = \begin{bmatrix} \sum m_i \left({}^{b}r_{i/G,y}{}^{2} + {}^{b}r_{i/G,z}{}^{2}\right) & -\sum m_i\, {}^{b}r_{i/G,x}\, {}^{b}r_{i/G,y} & -\sum m_i\, {}^{b}r_{i/G,x}\, {}^{b}r_{i/G,z} \\ -\sum m_i\, {}^{b}r_{i/G,x}\, {}^{b}r_{i/G,y} & \sum m_i \left({}^{b}r_{i/G,x}{}^{2} + {}^{b}r_{i/G,z}{}^{2}\right) & -\sum m_i\, {}^{b}r_{i/G,y}\, {}^{b}r_{i/G,z} \\ -\sum m_i\, {}^{b}r_{i/G,x}\, {}^{b}r_{i/G,z} & -\sum m_i\, {}^{b}r_{i/G,y}\, {}^{b}r_{i/G,z} & \sum m_i \left({}^{b}r_{i/G,x}{}^{2} + {}^{b}r_{i/G,y}{}^{2}\right) \end{bmatrix}$$

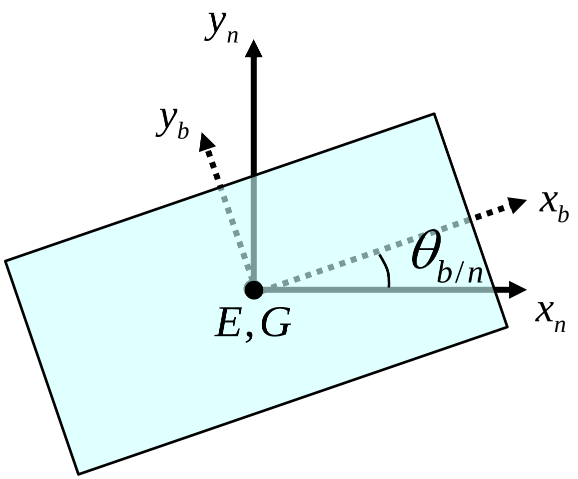
1) Bedford, A., Fowler, W., Engineering Mechanics Dynamics 5th edition, Prentice Hall, 2008, p.405

2) Ohanian, H. C., Physics 2nd edition, expended, Norton & Company Inc., 1989, p.304

Orientation of the rigid body in 2 dimensional motion

$${}^{b}\mathbf{I}_{G}\,\boxed{{}^{b}\dot{\boldsymbol{\omega}}_{b/n}} = {}^{b}\mathbf{M}_{G}$$

- **The rigid body is rotated with the angle $\theta_{b/n}$ about z_b-axis with respect to n-frame.**



For determining the orientation of the b-frame, body fixed frame, We consider a general situation in which two coordinate systems, b-frame and n-frame, have a common origin.

$\theta_{b/n}$ **: Rotational angle of the b-frame with respect to n-frame**

n-frame: Inertial reference frame

Point E: Origin of the inertial frame(n-frame)

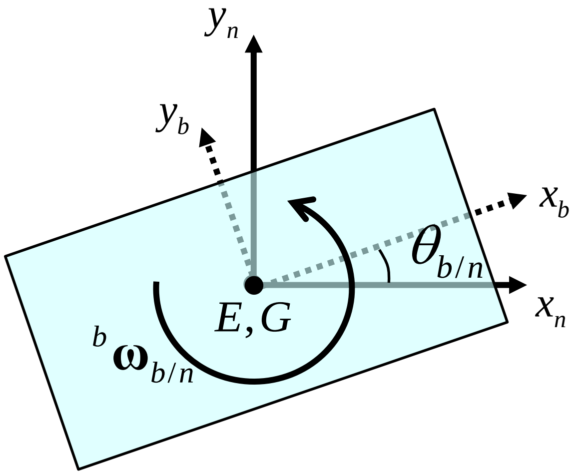
b-frame: Body fixed frame

Point G: Center of mass,
Origin of the body-fixed frame(b-frame)

Orientation and angular velocity of the rigid body in 2 dimensional motion

$${}^{b}\mathbf{I}_{G}\,\boxed{{}^{b}\dot{\boldsymbol{\omega}}_{b/n}} = {}^{b}\mathbf{M}_{G}$$

- The rigid body is rotating with an angular velocity ${}^{b}\boldsymbol{\omega}_{b/n}$ about z_b-axis with respect to n-frame



$\theta_{b/n}$: Rotational angle of the b-frame with respect to n-frame

${}^{b}\boldsymbol{\omega}_{b/n}$: Angular velocity of the b-frame with respect to n-frame decomposed in b-frame

n-frame: Inertial frame $x_n\, y_n\, z_n$

Point E: Origin of the inertial frame(n-frame)

b-frame: Body fixed frame $x_b\, y_b\, z_b$

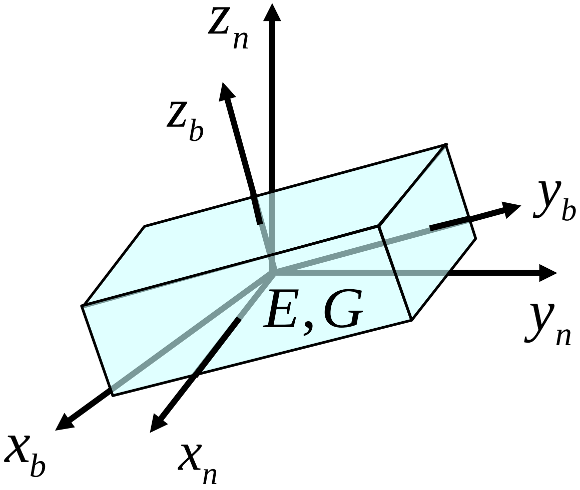
Point G: Center of mass,
Origin of the body-fixed frame(b-frame)

Relation between the time derivative of the rotational angle and angular velocity in 2 dimensional motion

$${}^{b}\boldsymbol{\omega}_{b/n} = \dot{\theta}_{b/n}\mathbf{k}_{b}$$

, where $\mathbf{k}_b$ is unit vector of the z_b-axis

Orientation of the rigid body in 3 dimensional motion



n-frame: Inertial frame $x_n\, y_n\, z_n$

Point E: Origin of the inertial frame(n-frame)

b-frame: Body fixed frame $x_b\, y_b\, z_b$

Point G: Center of mass,
Origin of the body-fixed frame(b-frame)

→ For describing the orientation of the rigid body in 3 dimensional motion, Euler angles are used.

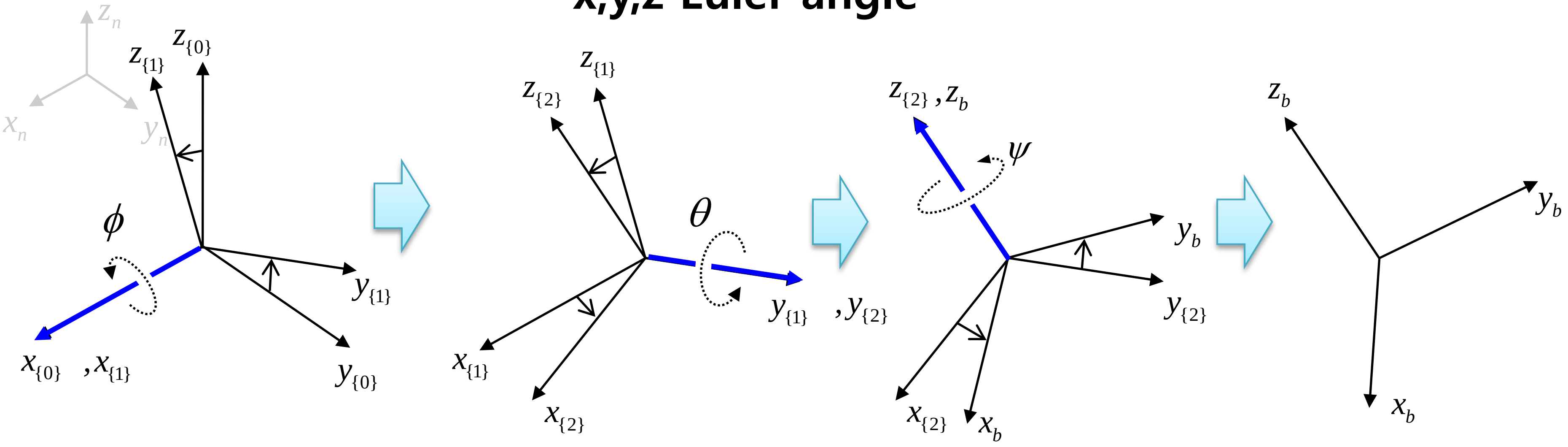
$$\gamma = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

Three independent direction angles, Euler angles, treat this matter as a specific sequence of rotations.

Euler (Eulerian) angles

x,y,z Euler angle: $\gamma = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$

x,y,z Euler angle



Rotation of an angle ϕ about the $x_{\{0\}}$ axis

Rotation of an angle θ about the $y_{\{1\}}$ axis

Rotation of an angle ψ about the $z_{\{2\}}$ axis

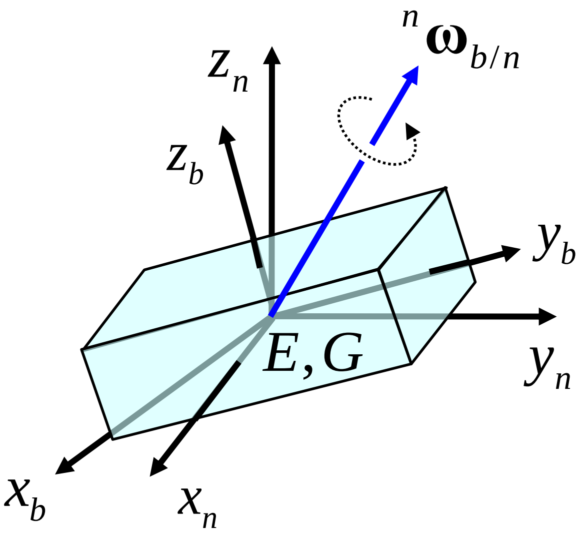
By using Euler angles(three successive rotations), any orientation of the b-frame can be represented.

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 68

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Angular velocity of the rigid body in 3 dimensional motion

- The rigid body is rotating with the angular velocity ${}^{n}\boldsymbol{\omega}_{b/n}$.



n-frame: Inertial frame $x_n\, y_n\, z_n$

Point E: Origin of the inertial frame(n-frame)

b-frame: Body fixed frame $x_b\, y_b\, z_b$

Point G: Center of mass,
Origin of the body-fixed frame(b-frame)

→ The angular velocity vector ${}^{n}\boldsymbol{\omega}_{b/n}$ is not equal to derivative of the Euler angles.

$$ {}^{n}\boldsymbol{\omega}_{b/n} \neq \dot{\boldsymbol{\gamma}} $$

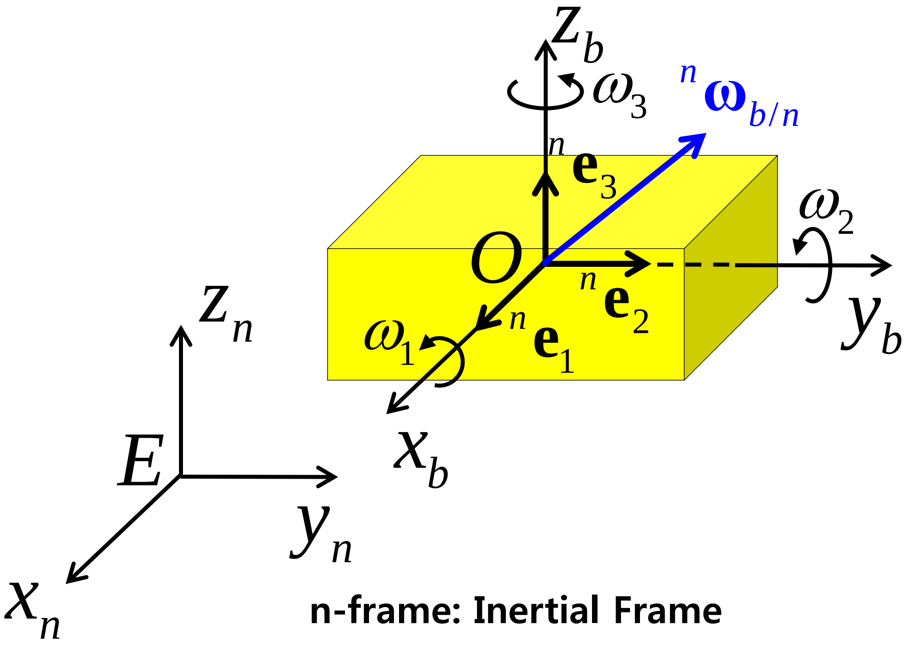
Then how can we calculate the angular velocity vector ${}^{n}\boldsymbol{\omega}_{b/n}$?

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, pp. 127

Angular Velocity

An angular velocity $^{n}\boldsymbol{\omega}_{b/n}$ is the sum of simple rotations described by angular velocities $\omega_m\ {}^{n}\mathbf{e}_m$, where $^{n}\mathbf{e}_m$ is a unit vector parallel to the respective rotation axis, in accord with the right-hand rule.

$$^{n}\boldsymbol{\omega}_{b/n} = \sum_{m=1}^{3} \omega_m\ {}^{n}\mathbf{e}_m$$



$$^{n}\boldsymbol{\omega} = \omega_1\ {}^{n}\mathbf{e}_1 + \omega_2\ {}^{n}\mathbf{e}_2 + \omega_3\ {}^{n}\mathbf{e}_3 = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

where $^{n}\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, {}^{n}\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, {}^{n}\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

n-frame: Inertial Frame

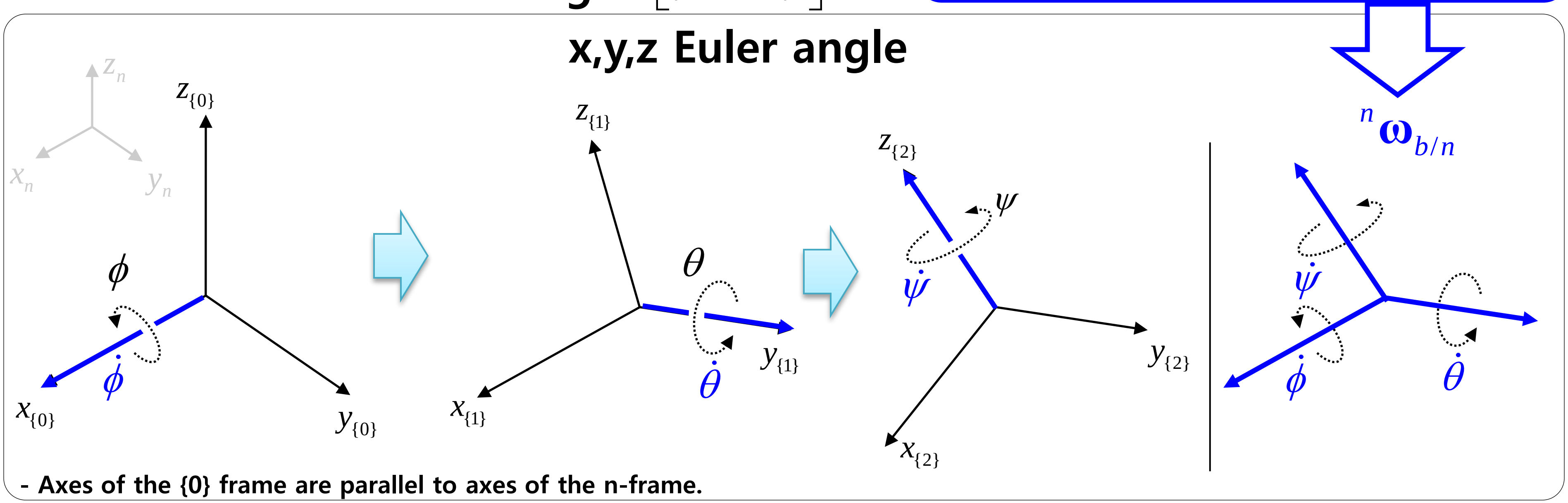
b-frame: Body Fixed Frame

Relation between the time derivative of the Euler angle and angular velocity in 3 dimensional motion

x,y,z Euler angle at time: $\begin{bmatrix}\phi & \theta & \psi\end{bmatrix}^T$

Time derivative of Euler angle: $\begin{bmatrix}\dot{\phi} & \dot{\theta} & \dot{\psi}\end{bmatrix}^T$

An angular velocity ${}^{n}\boldsymbol{\omega}_{b/n}$ is the sum of simple rotations.


x,y,z Euler angle

- Axes of the {0} frame are parallel to axes of the n-frame.

The time derivative of Euler angle $\begin{bmatrix}\dot{\phi} & \dot{\theta} & \dot{\psi}\end{bmatrix}^T$ is the angular velocity components directed along ${}^{n}\mathbf{x}_{\{0\}}, {}^{n}\mathbf{y}_{\{1\}}, {}^{n}\mathbf{z}_{\{2\}}$ axes.

$${}^{n}\boldsymbol{\omega}_{b/n} = {}^{n}\mathbf{x}_{\{0\}}\dot{\phi} + {}^{n}\mathbf{y}_{\{1\}}\dot{\theta} + {}^{n}\mathbf{z}_{\{2\}}\dot{\psi}$$

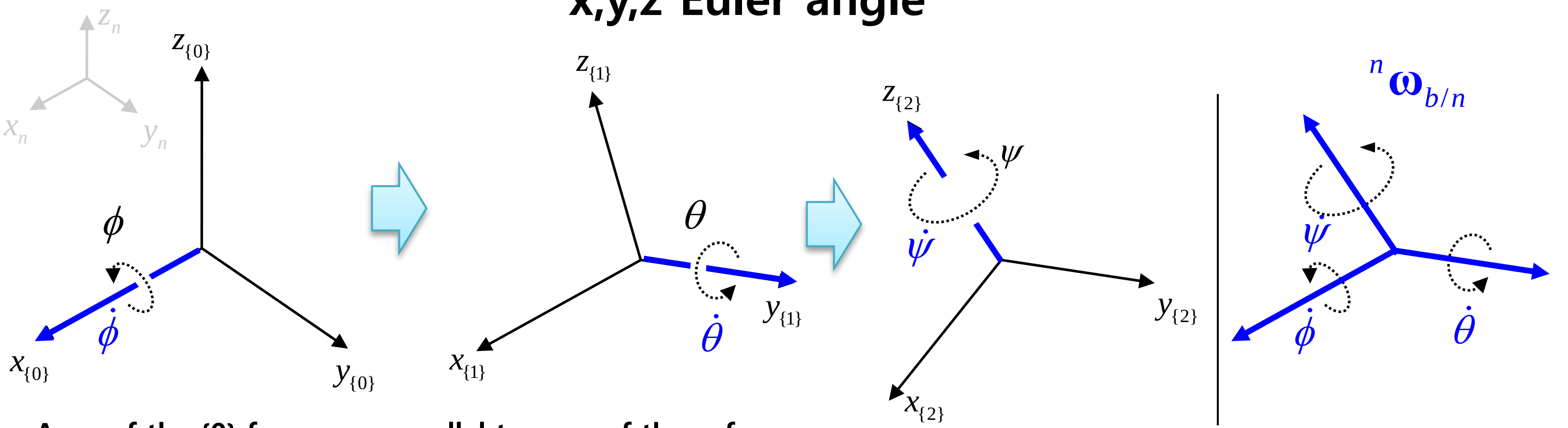
Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 68

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Relation between the time derivative of the Euler angle and angular velocity in 3 dimensional motion

x,y,z Euler angle at time: $\begin{bmatrix}\phi & \theta & \psi\end{bmatrix}^T$ **Time derivative of Euler angle:** $\begin{bmatrix}\dot{\phi} & \dot{\theta} & \dot{\psi}\end{bmatrix}^T$

x,y,z Euler angle



- Axes of the {0} frame are parallel to axes of the n-frame.

$$
{}^{n}\boldsymbol{\omega}_{b/n} = {}^{n}\mathbf{x}_{\{0\}}\dot{\phi} + {}^{n}\mathbf{y}_{\{1\}}\dot{\theta} + {}^{n}\mathbf{z}_{\{2\}}\dot{\psi}
$$

$$
= {}^{n}\mathbf{R}_{\{0\}}\,{}^{\{0\}}\mathbf{x}_{\{0\}}\dot{\phi} + {}^{n}\mathbf{R}_{\{0\}}\,{}^{\{0\}}\mathbf{R}_{\{1\}}\,{}^{\{1\}}\mathbf{y}_{\{1\}}\dot{\theta} + {}^{n}\mathbf{R}_{\{0\}}\,{}^{\{0\}}\mathbf{R}_{\{1\}}\,{}^{\{1\}}\mathbf{R}_{\{2\}}\,{}^{\{2\}}\mathbf{z}_{\{2\}}\dot{\psi}
$$

$$
= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\dot{\phi} + \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&\cos\phi&-\sin\phi\\0&\sin\phi&\cos\phi\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\dot{\theta} + \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&\cos\phi&-\sin\phi\\0&\sin\phi&\cos\phi\end{bmatrix}\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\dot{\psi}
$$

$$
= \begin{bmatrix}1\\0\\0\end{bmatrix}\dot{\phi} + \begin{bmatrix}0\\\cos\phi\\\sin\phi\end{bmatrix}\dot{\theta} + \begin{bmatrix}\sin\theta\\-\sin\phi\cos\theta\\\cos\phi\cos\theta\end{bmatrix}\dot{\psi} = \underbrace{\begin{bmatrix}1&0&\sin\theta\\0&\cos\phi&-\sin\phi\cos\theta\\0&\sin\phi&\cos\phi\cos\theta\end{bmatrix}}_{{}^{n}\mathbf{G}}\underbrace{\begin{bmatrix}\dot{\phi}\\\dot{\theta}\\\dot{\psi}\end{bmatrix}}_{\dot{\boldsymbol{\gamma}}}
$$

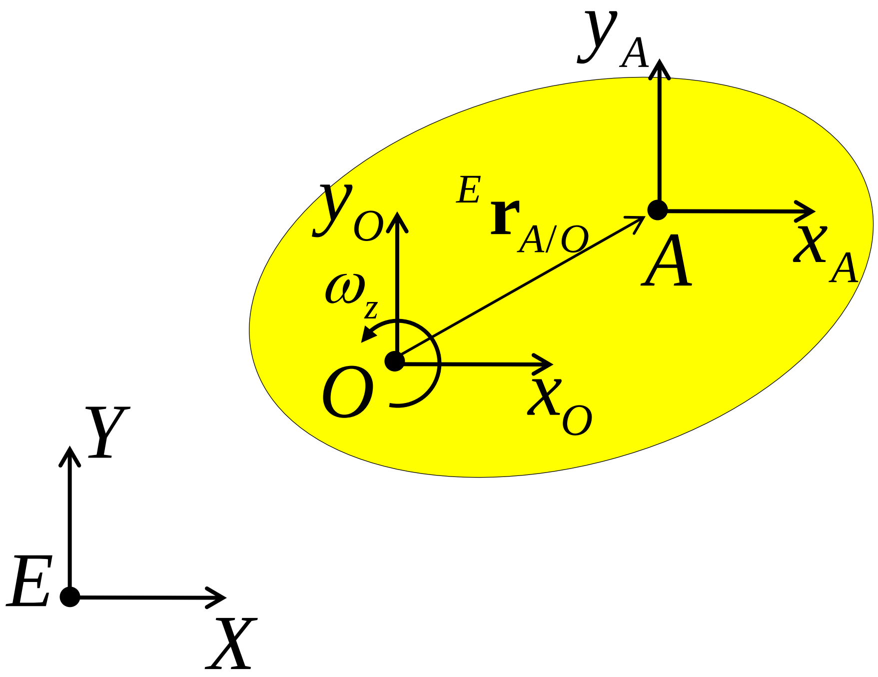
$$
\boxed{{}^{n}\boldsymbol{\omega}_{b/n} = {}^{n}\mathbf{G}\dot{\boldsymbol{\gamma}}}
$$

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 68
Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Reference)
Euler Angle

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, pp. 127

Angular Velocity



Linear Velocity Vector of Point O

$$ {}^E\mathbf{v}_{O/E} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T $$

Angular Velocity Vector of O-frame

$$ {}^E\boldsymbol{\omega}_{O/E} = \begin{bmatrix} 0 & 0 & \omega_z \end{bmatrix}^T $$

Linear Velocity Vector of Point A

$$ {}^E\mathbf{v}_{A/E} = {}^E\boldsymbol{\omega}_{O/E} \times {}^E\mathbf{r}_{A/O} $$

Angular Velocity Vector of A-frame

E-frame: Inertial Frame

O-frame: Body Fixed Frame

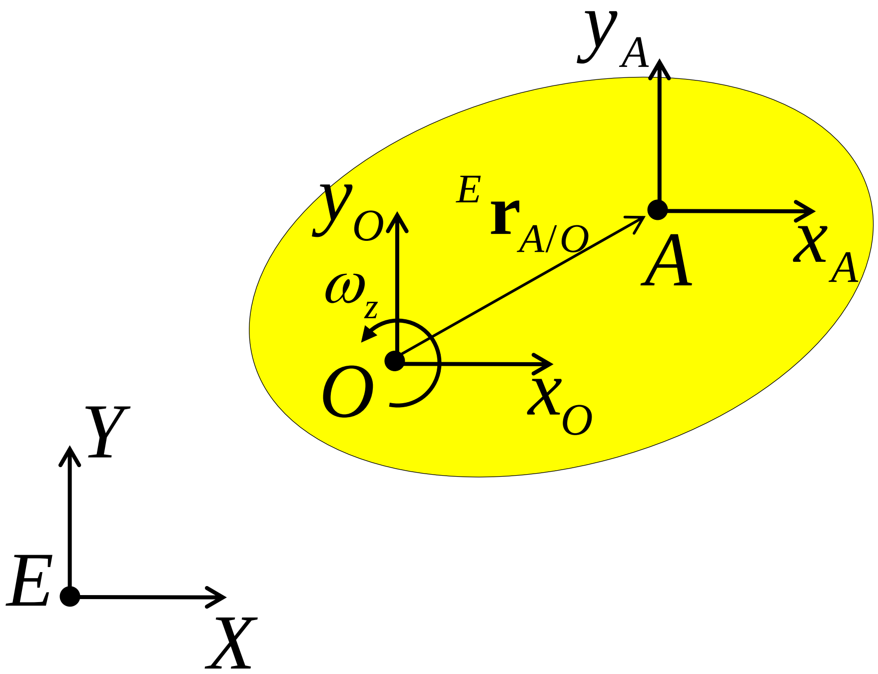
Point O: Pivot(stationary) Point

A-frame: Body Fixed Frame

Point A: Arbitrary Point on the Rigid Body

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, pp. 127

Angular Velocity



Linear Velocity Vector of Point O

$$ {}^E\mathbf{v}_{O/E} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T $$

Angular Velocity Vector of O-frame

$$ {}^E\boldsymbol{\omega}_{O/E} = \begin{bmatrix} 0 & 0 & \omega_z \end{bmatrix}^T $$

Linear Velocity Vector of Point A

$$ {}^E\mathbf{v}_{A/E} = {}^E\boldsymbol{\omega}_{O/E} \times {}^E\mathbf{r}_{A/O} $$

Angular Velocity Vector of A-frame

$$ {}^E\boldsymbol{\omega}_{A/E} = \begin{bmatrix} 0 & 0 & \omega_z \end{bmatrix}^T $$

E-frame: Inertial Frame

O-frame: Body Fixed Frame

Point O: Pivot(stationary) Point

A-frame: Body Fixed Frame

Point A: Arbitrary Point on the Rigid Body

Angular Velocity Vector of Arbitrary Body Fixed frame

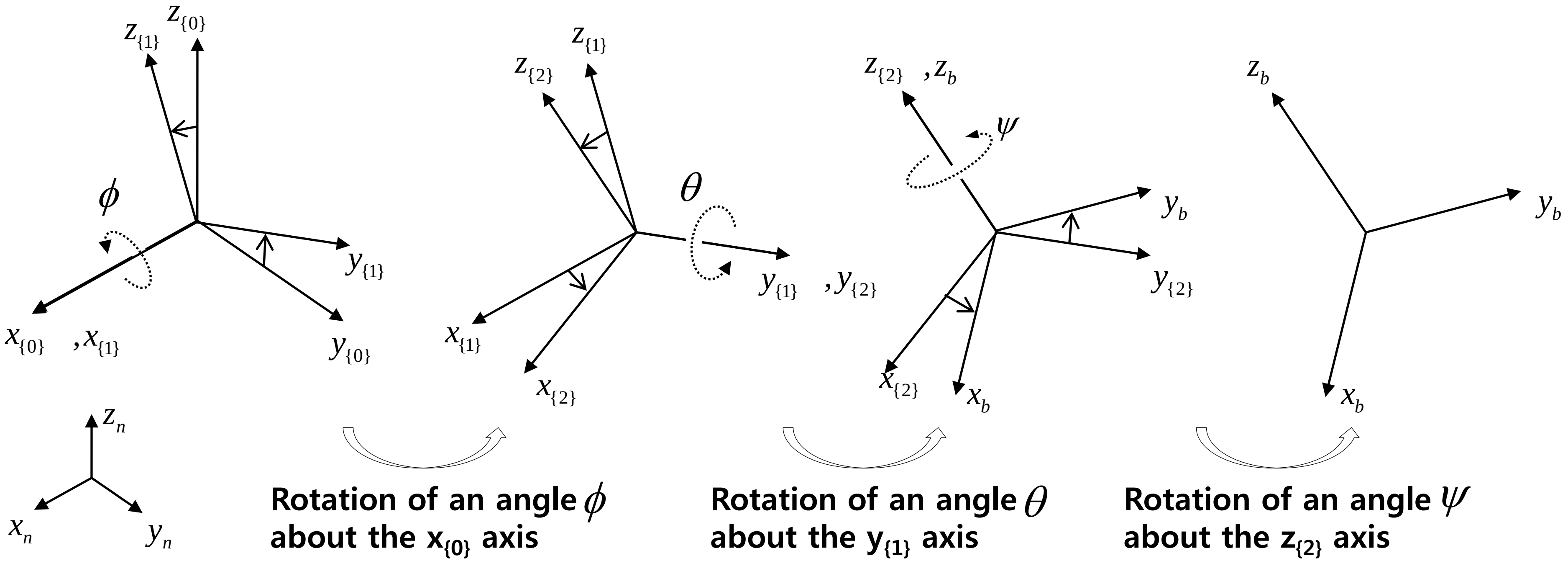
$$ {}^E\boldsymbol{\omega}_{P/E} = \begin{bmatrix} 0 & 0 & \omega_z \end{bmatrix}^T $$

Orientation of the rigid body in spatial motion - Euler angle

One of the most common and widely used parameters in describing reference orientations are the three independent Euler angle.
The transformation between two coordinate systems(Inertial frame and body fixed frame) can be carried out by means of three successive rotations performed in a given sequence.

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 63

x,y,z Euler angle $[\phi \quad \theta \quad \psi]^T$



Rotation of an angle ϕ about the $x_{\{0\}}$ axis

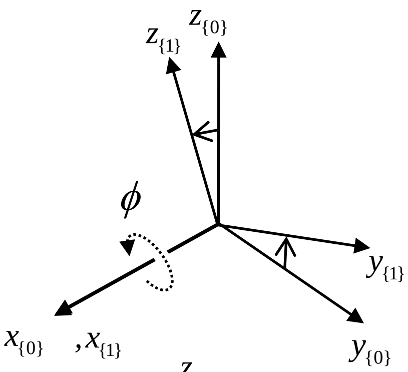
Rotation of an angle θ about the $y_{\{1\}}$ axis

Rotation of an angle ψ about the $z_{\{2\}}$ axis

- Axes of the E' frame are parallel to axes of the E-frame.

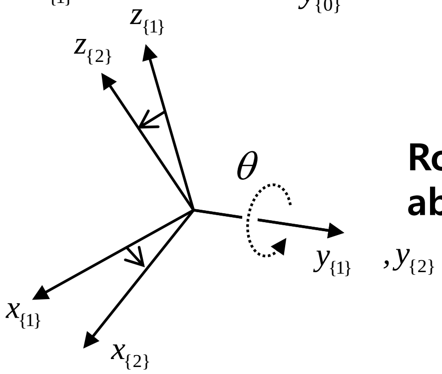
Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Rotation transformation in spatial motion



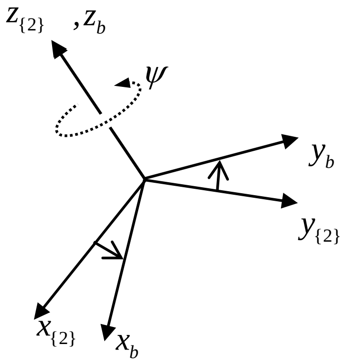
Rotation of an angle ϕ about the $x_{\{0\}}$-axis

$$ {}^{\{0\}}\mathbf{r}_{P/O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} {}^{\{1\}}\mathbf{r}_{P/O} $$



Rotation of an angle θ about the $y_{\{1\}}$-axis

$$ {}^{\{1\}}\mathbf{r}_{P/O} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} {}^{\{2\}}\mathbf{r}_{P/O} $$

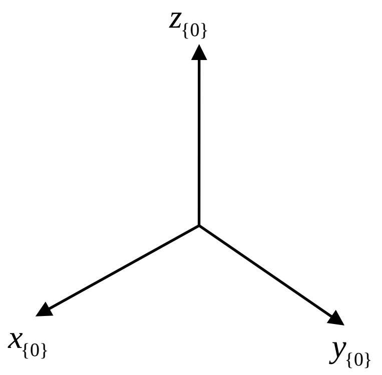


Rotation of an angle ψ about the $z_{\{2\}}$ axis

$$ {}^{\{2\}}\mathbf{r}_{P/O} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{b}\mathbf{r}_{P/O} $$

Rotation transformation in spatial motion

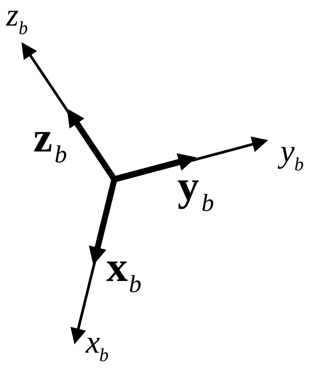
x,y,z Euler angle $[\phi \quad \theta \quad \psi]^T$



Rotation of an angle ϕ about the x axis

Rotation of an angle θ about the y axis

Rotation of an angle ψ about the z axis

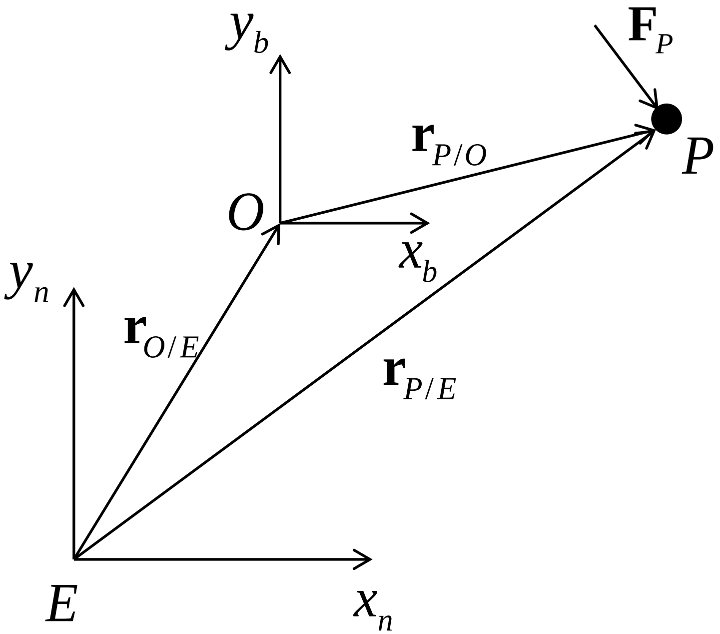


$$ {}^{\{0\}}\mathbf{r}_{P/O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} {}^{\{1\}}\mathbf{r}_{P/O} \curvearrowleft {}^{\{1\}}\mathbf{r}_{P/O} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} {}^{\{2\}}\mathbf{r}_{P/O} \curvearrowleft {}^{\{2\}}\mathbf{r}_{P/O} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{b}\mathbf{r}_{P/O} $$

$$ {}^{\{0\}}\mathbf{r}_{P/O} = \begin{array}{c} \left[\begin{array}{ccc} \cos\theta\cos\psi & -\cos\theta\sin\psi & \sin\theta \\ \sin\phi\sin\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & -\sin\phi\cos\theta \\ -\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi + \sin\phi\cos\psi & \cos\phi\cos\theta \end{array} \right] \\ \begin{array}{ccc} \qquad\qquad {}^{\{0\}}\mathbf{x}_b \qquad\qquad & \qquad\qquad {}^{\{0\}}\mathbf{y}_b \qquad\qquad & \qquad {}^{\{0\}}\mathbf{z}_b \end{array} \end{array} {}^{b}\mathbf{r}_{P/O} $$

Reference)
Coordinate Transformation

Inertial reference frame



$$m_P\ddot{\mathbf{r}}_{P/E} = \mathbf{F}_P$$

This equation is valid

$$m_P\ddot{\mathbf{r}}_{P/O} = \mathbf{F}_P$$

Is this equation valid?

Yes, it is valid.

$$\mathbf{r}_{P/E} = \mathbf{r}_{O/E} + \mathbf{r}_{P/O}$$

Time derivative

$$\dot{\mathbf{r}}_{P/E} = \dot{\mathbf{r}}_{O/E} + \dot{\mathbf{r}}_{P/O}$$

Time derivative

$$\ddot{\mathbf{r}}_{P/E} = \cancel{\ddot{\mathbf{r}}_{O/E}} + \ddot{\mathbf{r}}_{P/O}$$

$$\ddot{\mathbf{r}}_{P/E} = \ddot{\mathbf{r}}_{P/O}$$

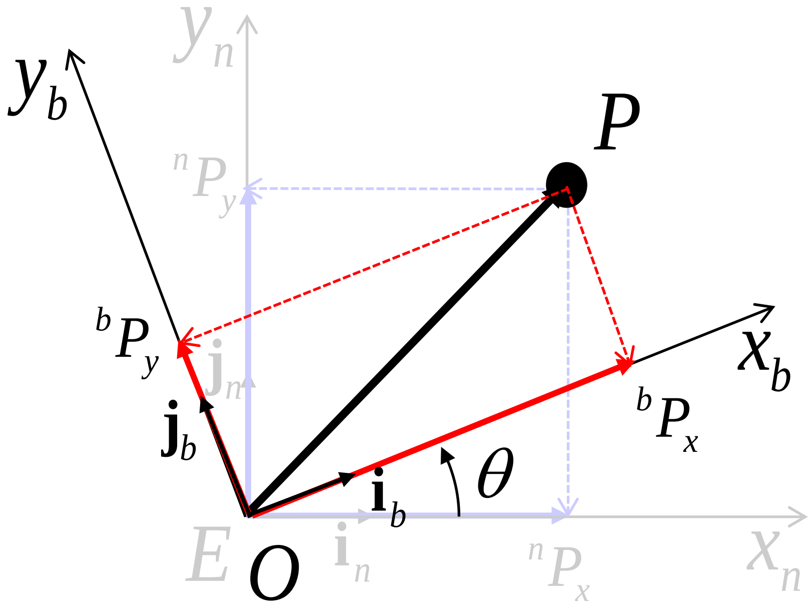
$n-\mathit{frame}$: **absolute reference frame**

$b-\mathit{frame}$: **moving(not rotating) reference frame with constant velocity**

E : **Origin of the n-frame**

O : **Origin of the b-frame**

Coordinates transformation of the vector

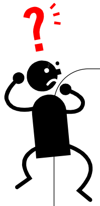


$$\mathbf{P} = {}^nP_x\,\mathbf{i}_n + {}^nP_y\,\mathbf{j}_n$$

$$\mathbf{P} = {}^bP_x\,\mathbf{i}_b + {}^bP_y\,\mathbf{j}_b$$

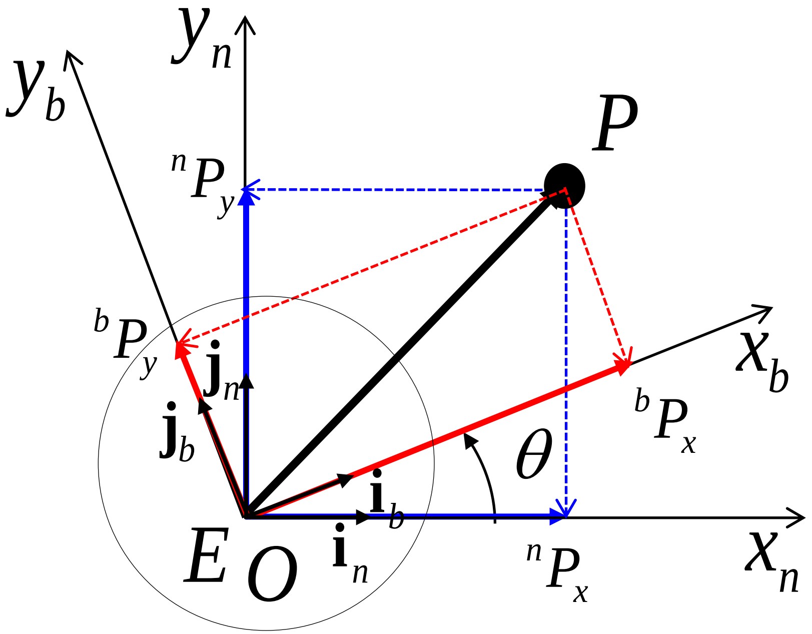
$$\Rightarrow \quad {}^nP_x\,\mathbf{i}_n + {}^nP_y\,\mathbf{j}_n = {}^bP_x\,\mathbf{i}_b + {}^bP_y\,\mathbf{j}_b$$

The same vector, different components



How to change the vector component from one frame to another frame?

Coordinates transformation of the vector



$$\mathbf{P} = {}^nP_x\,\mathbf{i}_n + {}^nP_y\,\mathbf{j}_n$$

$$\mathbf{P} = {}^bP_x\,\mathbf{i}_b + {}^bP_y\,\mathbf{j}_b$$

$${}^nP_x\,\mathbf{i}_n + {}^nP_y\,\mathbf{j}_n = {}^bP_x\,\mathbf{i}_b + {}^bP_y\,\mathbf{j}_b$$

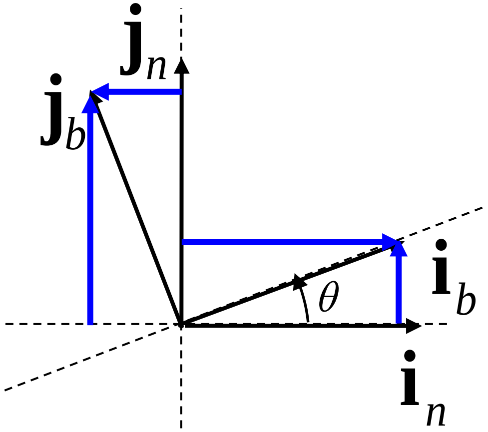
$$\begin{cases} {}^bP_x\,\mathbf{i}_b = {}^bP_x\,(+\cos\theta\mathbf{i}_n + \sin\theta\mathbf{j}_n) \\ {}^bP_y\,\mathbf{j}_b = {}^bP_y\,(-\sin\theta\mathbf{i}_n + \cos\theta\mathbf{j}_n) \end{cases}$$

$${}^nP_x\,\mathbf{i}_n + {}^nP_y\,\mathbf{j}_n = {}^bP_x\,(+\cos\theta\mathbf{i}_n + \sin\theta\mathbf{j}_n)$$

$$+\,{}^bP_y\,(-\sin\theta\mathbf{i}_n + \cos\theta\mathbf{j}_n)$$

$$=({}^bP_x\cos\theta - {}^bP_y\sin\theta)\mathbf{i}_n$$

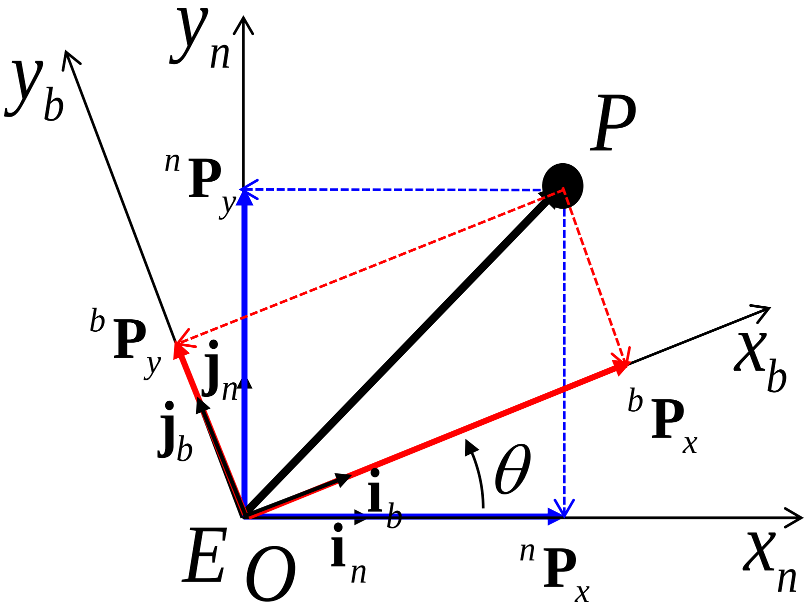
$$+({}^bP_x\sin\theta + {}^bP_y\cos\theta)\mathbf{j}_n$$



$$\mathbf{i}_b = \cos\theta\mathbf{i}_n + \sin\theta\mathbf{j}_n$$

$$\mathbf{j}_b = -\sin\theta\mathbf{i}_n + \cos\theta\mathbf{j}_n$$

Coordinates transformation of the vector



$$\mathbf{P} = {}^nP_x\,\mathbf{i}_n + {}^nP_y\,\mathbf{j}_n$$

$$\mathbf{P} = {}^bP_x\,\mathbf{i}_b + {}^bP_y\,\mathbf{j}_b$$

$${}^nP_x\,\mathbf{i}_n + {}^nP_y\,\mathbf{j}_n = {}^bP_x\,\mathbf{i}_b + {}^bP_y\,\mathbf{j}_b$$

$${}^nP_x\,\mathbf{i}_n + {}^nP_y\,\mathbf{j}_n = ({}^bP_x\cos\theta - {}^bP_y\sin\theta)\mathbf{i}_n + ({}^bP_x\sin\theta + {}^bP_y\cos\theta)\mathbf{j}_n$$

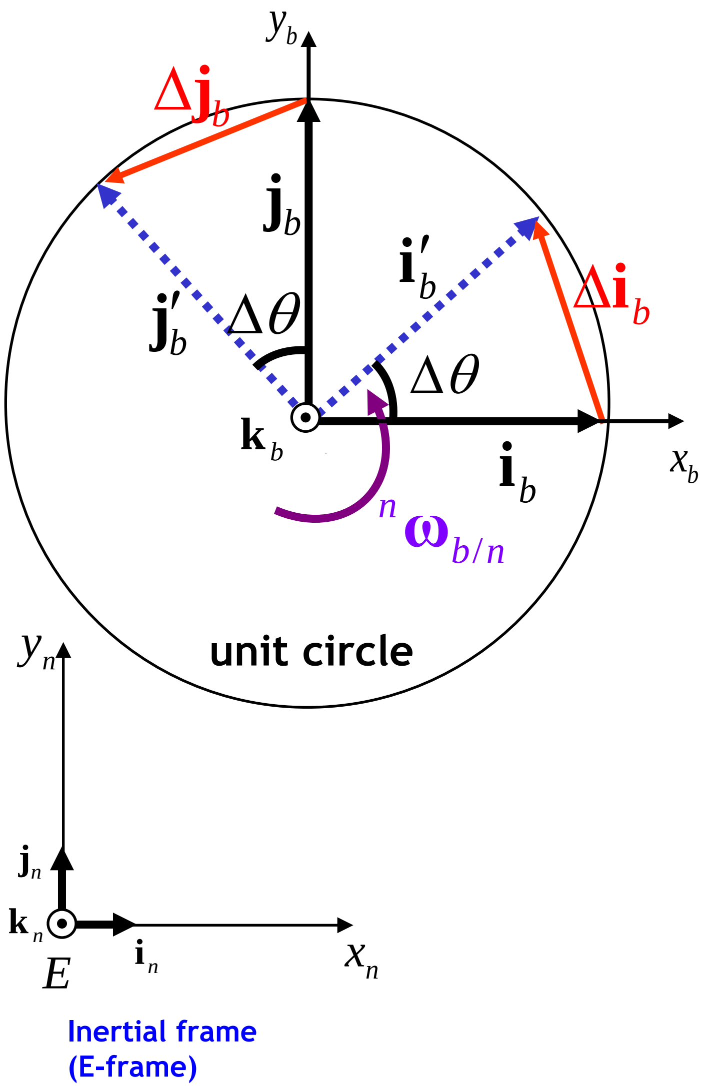
$$\begin{bmatrix} {}^nP_x \\ {}^nP_y \end{bmatrix}[\mathbf{i}_n \quad \mathbf{j}_n] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} {}^bP_x \\ {}^bP_y \end{bmatrix}[\mathbf{i}_n \quad \mathbf{j}_n]$$

$$\begin{bmatrix} {}^nP_x \\ {}^nP_y \end{bmatrix} = \underbrace{\boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}}_{{}^n\mathbf{R}_b}\begin{bmatrix} {}^bP_x \\ {}^bP_y \end{bmatrix} \Rightarrow \begin{bmatrix} {}^nP_x \\ {}^nP_y \end{bmatrix} = {}^n\mathbf{R}_b\begin{bmatrix} {}^bP_x \\ {}^bP_y \end{bmatrix}$$

Time Derivative of a Rotating unit vector

$$\omega = \frac{d\theta}{dt}$$

$$(\boldsymbol{\omega} = \dot{\theta}\mathbf{a} = \omega\mathbf{a})$$



Small Change in $\mathbf{i}_b$: $d\mathbf{i}_b$

$$\begin{cases} \text{Magnitude: } \lim_{\Delta\theta\to 0}|\Delta\mathbf{i}_b| = \lim_{\Delta\theta\to 0}\left(|\mathbf{i}_b|\cdot\Delta\theta\right) = d\theta \\ \text{Direction: } \Delta\theta \to 0, \text{ direction of } \Delta\mathbf{i}_b \text{ converges to the direction of } \mathbf{j}_b. \end{cases}$$

$$d\mathbf{i}_n = \lim_{\Delta\theta\to 0}\Delta\mathbf{i}_n = d\theta\,\mathbf{j}_n$$

Small Change in $\mathbf{j}_b$: $d\mathbf{j}_b$

$$\begin{cases} \text{Magnitude: } \lim_{\Delta\theta\to 0}|\Delta\mathbf{j}_b| = \lim_{\Delta\theta\to 0}\left(|\mathbf{j}_b|\cdot\Delta\theta\right) = d\theta \\ \text{Direction: } \Delta\theta \to 0, \text{ direction of } \Delta\mathbf{j}_b \text{ converges to the direction of } -\mathbf{i}_b. \end{cases}$$

$$d\mathbf{j}_b = \lim_{\Delta\theta\to 0}\Delta\mathbf{j}_b = -d\theta\,\mathbf{i}_b$$

Time derivative of a rotating unit vector

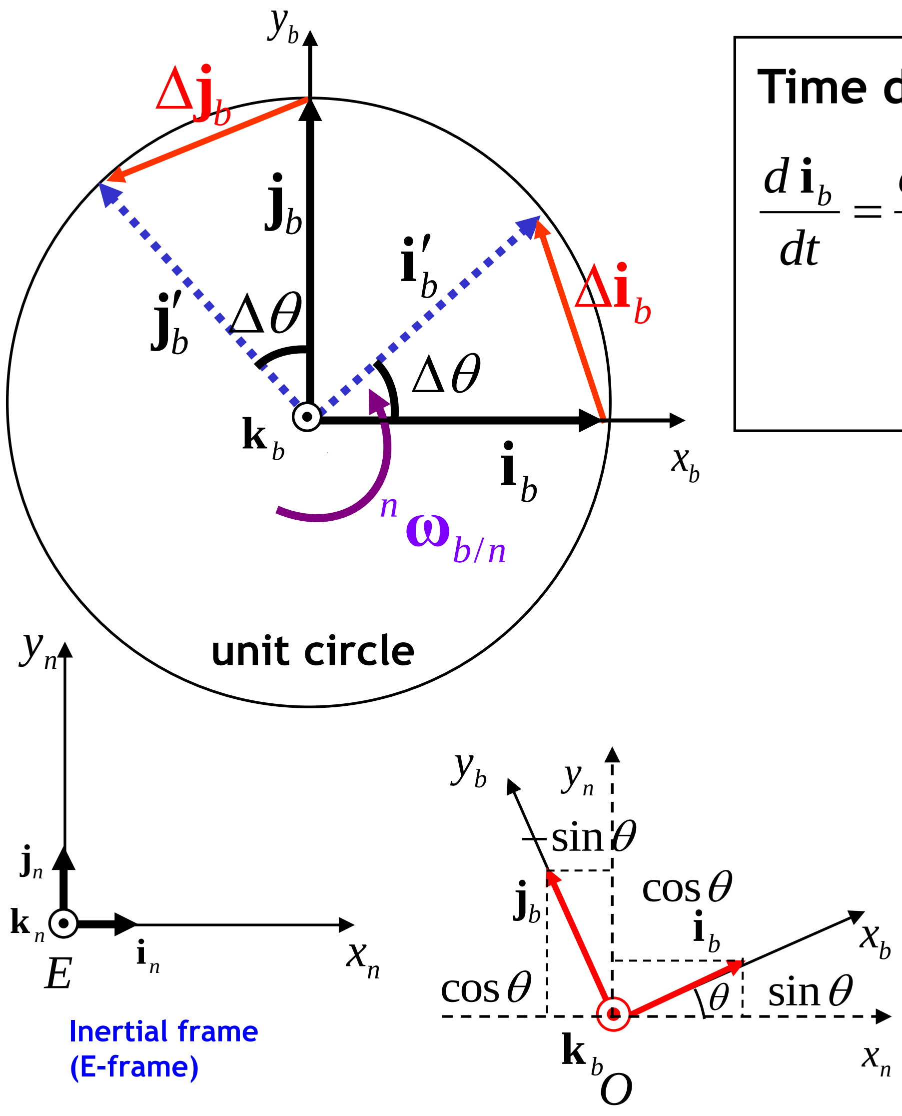
$$\frac{d\,\mathbf{i}_b}{dt} = \frac{d\theta\,\mathbf{j}_b}{dt} = \omega\,\mathbf{j}_b$$
$$= \omega_{b/n}(\mathbf{k}_b \times \mathbf{i}_b)$$
$$= {}^{n}\boldsymbol{\omega}_{b/n} \times \mathbf{i}_b$$

$$\frac{d\mathbf{j}_b}{dt} = -\frac{d\theta}{dt}\mathbf{i}_b = -\omega\,\mathbf{i}_b$$
$$= \omega_{b/n}(\mathbf{k}_b \times \mathbf{j}_b)$$
$$= {}^{n}\boldsymbol{\omega}_{b/n} \times \mathbf{j}_b$$

회전하는 단위 벡터의 시간에 대한 미분

$$\omega = \frac{d\theta}{dt}$$

$$(\boldsymbol{\omega} = \dot{\theta}\mathbf{a} = \omega\mathbf{a})$$



Time derivative of a rotating unit vector

$$\frac{d\,\mathbf{i}_b}{dt} = \frac{d\theta\,\mathbf{j}_b}{dt} = \omega\,\mathbf{j}_b = \omega_{b/n}(\mathbf{k}_b \times \mathbf{i}_b) = {}^{n}\boldsymbol{\omega}_{b/n} \times \mathbf{i}_b$$

$$\frac{d\mathbf{j}_b}{dt} = -\frac{d\theta}{dt}\mathbf{i}_b = -\omega\,\mathbf{i}_b = \omega_{b/n}(\mathbf{k}_b \times \mathbf{j}_b) = {}^{n}\boldsymbol{\omega}_{b/n} \times \mathbf{j}_b$$

$${}^{n}\mathbf{R}_b(\theta) = \begin{bmatrix}\mathbf{i}_n & \mathbf{j}_n\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}\mathbf{i}_b & \mathbf{j}_b\end{bmatrix}$$

※ Derivative of a Rotation Matrix

$$\frac{d\,{}^{n}\mathbf{R}_b(\theta)}{dt} = \begin{bmatrix}\dfrac{d\mathbf{i}_b}{dt} & \dfrac{d\mathbf{j}_b}{dt}\end{bmatrix}$$

$$= \begin{bmatrix}{}^{n}\boldsymbol{\omega}_{b/n} \times \mathbf{i}_b & {}^{n}\boldsymbol{\omega}_{b/n} \times \mathbf{j}_b\end{bmatrix}$$

$$= {}^{n}\boldsymbol{\omega}_{b/n} \times \begin{bmatrix}\mathbf{i}_b & \mathbf{j}_b\end{bmatrix}$$

$$= {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{R}_b$$

Naval Architecture & Ocean Engineering

Topics in ship design automation

3. Multibody Dynamics

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September, 2010

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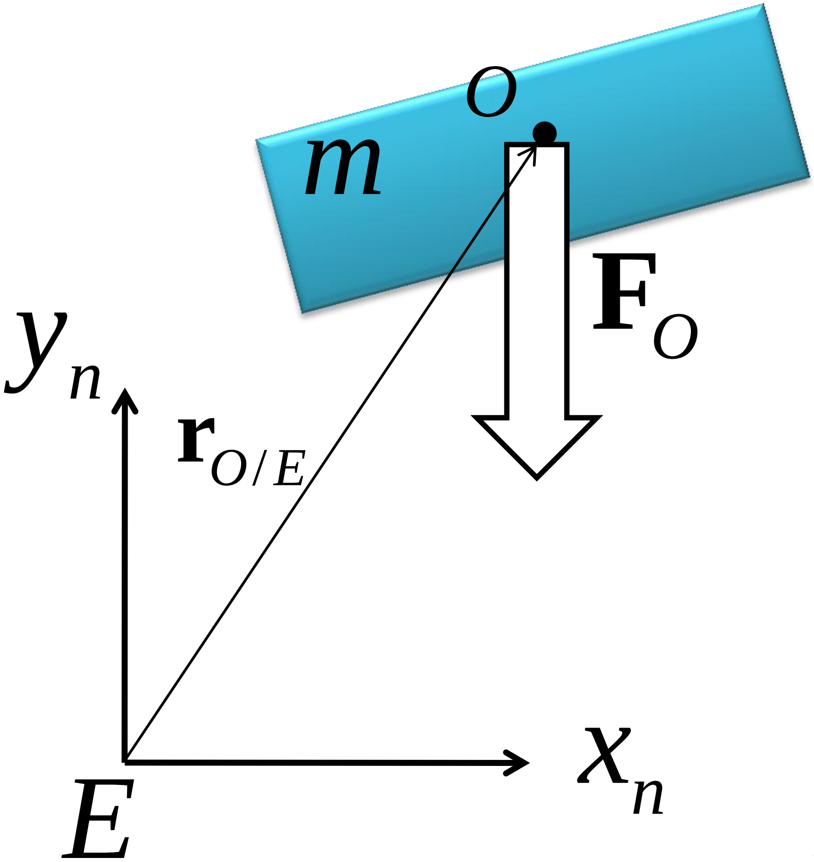
3.1 Introduction to Multibody Dynamics

Multibody System Dynamics

'Multibody System Dynamics' is the discipline describing the dynamic behavior of multibody system which consists of interconnected rigid bodies.



n-frame: inertial reference frame
O: center of mass of the rigid body

Given : Resultant Force

Find : Motion
(Position, Velocity, Acceleration)

Law of Nature : Newton's 2nd Law

$$m\ \ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O$$

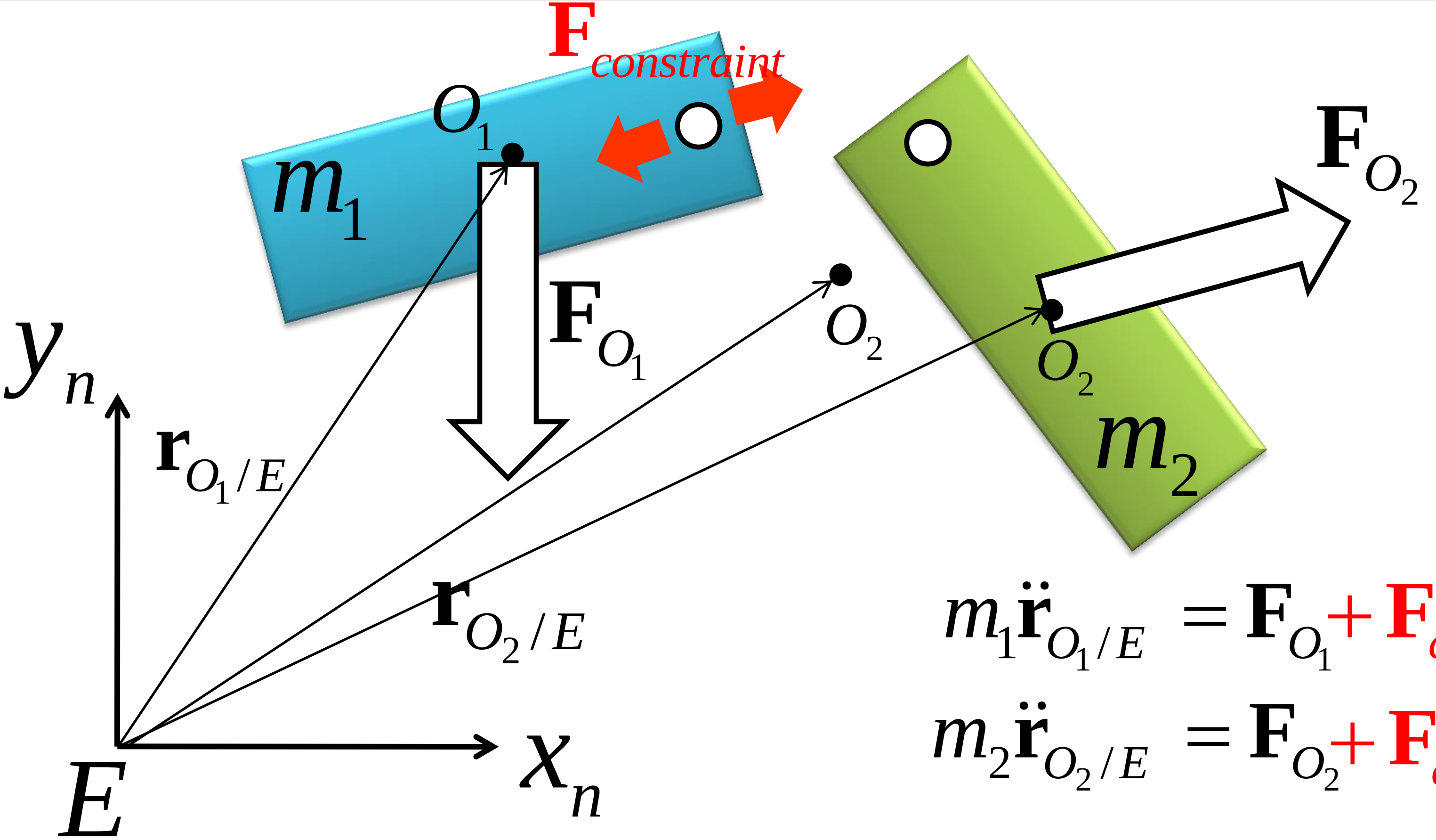
➡ **Equation of Motion**

,where $\mathbf{r}_{O/E} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad x, y : coordinates$

'Reference frame' should be specified to define the position

Multibody System Dynamics

'Multibody System Dynamics' is the discipline describing the dynamic behavior of multibody system which consists of interconnected rigid bodies.



$$m_1\ddot{\mathbf{r}}_{O_1/E} = \mathbf{F}_{O_1} + \mathbf{F}^1_{constraint}$$

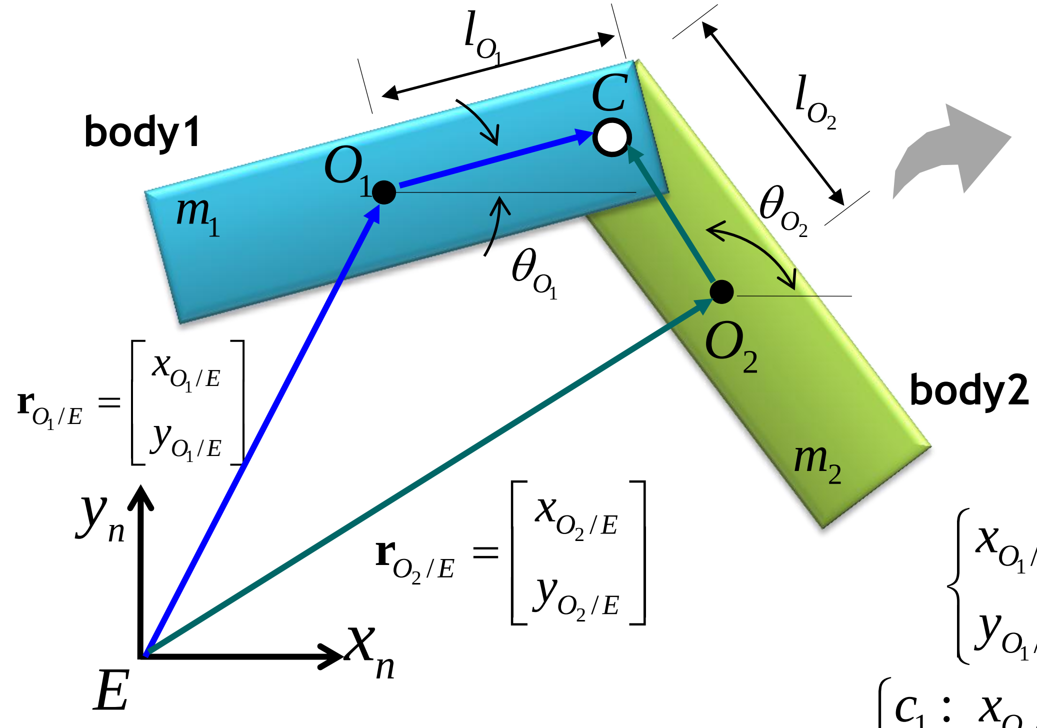
$$m_2\ddot{\mathbf{r}}_{O_2/E} = \mathbf{F}_{O_2} + \mathbf{F}^2_{constraint}$$

$$\mathbf{F}^1_{constraint} = -\mathbf{F}^2_{constraint}$$

n-frame: inertial reference frame
O: center of mass of the rigid body

Multibody System Dynamics

'Multibody System Dynamics' is the discipline describing the dynamic behavior of multibody system which consists of interconnected rigid bodies.



$$\mathbf{r}_{O_1/E} = \begin{bmatrix} x_{O_1/E} \\ y_{O_1/E} \end{bmatrix}$$

$$\mathbf{r}_{O_2/E} = \begin{bmatrix} x_{O_2/E} \\ y_{O_2/E} \end{bmatrix}$$

Constraint force is caused by kinematic constraint

Kinematic constraint

$$\mathbf{r}_{O_1/E} + \mathbf{r}_{C/O_1} = \mathbf{r}_{O_2/E} + \mathbf{r}_{C/O_2}$$

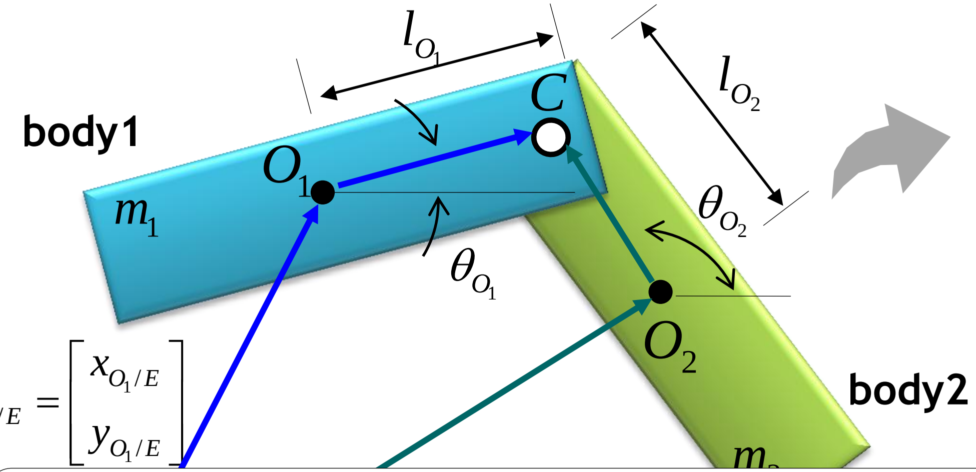
$$\begin{cases} x_{O_1/E} + l_{O_1} \cos\theta_{O_1} = x_{O_2/E} + l_{O_2} \cos\theta_{O_2} \\ y_{O_1/E} + l_{O_1} \sin\theta_{O_1} = y_{O_2/E} + l_{O_2} \sin\theta_{O_2} \end{cases}$$

$$\Rightarrow \begin{cases} c_1 : \ x_{O_1/E} + l_{O_1} \cos\theta_{O_1} - x_{O_2/E} - l_{O_2} \cos\theta_{O_2} = 0 \\ c_2 : \ y_{O_1/E} + l_{O_1} \sin\theta_{O_1} - y_{O_2/E} - l_{O_2} \sin\theta_{O_2} = 0 \end{cases}$$

n-frame: inertial reference frame
O: center of mass of the rigid body

Multibody System Dynamics

‘Multibody System Dynamics’ is the discipline describing the dynamic behavior of multibody system which consists of interconnected rigid bodies.



$$\mathbf{r}_{O_1/E} = \begin{bmatrix} x_{O_1/E} \\ y_{O_1/E} \end{bmatrix}$$

Constraint force is caused by kinematic constraint

Kinematic constraint

$$\mathbf{r}_{O_1/E} + \mathbf{r}_{C/O_1} = \mathbf{r}_{O_2/E} + \mathbf{r}_{C/O_2}$$

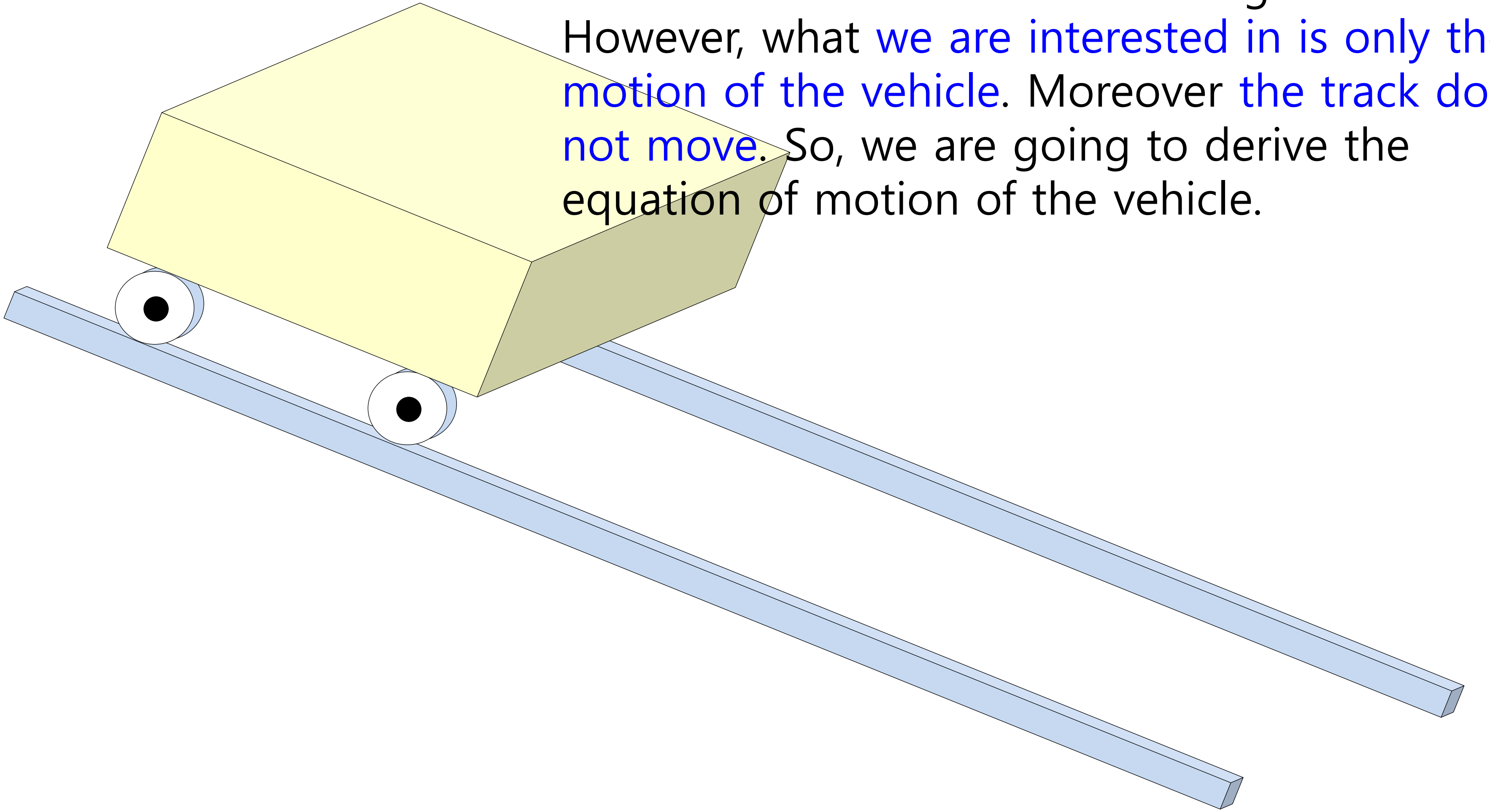
How can we derive the equations of motion for multibody system considering constraint force?

$$C_2 : \; y_{O_1/E} + l_{O_1}\sin\theta_{O_1} - y_{O_2/E} - l_{O_2}\sin\theta_{O_2} = 0$$

n-frame: inertial reference frame

O: center of mass of the rigid body

Vehicle constrained to move along straight track



This system consists of a inclined track and a vehicle constrained to move along the track. However, what we are interested in is only the motion of the vehicle. Moreover the track does not move. So, we are going to derive the equation of motion of the vehicle.

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

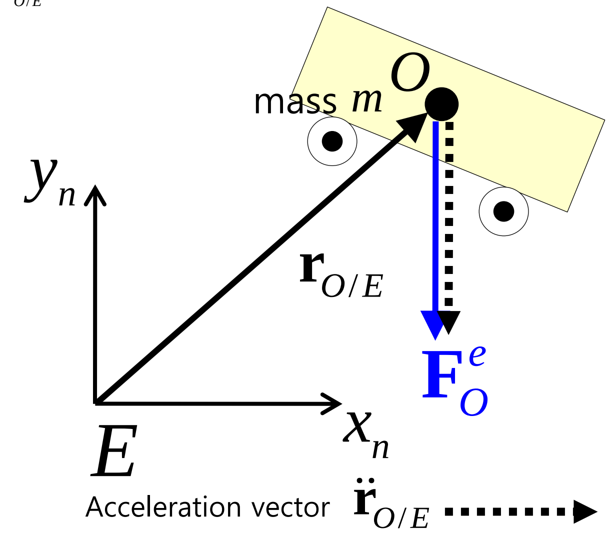
- Derivation of equations of motion for a free falling body by using Newton's 2nd law

A free falling body

$n-frame$: Inertial reference frame

O: Center of mass of the vehicle

$\mathbf{r}_{O/E}$: Position vector of the center of mass O



According to Newton's 2nd law

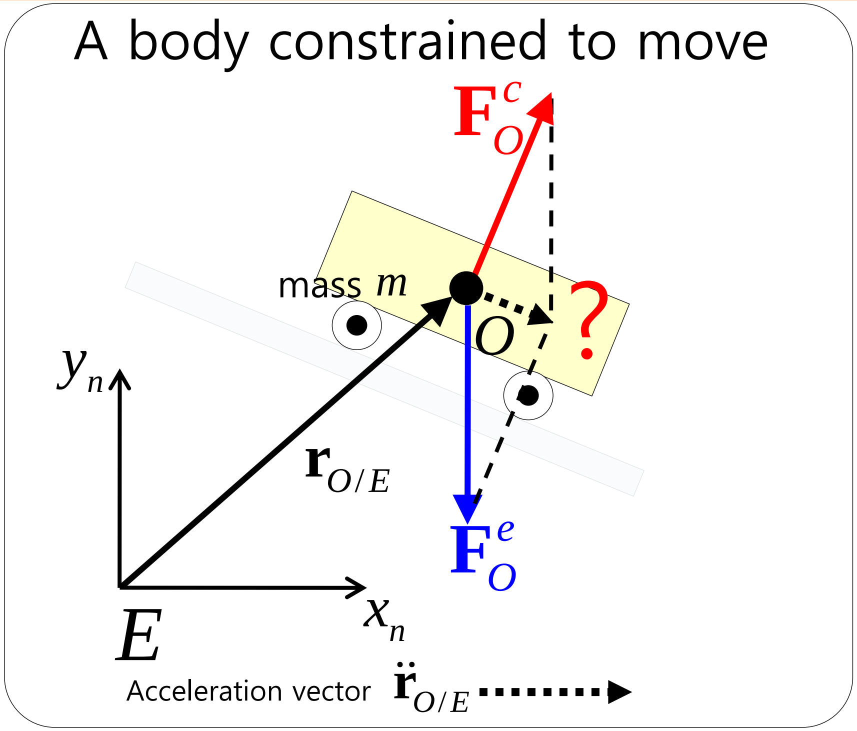
$$m\ddot{\mathbf{r}}_{O/E} = \sum \mathbf{F} \quad , where\ \mathbf{r}_{O/E} = \begin{bmatrix} x_{O/E} \\ y_{O/E} \end{bmatrix}, \sum \mathbf{F} = \begin{bmatrix} \sum F_x \\ \sum F_y \end{bmatrix}$$

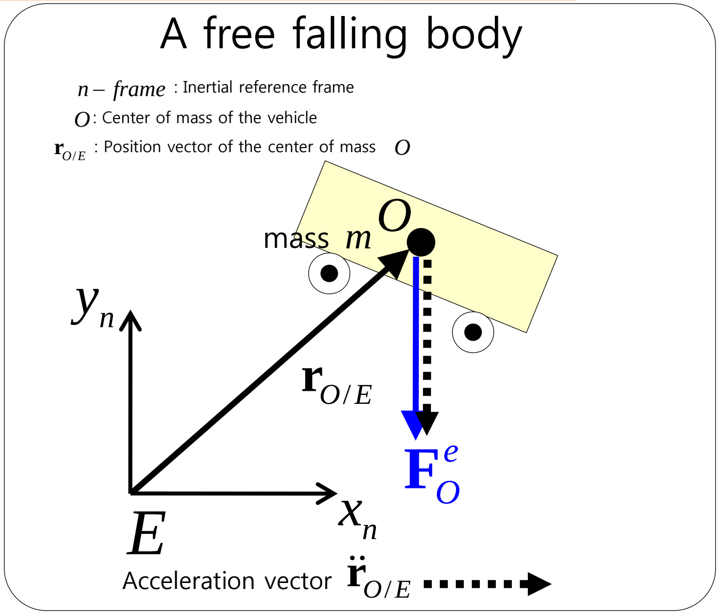
If the gravitational force $\mathbf{F}_O^e$ is the only force that acts on the vehicle, the vehicle will vertically fall down.

$$m\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track - Resultant force that acts on a constrained body


A body constrained to move


A free falling body

$n-frame$: Inertial reference frame

O: Center of mass of the vehicle

$\mathbf{r}_{O/E}$: Position vector of the center of mass O

According to Newton's 2nd law

$$m\ddot{\mathbf{r}}_{O/E} = \sum \mathbf{F} \text{ , where } \mathbf{r}_{O/E} = \begin{bmatrix} x_{O/E} \\ y_{O/E} \end{bmatrix}, \sum \mathbf{F} = \begin{bmatrix} \sum F_x \\ \sum F_y \end{bmatrix}$$

For the vehicle to move along the track,
there must be an additional force $\mathbf{F}_O^c$, besides the gravitational force $\mathbf{F}_O^e$.

$$\sum \mathbf{F} = \mathbf{F}_O^e + \mathbf{F}_O^c$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track - Matrix representation of the equations of motion

According to Newton's 2nd law

$$m\ddot{\mathbf{r}}_{O/E} = \sum \mathbf{F} \text{ , where } \mathbf{r}_{O/E} = \begin{bmatrix} x_{O/E} \\ y_{O/E} \end{bmatrix}, \sum \mathbf{F} = \begin{bmatrix} \sum F_x \\ \sum F_y \end{bmatrix}$$

⇩

$$m\ddot{x}_{O/E} = \sum F_x$$

$$m\ddot{y}_{O/E} = \sum F_y$$

⇩ Matrix representation

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_{O/E} \\ \ddot{y}_{O/E} \end{bmatrix} = \begin{bmatrix} \sum F_x \\ \sum F_y \end{bmatrix}$$

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \sum \mathbf{F} \text{ , where } \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \sum \mathbf{F} = \begin{bmatrix} \sum F_x \\ \sum F_y \end{bmatrix}$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

- Derivation of equations of motion for a constrained body by using Newton's 2nd law

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \sum \mathbf{F}, \text{where } \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

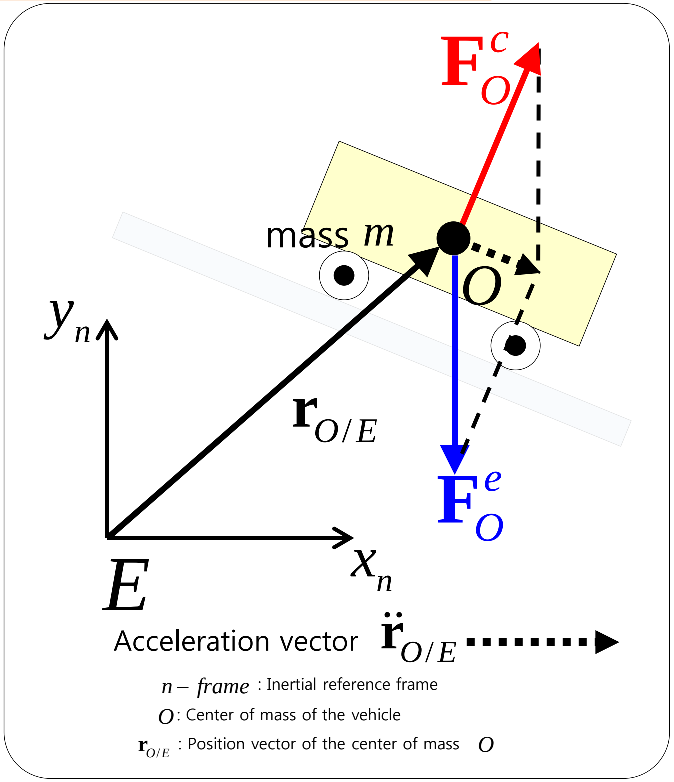
$$\sum \mathbf{F} = \mathbf{F}_O^e + \mathbf{F}_O^c$$

Equations of motion

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c$$

: external force

$\mathbf{F}_O^c$: constraint reaction force



Acceleration vector $\ddot{\mathbf{r}}_{O/E}$

$n-frame$: Inertial reference frame

O: Center of mass of the vehicle

$\mathbf{r}_{O/E}$: Position vector of the center of mass O

How to solve the equations of motion?

Find: $\ddot{\mathbf{r}}_{O/E}$ Given: $\mathbf{M}, \mathbf{F}_O^e$ Unknown: $\mathbf{F}_O^c$

We should know the constraint reaction force $\mathbf{F}_O^c$.

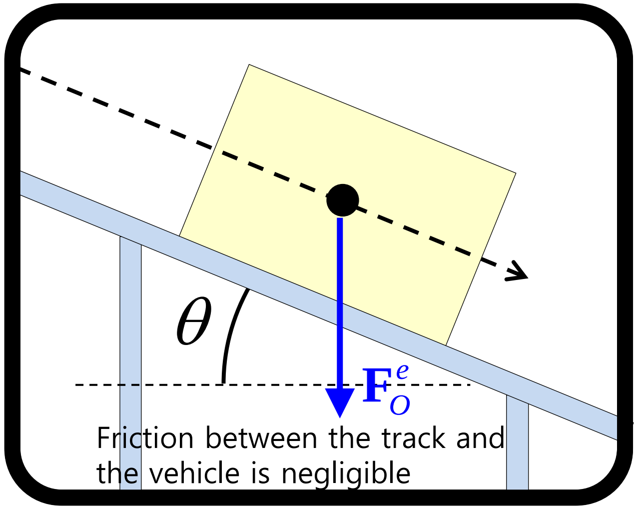
$\mathbf{F}_O^c$ is related with .

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

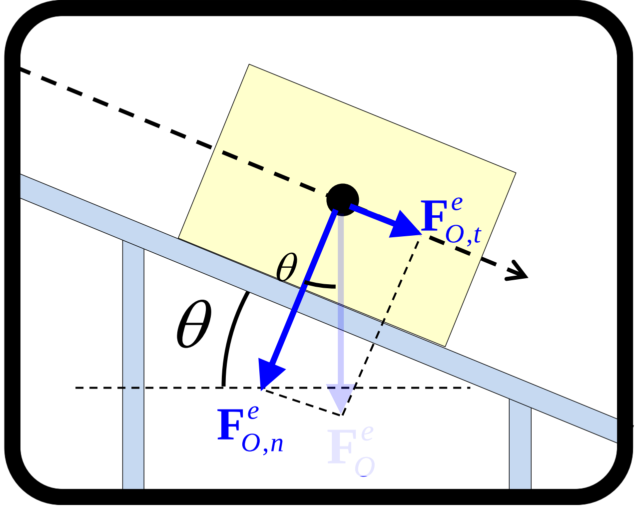
- Constraint reaction force

① External Force $\mathbf{F}_O^e$: Gravitational force

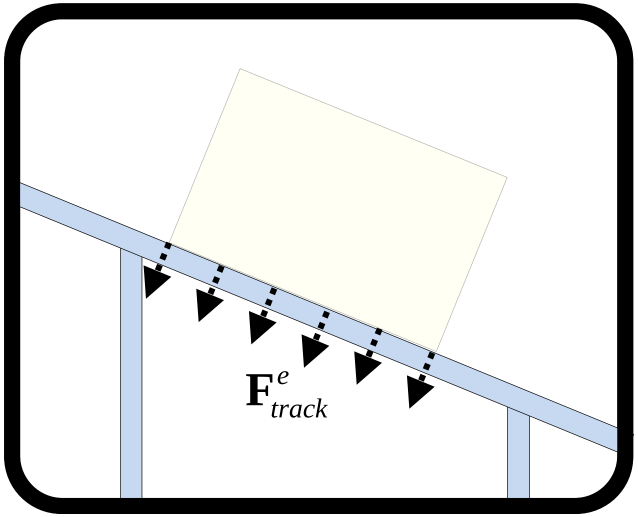


Friction between the track and the vehicle is negligible

② $\mathbf{F}_O^e$ is resolved into a normal vector and a tangential vector

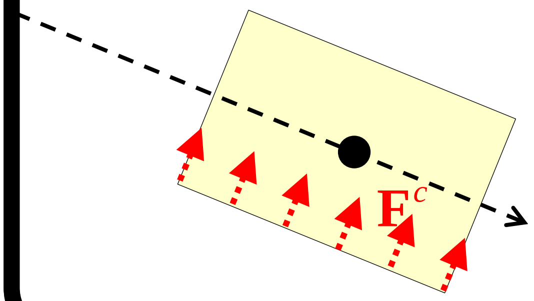


③ $\mathbf{F}_{O,n}^e$ causes a distributed force $\mathbf{F}_{track}^e$, that acts on the track.

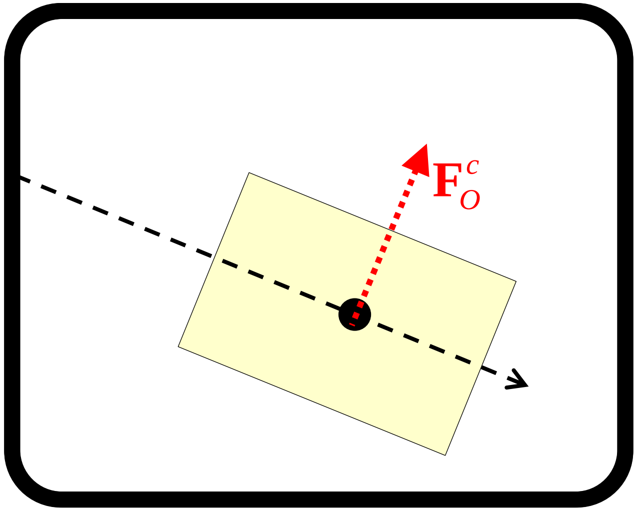


④ A reaction force acts on the vehicle(1).

According to Newton's 3rd law a reaction force $\mathbf{F}^c$ acts on the vehicle.



⑤ The reaction force can be considered to act on the center of mass of the vehicle.



According to Newton's 3rd law

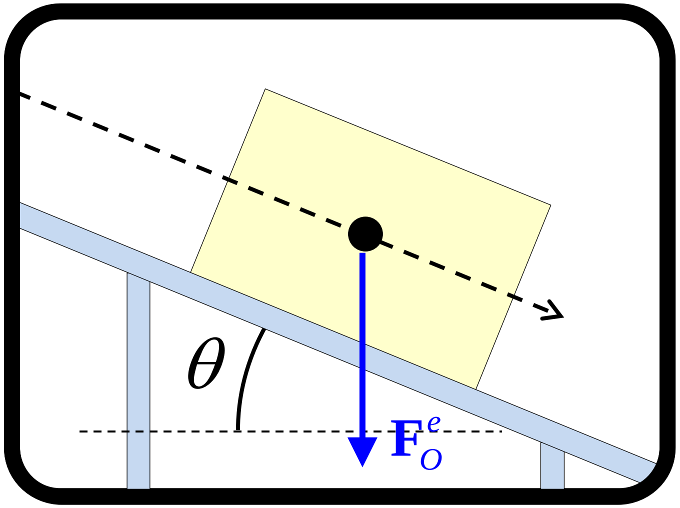
$\mathbf{F}_O^c$ is equal and opposite to $\mathbf{F}_{O,n}^e$

The constrained reaction force is perpendicular to the curve or surface along which the particle is constrained to move.

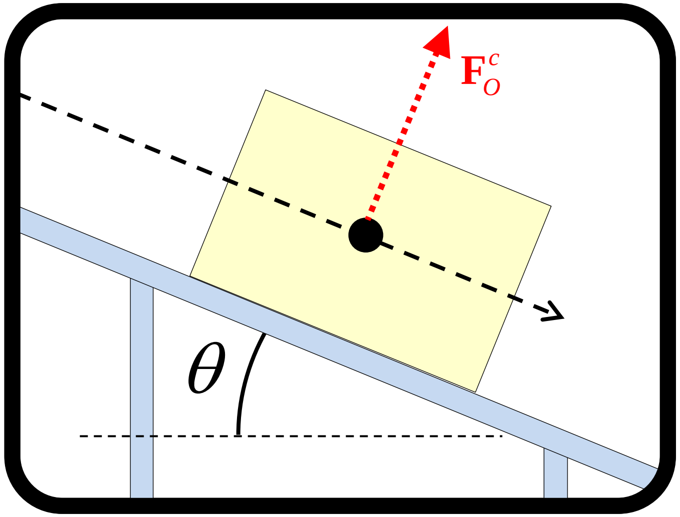
Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track - Free body diagram of the vehicle

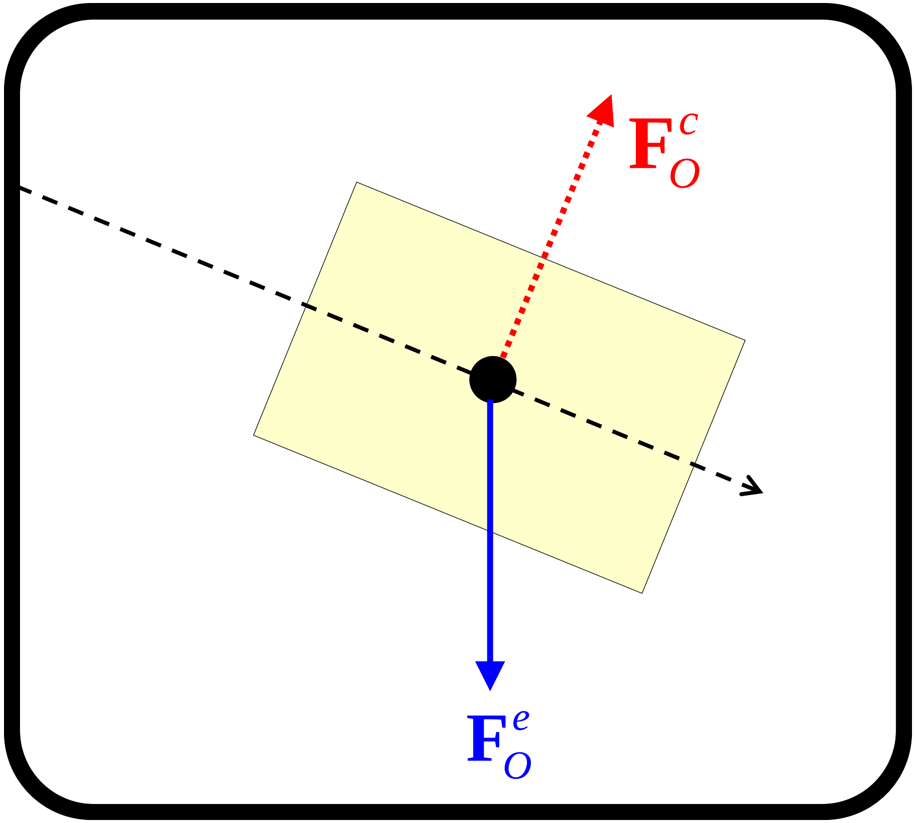
① External Force $\mathbf{F}_O^e$: Gravitational force



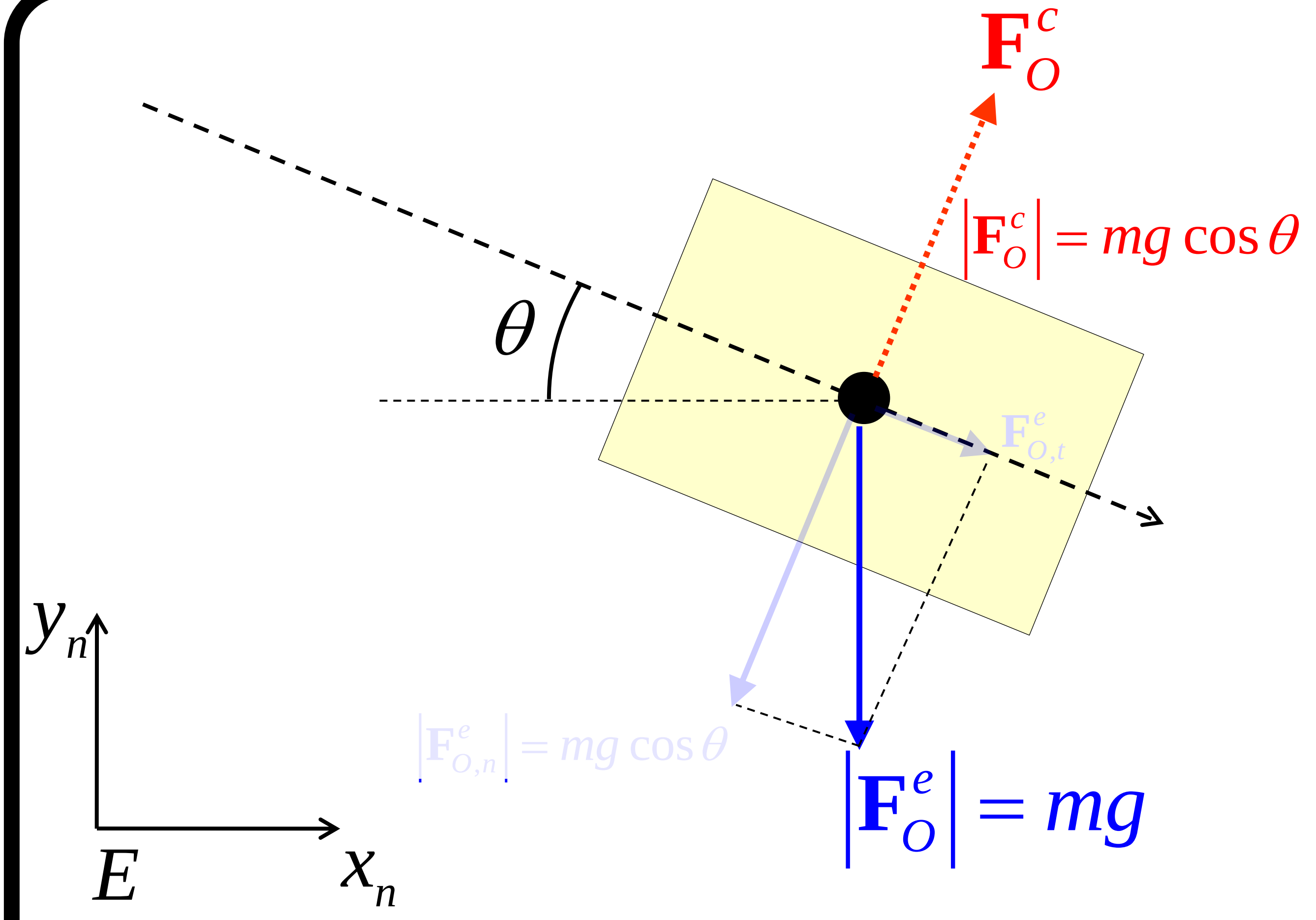
② $\mathbf{F}_O^c$ is the constrained reaction force.



③ Free body diagram of the vehicle.



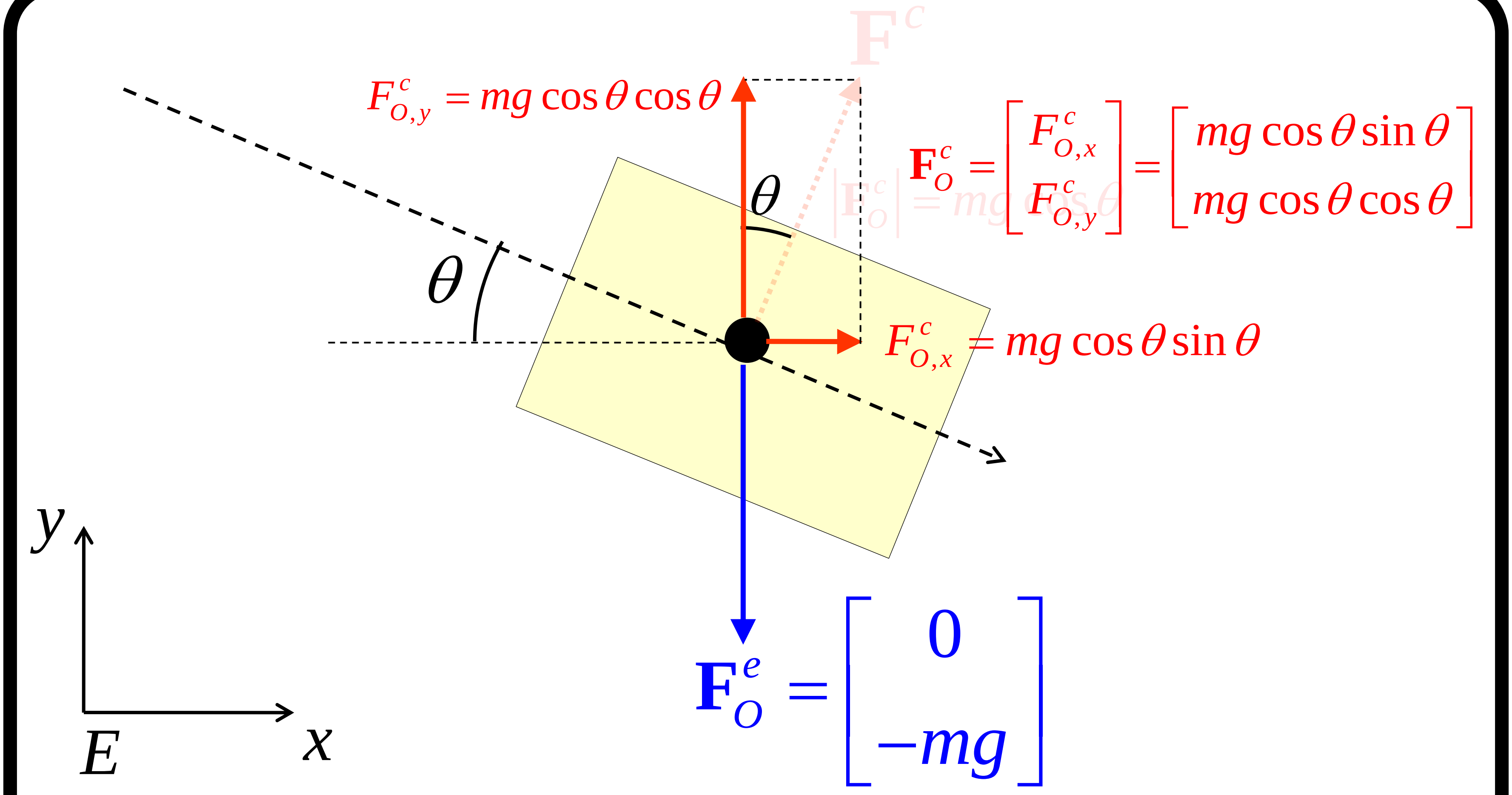
Vehicle constrained to move along the straight track - Free body diagram of the vehicle



$\mathbf{M}\boxed{\ddot{\mathbf{r}}_{O/E}} = \mathbf{F}_O^e + \mathbf{F}_O^c$ Since the position vector is defined in the n-frame, $\mathbf{F}_O^e$ and $\mathbf{F}_O^c$ are to be decomposed in the n-frame.

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 15 ~ 16

Vehicle constrained to move along the straight track - Free body diagram of the vehicle



$$F_{O,y}^c = mg\cos\theta\cos\theta$$

$$\mathbf{F}_O^c = \begin{bmatrix} F_{O,x}^c \\ F_{O,y}^c \end{bmatrix} = \begin{bmatrix} mg\cos\theta\sin\theta \\ mg\cos\theta\cos\theta \end{bmatrix}$$

$$F_{O,x}^c = mg\cos\theta\sin\theta$$

$$\mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

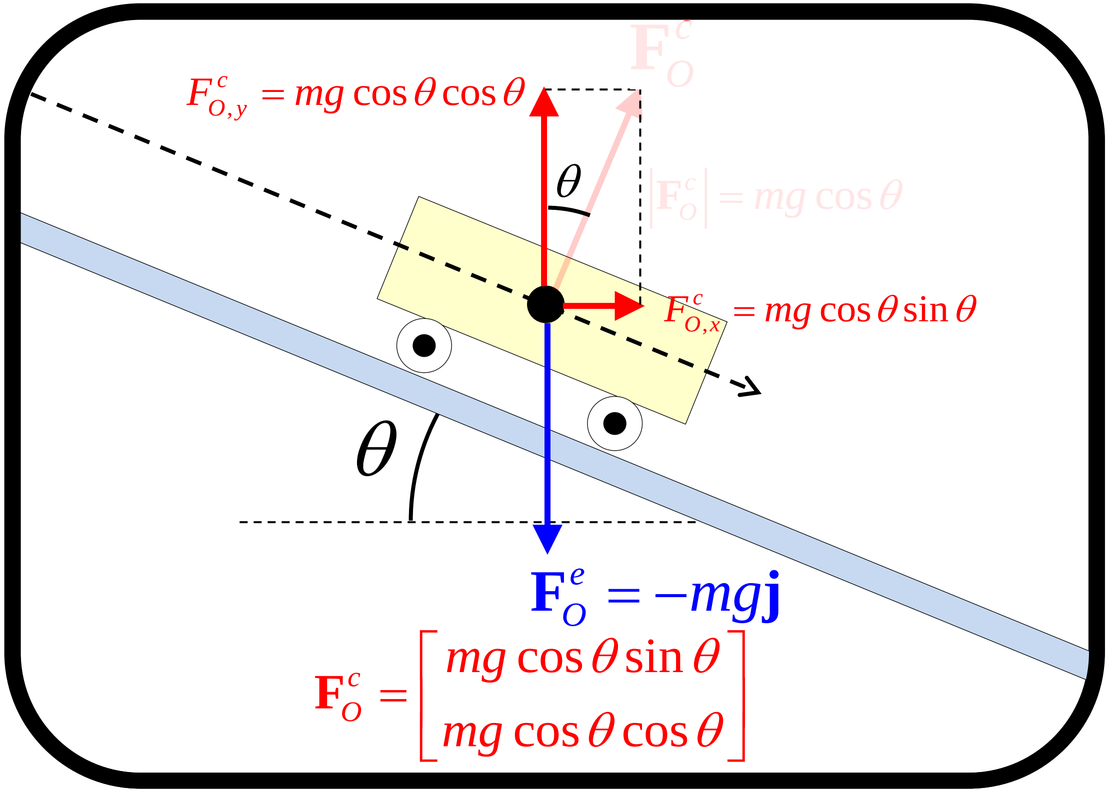
$$\mathbf{M}\boxed{\ddot{\mathbf{r}}_{O/E}} = \mathbf{F}_O^e + \mathbf{F}_O^c$$

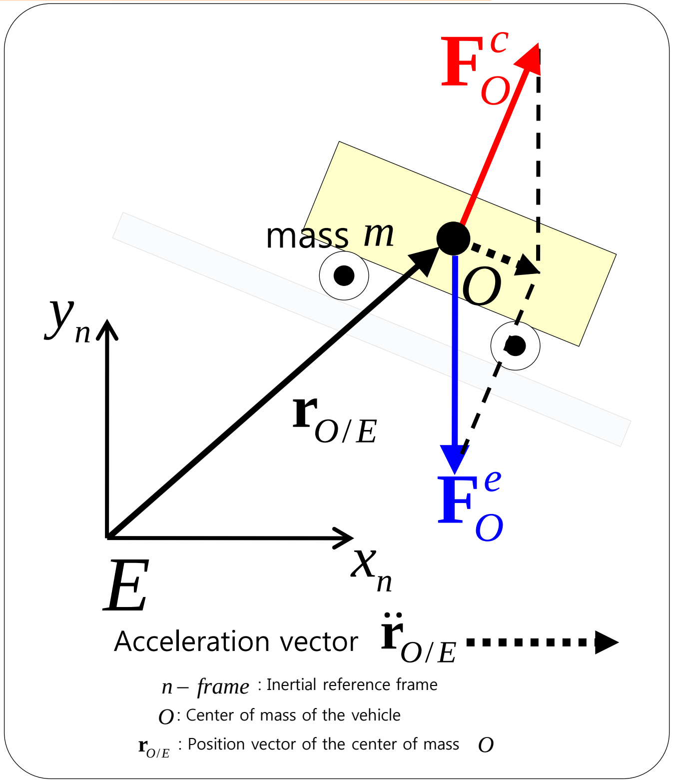
Since the position vector is defined in the n-frame, $\mathbf{F}_O^e$ and $\mathbf{F}_O^c$ are to be decomposed in the n-frame.

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

- Derivation of equations of motion for a constrained body by using Newton's 2nd law





$n-$ *frame* : Inertial reference frame

O: Center of mass of the vehicle

$\mathbf{r}_{O/E}$: Position vector of the center of mass O

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c, \text{ where } \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$\mathbf{F}_O^e$: External force

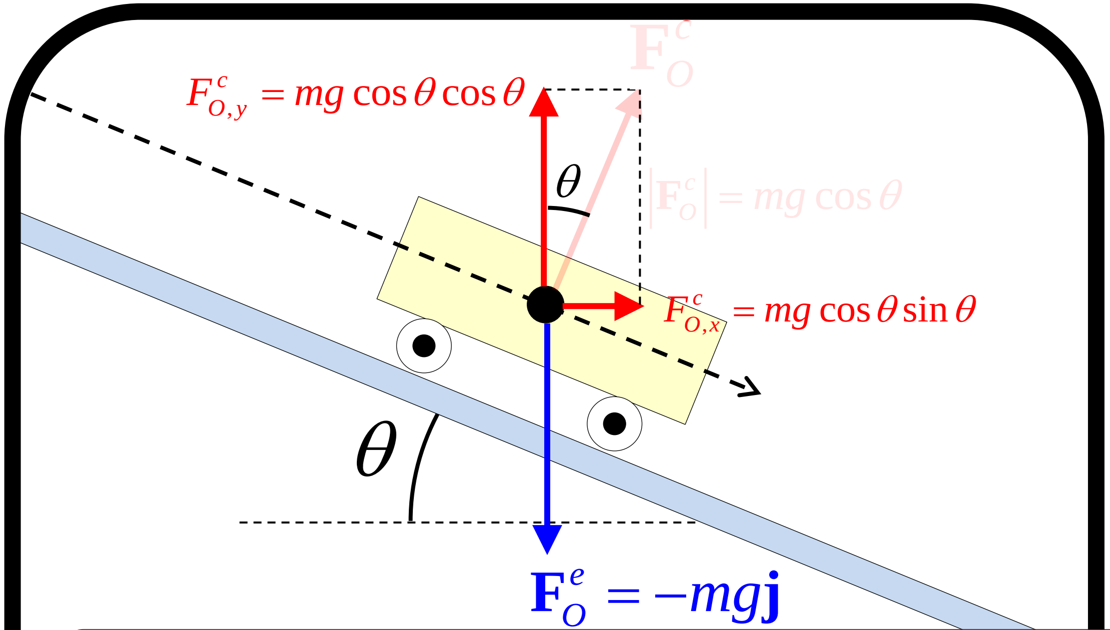
$\mathbf{F}_O^c$: Constraint reaction force

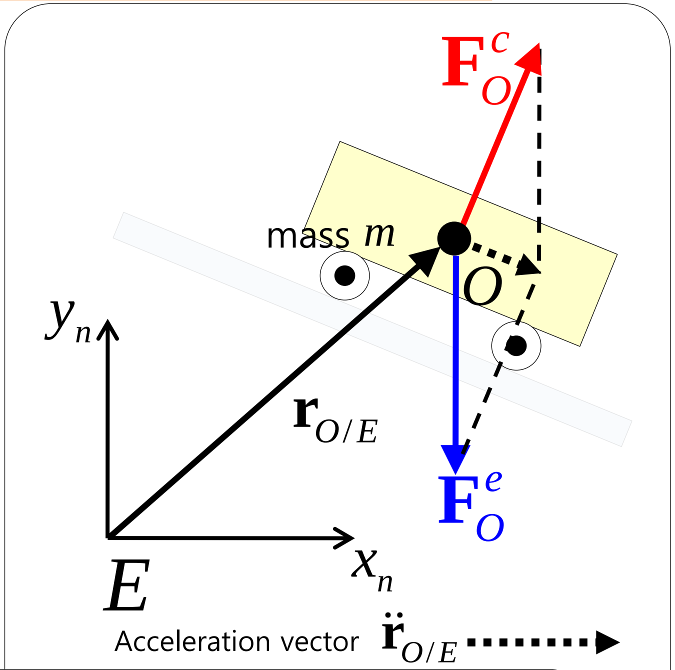
To solve the equations of motion, we should know the constraint reaction force.

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_{O/E} \\ \ddot{y}_{O/E} \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} mg\cos\theta\sin\theta \\ mg\cos\theta\cos\theta \end{bmatrix}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion for a constrained body by using Newton's 2nd law





Sometimes it is complicated to calculate the constraint reaction force.

Can we solve the equations of motion without calculating the constraint reaction force explicitly?

I

T

$$\begin{bmatrix} \\ 0 & m \end{bmatrix}\begin{bmatrix} \\ \ddot{y}_{O/E} \end{bmatrix} = \begin{bmatrix} \\ -mg \end{bmatrix} + \begin{bmatrix} \\ mg\cos\theta\cos\theta \end{bmatrix}$$

3.2 Embedding technique (Relative coordinate formulation)

Vehicle constrained to move along straight track - Problem Definition

$n-frame$: Inertial reference frame.

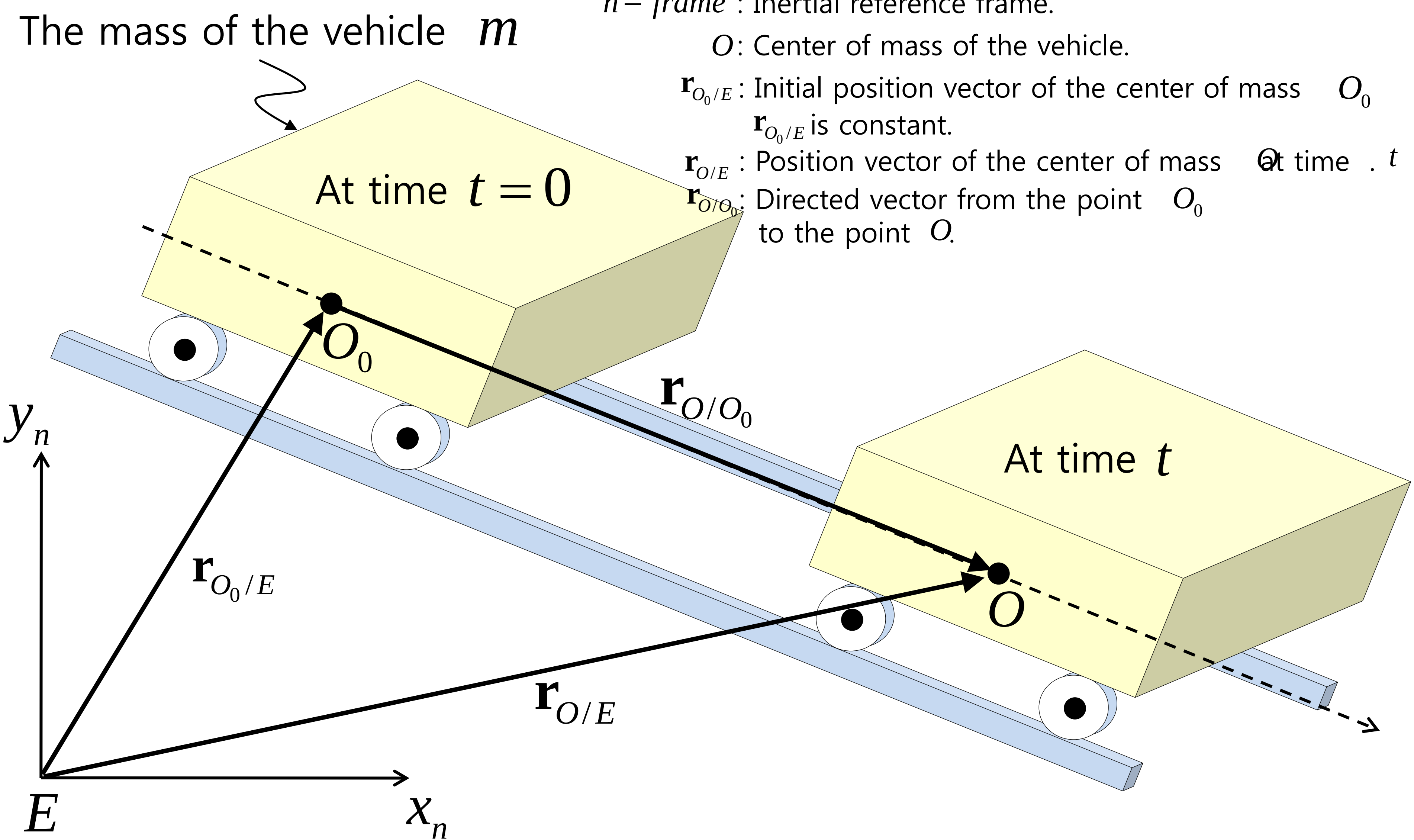
O: Center of mass of the vehicle.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0

$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t.

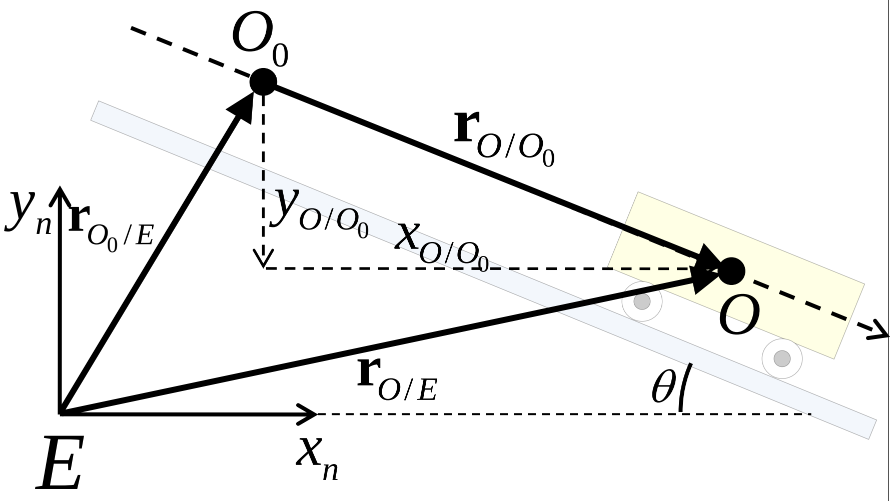
$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O.



Directed distance: A segment of a line having an indicated positive sense

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along straight track
- Kinematic constraint expressed by generalized coordinates



$n-frame$: Inertial reference frame.

O: Center of mass of the vehicle.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0.
$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t.

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O.

θ: Angle of inclination of the track. It is constant.

$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + \mathbf{r}_{O/O_0} \cdots (1)$$

$$\mathbf{r}_{O/E} = \begin{bmatrix} x_{O/E} \\ y_{O/E} \end{bmatrix}$$: The position vector of the vehicle.

$$\mathbf{r}_{O_0/E} = \begin{bmatrix} x_{O_0/E} \\ y_{O_0/E} \end{bmatrix}$$: The initial position vector of the vehicle, which is given

$$\mathbf{r}_{O/O_0} = \begin{bmatrix} x_{O/O_0} \\ y_{O/O_0} \end{bmatrix}$$: Directed vector from the point O_0 to the point O.

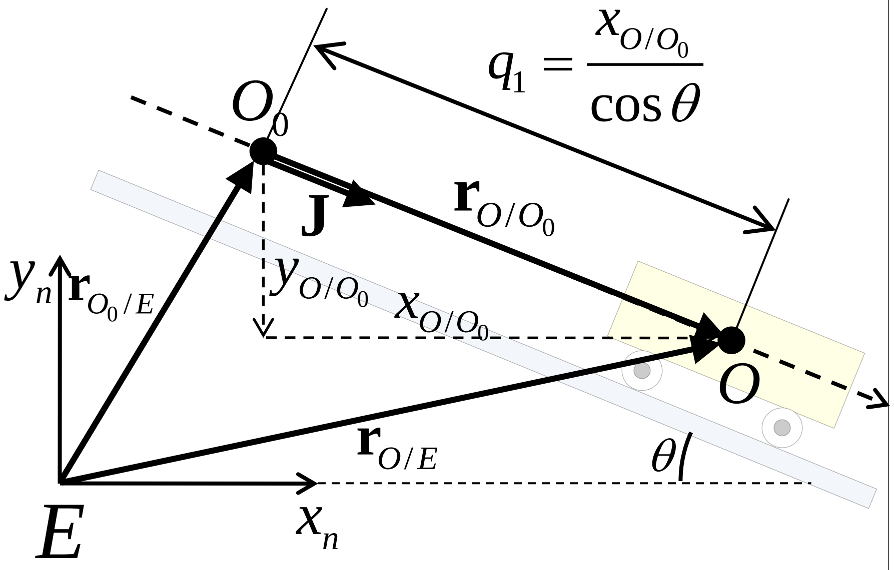
This vector should be parallel to the slop.

⇩

$$\frac{y_{O/O_0}}{x_{O/O_0}} = -\tan\theta \Rightarrow y_{O/O_0} = -\tan\theta \cdot x_{O/O_0} \cdots (2)$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along straight track
- Kinematic constraint expressed by generalized coordinates q_1



$n-frame$: Inertial reference frame.

O: Center of mass of the vehicle.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0.
$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t.

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O.

θ: Angle of inclination of the track. It is constant.

$$\begin{bmatrix} x_{O/O_0} \\ -\tan\theta\cdot x_{O/O_0} \end{bmatrix} = x_{O/O_0}\begin{bmatrix} 1 \\ -\tan\theta \end{bmatrix} = x_{O/O_0}\frac{\cos\theta}{\cos\theta}\begin{bmatrix} 1 \\ -\tan\theta \end{bmatrix} = x_{O/O_0}\frac{1}{\cos\theta}\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + \mathbf{r}_{O/O_0} \quad \cdots (1)$$

,where $\mathbf{r}_{O/E} = \begin{bmatrix} x_{O/E} \\ y_{O/E} \end{bmatrix}, \mathbf{r}_{O_0/E} = \begin{bmatrix} x_{O_0/E} \\ y_{O_0/E} \end{bmatrix}, \mathbf{r}_{O/O_0} = \begin{bmatrix} x_{O/O_0} \\ y_{O/O_0} \end{bmatrix}$

$$\frac{y_{O/O_0}}{x_{O/O_0}} = -\tan\theta \Rightarrow y_{O/O_0} = -\tan\theta\cdot x_{O/O_0} \quad \cdots (2)$$

From equations (1) and (2)

$$\begin{bmatrix} x_{O/E} \\ y_{O/E} \end{bmatrix} = \begin{bmatrix} x_{O_0/E} \\ y_{O_0/E} \end{bmatrix} + \begin{bmatrix} x_{O/O_0} \\ -\tan\theta\cdot x_{O/O_0} \end{bmatrix}$$

$$\begin{bmatrix} x_{O/E} \\ y_{O/E} \end{bmatrix} = \begin{bmatrix} x_{O_0/E} \\ y_{O_0/E} \end{bmatrix} + \underbrace{x_{O/O_0}\frac{1}{\cos\theta}}_{\rightarrow q_1}\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

Kinematic constraint

$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + q_1\mathbf{J}$$, where $\mathbf{J} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$

$$\mathbf{r}_{O/E} = \mathbf{r}_{O/E}(q_1)$$ $\because \mathbf{r}_{O_0/E}$ and $\mathbf{J}$ are constant.

Vehicle constrained to move along the straight track

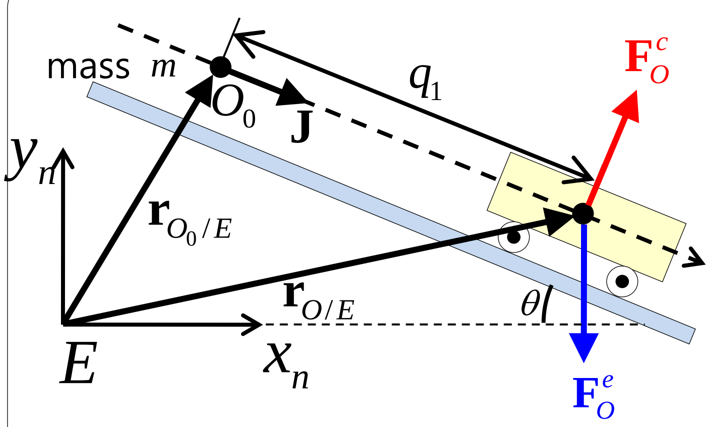
- Derivation of equations of motion without calculating the constraint reaction force(1/6)

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \sum \mathbf{F} \,, \sum \mathbf{F} = \mathbf{F}_O^e + \mathbf{F}_O^c$$

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c \quad , where\ \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$\mathbf{F}_O^e$: Known external force

$\mathbf{F}_O^c$: Unknown constraint reaction force



$n-frame$: Inertial reference frame.

O: Center of mass of the vehicle.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0

$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t .

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O.

θ: Angle of inclination of the track. It is constant.

Can we solve the equations of motion without calculating the constraint reaction force?

The constraint reaction force must be perpendicular to the track along which the vehicle is constrained to move!!

Kinematics $\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + q_1\mathbf{J}$

, where $\mathbf{J} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$

Taking the scalar product of both sides of the equations of motion with the vector $\mathbf{J}$ that is tangent to the track

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

- Derivation of equations of motion without calculating the constraint reaction force(2/6)

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c \quad , where\ \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

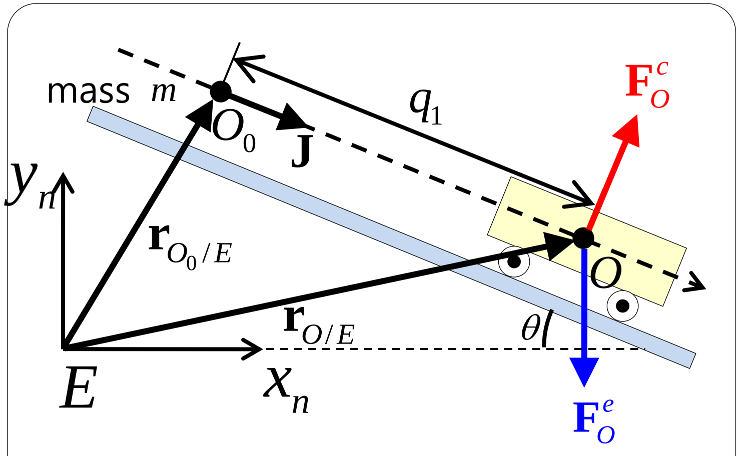
$\mathbf{F}_O^e$: Known external force

$\mathbf{F}_O^c$: Unknown constraint reaction force

The constraint reaction force $\mathbf{F}_O^c$ is perpendicular to the vector $\mathbf{J}$.

$$\boxed{\mathbf{J}\cdot\mathbf{M}\ddot{\mathbf{r}}_{O/E}} = \mathbf{J}\cdot\mathbf{F}_O^e + \cancel{\mathbf{J}\cdot\mathbf{F}_O^c}$$

Scalar product operation is not defined for matrix form.



$n-$ *frame* : Inertial reference frame.

O: Center of mass of the vehicle.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0.

$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t .

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O.

θ: Angle of inclination of the track. It is constant.

Kinematics $\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + q_1\mathbf{J}$

, where $\mathbf{J} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Scalar product of vectors - Matrix representation

1) Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons, Inc., p.346

✓ Scalar product of vectors

$$\mathbf{A} = \left(a_1, a_2, \cdots, a_n\right) \qquad \mathbf{B} = \left(b_1, b_2, \cdots, b_n\right)$$

$$\mathbf{A} \bullet \mathbf{B} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

✓ Matrix representation

$$\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\boxed{\mathbf{A} \bullet \mathbf{B} = \mathbf{A}^T \mathbf{B}}$$

$\mathbf{A}^T \mathbf{B}$ preserves the value of the scalar product of the vectors A and B

$$\mathbf{A}^T \mathbf{B} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion without calculating the constraint reaction force(3/6)

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c \quad , where\ \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$\mathbf{F}_O^e$: Known external force

$\mathbf{F}_O^c$: Unknown constraint reaction force

The constraint reaction force $\mathbf{F}_O^c$ is perpendicular to the vector $\mathbf{J}$.

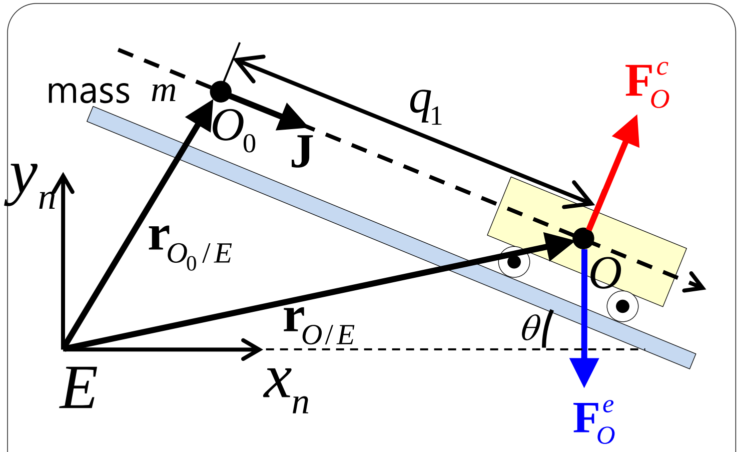
$$\boxed{\mathbf{J}\cdot\mathbf{M}\ddot{\mathbf{r}}_{O/E}} = \mathbf{J}\cdot\mathbf{F}_O^e + \cancel{\mathbf{J}\cdot\mathbf{F}_O^c}$$

Scalar product operation is not defined for matrix form.

$$\boxed{\mathbf{A}\cdot\mathbf{B} = \mathbf{A}^T\mathbf{B}}$$

$\mathbf{A}^T\mathbf{B}$ preserves the value of the scalar product of the vectors A and B

$$\mathbf{J}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{J}^T\mathbf{F}_O^e$$



$n-frame$: Inertial reference frame.

O: Center of mass of the vehicle.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0.

$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t .

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O.

θ: Angle of inclination of the track. It is constant.

Kinematics $\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + q_1\mathbf{J}$

, where $\mathbf{J} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

- Derivation of equations of motion without calculating the constraint reaction force(4/6)

$\mathbf{F}_O^e$: Known external force

$\mathbf{F}_O^c$: Unknown constraint reaction force

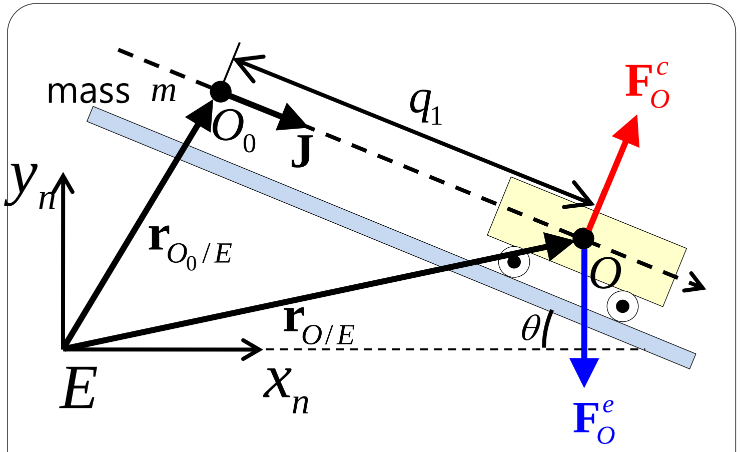
$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c \quad \text{, where } \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$$\mathbf{J}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{J}^T\mathbf{F}_O^e$$

$$\begin{bmatrix} J_x & J_y \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_{O/E} \\ \ddot{y}_{O/E} \end{bmatrix} = \begin{bmatrix} J_x & J_y \end{bmatrix}\begin{bmatrix} F_{O,x}^e \\ F_{O,y}^e \end{bmatrix}$$

$$J_x m\ddot{x}_{O/E} + J_y m\ddot{y}_{O/E} = J_x F_{O,x}^e + J_y F_{O,y}^e$$

2 variables $(x_{O/E}, y_{O/E})$, 1 equation



$n-$ *frame* : Inertial reference frame.

O: Center of mass of the vehicle.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0.
$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t .

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O.

θ: Angle of inclination of the track. It is constant.

Kinematics $\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + q_1\mathbf{J}$

, where $\mathbf{J} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$

To solve the equations of motion, we need one more equation. How can we get another equation?

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

- Derivation of equations of motion without calculating the constraint reaction force(5/6)

$\mathbf{F}_O^e$: Known external force

$\mathbf{F}_O^c$: Unknown constraint reaction force

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c \quad \text{, where } \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$$\mathbf{J}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{J}^T\mathbf{F}_O^e$$

$$J_x m\ddot{x}_{O/E} + J_y m\ddot{y}_{O/E} = J_x F_{O,x}^e + J_y F_{O,y}^e$$

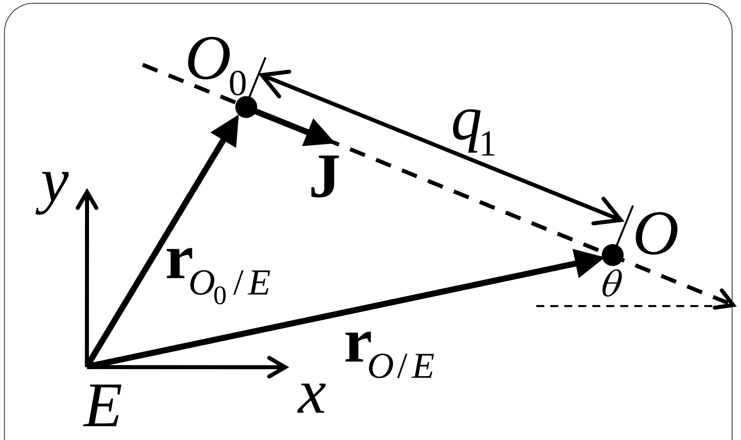
2 variables $(x_{O/E}, y_{O/E})$, 1 equation

3 variables $(x_{O/E}, y_{O/E}, q_1)$ 3 equations

$$x_{O/E} = x_{O_0/E} + q_1\cos\theta$$

$$y_{O/E} = y_{O_0/E} - q_1\sin\theta$$

3 variables $(x_{O/E}, y_{O/E}, q_1)$ 2 equations



Kinematics $\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + q_1\mathbf{J}$

, where $\mathbf{J} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$

Kinematics

$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + q_1\mathbf{J}$, where $\mathbf{J} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

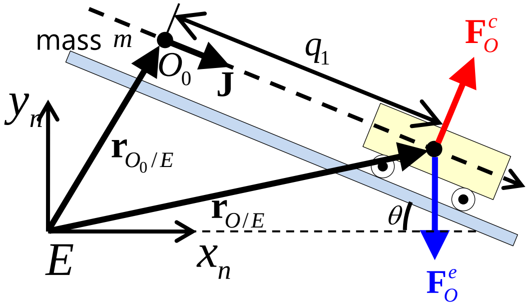
Vehicle constrained to move along the straight track

- Derivation of equations of motion without calculating the constraint reaction force(6/6)

According to Newton's 2nd law

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c$$

$$\text{, where } \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$



The constraint reaction force is suppressed

$$\mathbf{J}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{J}^T\mathbf{F}_O^e \cdots (1)$$

Kinematics

$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + q_1\mathbf{J}$$

The time derivative

$$\dot{\mathbf{r}}_{O/E} = \mathbf{J}\dot{q}_1$$

The time derivative

$$\ddot{\mathbf{r}}_{O/E} = \mathbf{J}\ddot{q}_1 \cdots (2)$$

Substituting (2) into (1)

$$\boxed{\mathbf{J}^T\mathbf{M}\mathbf{J}\ddot{q}_1 = \mathbf{J}^T\mathbf{F}_O^e}$$

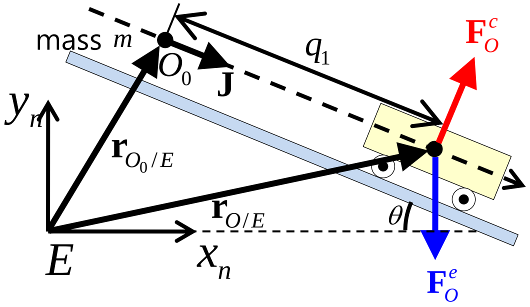
Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

- Derivation of equations of motion without calculating the constraint reaction force(6/6)

According to Newton's 2nd law

$$\underset{\text{known}}{\underline{\mathbf{M}}}\,\underset{\text{variable}}{\underline{\ddot{\mathbf{r}}_{O/E}}} = \underset{\text{known}}{\underline{\mathbf{F}_O^e}} + \underset{\text{unknown}}{\underline{\mathbf{F}_O^c}}$$



→ We can calculate matrix J from kinematics

Kinematics

$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + q_1\mathbf{J}$$

The time derivative

$$\dot{\mathbf{r}}_{O/E} = \mathbf{J}\dot{q}_1$$

The time derivative

$$\ddot{\mathbf{r}}_{O/E} = \mathbf{J}\ddot{q}_1$$

$$\mathbf{J} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

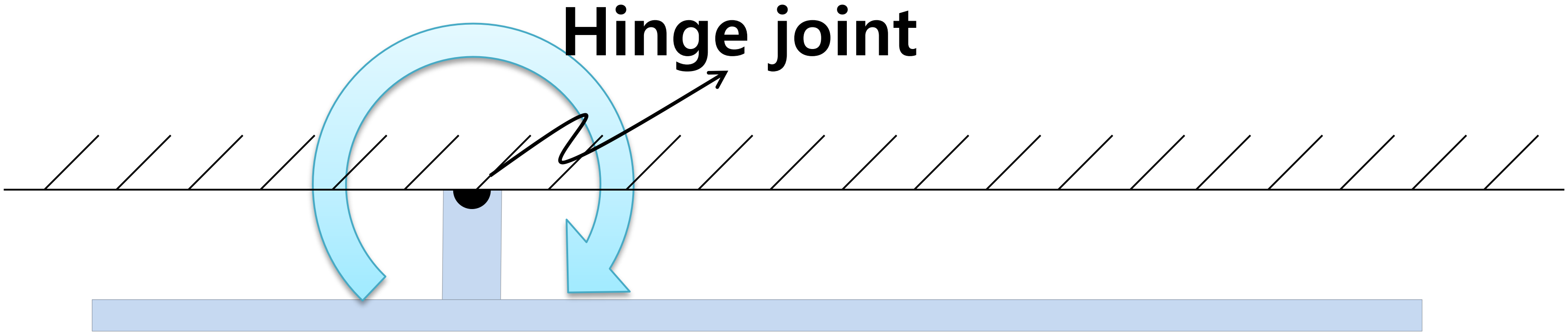
$$\underset{\text{known}}{\underline{\mathbf{J}^T}}\,\underset{\text{known}}{\underline{\mathbf{M}}}\,\underset{\text{known}}{\underline{\mathbf{J}}}\,\underset{\text{variable}}{\underline{\ddot{q}_1}} = \underset{\text{known}}{\underline{\mathbf{J}^T}}\,\underset{\text{known}}{\underline{\mathbf{F}_O^e}}$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

Suppose that the track is rotating about the hinge joint fixed on the ceiling

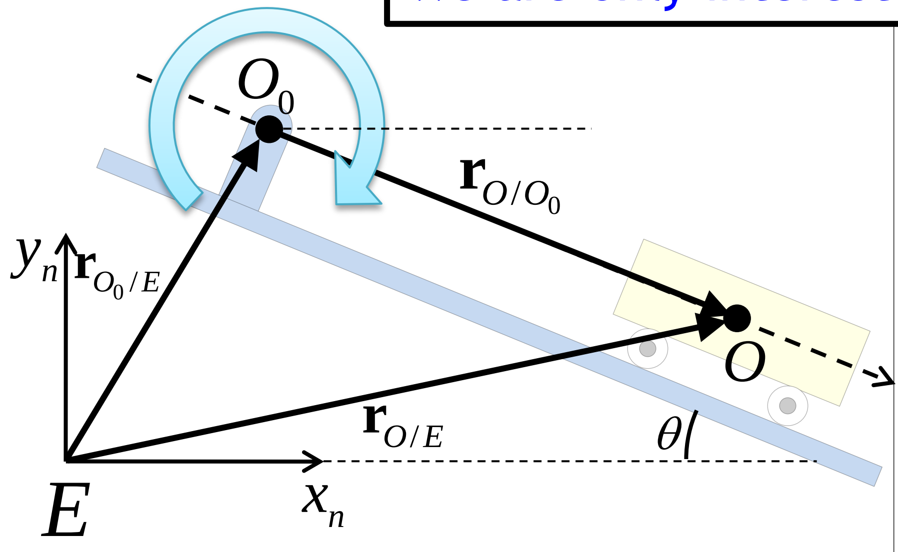


Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

Suppose that the track is rotating about axis-z through the hinge joint point O_0

We are only interested in the motion of the vehicle.



$n-frame$: Inertial reference frame.

O: Center of mass of the vehicle.

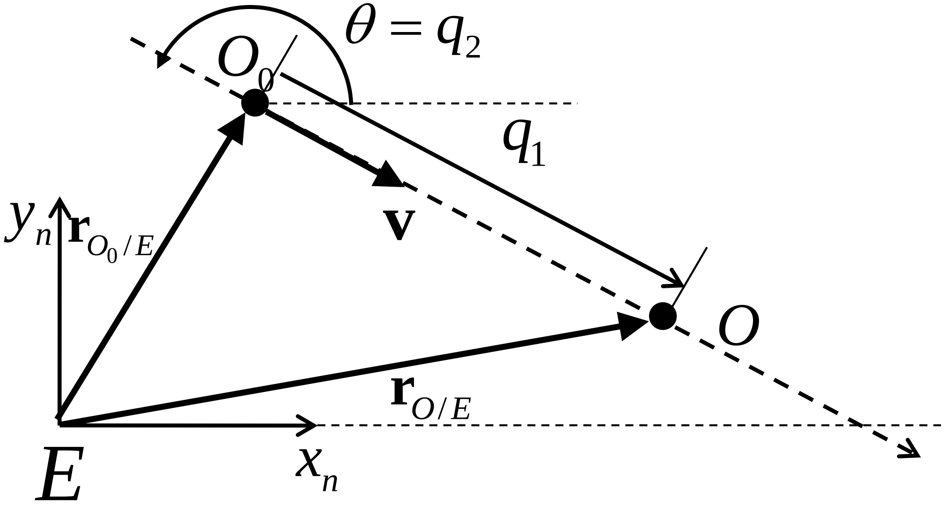
O_0: Center of mass of the track, Hinge joint.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0
$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t.

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O

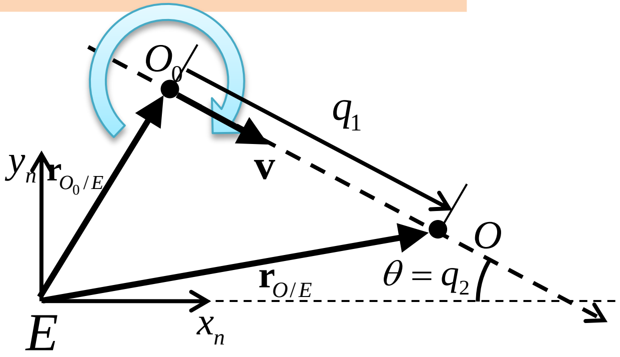
θ: Angle of inclination of the track.



$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + \mathbf{v}q_1 \quad , where\ \mathbf{v} = \begin{bmatrix} \cos q_2 \\ -\sin q_2 \end{bmatrix}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique



$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + \mathbf{v}q_1 \quad , where\ \mathbf{v} = \begin{bmatrix} \cos q_2 \\ -\sin q_2 \end{bmatrix}$$

$$x_{O/E} = x_{O_0/E} + q_1 \cos q_2$$

$$y_{O/E} = y_{O_0/E} - q_1 \sin q_2$$

Time derivative

$$\dot{x}_{O/E} = \dot{q}_1 \cos q_2 - q_1 \sin q_2 \cdot \dot{q}_2$$

$$\dot{y}_{O/E} = -\dot{q}_1 \sin q_2 - q_1 \cos q_2 \cdot \dot{q}_2$$

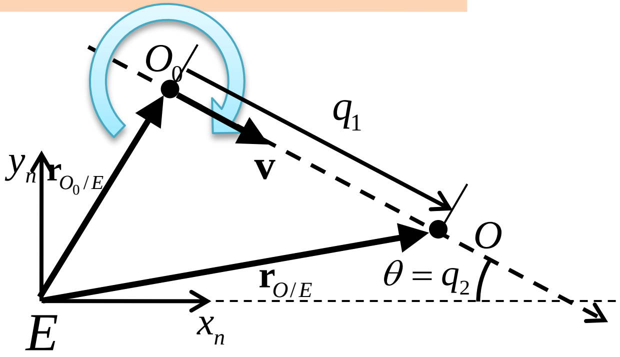
$$\underbrace{\begin{bmatrix} \dot{x}_{O/E} \\ \dot{y}_{O/E} \end{bmatrix}}_{\dot{\mathbf{r}}_{O/E}} = \underbrace{\begin{bmatrix} \cos q_2 & -q_1 \sin q_2 \\ -\sin q_2 & -q_1 \cos q_2 \end{bmatrix}}_{\mathbf{J}} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{\mathbf{q}}}$$

$$\dot{\mathbf{r}}_{O/E} = \mathbf{J}\dot{\mathbf{q}}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

$$\underbrace{\begin{bmatrix}\dot{x}_{O/E}\\ \dot{y}_{O/E}\end{bmatrix}}_{\dot{\mathbf{r}}_{O/E}}=\underbrace{\begin{bmatrix}\cos q_2 & -q_1\sin q_2\\ -\sin q_2 & -q_1\cos q_2\end{bmatrix}}_{\mathbf{J}}\underbrace{\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}}_{\dot{\mathbf{q}}}$$



$$\mathbf{J}=\begin{bmatrix}\cos q_2 & -q_1\sin q_2\\ -\sin q_2 & -q_1\cos q_2\end{bmatrix}$$

Time derivative

$$\dot{\mathbf{J}}=\begin{bmatrix}-\sin q_2\cdot\dot{q}_2 & -\dot{q}_1\sin q_2-q_1\cos q_2\cdot\dot{q}_2\\ -\cos q_2\cdot\dot{q}_2 & -\dot{q}_1\cos q_2+q_1\sin q_2\cdot\dot{q}_2\end{bmatrix}$$

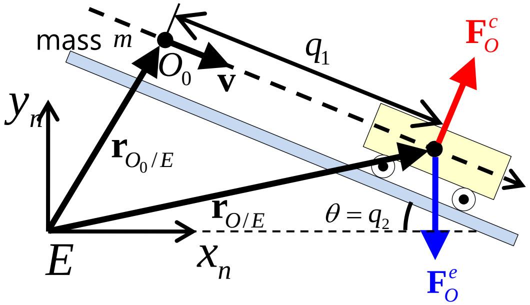
Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

According to Newton's 2nd law

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c$$

$$, where\ \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$



The constraint reaction force is suppressed

$$\mathbf{J}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{J}^T\mathbf{F}_O^e \cdots (1)$$

From Kinematics

$$\dot{\mathbf{r}}_{O/E} = \mathbf{J}\dot{\mathbf{q}}$$

The time derivative

$$\ddot{\mathbf{r}}_{O/E} = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} \cdots (2)$$

Substituting (2) into (1)

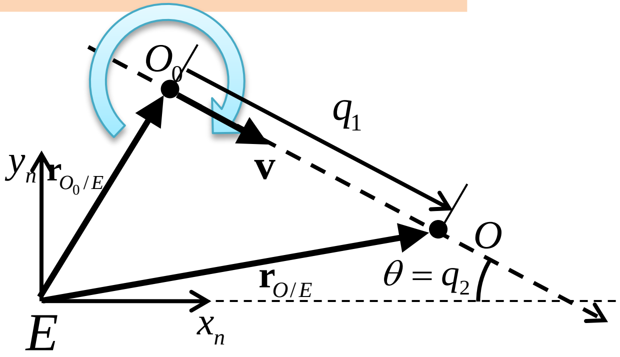
$$\mathbf{J}^T\mathbf{M}\mathbf{J}\ddot{\mathbf{q}} + \mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}} = \mathbf{J}^T\mathbf{F}_O^e$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

$$\underbrace{\boxed{\mathbf{J}^T\mathbf{M}\mathbf{J}}}_{\tilde{\mathbf{M}}}\ddot{\mathbf{q}}+\underbrace{\boxed{\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}}}_{\tilde{\mathbf{k}}}=\underbrace{\boxed{\mathbf{J}^T\mathbf{F}_O^e}}_{\tilde{\mathbf{F}}^e}$$



$$\tilde{\mathbf{M}} = \mathbf{J}^T\mathbf{M}\mathbf{J}$$

$$=\begin{bmatrix} \cos q_2 & -q_1\sin q_2 \\ -\sin q_2 & -q_1\cos q_2 \end{bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \cos q_2 & -q_1\sin q_2 \\ -\sin q_2 & -q_1\cos q_2 \end{bmatrix}$$

$$=\begin{bmatrix} \cos q_2 & -\sin q_2 \\ -q_1\sin q_2 & -q_1\cos q_2 \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \cos q_2 & -q_1\sin q_2 \\ -\sin q_2 & -q_1\cos q_2 \end{bmatrix}$$

$$=\begin{bmatrix} m\cos q_2 & -m\sin q_2 \\ -mq_1\sin q_2 & -mq_1\cos q_2 \end{bmatrix}\begin{bmatrix} \cos q_2 & -q_1\sin q_2 \\ -\sin q_2 & -q_1\cos q_2 \end{bmatrix}$$

$$=\begin{bmatrix} m\cos^2 q_2 + m\sin^2 q_2 & -mq_1\cos q_2\sin q_2 + mq_1\sin q_2\cos q_2 \\ -mq_1\sin q_2\cos q_2 + mq_1\cos q_2\sin q_2 & mq_1^2\cos^2 q_2 + mq_1^2\sin^2 q_2 \end{bmatrix} = \boxed{\begin{bmatrix} m & 0 \\ 0 & mq_1^2 \end{bmatrix}}$$

$$\mathbf{J} = \begin{bmatrix} \cos q_2 & -q_1\sin q_2 \\ -\sin q_2 & -q_1\cos q_2 \end{bmatrix}$$

$$\dot{\mathbf{J}} = \begin{bmatrix} -\sin q_2\cdot\dot{q}_2 & -\dot{q}_1\sin q_2 - q_1\cos q_2\cdot\dot{q}_2 \\ -\cos q_2\cdot\dot{q}_2 & -\dot{q}_1\cos q_2 + q_1\sin q_2\cdot\dot{q}_2 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$\mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

$$\mathbf{J}=\begin{bmatrix}\cos q_2 & -q_1\sin q_2\\ -\sin q_2 & -q_1\cos q_2\end{bmatrix}$$

$$\underbrace{\mathbf{J}^T\mathbf{M}\mathbf{J}}_{\tilde{\mathbf{M}}}\ddot{\mathbf{q}}+\underbrace{\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}}_{\tilde{\mathbf{k}}}=\underbrace{\mathbf{J}^T\mathbf{F}_O^e}_{\tilde{\mathbf{F}}^e}$$

$$\dot{\mathbf{J}}=\begin{bmatrix}-\sin q_2\cdot\dot{q}_2 & -\dot{q}_1\sin q_2-q_1\cos q_2\cdot\dot{q}_2\\ -\cos q_2\cdot\dot{q}_2 & -\dot{q}_1\cos q_2+q_1\sin q_2\cdot\dot{q}_2\end{bmatrix}$$

$$\mathbf{M}=\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\qquad \mathbf{F}^e=\begin{bmatrix}0\\ -mg\end{bmatrix}$$

$$\tilde{\mathbf{M}}=\begin{bmatrix}m & 0\\ 0 & mq_1^2\end{bmatrix}$$

$$\tilde{\mathbf{k}}=\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}$$

$$=\begin{bmatrix}\cos q_2 & -q_1\sin q_2\\ -\sin q_2 & -q_1\cos q_2\end{bmatrix}^T\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\begin{bmatrix}-\sin q_2\cdot\dot{q}_2 & -\dot{q}_1\sin q_2-q_1\cos q_2\cdot\dot{q}_2\\ -\cos q_2\cdot\dot{q}_2 & -\dot{q}_1\cos q_2+q_1\sin q_2\cdot\dot{q}_2\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}$$

$$\downarrow$$

$$=\begin{bmatrix}\cos q_2 & -\sin q_2\\ -q_1\sin q_2 & -q_1\cos q_2\end{bmatrix}\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\begin{bmatrix}-\sin q_2\cdot\dot{q}_2 & -\dot{q}_1\sin q_2-q_1\cos q_2\cdot\dot{q}_2\\ -\cos q_2\cdot\dot{q}_2 & -\dot{q}_1\cos q_2+q_1\sin q_2\cdot\dot{q}_2\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}$$

$$\downarrow$$

$$=\begin{bmatrix}m\cos q_2 & -m\sin q_2\\ -mq_1\sin q_2 & -mq_1\cos q_2\end{bmatrix}\begin{bmatrix}-\sin q_2\cdot\dot{q}_2 & -\dot{q}_1\sin q_2-q_1\cos q_2\cdot\dot{q}_2\\ -\cos q_2\cdot\dot{q}_2 & -\dot{q}_1\cos q_2+q_1\sin q_2\cdot\dot{q}_2\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}$$

$$\downarrow$$

$$=\begin{bmatrix}-m\cos q_2\sin q_2\cdot\\ -mq_1\sin q_2\left(-\sin q_2\cdot\right.\end{bmatrix}$$

[redacted]

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

$$\mathbf{J} = \begin{bmatrix} \cos q_2 & -q_1 \sin q_2 \\ -\sin q_2 & -q_1 \cos q_2 \end{bmatrix}$$

$$\dot{\mathbf{J}} = \begin{bmatrix} -\sin q_2 \cdot \dot{q}_2 & -\dot{q}_1 \sin q_2 - q_1 \cos q_2 \cdot \dot{q}_2 \\ -\cos q_2 \cdot \dot{q}_2 & -\dot{q}_1 \cos q_2 + q_1 \sin q_2 \cdot \dot{q}_2 \end{bmatrix}$$

$$\underbrace{\mathbf{J}^T\mathbf{M}\mathbf{J}}_{\tilde{\mathbf{M}}}\ddot{\mathbf{q}} + \underbrace{\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}}_{\tilde{\mathbf{k}}} = \underbrace{\mathbf{J}^T\mathbf{F}_O^e}_{\tilde{\mathbf{F}}^e}$$

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$$\tilde{\mathbf{k}} = \mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}$$

$$\tilde{\mathbf{M}} = \begin{bmatrix} m & 0 \\ 0 & mq_1^2 \end{bmatrix}$$

$= \Big[\; -m\cos q_2 \sin q_2 \cdot$ [redacted]
$\;\; -mq_1 \sin q_2 \left(-\sin q_2 \cdot\right.$

↓

$$= \begin{bmatrix} 0 & -m\cos q_2 \sin q_2 \cdot \dot{q}_1 - m\cos^2 q_2 \cdot q_1\dot{q}_2 + m\sin q_2 \cos q_2 \cdot \dot{q}_1 - m\sin^2 q_2 \cdot q_1\dot{q}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \end{bmatrix}$$

$= \Big[\; mq_1 \sin^2 q_2 \cdot$ [redacted]

↓

$$= \begin{bmatrix} 0 & -m\cos q_2 \sin q_2 \cdot \dot{q}_1 + m\sin q_2 \cos q_2 \cdot \dot{q}_1 - m\left(\sin^2 q_2 + \cos^2 q_2\right) \cdot q_1\dot{q}_2 \\ mq_1\left(\sin^2 q_2 + \cos^2 q_2\right)\dot{q}_2 & m\left(\sin^2 q_2 + \cos^2 q_2\right) \cdot q_1\dot{q}_1 + m\sin q_2 \cos q_2 \cdot q_1{}^2\dot{q}_2 - m\cos q_2 \sin q_2 \cdot q_1{}^2\dot{q}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

↓

$$= \begin{bmatrix} 0 & -mq_1\dot{q}_2 \\ mq_1\dot{q}_2 & mq_1\dot{q}_1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \boxed{\begin{bmatrix} -mq_1\dot{q}_2{}^2 \\ 2mq_1\dot{q}_1\dot{q}_2 \end{bmatrix}}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

$$\mathbf{J}=\begin{bmatrix}\cos q_2 & -q_1\sin q_2\\ -\sin q_2 & -q_1\cos q_2\end{bmatrix}$$

$$\dot{\mathbf{J}}=\begin{bmatrix}-\sin q_2\cdot\dot q_2 & -\dot q_1\sin q_2-q_1\cos q_2\cdot\dot q_2\\ -\cos q_2\cdot\dot q_2 & -\dot q_1\cos q_2+q_1\sin q_2\cdot\dot q_2\end{bmatrix}$$

$$\mathbf{M}=\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\qquad \mathbf{F}^e=\begin{bmatrix}0\\ -mg\end{bmatrix}$$

$$\tilde{\mathbf{M}}=\begin{bmatrix}m & 0\\ 0 & mq_1^2\end{bmatrix},\ \tilde{\mathbf{k}}=\begin{bmatrix}-mq_1\dot q_2{}^2\\ 2mq_1\dot q_1\dot q_2\end{bmatrix}$$

$$\underbrace{\boxed{\mathbf{J}^T\mathbf{M}\mathbf{J}}}_{\tilde{\mathbf{M}}}\ddot{\mathbf{q}}+\underbrace{\boxed{\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}}}_{\tilde{\mathbf{k}}}=\underbrace{\boxed{\mathbf{J}^T\mathbf{F}_O^e}}_{\tilde{\mathbf{F}}^e}$$

$$\tilde{\mathbf{F}}^e=\mathbf{J}^T\mathbf{F}^e$$

$$=\begin{bmatrix}\cos q_2 & -q_1\sin q_2\\ -\sin q_2 & -q_1\cos q_2\end{bmatrix}^T\begin{bmatrix}0\\ -mg\end{bmatrix}$$

$$=\begin{bmatrix}\cos q_2 & -\sin q_2\\ -q_1\sin q_2 & -q_1\cos q_2\end{bmatrix}\begin{bmatrix}0\\ -mg\end{bmatrix}$$

$$=\begin{bmatrix}0+mg\sin q_2\\ 0+mgq_1\cos q_2\end{bmatrix}$$

$$=\boxed{\begin{bmatrix}mg\sin q_2\\ mgq_1\cos q_2\end{bmatrix}}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

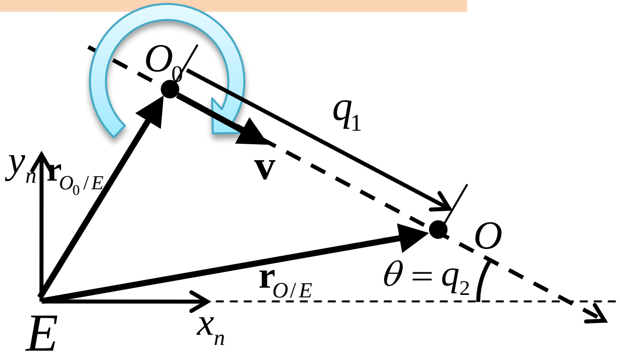
$$\underbrace{\boxed{\mathbf{J}^T\mathbf{M}\mathbf{J}}}_{\tilde{\mathbf{M}}}\ddot{\mathbf{q}}+\underbrace{\boxed{\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}}}_{\tilde{\mathbf{k}}}=\underbrace{\boxed{\mathbf{J}^T\mathbf{F}_O^e}}_{\tilde{\mathbf{F}}^e}$$

$$\tilde{\mathbf{M}}=\begin{bmatrix} m & 0 \\ 0 & m{q_1}^2 \end{bmatrix},\tilde{\mathbf{k}}=\begin{bmatrix} -mq_1{\dot{q}_2}^2 \\ 2mq_1\dot{q}_1\dot{q}_2 \end{bmatrix},\tilde{\mathbf{F}}^e=\begin{bmatrix} mg\sin q_2 \\ mgq_1\cos q_2 \end{bmatrix}$$

$$\begin{bmatrix} m & 0 \\ 0 & m{q_1}^2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}+\begin{bmatrix} -mq_1{\dot{q}_2}^2 \\ 2mq_1\dot{q}_1\dot{q}_2 \end{bmatrix}=\begin{bmatrix} mg\sin q_2 \\ mgq_1\cos q_2 \end{bmatrix}$$

$$m\ddot{q}_1-mq_1{\dot{q}_2}^2=mg\sin q_2 \text{ .... } (1)$$

$$m{q_1}^2\ddot{q}_2+2mq_1\dot{q}_1\dot{q}_2=mgq_1\cos q_2 \text{ .. } (2)$$



$$\mathbf{J}=\begin{bmatrix} \cos q_2 & -q_1\sin q_2 \\ -\sin q_2 & -q_1\cos q_2 \end{bmatrix}$$

$$\dot{\mathbf{J}}=\begin{bmatrix} -\sin q_2\cdot\dot{q}_2 & -\dot{q}_1\sin q_2-q_1\cos q_2\cdot\dot{q}_2 \\ -\cos q_2\cdot\dot{q}_2 & -\dot{q}_1\cos q_2+q_1\sin q_2\cdot\dot{q}_2 \end{bmatrix}$$

$$\mathbf{M}=\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

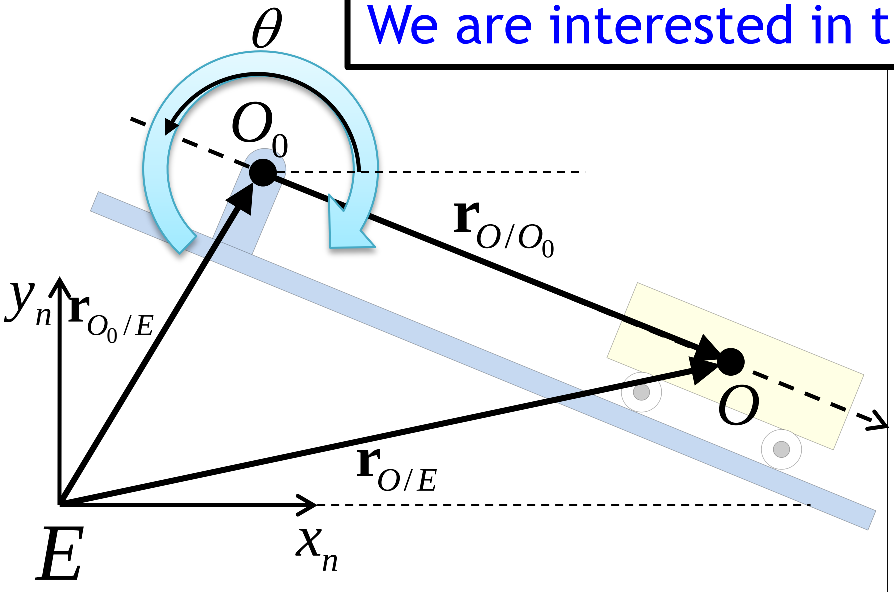
$$\mathbf{F}^e=\begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

Suppose that the track is rotating about axis-z through the hinge joint point O$_0$

We are interested in the motion of the vehicle and the track.



$n-frame$: Inertial reference frame.

O: Center of mass of the vehicle.

O_0: Center of mass of the track, Hinge joint.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0

$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t.

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O

θ: Angle of inclination of the track.

Equations of motion of the vehicle

$$m_v \ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c$$

Equations of motion of the track

$$m_t \ddot{\mathbf{r}}_{O_0/E} = \mathbf{F}_{O_0}^e + \mathbf{F}_{O_0}^c$$

$$I_t \ddot{\theta} = \mathbf{M}_{O_0}^e + \mathbf{M}_{O_0}^c$$

⇩

Matrix form

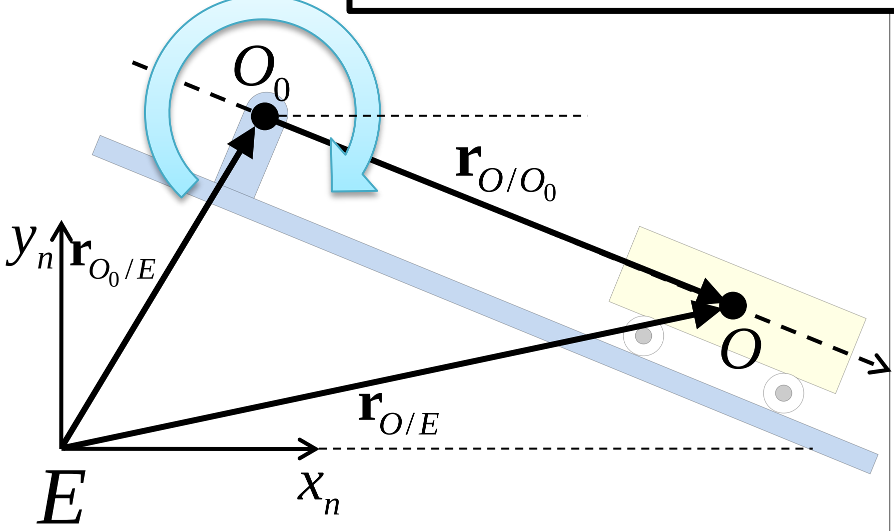
$$\underbrace{\begin{bmatrix} m_v & & & & \\ & m_v & & & \\ & & m_t & & \\ & & & m_t & \\ & & & & I_t \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_{O/E} \\ \ddot{y}_{O/E} \\ \ddot{x}_{O_0/E} \\ \ddot{y}_{O_0/E} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{\mathbf{r}}} = \underbrace{\begin{bmatrix} F_{O,x}^e \\ F_{O,y}^e \\ F_{O_0,x}^e \\ F_{O_0,y}^e \\ M_{O_0}^e \end{bmatrix}}_{\mathbf{F}^e} + \underbrace{\begin{bmatrix} F_{O,x}^c \\ F_{O,y}^c \\ F_{O_0,x}^c \\ F_{O_0,y}^c \\ M_{O_0}^c \end{bmatrix}}_{\mathbf{F}^c}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

Suppose that the track is rotating about axis-z through the hinge joint point O_0

We are interested in the motion of the vehicle and the track.



$n - frame$: Inertial reference frame.

O: Center of mass of the vehicle.

O_0: Center of mass of the track, Hinge joint.

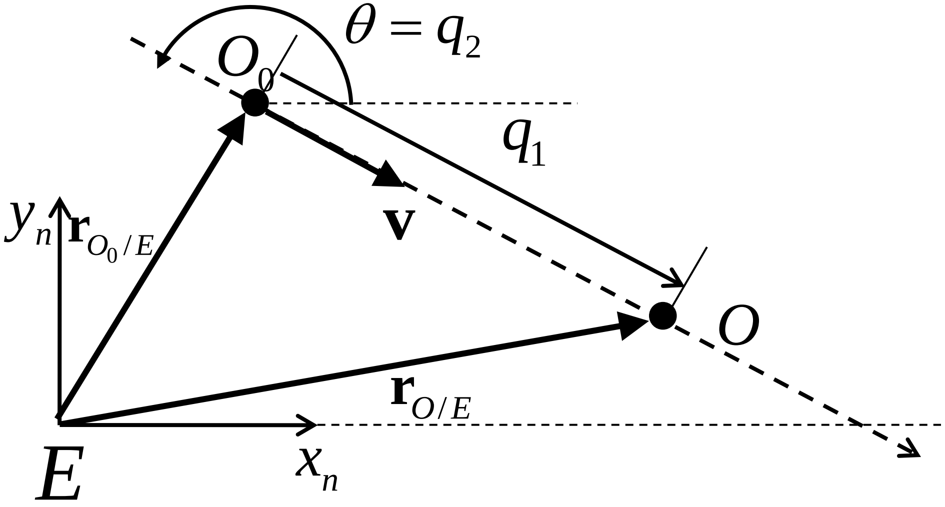
$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0

$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t.

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O.

θ: Angle of inclination of the track.



Position of the vehicle

$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + \mathbf{v}q_1 \quad , where \ \mathbf{v} = \begin{bmatrix} -\cos q_2 \\ -\sin q_2 \end{bmatrix}$$

Position of the track

$\mathbf{r}_{O_o/E}$ **: constant**

Orientation of the track

$\theta = q_2$

Vehicle constrained to move along the straight track

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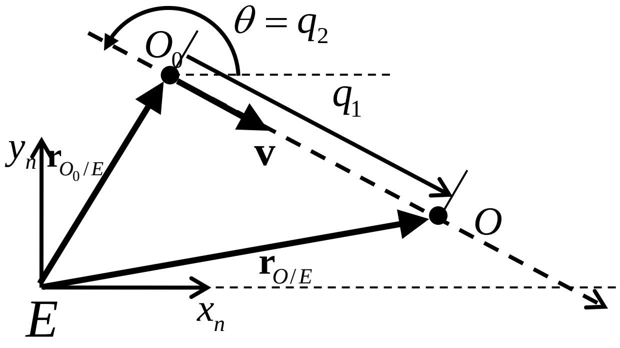


Position of the vehicle

$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + \mathbf{v} q_1 \quad , where \ \mathbf{v} = \begin{bmatrix} -\cos q_2 \\ -\sin q_2 \end{bmatrix}$$

Position of the track

$\mathbf{r}_{O_o/E}$ **: constant**

Orientation of the track

$$\theta = q_2$$

$$\boxed{\begin{aligned} x_{O/E} &= x_{O_0/E} - q_1 \cos q_2 \\ y_{O/E} &= y_{O_0/E} - q_1 \sin q_2 \end{aligned}}$$

$$\boxed{\begin{aligned} x_{O_0/E} &= x_{O_0/E} \\ y_{O_0/E} &= y_{O_0/E} \end{aligned}}$$

$$\boxed{\theta = q_2}$$

Time derivative

$$\dot{x}_{O/E} = -\dot{q}_1 \cos q_2 + q_1 \sin q_2 \cdot \dot{q}_2$$

$$\dot{y}_{O/E} = -\dot{q}_1 \sin q_2 - q_1 \cos q_2 \cdot \dot{q}_2$$

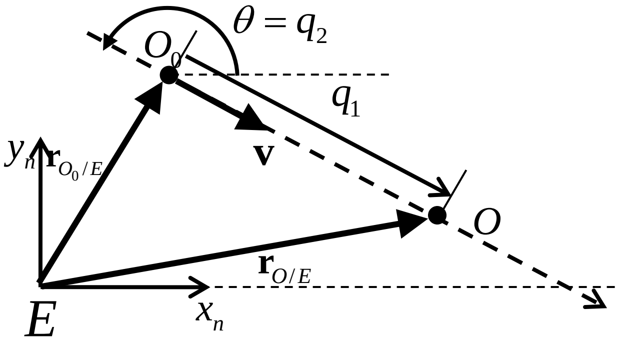
$$\dot{x}_{O_0/E} = 0$$

$$\dot{y}_{O_0/E} = 0$$

$$\dot{\theta} = \dot{q}_2$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique



$$\dot{x}_{O/E} = -\dot{q}_1 \cos q_2 + q_1 \sin q_2 \cdot \dot{q}_2$$

$$\dot{y}_{O/E} = -\dot{q}_1 \sin q_2 - q_1 \cos q_2 \cdot \dot{q}_2$$

$$\dot{x}_{O_0/E} = 0$$

$$\dot{y}_{O_0/E} = 0$$

$$\dot{\theta} = \dot{q}_2$$

$$\underbrace{\begin{bmatrix} \dot{x}_{O/E} \\ \dot{y}_{O/E} \\ \dot{x}_{O_0/E} \\ \dot{y}_{O_0/E} \\ \dot{\theta} \end{bmatrix}}_{\dot{\mathbf{r}}} = \underbrace{\begin{bmatrix} -\cos q_2 & +q_1 \sin q_2 \\ -\sin q_2 & -q_1 \cos q_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{J}} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{\mathbf{q}}}$$

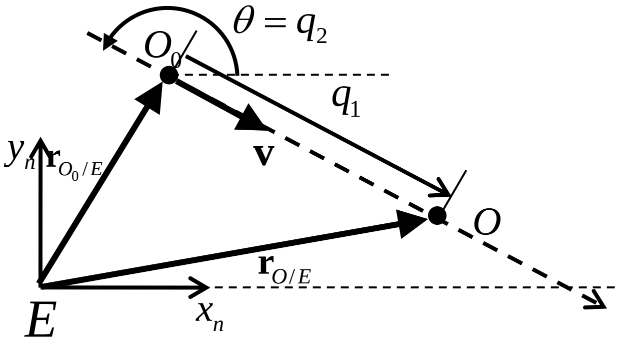
$$\dot{\mathbf{r}} = \mathbf{J}\dot{\mathbf{q}}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

$$\underbrace{\begin{bmatrix}\dot{x}_{O/E}\\ \dot{y}_{O/E}\\ \dot{x}_{O_0/E}\\ \dot{y}_{O_0/E}\\ \dot{\theta}\end{bmatrix}}_{\dot{\mathbf{r}}} = \underbrace{\begin{bmatrix} -\cos q_2 & +q_1 \sin q_2\\ -\sin q_2 & -q_1 \cos q_2\\ 0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}}_{\mathbf{J}} \underbrace{\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}}_{\dot{\mathbf{q}}}$$

$$\dot{\mathbf{r}} = \mathbf{J}\dot{\mathbf{q}}$$



$$\mathbf{J} = \begin{bmatrix} -\cos q_2 & +q_1 \sin q_2\\ -\sin q_2 & -q_1 \cos q_2\\ 0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix} \xrightarrow{\text{Time derivative}} \dot{\mathbf{J}} = \begin{bmatrix} \sin q_2 \cdot \dot{q}_2 & \dot{q}_1 \sin q_2 + q_1 \cos q_2 \cdot \dot{q}_2\\ -\cos q_2 \cdot \dot{q}_2 & -\dot{q}_1 \cos q_2 + q_1 \sin q_2 \cdot \dot{q}_2\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

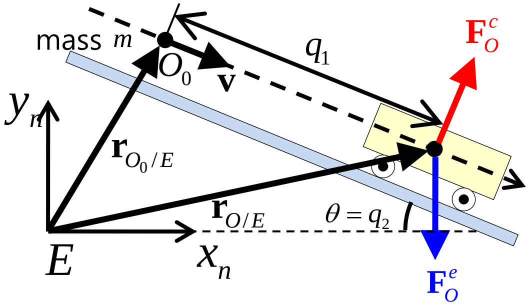
Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

According to Newton's 2nd law

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c$$

$$, where\ \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$



The constraint reaction force is suppressed

$$\mathbf{J}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{J}^T\mathbf{F}_O^e \cdots (1)$$

From Kinematics

$$\dot{\mathbf{r}}_{O/E} = \mathbf{J}\dot{\mathbf{q}}$$

The time derivative

$$\ddot{\mathbf{r}}_{O/E} = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} \cdots (2)$$

Substituting (2) into (1)

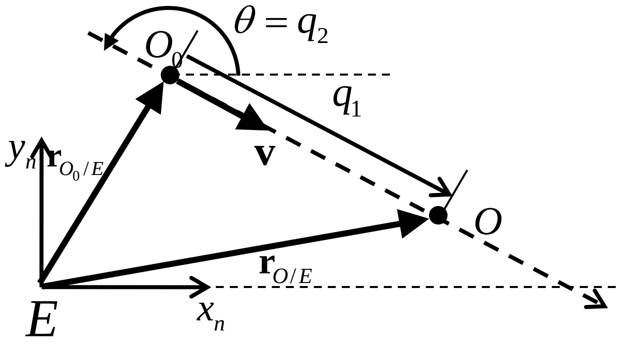
$$\mathbf{J}^T\mathbf{M}\mathbf{J}\ddot{\mathbf{q}} + \mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}} = \mathbf{J}^T\mathbf{F}_O^e$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using imbedding technique

$$\tilde{\mathbf{M}}\ddot{\mathbf{q}} + \tilde{\mathbf{k}}\dot{\mathbf{q}} = \tilde{\mathbf{F}}^e$$

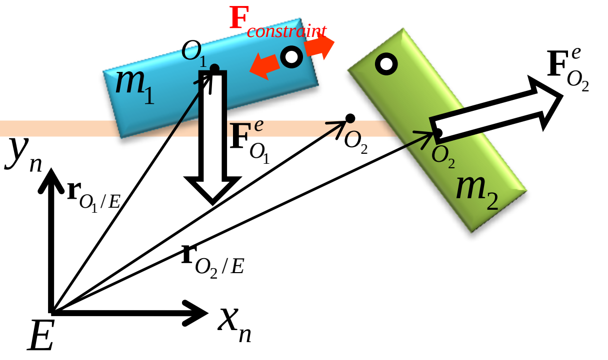


, **where** $\tilde{\mathbf{M}} = \mathbf{J}^T\mathbf{M}\mathbf{J},\ \tilde{\mathbf{k}} = \mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}},\ \tilde{\mathbf{F}}^e = \mathbf{J}^T\mathbf{F}^e_O$

$$\mathbf{J} = \begin{bmatrix} -\cos q_2 & +q_1 \sin q_2 \\ -\sin q_2 & -q_1 \cos q_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \dot{\mathbf{J}} = \begin{bmatrix} \sin q_2 \cdot \dot{q}_2 & \dot{q}_1 \sin q_2 + q_1 \cos q_2 \cdot \dot{q}_2 \\ -\cos q_2 \cdot \dot{q}_2 & -\dot{q}_1 \cos q_2 + q_1 \sin q_2 \cdot \dot{q}_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m_v & & & & \\ & m_v & & & \\ & & m_t & & \\ & & & m_t & \\ & & & & I_t \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} F^e_{O,x} \\ F^e_{O,y} \\ F^e_{O_0,x} \\ F^e_{O_0,y} \\ M^e_{O_0} \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Multibody System Dynamics



$$\mathbf{M}_1\ddot{\mathbf{r}}_{O_1/E} = \mathbf{F}^e_{O_1} + \mathbf{F}^1_{constraint} \quad , where\ \mathbf{M}_1 = \begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix}$$

$$\mathbf{M}_2\ddot{\mathbf{r}}_{O_2/E} = \mathbf{F}^e_{O_2} + \mathbf{F}^2_{constraint} \quad , where\ \mathbf{M}_2 = \begin{bmatrix} m_2 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}_1 & 0 \\ 0 & \mathbf{M}_2 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_{O_1/E} \\ \ddot{\mathbf{r}}_{O_2/E} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e_{O_1} \\ \mathbf{F}^e_{O_2} \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{F}^1_{constraint}} \\ \underline{\mathbf{F}^2_{constraint}} \end{bmatrix}$$

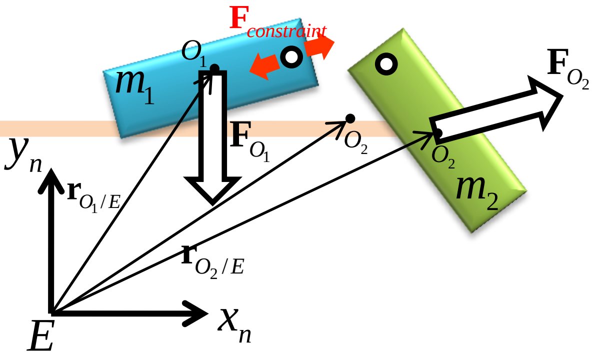
If constraint force is known and the variables are independent, these equations can be solved row by row.

Diagonal Matrix

Because of kinematic constraint, these **are not independent variables**

Constraint force is caused by kinematic constraint, and the constraint force is unknown.

Multibody System Dynamics



$$\mathbf{M}_1\ddot{\mathbf{r}}_{O_1/E} = \mathbf{F}^e_{O_1} + \mathbf{F}^1_{constraint} \quad , where\ \mathbf{M}_1 = \begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix}$$

$$\mathbf{M}_2\ddot{\mathbf{r}}_{O_2/E} = \mathbf{F}^e_{O_2} + \mathbf{F}^2_{constraint} \quad , where\ \mathbf{M}_2 = \begin{bmatrix} m_2 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\mathbf{M}\ddot{\mathbf{r}} = \mathbf{F}_O + \mathbf{F}_{constraint} \quad , where\ \mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & 0 \\ 0 & \mathbf{M}_2 \end{bmatrix}, \ddot{\mathbf{r}} = \begin{bmatrix} \ddot{\mathbf{r}}_{O_1/E} \\ \ddot{\mathbf{r}}_{O_2/E} \end{bmatrix}, \mathbf{F}_O = \begin{bmatrix} \mathbf{F}_{O_1} \\ \mathbf{F}_{O_2} \end{bmatrix}, \mathbf{F}_{constraint} = \begin{bmatrix} \mathbf{F}^1_{constraint} \\ \mathbf{F}^2_{constraint} \end{bmatrix}$$

From kinematic constraint ⇨ $\dot{\mathbf{r}} = \mathbf{J}\dot{\mathbf{q}}$

$\mathbf{J}$: Velocity transformation matrix

$\mathbf{q}$: Generalized coordinate

$$\underbrace{\mathbf{J}^T\mathbf{M}\mathbf{J}}_{\tilde{\mathbf{M}}}\ddot{\mathbf{q}} + \underbrace{\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}}_{\tilde{\mathbf{k}}} = \underbrace{\mathbf{J}^T\mathbf{F}^e_O}_{\tilde{\mathbf{F}}^e}$$

In general, since $\tilde{\mathbf{M}}$ is not diagonal matrix, all equations should be solved simultaneously.

3.3 Absolute coordinate formulation

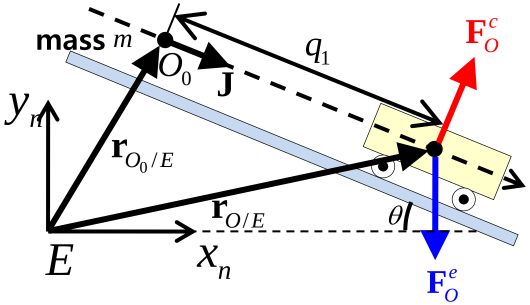
Vehicle constrained to move along the straight track

- Derivation of equations of motion using embedding technique

According to Newton's 2nd law

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c$$

$$, where\ \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}_O^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$



The constraint reaction force is suppressed

Kinematics

$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + q_1\mathbf{J}$$

The time derivative

$$\dot{\mathbf{r}}_{O/E} = \mathbf{J}\dot{q}_1$$

The time derivative

$$\ddot{\mathbf{r}}_{O/E} = \mathbf{J}\ddot{q}_1 \cdots (2)$$

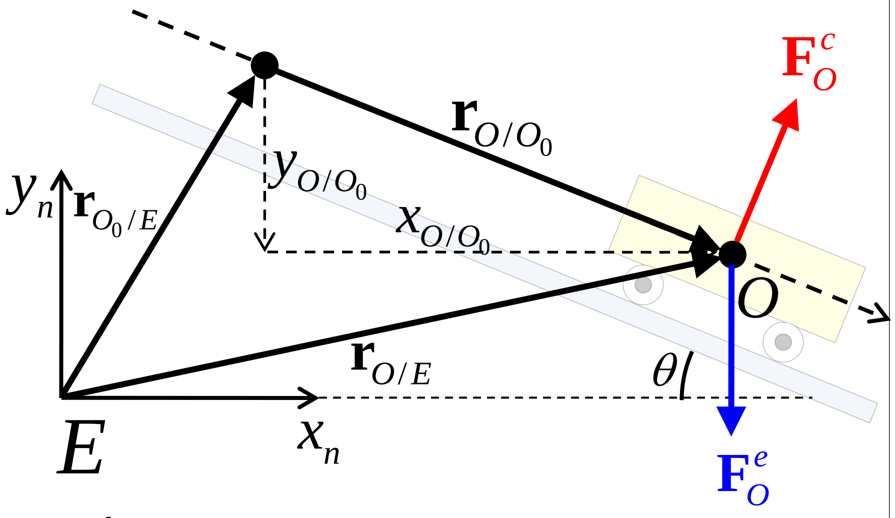
$$\mathbf{J}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{J}^T\mathbf{F}_O^e \cdots (1)$$

What if we want to express the equations of motion in terms of the variable $x_{O/E}$ and $y_{O/E}$?

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 13 ~ 16

Vehicle constrained to move along straight track

- Kinematic constraint



$n-frame$: Inertial reference frame.

O: Center of mass of the vehicle.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0.
$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t.

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0
to the point O.

θ: Angle of inclination of the track. It is constant.

$$\frac{y_{O/O_0}}{x_{O/O_0}} = -\tan\theta \Rightarrow y_{O/O_0} = -x_{O/O_0}\tan\theta \cdots \text{(1)}$$

$$\boxed{y_{O/O_0} + x_{O/O_0}\tan\theta = 0}$$

Constraint represented by $y_{O/O_0}, x_{O/O_0}$

$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + \mathbf{r}_{O/O_0}$$

$$\Downarrow$$

$$\begin{bmatrix} x_{O/E} \\ y_{O/E} \end{bmatrix} = \begin{bmatrix} x_{O_0/E} \\ y_{O_0/E} \end{bmatrix} + \begin{bmatrix} x_{O/O_0} \\ y_{O/O_0} \end{bmatrix}$$

$$\Downarrow$$

$$y_{O/E} = y_{O_0/E} + y_{O/O_0}, \qquad x_{O/E} = x_{O_0/E} + x_{O/O_0}$$

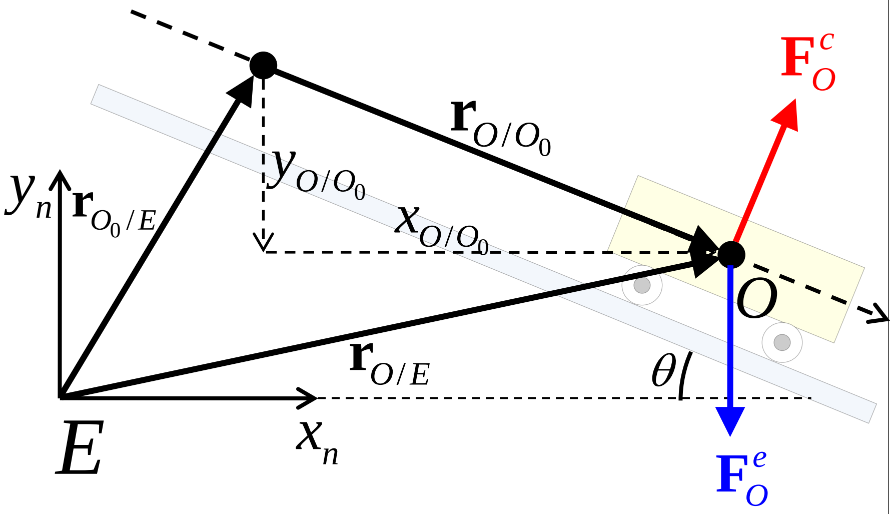
$\downarrow$ By using eq. (1) $\qquad \downarrow$

$$y_{O/E} = y_{O_0/E} - x_{O/O_0}\tan\theta, \qquad x_{O/E} - x_{O_0/E} = x_{O/O_0}$$

Continue..

Vehicle constrained to move along straight track

- Kinematic constraint



$n-frame$: Inertial reference frame.

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$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0.
$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t.

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0
to the point O.

θ: Angle of inclination of the track. It is constant.

$$\frac{y_{O/O_0}}{x_{O/O_0}} = -\tan\theta \Rightarrow y_{O/O_0} = -x_{O/O_0}\tan\theta \cdots \textbf{(1)}$$

$$\boxed{y_{O/O_0} + x_{O/O_0}\tan\theta = 0}$$

Constraint represented by $y_{O/O_0}, x_{O/O_0}$

$$\mathbf{r}_{O/E} = \mathbf{r}_{O_0/E} + \mathbf{r}_{O/O_0}$$

Continue..

$$y_{O/E} = y_{O_0/E} - \boxed{x_{O/O_0}}\tan\theta, \qquad x_{O/E} - x_{O_0/E} = \boxed{x_{O/O_0}}$$

$$y_{O/E} = y_{O_0/E} - (x_{O/E} - x_{O_0/E})\tan\theta$$

$$y_{O/E} = y_{O_0/E} - x_{O/E}\tan\theta + x_{O_0/E}\tan\theta$$

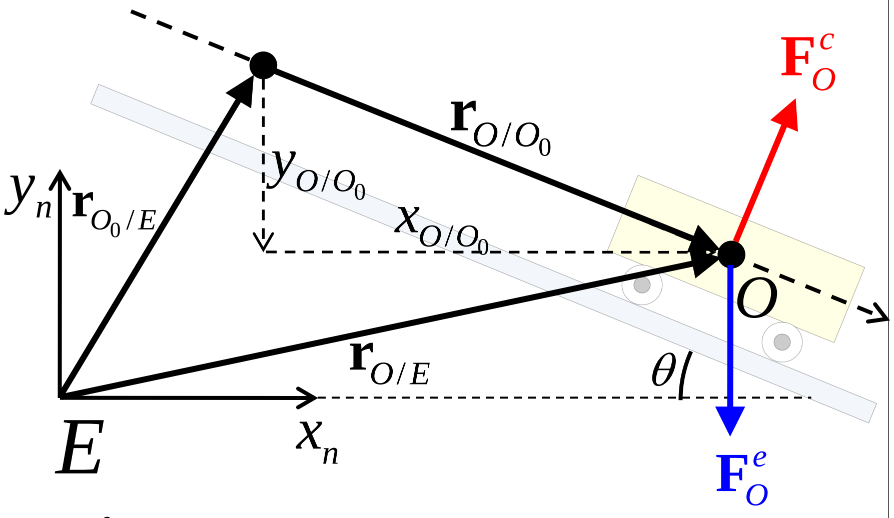
$$y_{O/E} + x_{O/E}\tan\theta - x_{O_0/E}\tan\theta - y_{O_0/E} = 0$$

$$\boxed{y_{O/E} + x_{O/E}\tan\theta + A = 0}\text{, where } A = -x_{O_0/E}\tan\theta - y_{O_0/E}$$

Constraint represented by $y_{O/E}, x_{O/E}$

Vehicle constrained to move along straight track

- Derivation of equations of motion expressed by Cartesian coordinates(1/4)



$n-frame$: Inertial reference frame.

O: Center of mass of the vehicle.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0.
$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t.

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0
to the point O.

θ: Angle of inclination of the track. It is constant.

$$\boxed{y_{O/O_0} + x_{O/O_0} \tan\theta = 0}$$

$$\boxed{y_{O/E} + x_{O/E} \tan\theta + A = 0}$$

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c \quad , where\ \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

The constraint reaction force $\mathbf{F}_O^c$ is perpendicular to the vector $\mathbf{r}_{O/O_0}$

$$\mathbf{r}_{O/O_0} \cdot \mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{r}_{O/O_0} \cdot \mathbf{F}_O^e + \cancel{\mathbf{r}_{O/O_0} \cdot \mathbf{F}_O^c}$$

Scalar product of vectors
- Matrix representation

1) Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons, Inc., p.346

✓ Scalar product of vectors

$$\mathbf{A} = (a_1, a_2, \cdots, a_n) \qquad \mathbf{B} = (b_1, b_2, \cdots, b_n)$$

$$\mathbf{A} \bullet \mathbf{B} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

✓ Matrix representation

$$\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

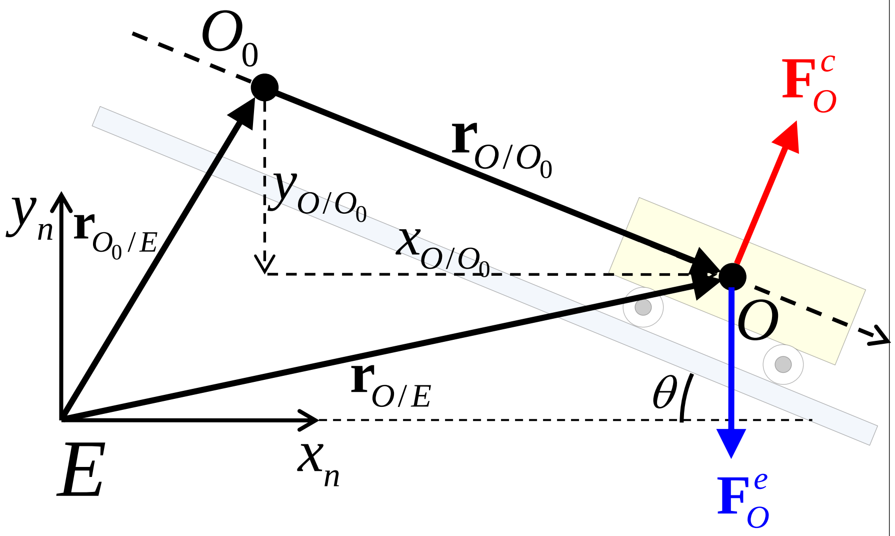
$$\boxed{\mathbf{A} \bullet \mathbf{B} = \mathbf{A}^T \mathbf{B}}$$

$\mathbf{A}^T \mathbf{B}$ preserves the value of the scalar product of the vectors A and B

$$\mathbf{A}^T \mathbf{B} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

Vehicle constrained to move along straight track

- Derivation of equations of motion expressed by Cartesian coordinates(1/4)



$n-frame$: Inertial reference frame.

O: Center of mass of the vehicle.

$\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0.
$\mathbf{r}_{O_0/E}$ is constant.

$\mathbf{r}_{O/E}$: Position vector of the center of mass O at time t.

$\mathbf{r}_{O/O_0}$: Directed vector from the point O_0 to the point O.

θ: Angle of inclination of the track. It is constant.

$$\boxed{y_{O/O_0} + x_{O/O_0} \tan\theta = 0}$$

$$\boxed{y_{O/E} + x_{O/E} \tan\theta + A = 0}$$

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{F}_O^e + \mathbf{F}_O^c \quad \text{, where } \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

The constraint reaction force $\mathbf{F}_O^c$ is perpendicular to the vector $\mathbf{r}_{O/O_0}$

$$\mathbf{r}_{O/O_0} \cdot \mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{r}_{O/O_0} \cdot \mathbf{F}^e + \cancel{\mathbf{r}_{O/O_0} \cdot \mathbf{F}^c}$$

$$\boxed{\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^T \mathbf{B}}$$

$$\boxed{\mathbf{r}_{O/O_0}{}^T \mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{r}_{O/O_0}{}^T \mathbf{F}_O^e}$$

, where $\mathbf{r}_{O/O_0} = \begin{bmatrix} x_{O/O_0} \\ y_{O/O_0} \end{bmatrix}, \dfrac{y_{O/O_0}}{x_{O/O_0}} = -\tan\theta$

Vehicle constrained to move along straight track

- Derivation of equations of motion expressed by Cartesian coordinates(2/4)

$$\boxed{\mathbf{r}_{O/O_0}{}^T(\mathbf{M}\ddot{\mathbf{r}}_{O/E}-\mathbf{F}_O^e)=0}$$

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E}=\mathbf{F}_O^e+\mathbf{F}_O^c \quad ,\text{where } \mathbf{M}=\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e=\begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

The constraint reaction force $\mathbf{F}_O^c$ is perpendicular to the vector $\mathbf{r}_{O/O_0}$

Kinematic constraint

$$\boxed{y_{O/O_0}+x_{O/O_0}\tan\theta=0}$$

$$\boxed{y_{O/E}+x_{O/E}\tan\theta+A=0}$$

$$\boxed{\mathbf{r}_{O/O_0}{}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E}=\mathbf{r}_{O/O_0}{}^T\mathbf{F}_O^e}$$

, where $\mathbf{r}_{O/O_0}=\begin{bmatrix} x_{O/O_0} \\ y_{O/O_0} \end{bmatrix}$

$$\mathbf{r}_{O/O_0}{}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E}-\mathbf{r}_{O/O_0}{}^T\mathbf{F}_O^e=0$$

$$\boxed{\mathbf{r}_{O/O_0}{}^T(\mathbf{M}\ddot{\mathbf{r}}_{O/E}-\mathbf{F}_O^e)=0}$$

$$\begin{bmatrix} x_{O/O_0} & y_{O/O_0} \end{bmatrix}\left(\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_{O/E} \\ \ddot{y}_{O/E} \end{bmatrix}-\begin{bmatrix} 0 \\ -mg \end{bmatrix}\right)=0$$

$$\begin{bmatrix} x_{O/O_0} & y_{O/O_0} \end{bmatrix}\begin{bmatrix} m\ddot{x}_{O/E} \\ m\ddot{y}_{O/E}+mg \end{bmatrix}=0$$

$$\boxed{x_{O/O_0}(m\ddot{x}_{O/E})+y_{O/O_0}(m\ddot{y}_{O/E}+mg)=0 \cdots\text{(1)}}$$

Vehicle constrained to move along straight track

- Derivation of equations of motion expressed by Cartesian coordinates(3/4)

$$\boxed{\mathbf{r}_{O/O_0}{}^T(\mathbf{M}\ddot{\mathbf{r}}_{O/E}-\mathbf{F}_O^e)=0}$$

$$\mathbf{M}\ddot{\mathbf{r}}_{O/E}=\mathbf{F}_O^e+\mathbf{F}_O^c \quad \text{, where } \mathbf{M}=\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e=\begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

The constraint reaction force $\mathbf{F}_O^c$ is perpendicular to the vector $\mathbf{r}_{O/O_0}$

Kinematic constraint

$$\boxed{y_{O/O_0}+x_{O/O_0}\tan\theta=0}$$

$$\boxed{y_{O/E}+x_{O/E}\tan\theta+A=0}$$

$$\boxed{\mathbf{r}_{O/O_0}{}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E}=\mathbf{r}_{O/O_0}{}^T\mathbf{F}_O^e}\text{, where } \mathbf{r}_{O/O_0}=\begin{bmatrix} x_{O/O_0} \\ y_{O/O_0} \end{bmatrix}$$

$$\boxed{x_{O/O_0}(m\ddot{x}_{O/E})+y_{O/O_0}(m\ddot{y}_{O/E}+mg)=0 \cdots\text{(1)}}$$

If x_{O/O_0} and y_{O/O_0} were independent of each other, $m\ddot{x}_{O/E}$ and $m\ddot{y}_{O/E}+mg$ should be zero to satisfy the equation (1), and $m\ddot{x}_{O/E}=0,\ m\ddot{y}_{O/E}+mg=0$ will be the equations of motion

This, however, is not the case, because of the kinematic relation

$$\boxed{x_{O/O_0}\tan\theta+y_{O/O_0}=0}$$

By eliminating one of two variables from the equation (1), we can consider another variable as a free variable.

Vehicle constrained to move along straight track

$\mathbf{r}_{O/O_0}{}^T(\mathbf{M}\ddot{\mathbf{r}}_{O/E} - \mathbf{F}_O^e) = 0$

- Derivation of equations of motion expressed by Cartesian coordinates(4/4)

$$\mathbf{r}_{O/O_0}{}^T \mathbf{M}\ddot{\mathbf{r}}_{O/E} = \mathbf{r}_{O/O_0}{}^T \mathbf{F}_O^e$$

, where $\mathbf{r}_{O/O_0} = \begin{bmatrix} x_{O/O_0} \\ y_{O/O_0} \end{bmatrix}$

Kinematic constraint

$y_{O/O_0} + x_{O/O_0} \tan\theta = 0$

$y_{O/E} + x_{O/E} \tan\theta + A = 0$

$$x_{O/O_0}(m\ddot{x}_{O/E}) + y_{O/O_0}(m\ddot{y}_{O/E} + mg) = 0 \quad \cdots(1)$$

$$x_{O/O_0} \tan\theta + y_{O/O_0} = 0 \quad \cdots(2)$$

Eliminate y_{O/O_0}

(1) - (2)× $(m\ddot{y}_{O/E} + mg)$

$$x_{O/O_0}(\tan\theta(m\ddot{y}_{O/E} + mg) - m\ddot{x}_{O/E}) = 0 \quad \cdots(1)'$$

x_{O/O_0} **is a free variable.**

$$\tan\theta(m\ddot{y}_{O/E} + mg) - m\ddot{x}_{O/E} = 0 \quad \cdots(1)''$$

2 variables$(x_{O/E}, y_{O/E})$**, 1 equation**

From the kinematic relation

$$y_{O/E} + \tan\theta \cdot x_{O/E} + A = 0$$

, where $A = -y_{O_0/E} - \tan\theta \cdot x_{O_0/E}$

The time derivative

$$\dot{x}_{O/E} \tan\theta + \dot{y}_{O/E} = 0$$

The time derivative

$$\ddot{x}_{O/E} \tan\theta + \ddot{y}_{O/E} = 0$$

2 variables$(x_{O/E}, y_{O/E})$**, 1 equation**

We can solve the equations of motion

Vehicle constrained to move along straight track

- Derivation of equations of motion expressed by Cartesian coordinates(4/4)

$$\mathbf{r}_{O/O_0}{}^T(\mathbf{M}\ddot{\mathbf{r}}_{O/E}-\mathbf{F}_O^e)=0$$

$$\boxed{\mathbf{r}_{O/O_0}{}^T\mathbf{M}\ddot{\mathbf{r}}_{O/E}=\mathbf{r}_{O/O_0}{}^T\mathbf{F}_O^e}$$

, where $\mathbf{r}_{O/O_0}=\begin{bmatrix} x_{O/O_0} \\ y_{O/O_0} \end{bmatrix}$

Kinematic constraint

$$\boxed{y_{O/O_0}+x_{O/O_0}\tan\theta=0}$$

$$\boxed{y_{O/E}+x_{O/E}\tan\theta+A=0}$$

$$\boxed{x_{O/O_0}(m\ddot{x}_{O/E})+y_{O/O_0}(m\ddot{y}_{O/E}+mg)=0 \;\cdots(1)}$$

$$\boxed{x_{O/O_0}\tan\theta+y_{O/O_0}=0 \qquad \cdots(2)}$$

Eliminate y_{O/O_0}

(1) - (2)× $(m\ddot{y}_{O/E}+mg)$

$$x_{O/O_0}(\tan\theta(m\ddot{y}_{O/E}+mg)-m\ddot{x}_{O/E})=0 \;\cdots(1)'$$

From the kinematic relation

$$\boxed{y_{O/E}+\tan\theta\cdot x_{O/E}+A=0}$$

, where $A=-y_{O_0/E}-\tan\theta\cdot x_{O_0/E}$

The time derivative

$$\dot{x}_{O/E}\tan\theta+\dot{y}_{O/E}=0$$

Which variables should be eliminated among dependant variables? Do we have to decide?

We can solve the equations of motion

Vehicle constrained to move along straight track - Equations of motions

$\mathbf{r}_{O/O_0}{}^T(\mathbf{M}\ddot{\mathbf{r}}_{O/E} - \mathbf{F}_O^e) = 0$

$$x_{O/O_0}(m\ddot{x}_{O/E}) + y_{O/O_0}(m\ddot{y}_{O/E} + mg) = 0 \quad \cdots (1)$$

$$x_{O/O_0}\tan\theta + y_{O/O_0} = 0 \quad \cdots (2)$$

Kinematic constraint

$y_{O/O_0} + x_{O/O_0}\tan\theta = 0$

$y_{O/E} + x_{O/E}\tan\theta + A = 0$

To eliminate y_{O/O_0} , we multiplied the equation (2) by $(m\ddot{y}_{O/E} + mg)$ in previous page.

$$(1) - (2) \times (m\ddot{y}_{O/E} + mg)$$

What if we didn't decide which variable will be eliminated yet?

Then, we multiply an '**undetermined multiplier** λ'.

$$(1) + (2) \times \lambda$$

Vehicle constrained to move along straight track - Equations of motions

$$\boxed{\mathbf{r}_{O/O_0}{}^{T}(\mathbf{M}\ddot{\mathbf{r}}_{O/E}-\mathbf{F}_{O}^{e})=0}$$

$$\boxed{x_{O/O_0}(m\ddot{x}_{O/E})+y_{O/O_0}(m\ddot{y}_{O/E}+mg)=0} \text{ ▪▪▪(1)}$$

$$\boxed{x_{O/O_0}\tan\theta+y_{O/O_0}=0} \text{ ▪▪▪(2)}$$

(1) + (2)× λ

$$x_{O/O_0}\boxed{(\tan\theta\cdot\lambda+m\ddot{x}_{O/E})}+y_{O/O_0}\boxed{(m\ddot{y}_{O/E}+mg+\lambda)}=0 \text{ ▪▪▪(1)'}$$

Now, we can choose λ so that the factor multiplying y_{O/O_0} shall vanish.

$$\boxed{m\ddot{y}_{O/E}+mg+\lambda}=0 \text{ ▪▪▪(1-1)'}$$

Because $y_{O/E}$ is eliminated, $x_{O/E}$ is a free variable.

$$\boxed{\tan\theta\cdot\lambda+m\ddot{x}_{O/E}}=0 \text{ ▪▪▪(1-2)'}$$

From (2) we can get an additional equation.

$$\ddot{x}_{O/E}\tan\theta+\ddot{y}_{O/E}=0 \text{ ▪▪▪(2)'}$$

3 variables $(x_{O/E}, y_{O/E}, \lambda)$, 3 equation

The equations of motion are expressed in terms of $\ddot{x}_{O/E}$, $\ddot{y}_{O/E}$, and λ

We call λ as "Lagrange Multiplier"

Vehicle constrained to move along straight track - Equations of motions

Kinematic constraint

$$\boxed{y_{O/O_0} + x_{O/O_0}\tan\theta = 0}$$

$$\boxed{y_{O/E} + x_{O/E}\tan\theta + A = 0}$$

$$\boxed{x_{O/O_0}(m\ddot{x}_{O/E}) + y_{O/O_0}(m\ddot{y}_{O/E} + mg) = 0} \quad \cdots(1)$$

$$\boxed{x_{O/O_0}\tan\theta + y_{O/O_0} = 0} \quad \cdots(2)$$

$$\boxed{\mathbf{r}_{O/O_0}{}^T(\mathbf{M}\ddot{\mathbf{r}}_{O/E} - \mathbf{F}_O^e) = 0}$$

$$\boxed{\mathbf{r}_{O/O_0}{}^T\nabla C(x_{O/E}, y_{O/E})^T = 0}$$

$$\boxed{x_{O/O_0}\tan\theta + y_{O/O_0} = 0}$$

Matrix representation

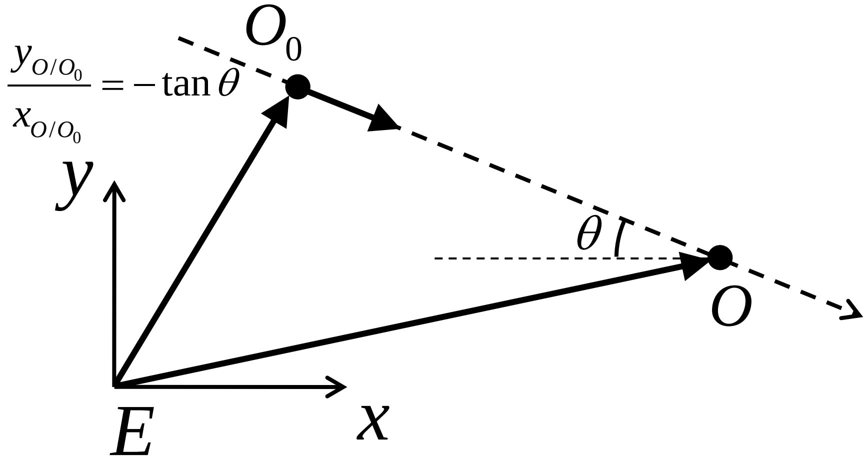
$$\begin{bmatrix} x_{O/O_0} & y_{O/O_0} \end{bmatrix}\begin{bmatrix} \tan\theta \\ 1 \end{bmatrix} = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$\mathbf{r}_{O/E}{}^T \qquad \nabla C(x_{O/E}, y_{O/E})^T$$

$$\Downarrow$$

$$\boxed{\mathbf{r}_{O/O_0}{}^T\nabla C(x_{O/E}, y_{O/E})^T = 0}$$



$$y_{O/E} + \tan\theta \cdot x_{O/E} + A = 0,$$ **where** $A = -y_{O/E} - \tan\theta \cdot x_{O/E}$

$$C(x_{O/E}, y_{O/E}) = 0$$

Gradient of the function C

$$\nabla C(x_{O/E}, y_{O/E}) = \begin{bmatrix} \frac{\partial C(x_{O/E}, y_{O/E})}{\partial x_{O/E}} & \frac{\partial C(x_{O/E}, y_{O/E})}{\partial y_{O/E}} \end{bmatrix} = \begin{bmatrix} \tan\theta & 1 \end{bmatrix}$$

Vehicle constrained to move along straight track - Equations of motions

$$x_{O/E}\tan\theta + y_{O/E} + A = 0$$

$$C(x_{O/E}, y_{O/E}) = 0$$

$$\mathbf{r}_{O/O_0} = \begin{bmatrix} x_{O/O_0} \\ y_{O/O_0} \end{bmatrix}$$

$$\boxed{x_{O/O_0}(m\ddot{x}_{O/E}) + y_{O/O_0}(m\ddot{y}_{O/E} + mg) = 0 \cdots (1)}$$

$$\boxed{x_{O/O_0}\tan\theta + y_{O/O_0} = 0 \qquad \cdots (2)}$$

$$\boxed{\mathbf{r}_{O/O_0}{}^{T}(\mathbf{M}\ddot{\mathbf{r}}_{O/E} - \mathbf{F}_{O}^{e}) = 0}$$

$$\boxed{\mathbf{r}_{O/O_0}{}^{T}\nabla C\left(x_{O/E}, y_{O/E}\right)^{T} = 0}$$

$$\nabla C(x_{O/E}, y_{O/E}) = [\tan\theta \quad 1]$$

(1) + (2)×λ

$$x_{O/O_0}\boxed{(\tan\theta\cdot\lambda + m\ddot{x}_{O/E})} + y_{O/O_0}\boxed{(m\ddot{y}_{O/E} + mg + \lambda)} = 0$$

$$\boxed{\tan\theta\cdot\lambda + m\ddot{x}_{O/E} = 0,\ m\ddot{y}_{O/E} + mg + \lambda = 0}$$

(1) + (2)× λ

$$\mathbf{r}_{O/O_0}{}^{T}(\mathbf{M}\ddot{\mathbf{r}}_{O/E} - \mathbf{F}_{O}^{e}) + \mathbf{r}_{O/O_0}{}^{T}\nabla C\left(x_{O/E}, y_{O/E}\right)^{T}\lambda = 0$$

$$\mathbf{r}_{O/O_0}{}^{T}\boxed{(\mathbf{M}\ddot{\mathbf{r}}_{O/E} - \mathbf{F}_{O}^{e} + \nabla C\left(x_{O/E}, y_{O/E}\right)^{T}\lambda)} = 0$$

$$\boxed{\mathbf{M}\ddot{\mathbf{r}}_{O/E} - \mathbf{F}_{O}^{e} + \nabla C\left(x_{O/E}, y_{O/E}\right)^{T}\lambda = 0}$$

From (2) we can get an additional equation.

$$\boxed{\ddot{x}_{O/E}\tan\theta + \ddot{y}_{O/E} = 0}$$

$$\boxed{\nabla C\left(x_{O/E}, y_{O/E}\right)\ddot{\mathbf{r}}_{O/E} = 0}$$

Matrix representation

$$\begin{bmatrix} m & 0 & \tan\theta \\ 0 & m & 1 \\ \tan\theta & 1 & 0 \end{bmatrix}\begin{bmatrix} \ddot{x}_{O/E} \\ \ddot{y}_{O/E} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}$$

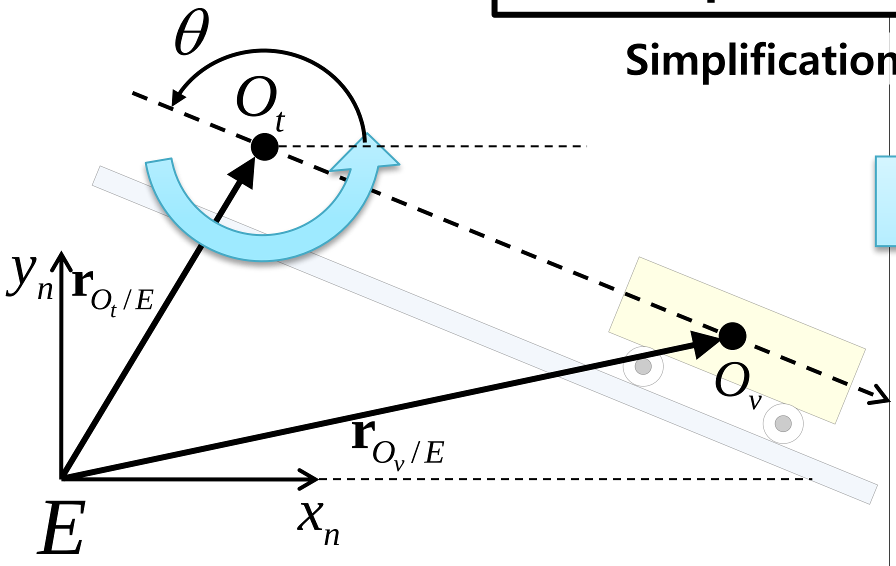
Matrix representation

$$\begin{bmatrix} \mathbf{M} & \nabla C\left(x_{O/E}, y_{O/E}\right)^{T} \\ \nabla C\left(x_{O/E}, y_{O/E}\right) & 0 \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{r}}_{O/E} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{O}^{e} \\ 0 \end{bmatrix}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

Suppose that the track is rotating about space fixed point O_0



$n-frame$: Inertial reference frame.

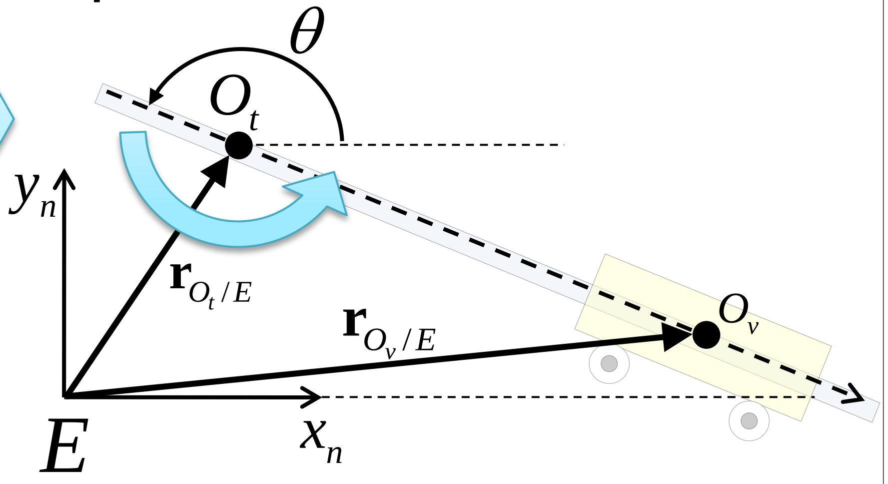
O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

Simplification of the problem



$n-frame$: Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

Absolute coordinate formulation (Augmented Formulation)

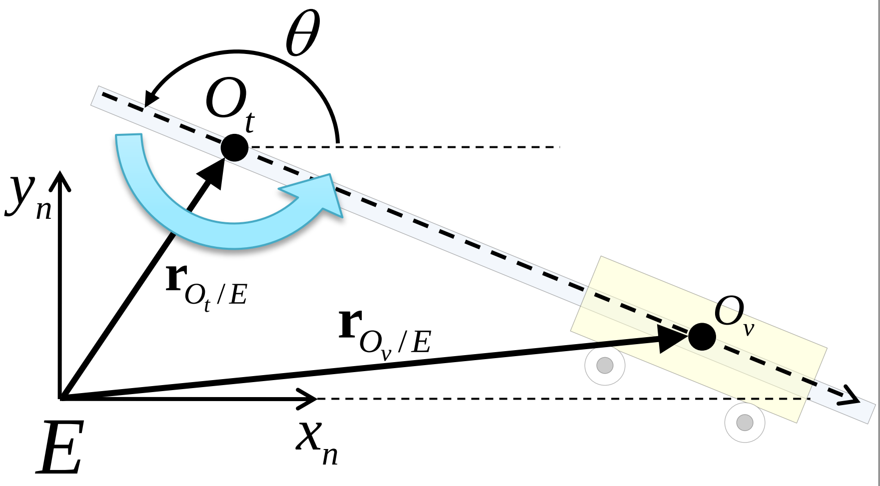
$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_{\mathbf{r}}^{T} \\ \mathbf{C}_{\mathbf{r}} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{e} \\ \mathbf{F}^{d} \end{bmatrix}$$

Constraint: $\mathbf{C}(\mathbf{r},t)=0$

Differentiation twice $\mathbf{C}_{\mathbf{r}}\ddot{\mathbf{r}}+\underbrace{\boxed{(\mathbf{C}_{\mathbf{r}}\dot{\mathbf{r}})_{\mathbf{r}}\dot{\mathbf{r}}}}_{-\mathbf{F}^{d}}=0$

λ : Lagrange Multiplier [Ahmed A. Shabana, 1992, p. 192 ~ 197]

- Given: F^e, F^d
- Find : $\ddot{\mathbf{r}}$, $\boldsymbol{\lambda}$ ($-\mathbf{C}_{\mathbf{r}}^{T}\cdot\boldsymbol{\lambda}$ is constraint force)



$n-frame$: Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

To construct equations of motion by using augmented formulation, we have to know the terms in following differential algebraic equation.

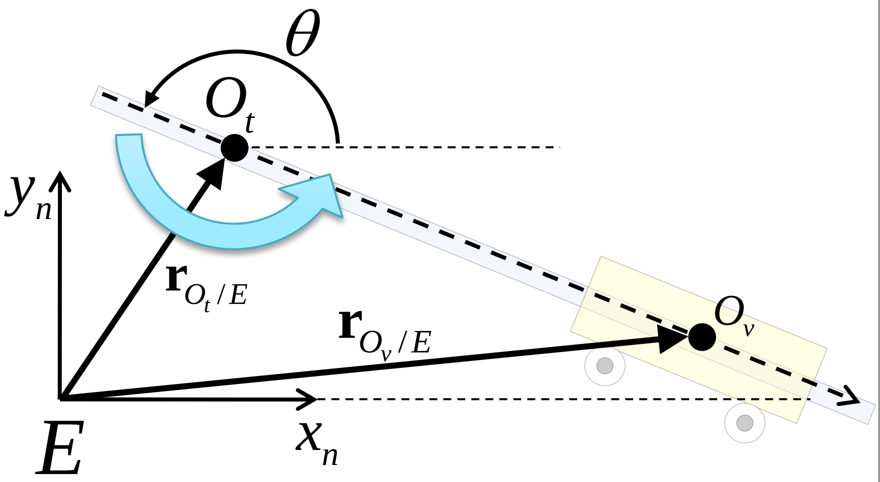
$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_{\mathbf{r}}^{T} \\ \mathbf{C}_{\mathbf{r}} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{e} \\ \mathbf{F}^{d} \end{bmatrix}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

$$\begin{bmatrix} \mathbf{M} & \mathbf{C_r}^T \\ \mathbf{C_r} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e \\ \mathbf{F}^d \end{bmatrix}$$

$$\mathbf{F}^d = -(\mathbf{C_r}\dot{\mathbf{r}})_{\mathbf{r}}\dot{\mathbf{r}}$$



$n-frame$: Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

Equations of motion of the vehicle.

$$\begin{bmatrix} m_v & 0 \\ 0 & m_v \end{bmatrix} \begin{bmatrix} \ddot{x}_{O_v/E} \\ \ddot{y}_{O_v/E} \end{bmatrix} = \begin{bmatrix} F^e_{O_v,x} \\ F^e_{O_v,y} \end{bmatrix} + \begin{bmatrix} F^c_{O_v,x} \\ F^c_{O_v,y} \end{bmatrix}$$

m_v : Mass of the vehicle

$\mathbf{r}_{O_v/E} = \begin{bmatrix} x_{O_v/E} \\ y_{O_v/E} \end{bmatrix}$: Position vector of the center of mass of the vehicle.

$\mathbf{F}^e_{O_v} = \begin{bmatrix} F^e_{O_v,x} \\ F^e_{O_v,y} \end{bmatrix}$: external force exerted on the center of mass of the vehicle

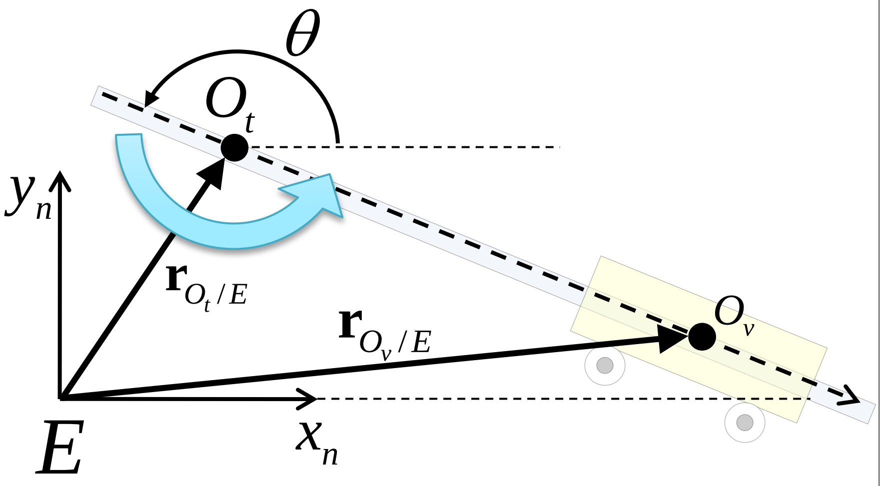
$\mathbf{F}^c_{O_v} = \begin{bmatrix} F^c_{O_v,x} \\ F^c_{O_v,y} \end{bmatrix}$: constraint force exerted on the center of mass of the vehicle

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

$$\begin{bmatrix} \mathbf{M} & \mathbf{C_r}^T \\ \mathbf{C_r} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e \\ \mathbf{F}^d \end{bmatrix}$$

$$\mathbf{F}^d = -(\mathbf{C_r}\dot{\mathbf{r}})_\mathbf{r}\,\dot{\mathbf{r}}$$



$n-frame$: Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

Equations of motion of the track.

$$\begin{bmatrix} m_t & 0 \\ 0 & m_t \end{bmatrix} \begin{bmatrix} \ddot{x}_{O_t/E} \\ \ddot{y}_{O_t/E} \end{bmatrix} = \begin{bmatrix} F^e_{O_t,x} \\ F^e_{O_t,y} \end{bmatrix} + \begin{bmatrix} F^c_{O_t,x} \\ F^c_{O_t,y} \end{bmatrix}$$

$$I_t\ddot{\theta} = M^e_{O_t} + M^c_{O_t}$$

m_t, I_t: Mass and mass moment of inertia of the track

$\mathbf{r}_{O_t/E} = \begin{bmatrix} x_{O_t/E} \\ y_{O_t/E} \end{bmatrix}$: Position vector of the track.

θ : Rotational angle of the track.

$\mathbf{F}^e_{O_t} = \begin{bmatrix} F^e_{O_t,x} \\ F^e_{O_t,y} \\ M^e_{O_t} \end{bmatrix}$: external force and moment exerted on the center of mass of the track

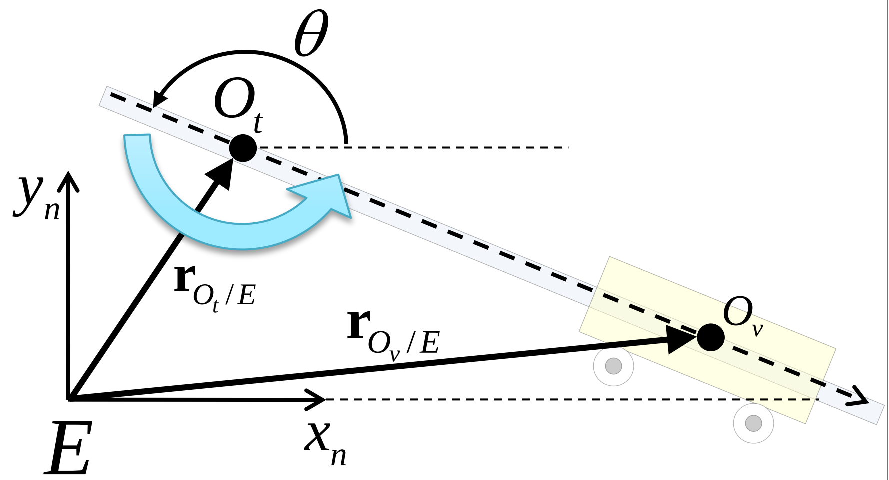
$\mathbf{F}^c_{O_t} = \begin{bmatrix} F^c_{O_t,x} \\ F^c_{O_t,y} \\ M^c_{O_t} \end{bmatrix}$: constraint force and moment exerted on the center of mass of the track

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

$$\begin{bmatrix} \boxed{\mathbf{M}} & \mathbf{C_r}^T \\ \mathbf{C_r} & 0 \end{bmatrix} \begin{bmatrix} \boxed{\ddot{\mathbf{r}}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{F}^e} \\ \mathbf{F}^d \end{bmatrix}$$

$$\mathbf{F}^d = -(\mathbf{C_r}\dot{\mathbf{r}})_\mathbf{r}\,\dot{\mathbf{r}}$$



$n-frame$: Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

Equations of motion of the vehicle.

$$\begin{bmatrix} m_v & 0 \\ 0 & m_v \end{bmatrix} \begin{bmatrix} \ddot{x}_{O_v/E} \\ \ddot{y}_{O_v/E} \end{bmatrix} = \begin{bmatrix} F^e_{O_v,x} \\ F^e_{O_v,y} \end{bmatrix} + \begin{bmatrix} F^c_{O_v,x} \\ F^c_{O_v,y} \end{bmatrix}$$

Equations of motion of the track.

$$\begin{bmatrix} m_t & 0 \\ 0 & m_t \end{bmatrix} \begin{bmatrix} \ddot{x}_{O_t/E} \\ \ddot{y}_{O_t/E} \end{bmatrix} = \begin{bmatrix} F^e_{O_t,x} \\ F^e_{O_t,y} \end{bmatrix} + \begin{bmatrix} F^c_{O_t,x} \\ F^c_{O_t,y} \end{bmatrix}$$

$$I_t\ddot{\theta} = M^e_{O_t} + M^c_{O_t}$$

Combining three equations of motion

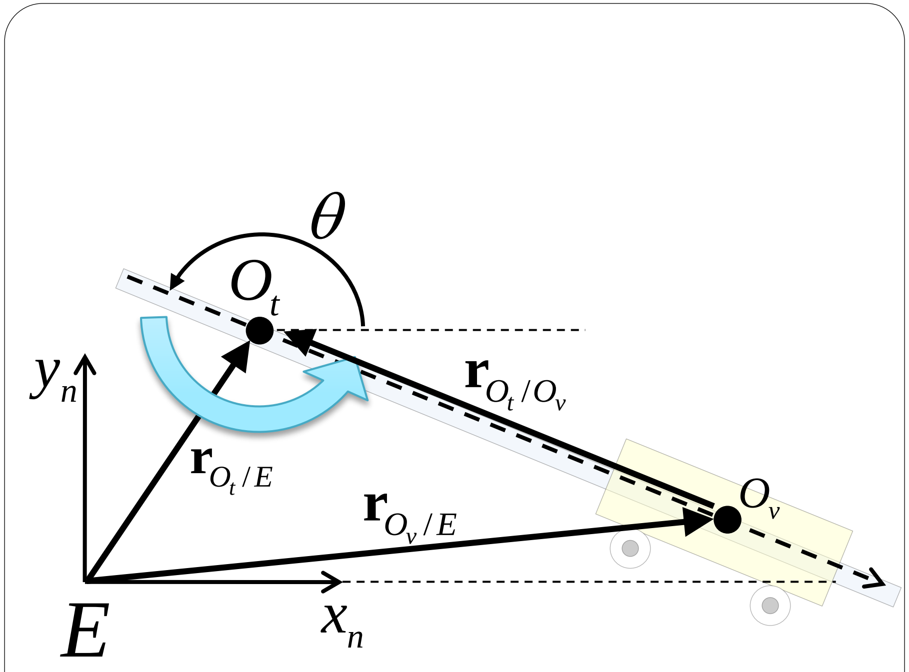
$$\underbrace{\boxed{\begin{bmatrix} m_v & & & & \\ & m_v & & & \\ & & m_t & & \\ & & & m_t & \\ & & & & I_t \end{bmatrix}}}_{\mathbf{M}} \underbrace{\boxed{\begin{bmatrix} \ddot{x}_{O_v/E} \\ \ddot{y}_{O_v/E} \\ \ddot{x}_{O_t/E} \\ \ddot{y}_{O_t/E} \\ \ddot{\theta} \end{bmatrix}}}_{\ddot{\mathbf{r}}} = \underbrace{\boxed{\begin{bmatrix} F^e_{O_v,x} \\ F^e_{O_v,y} \\ F^e_{O_t,x} \\ F^e_{O_t,y} \\ M^e_{O_t} \end{bmatrix}}}_{\mathbf{F}^e} + \begin{bmatrix} F^c_{O_v,x} \\ F^c_{O_v,y} \\ F^c_{O_t,x} \\ F^c_{O_t,y} \\ M^c_{O_t} \end{bmatrix}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

$$\begin{bmatrix} \boxed{\mathbf{M}} & \mathbf{C}_{\mathbf{r}}^{T} \\ \mathbf{C}_{\mathbf{r}} & 0 \end{bmatrix} \begin{bmatrix} \boxed{\ddot{\mathbf{r}}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{F}^{e}} \\ \mathbf{F}^{d} \end{bmatrix}$$

$$\mathbf{F}^{d} = -\left(\mathbf{C}_{\mathbf{r}}\dot{\mathbf{r}}\right)_{\mathbf{r}}\dot{\mathbf{r}}$$



$n-frame$: Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

Kinematical constraint

$$\mathbf{r}_{O_t/E} = \begin{bmatrix} x_{O_t/E} \\ y_{O_t/E} \end{bmatrix} = \begin{bmatrix} x_{const.} \\ y_{const.} \end{bmatrix}$$

$$\boxed{x_{O_t/E} = x_{const.}}$$

$$\boxed{y_{O_t/E} = y_{const.}}$$

$$\frac{y_{O_t/O_v}}{x_{O_t/O_v}} = \tan\theta \quad , where\ \mathbf{r}_{O_t/O_v} = \begin{bmatrix} x_{O_t/O_v} \\ y_{O_t/O_v} \end{bmatrix}$$

$$\mathbf{r}_{O_t/O_v} = \mathbf{r}_{O_t/E} - \mathbf{r}_{O_v/E}$$

$$\begin{bmatrix} x_{O_t/O_v} \\ y_{O_t/O_v} \end{bmatrix} = \begin{bmatrix} x_{O_t/E} \\ y_{O_t/E} \end{bmatrix} - \begin{bmatrix} x_{O_v/E} \\ y_{O_v/E} \end{bmatrix} = \begin{bmatrix} x_{const.} \\ y_{const.} \end{bmatrix} - \begin{bmatrix} x_{O_v/E} \\ y_{O_v/E} \end{bmatrix}$$

$$x_{O_t/O_v} = x_{const.} - x_{O_v/E}$$

$$y_{O_t/O_v} = y_{const.} - y_{O_v/E}$$

$$\boxed{\frac{y_{const.} - y_{O_v/E}}{x_{const.} - x_{O_v/E}} = \tan\theta}$$

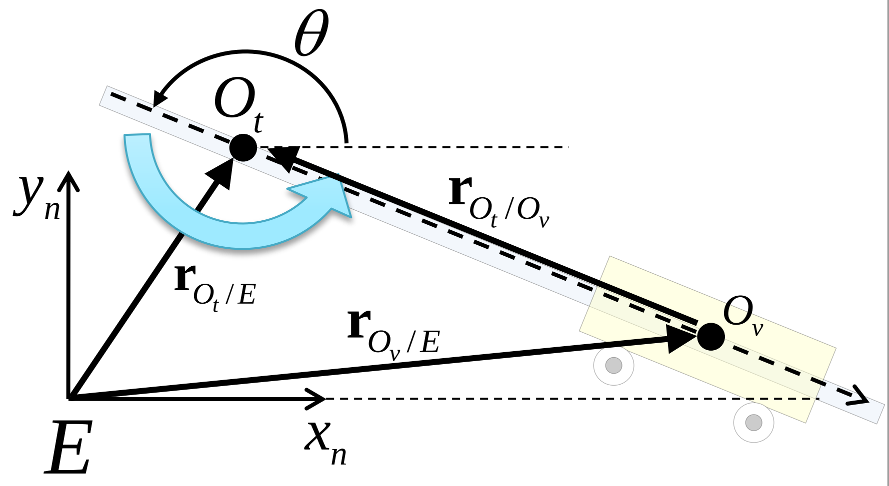
Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

$$\begin{bmatrix} \boxed{\mathbf{M}} & \mathbf{C_r}^T \\ \mathbf{C_r} & 0 \end{bmatrix} \begin{bmatrix} \boxed{\ddot{\mathbf{r}}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{F}^e} \\ \mathbf{F}^d \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} x_{O_v/E} & y_{O_v/E} & x_{O_t/E} & y_{O_t/E} & \theta \end{bmatrix}^T$$

$$\mathbf{F}^d = -(\mathbf{C_r}\dot{\mathbf{r}})_\mathbf{r}\,\dot{\mathbf{r}}$$



$n-frame$: Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

Kinematical constraint

$$\boxed{x_{O_t/E} = x_{const.}}$$

$$x_{O_t/E} - x_{const.} = 0 \Rightarrow C_1(\mathbf{r}) = 0$$

$$\boxed{y_{O_t/E} = y_{const.}}$$

$$y_{O_t/E} - y_{const.} = 0 \Rightarrow C_2(\mathbf{r}) = 0$$

$$\boxed{\frac{y_{const.} - y_{O_v/E}}{x_{const.} - x_{O_v/E}} = \tan\theta}$$

$$y_{const.} - y_{O_v/E} = (x_{const.} - x_{O_v/E})\tan\theta$$

$$y_{const.} - y_{O_v/E} - (x_{const.} - x_{O_v/E})\tan\theta = 0 \Rightarrow C_3(\mathbf{r}) = 0$$

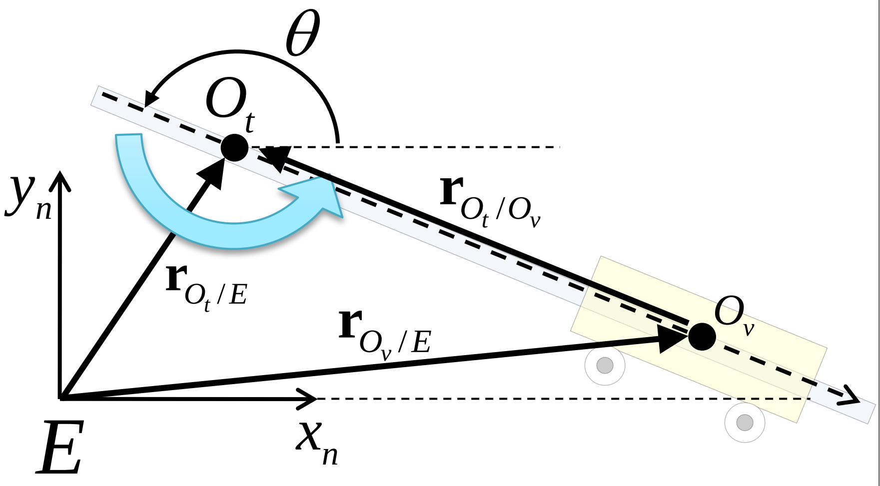
Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

$$\mathbf{r} = \begin{bmatrix} x_{O_v/E} & y_{O_v/E} & x_{O_t/E} & y_{O_t/E} & \theta \end{bmatrix}^T$$

$$\begin{bmatrix} \boxed{\mathbf{M}} & \mathbf{C_r}^T \\ \mathbf{C_r} & 0 \end{bmatrix} \begin{bmatrix} \boxed{\ddot{\mathbf{r}}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{F}^e} \\ \mathbf{F}^d \end{bmatrix}$$

$$\mathbf{F}^d = -\left(\mathbf{C_r}\dot{\mathbf{r}}\right)_{\mathbf{r}} \dot{\mathbf{r}}$$



$n-frame$: Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

Kinematical constraint

$$C_1(\mathbf{r}) = 0 \quad , where\ C_1(\mathbf{r}) = x_{O_t/E} - x_{const.}$$

$$C_2(\mathbf{r}) = 0 \quad , where\ C_2(\mathbf{r}) = y_{O_t/E} - y_{const.}$$

$$C_3(\mathbf{r}) = 0 \quad , where\ C_3(\mathbf{r}) = y_{const.} - y_{O_v/E} - (x_{const.} - x_{O_v/E}) \tan\theta$$

$$\mathbf{C}(\mathbf{r}) = \begin{bmatrix} C_1(\mathbf{r}) \\ C_2(\mathbf{r}) \\ C_3(\mathbf{r}) \end{bmatrix}$$

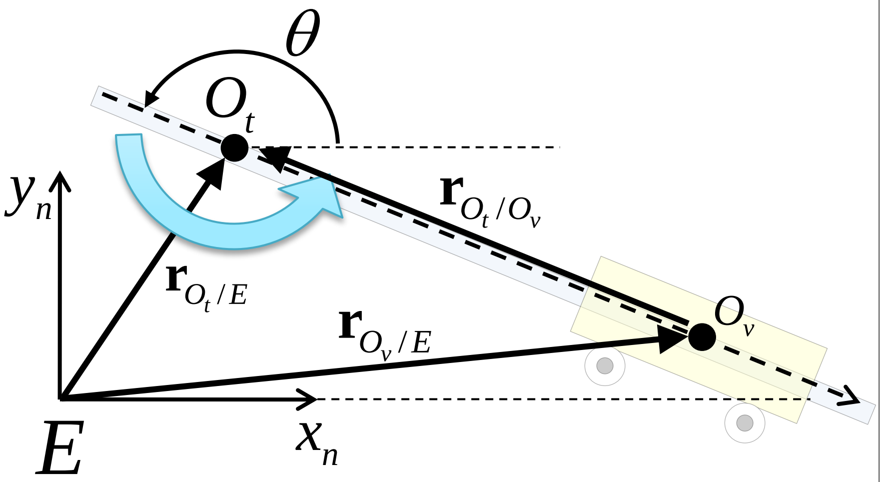
Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

$$\begin{bmatrix} \boxed{\mathbf{M}} & \boxed{\mathbf{C_r}^T} \\ \boxed{\mathbf{C_r}} & 0 \end{bmatrix} \begin{bmatrix} \boxed{\ddot{\mathbf{r}}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{F}^e} \\ \mathbf{F}^d \end{bmatrix}$$

$$\mathbf{F}^d = -\left(\mathbf{C_r}\dot{\mathbf{r}}\right)_{\mathbf{r}}\dot{\mathbf{r}}$$

$$\mathbf{r} = \begin{bmatrix} x_{O_v/E} & y_{O_v/E} & x_{O_t/E} & y_{O_t/E} & \theta \end{bmatrix}^T$$



$n-frame$: **Inertial reference frame.**

O_v : **Center of mass of the vehicle.**

O_t : **Center of mass of the track.**

$\mathbf{r}_{O_t/E}$ **is constant.**

θ: **Angle of inclination of the track.**

Kinematical constraint

$$\mathbf{C}(\mathbf{r}) = \begin{bmatrix} C_1(\mathbf{r}) \\ C_2(\mathbf{r}) \\ C_3(\mathbf{r}) \end{bmatrix}, where\ C_1(\mathbf{r}) = x_{O_t/E} - x_{const.}$$

$$C_2(\mathbf{r}) = y_{O_t/E} - y_{const.}$$

$$C_3(\mathbf{r}) = y_{const.} - y_{O_v/E} - (x_{const.} - x_{O_v/E})\tan\theta$$

$$\mathbf{C_r}(\mathbf{r}) = \begin{bmatrix} \frac{\partial C_1}{\partial x_{O_v/E}} & \frac{\partial C_1}{\partial y_{O_v/E}} & \frac{\partial C_1}{\partial x_{O_t/E}} & \frac{\partial C_1}{\partial y_{O_t/E}} & \frac{\partial C_1}{\partial \theta} \\ \frac{\partial C_2}{\partial x_{O_v/E}} & \frac{\partial C_2}{\partial y_{O_v/E}} & \frac{\partial C_2}{\partial x_{O_t/E}} & \frac{\partial C_2}{\partial y_{O_t/E}} & \frac{\partial C_2}{\partial \theta} \\ \frac{\partial C_3}{\partial x_{O_v/E}} & \frac{\partial C_3}{\partial y_{O_v/E}} & \frac{\partial C_3}{\partial x_{O_t/E}} & \frac{\partial C_3}{\partial y_{O_t/E}} & \frac{\partial C_3}{\partial \theta} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\tan\theta & -1 & 0 & 0 & -(x_{const.} - x_{O_v/E})\sec^2\theta \end{bmatrix}}$$

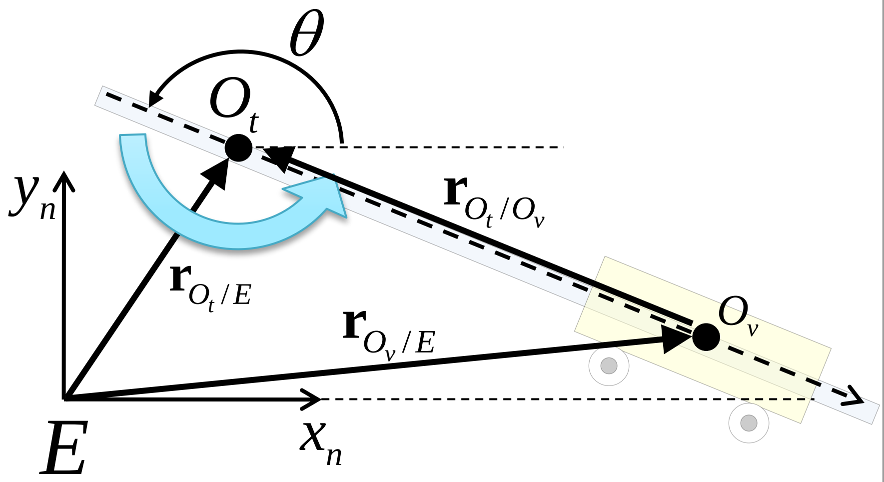
Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

$$\mathbf{r} = \begin{bmatrix} x_{O_v/E} & y_{O_v/E} & x_{O_t/E} & y_{O_t/E} & \theta \end{bmatrix}^T$$

$$\left[\begin{array}{c:c} \boxed{\mathbf{M}} & \boxed{\mathbf{C_r}^T} \\ \hdashline \boxed{\mathbf{C_r}} & 0 \end{array}\right] \left[\begin{array}{c} \boxed{\ddot{\mathbf{r}}} \\ \hdashline \boldsymbol{\lambda} \end{array}\right] = \left[\begin{array}{c} \boxed{\mathbf{F}^e} \\ \hdashline \mathbf{F}^d \end{array}\right]$$

$$\mathbf{F}^d = -(\mathbf{C_r}\dot{\mathbf{r}})_{\mathbf{r}}\dot{\mathbf{r}}$$



$n-frame$: Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

$$\mathbf{F}^d = -(\mathbf{C_r}\dot{\mathbf{r}})_{\mathbf{r}}\dot{\mathbf{r}}$$

$$\mathbf{C_r}\dot{\mathbf{r}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\tan\theta & -1 & 0 & 0 & -(x_{const.} - x_{O_v/E})\sec^2\theta \end{bmatrix} \begin{bmatrix} \dot{x}_{O_v/E} \\ \dot{y}_{O_v/E} \\ \dot{x}_{O_t/E} \\ \dot{y}_{O_t/E} \\ \dot{\theta} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_{O_t/E} \\ \dot{y}_{O_t/E} \\ -\tan\theta\cdot\dot{x}_{O_v/E} - \dot{y}_{O_v/E} - (x_{const.} - x_{O_v/E})\sec^2\theta\cdot\dot{\theta} \end{bmatrix}$$

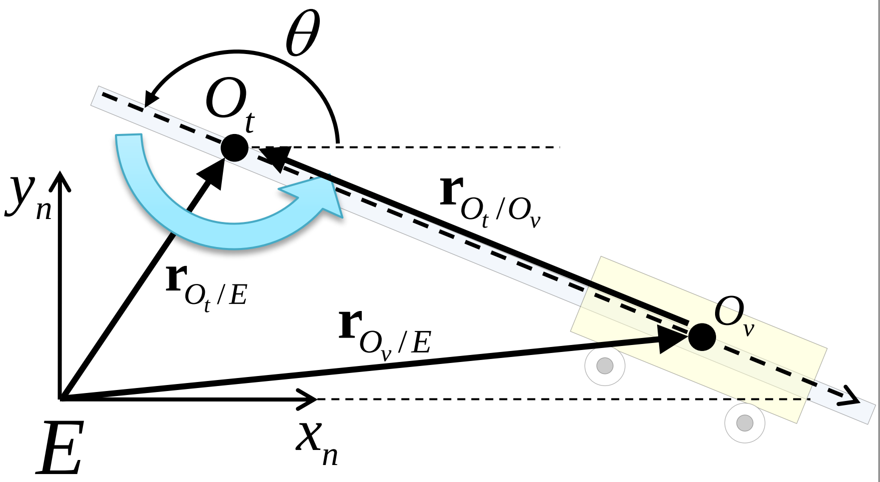
Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation

$$\mathbf{r} = \begin{bmatrix} x_{O_v/E} & y_{O_v/E} & x_{O_t/E} & y_{O_t/E} & \theta \end{bmatrix}^T$$

$$\begin{bmatrix} \boxed{\mathbf{M}} & \boxed{\mathbf{C_r}^T} \\ \boxed{\mathbf{C_r}} & 0 \end{bmatrix} \begin{bmatrix} \boxed{\ddot{\mathbf{r}}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{F}^e} \\ \boxed{\mathbf{F}^d} \end{bmatrix}$$

$$\mathbf{F}^d = -(\mathbf{C_r}\dot{\mathbf{r}})_\mathbf{r}\,\dot{\mathbf{r}}$$



$n-$ *frame* : Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

$$\mathbf{F}^d = -\boxed{(\mathbf{C_r}\dot{\mathbf{r}})_\mathbf{r}}\,\boxed{\dot{\mathbf{r}}} \qquad \dot{\mathbf{r}} = \begin{bmatrix} \dot{x}_{O_v/E} & \dot{y}_{O_v/E} & \dot{x}_{O_t/E} & \dot{y}_{O_t/E} & \dot{\theta} \end{bmatrix}^T$$

$$\mathbf{C_r}\dot{\mathbf{r}} = \begin{bmatrix} \dot{x}_{O_t/E} \\ \dot{y}_{O_t/E} \\ -\tan\theta\cdot\dot{x}_{O_v/E} - \dot{y}_{O_v/E} - (x_{const.} - x_{O_v/E})\sec^2\theta\cdot\dot{\theta} \end{bmatrix}$$

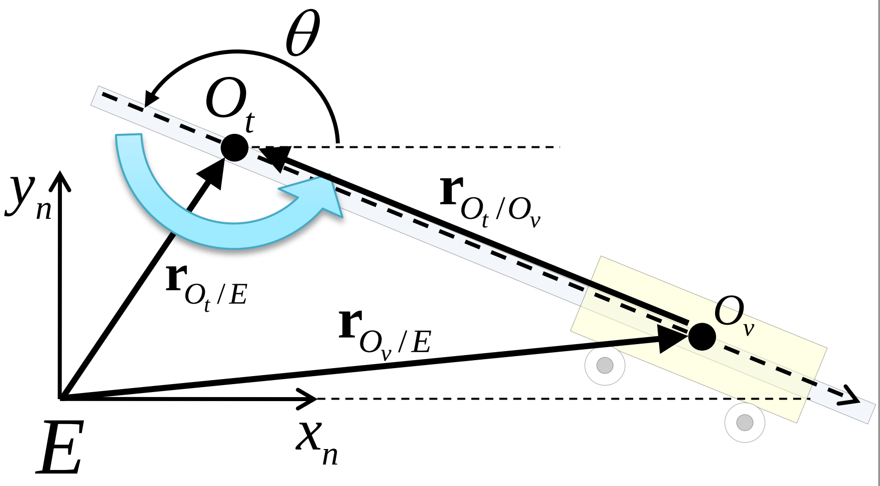
$$\Downarrow \qquad \frac{d\tan\theta}{d\theta} = \sec^2\theta, \frac{d\sec\theta}{d\theta} = \sec\theta\cdot\tan\theta$$

$$(\mathbf{C_r}\dot{\mathbf{r}})_\mathbf{r} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sec^2\theta\cdot\dot{\theta} & \text{[redacted]} & & & \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sec^2\theta\cdot\dot{\theta} & 0 & 0 & 0 & -\sec^2\theta\cdot\dot{x}_{O_v/E} - (x_{const.} - x_{O_v/E})2\sec^2\theta\cdot\tan\theta\cdot\dot{\theta} \end{bmatrix}}$$

Vehicle constrained to move along the straight track

- Derivation of equations of motion by using augmented formulation



$n-frame$: Inertial reference frame.

O_v : Center of mass of the vehicle.

O_t : Center of mass of the track.

$\mathbf{r}_{O_t/E}$ is constant.

θ: Angle of inclination of the track.

Absolute coordinate formulation (Augmented Formulation)

$$\begin{bmatrix} \mathbf{M} & \mathbf{C_r}^T \\ \mathbf{C_r} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e \\ \mathbf{F}^d \end{bmatrix}$$

Constraint: $\mathbf{C}(\mathbf{r},t)=0$

Differentiation twice $\mathbf{C_r}\ddot{\mathbf{r}}+\underbrace{\boxed{(\mathbf{C_r}\dot{\mathbf{r}})_\mathbf{r}\dot{\mathbf{r}}}}_{-\mathbf{F}^d}=0$

λ : Lagrange Multiplier

[Ahmed A. Shabana, 1992, p. 192 ~ 197]

- Given: F^e, F^d
- Find : $\ddot{\mathbf{r}}$, $\boldsymbol{\lambda}$ ($-\mathbf{C}_\mathbf{r}^T\cdot\boldsymbol{\lambda}$ is constraint force)

$$\mathbf{M}=\begin{bmatrix} m_v &&&& \\ & m_v &&& \\ && m_t && \\ &&& m_t & \\ &&&& I_t \end{bmatrix}, \ddot{\mathbf{r}}=\begin{bmatrix} \ddot{x}_{O_v/E} \\ \ddot{y}_{O_v/E} \\ \ddot{x}_{O_t/E} \\ \ddot{y}_{O_t/E} \\ \ddot{\theta} \end{bmatrix}, \mathbf{F}^e=\begin{bmatrix} F^e_{O_v,x} \\ F^e_{O_v,y} \\ F^e_{O_t,x} \\ F^e_{O_t,y} \\ M^e_{O_t} \end{bmatrix}$$

$$\mathbf{C_r}(\mathbf{r})=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\tan\theta & -1 & 0 & 0 & -(x_{const.}-x_{O_v/E})\sec^2\theta \end{bmatrix}$$

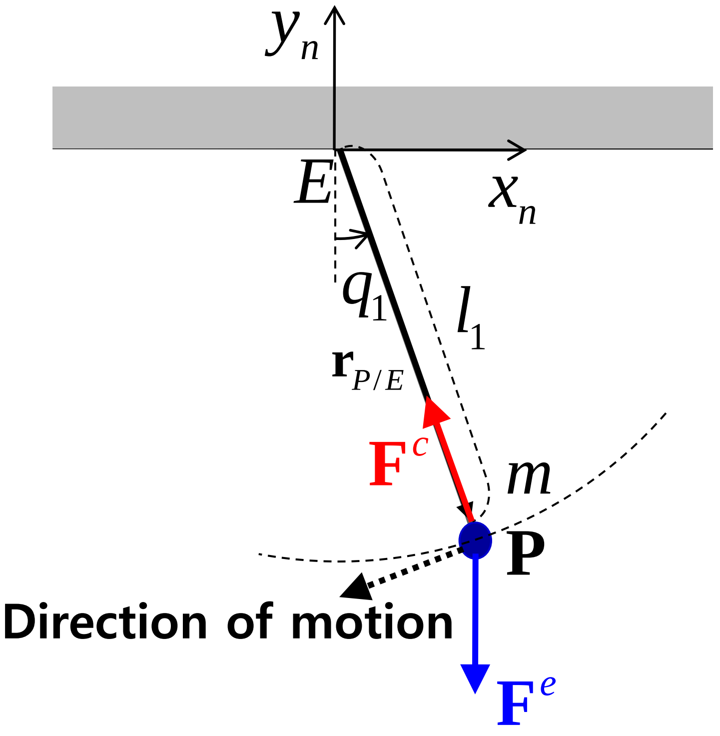
$$(\mathbf{C_r}\dot{\mathbf{r}})_\mathbf{r}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sec^2\theta\cdot\dot{\theta} & 0 & 0 & 0 & -\sec^2\theta\cdot\dot{x}_{O_v/E}-(x_{const.}-x_{O_v/E})2\sec^2\theta\cdot\tan\theta\cdot\dot{\theta} \end{bmatrix}$$

$$\dot{\mathbf{r}}=\begin{bmatrix} \dot{x}_{O_v/E} & \dot{y}_{O_v/E} & \dot{x}_{O_t/E} & \dot{y}_{O_t/E} & \dot{\theta} \end{bmatrix}^T$$

3.4 Relative coordinate formulation
Example of Pendulum

Example of the pendulum

2-D Pendulum



$n-frame$: Inertial reference frame

m : mass of particle P

$\mathbf{r}_{P/E}$: Position vector of the particle P

$$\mathbf{M}\ddot{\mathbf{r}}_{P/E} = \sum \mathbf{F}, \text{ where } \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$$\sum \mathbf{F} = \mathbf{F}^e + \mathbf{F}^c$$

Equations of motions

$$\mathbf{M}\ddot{\mathbf{r}}_{P/E} = \mathbf{F}^e + \mathbf{F}^c$$

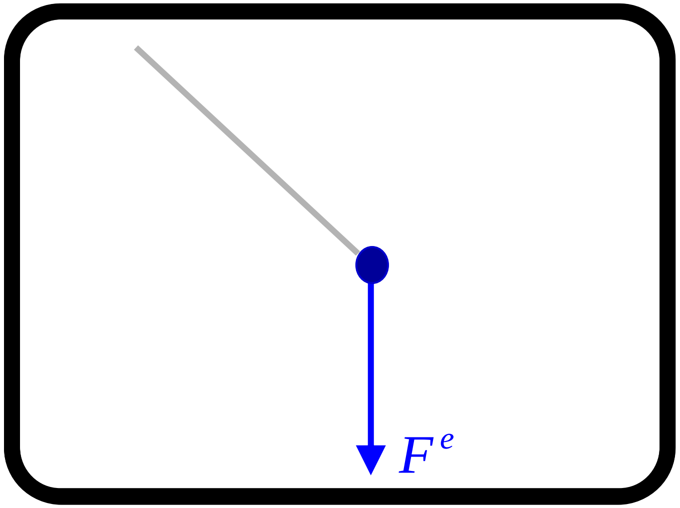
$\mathbf{F}^e$: Known external force

$\mathbf{F}^c$: Unknown constraint reaction force

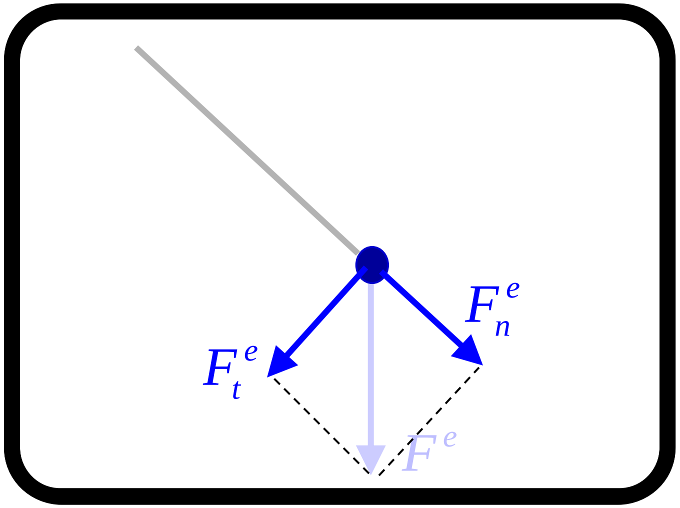
To solve the equations of motion, we should know the constraint reaction force.

Example of the pendulum
- Constraint reaction force

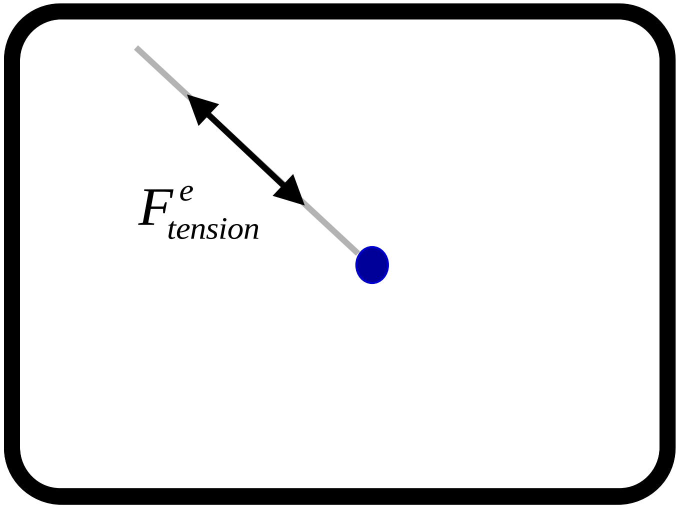
① External Force $\mathbf{F}^e$:
Gravitational force



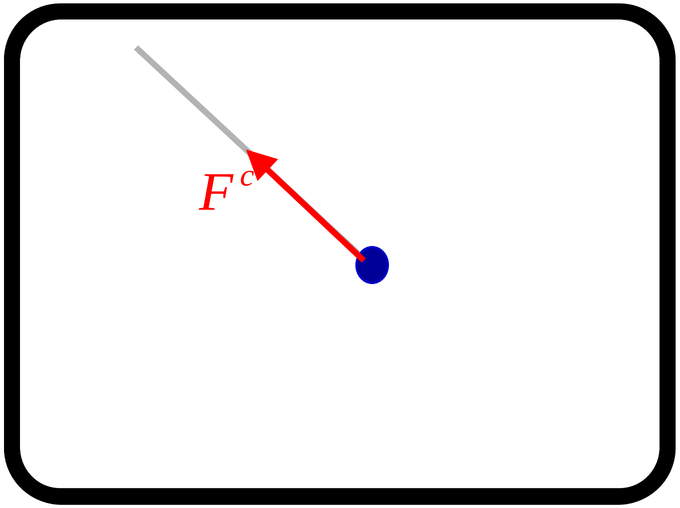
② $\mathbf{F}^e$ is resolved into a normal vector and a tangential vector



③ $\mathbf{F}_n^e$ causes
a tension of the wire $\mathbf{F}_{tension}^e$

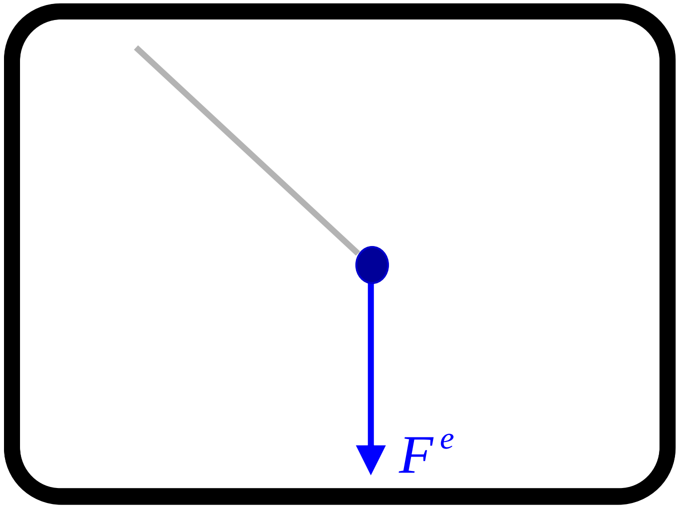


④$\mathbf{F}_{tension}^e$ causes
the constrained reaction force $\mathbf{F}^c$.

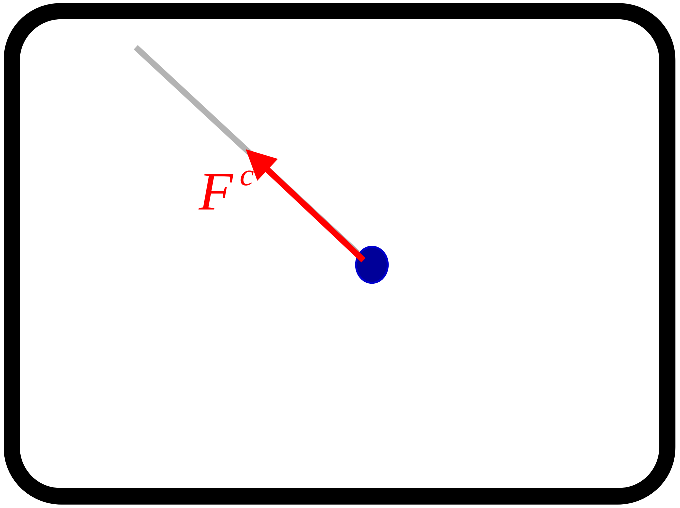


Example of the pendulum
- Free body diagram of the particle

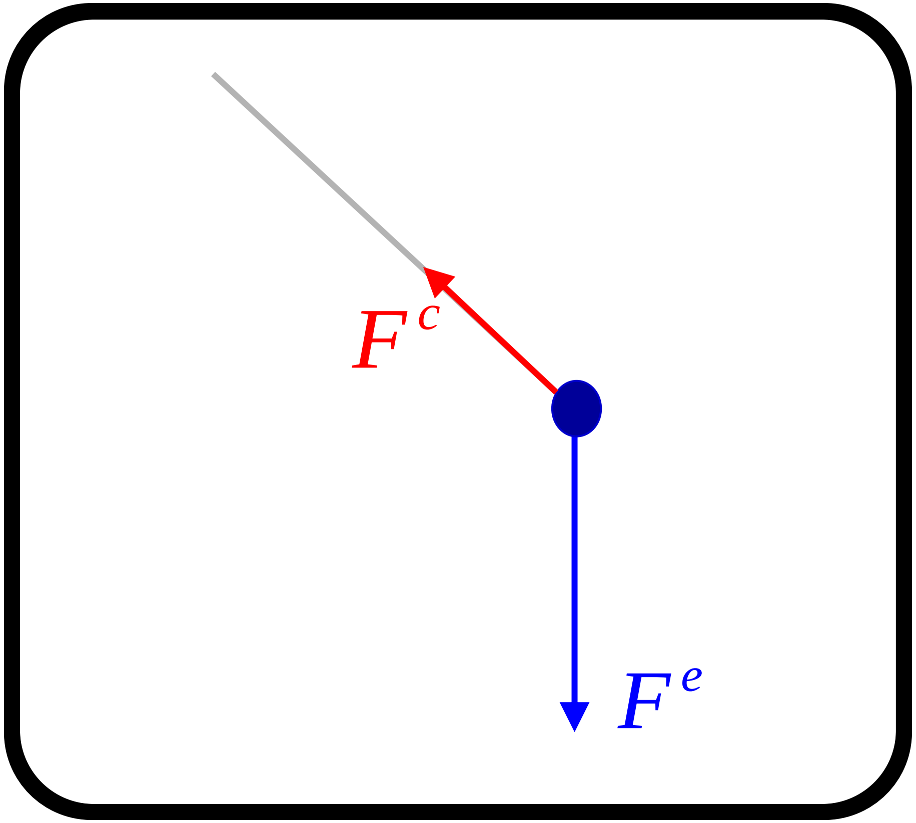
① External Force $\mathbf{F}^e$: Gravitational force



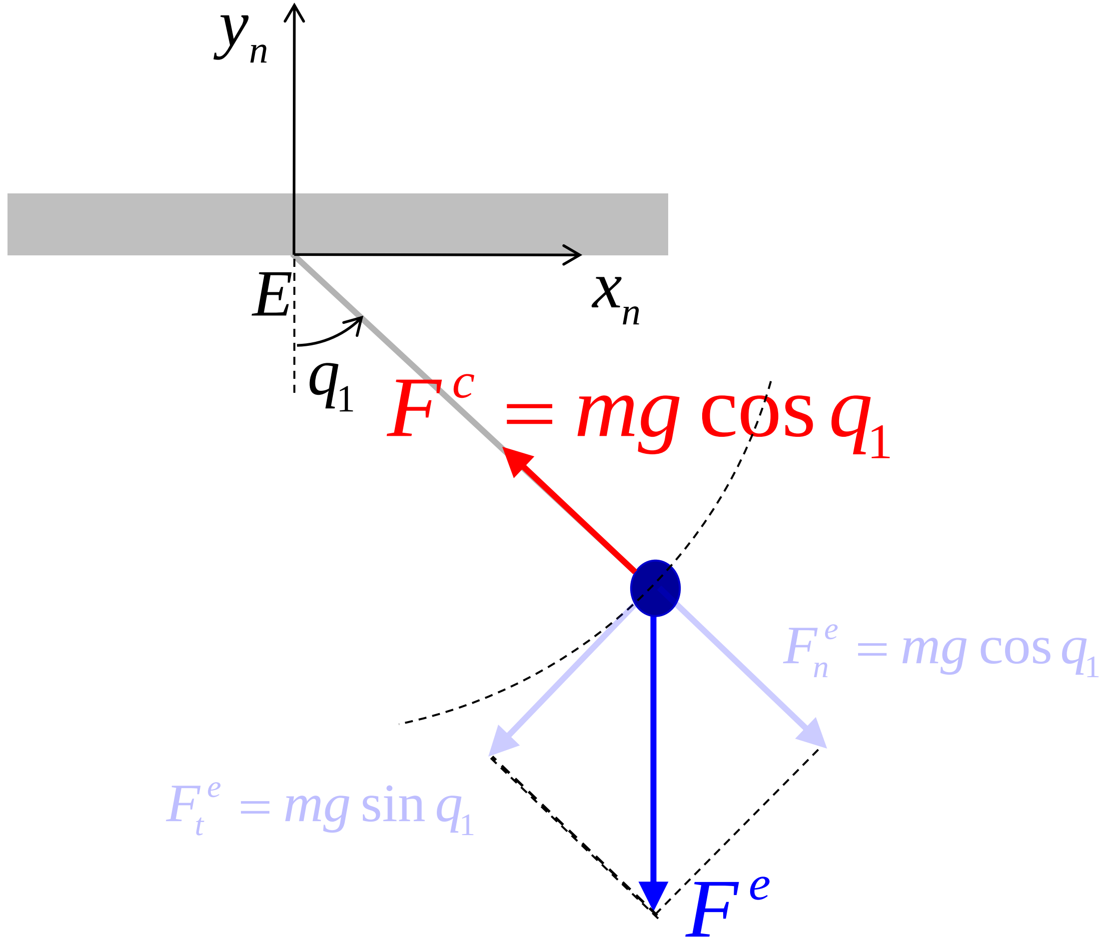
② $\mathbf{F}^c$ is the constrained reaction force.



③ Free body diagram of the vehicle.



Example of the pendulum
- Free body diagram of the particle

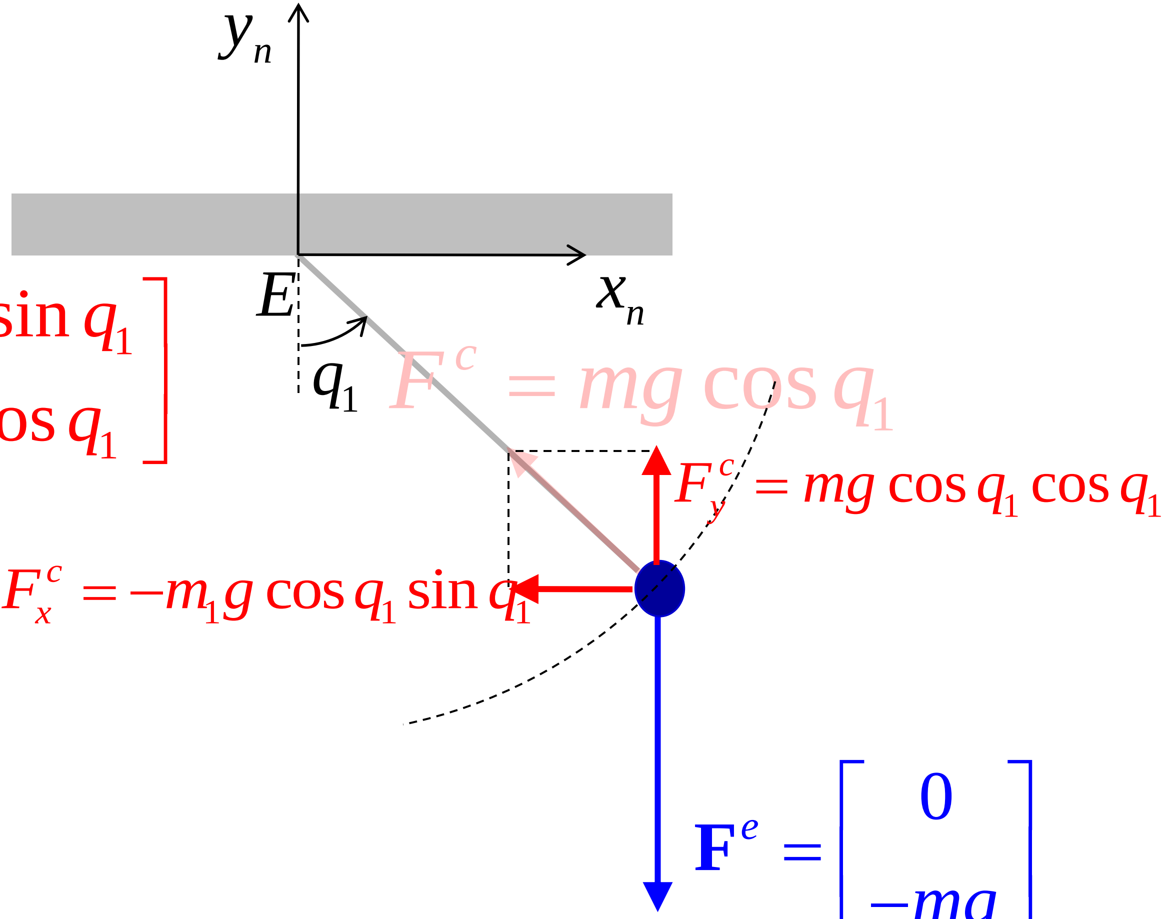


$$\mathbf{M}\boxed{\ddot{\mathbf{r}}_{P/E}} = \mathbf{F}^e + \mathbf{F}^c$$

Since the position vector is defined in n-frame,
$\mathbf{F}^e$ and $\mathbf{F}^c$ are to be decomposed in terms of unit vector along the coordinates axis

Example of the pendulum
- Free body diagram of the particle

$$\mathbf{F}^c = \begin{bmatrix} -mg\cos q_1 \sin q_1 \\ mg\cos q_1 \cos q_1 \end{bmatrix}$$



$$F_y^c = mg\cos q_1 \cos q_1$$

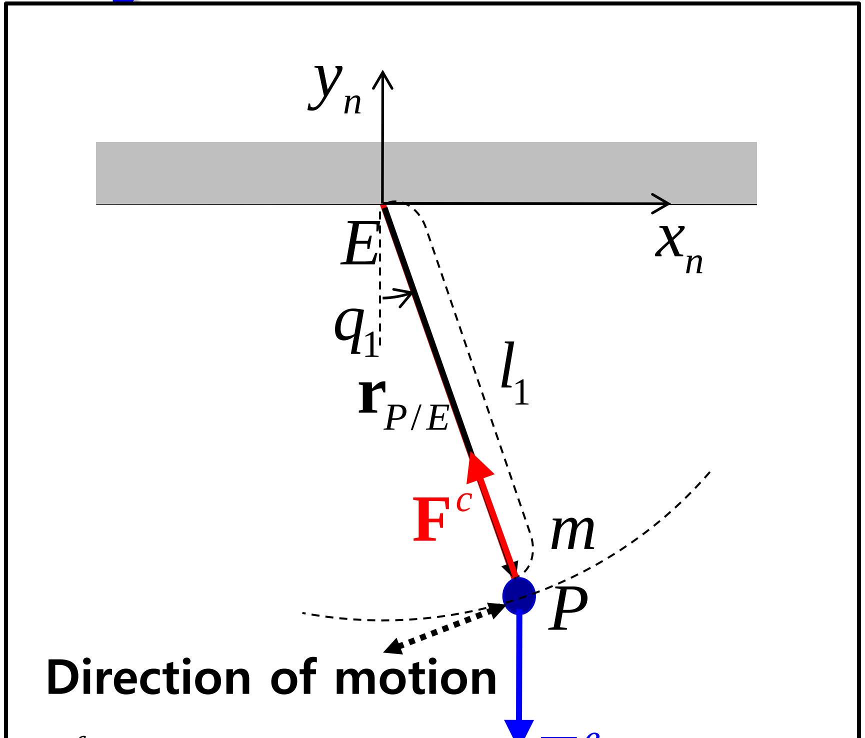
$$F_x^c = -m_1 g\cos q_1 \sin q_1$$

$$\mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$$\mathbf{M}\ddot{\mathbf{r}}_{P/E} = \mathbf{F}^e + \mathbf{F}^c$$

Since the position vector is defined in n-frame, $\mathbf{F}^e$ and $\mathbf{F}^c$ are to be decomposed in terms of unit vector along the coordinates axis

Example of the pendulum - Dynamics



$n-frame$: Inertial renerence frame

m : mass of particle P

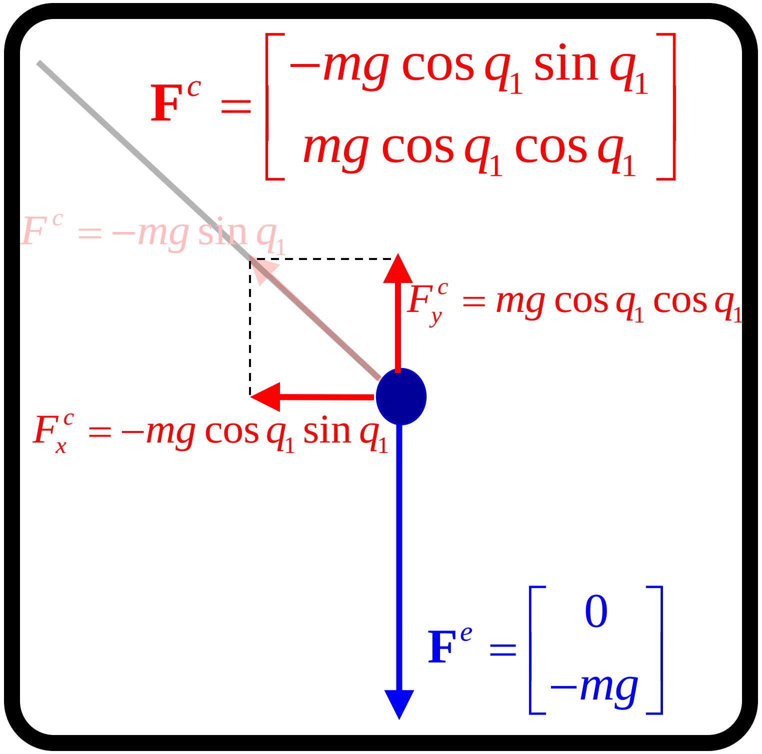
$\mathbf{r}_{P/E}$: Position vector of the particle P

$$\mathbf{M}\ddot{\mathbf{r}}_{P/E} = \mathbf{F}^e + \mathbf{F}^c \text{, where } \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$\mathbf{F}^e$: Known external force

$\mathbf{F}^c$: Unknown constraint reaction force

To solve the equations of motion, we should know the constraint reaction force.



$$\mathbf{F}^c = \begin{bmatrix} -mg\cos q_1 \sin q_1 \\ mg\cos q_1 \cos q_1 \end{bmatrix}$$

$$F^c = -mg\sin q_1$$

$$F_y^c = mg\cos q_1 \cos q_1$$

$$F_x^c = -mg\cos q_1 \sin q_1$$

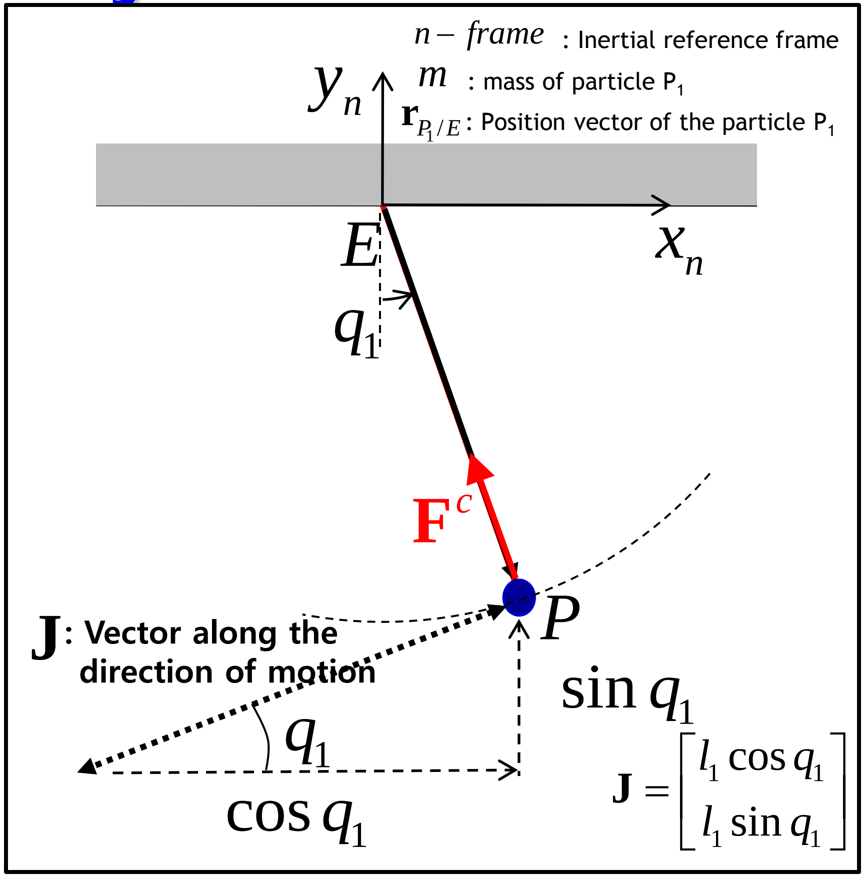
$$\mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$$\mathbf{M}\ddot{\mathbf{r}}_{P/E} = \mathbf{F}^e + \mathbf{F}^c$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_P \\ \ddot{y}_P \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} -mg\cos q_1 \sin q_1 \\ mg\cos q_1 \cos q_1 \end{bmatrix}$$

3 variables (x_P, y_P, θ), 2 equations

Example of the pendulum - Dynamics



$$\mathbf{M}\ddot{\mathbf{r}}_{P/E}=\mathbf{F}^e+\mathbf{F}^c \quad ,where\ \mathbf{M}=\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix},\mathbf{F}^e=\begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$\mathbf{F}^e$: Known external force

$\mathbf{F}^c$: Unknown constraint reaction force

The constraint reaction force $\mathbf{F}^c$ is perpendicular to the vector $\mathbf{u}_1$

$$\mathbf{J}\cdot\mathbf{M}\ddot{\mathbf{r}}_{P/E}=\mathbf{J}\cdot\mathbf{F}^e+\cancel{\mathbf{J}\cdot\mathbf{F}^c}$$

$$\boxed{\mathbf{A}\cdot\mathbf{B}=\mathbf{A}^T\mathbf{B}}$$

$$\mathbf{J}^T\mathbf{M}\ddot{\mathbf{r}}_{P/E}=\mathbf{J}^T\mathbf{F}^e \quad \text{- ①}$$

3 variables (x_P, y_P, q_1), 1 equation

From kinematic relation

$$x_P=l_1\sin q_1,\quad y_P=-l_1\cos q_1 \quad \text{- ②}$$

3 variables (x_P, y_P, q_1), 2 equations

3 variables (x_P, y_P, q_1), 3 equations

Example of the pendulum - Dynamics

$$\boxed{\mathbf{J}^T\mathbf{M}\ddot{\mathbf{r}}_{P/E} = \mathbf{J}^T\mathbf{F}^e \text{ - ①}}$$

From kinematic relation

$$x_P = l_1 \sin q_1, \quad y_P = -l_1 \cos q_1$$

⇩ **Time derivative**

$$\dot{x}_P = l_1 \cos q_1 \cdot \dot{q}_1$$

$$\dot{y}_P = l_1 \sin q_1 \cdot \dot{q}_1$$

⇩ **Matrix Representation**

$$\underbrace{\boxed{\begin{bmatrix} \dot{x}_P \\ \dot{y}_P \end{bmatrix}}}_{\dot{\mathbf{r}}_{P/E}} = \underbrace{\boxed{\begin{bmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \end{bmatrix}}}_{\mathbf{J}} \dot{q}_1 \quad \Rightarrow \quad \dot{\mathbf{r}}_{P/E} = \mathbf{J}\dot{q}_1 \quad , where\ \mathbf{J} = \begin{bmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \end{bmatrix}$$

⇩ **Time derivative**

$$\boxed{\ddot{\mathbf{r}}_{P/E} = \mathbf{J}\ddot{q}_1 + \dot{\mathbf{J}}\dot{q}_1 \text{ - ②}}$$

Example of the pendulum - Dynamics

$$\mathbf{J}^T \mathbf{M} \ddot{\mathbf{r}}_{P/E} = \mathbf{J}^T \mathbf{F}^e \text{ - ①}$$

$$\ddot{\mathbf{r}}_{P/E} = \mathbf{J}\ddot{q}_1 + \dot{\mathbf{J}}\dot{q}_1 \text{ - ②}$$

$$, where \ \mathbf{J} = \begin{bmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \end{bmatrix} \Rightarrow \dot{\mathbf{J}} = \begin{bmatrix} -l_1 \sin q_1 \cdot \dot{q}_1 \\ l_1 \cos q_1 \cdot \dot{q}_1 \end{bmatrix}$$

Substituting Eq. ② into Eq. ①

$$\mathbf{J}^T \mathbf{M} (\mathbf{J}\ddot{q}_1 + \dot{\mathbf{J}}\dot{q}_1) = \mathbf{J}^T \mathbf{F}^e$$

$$\boxed{\mathbf{J}^T \mathbf{M} \mathbf{J}\ddot{q}_1 + \mathbf{J}^T \mathbf{M} \dot{\mathbf{J}}\dot{q}_1 = \mathbf{J}^T \mathbf{F}^e}$$

Example of the pendulum
- Dynamics

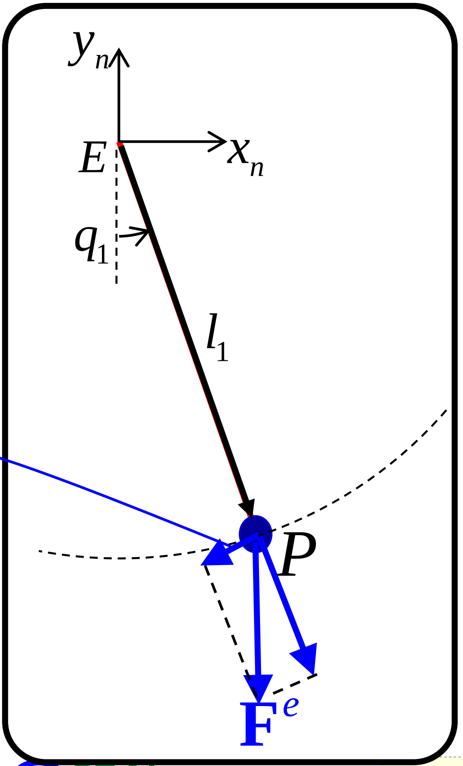
$$\boxed{\mathbf{J}^T\mathbf{M}\mathbf{J}\ddot{q}_1 + \mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{q}_1 = \mathbf{J}^T\mathbf{F}^e}$$

$$\mathbf{J} = \begin{bmatrix} l_1\cos q_1 \\ l_1\sin q_1 \end{bmatrix},\ \dot{\mathbf{J}} = \begin{bmatrix} -l_1\sin q_1\cdot\dot{q}_1 \\ l_1\cos q_1\cdot\dot{q}_1 \end{bmatrix},\ \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix},\ \mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$$\mathbf{J}^T\mathbf{M}\mathbf{J}\ddot{q}_1 = \begin{bmatrix} l_1\cos q_1 & l_1\sin q_1 \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} l_1\cos q_1 \\ l_1\sin q_1 \end{bmatrix}\ddot{q}_1 = ml_1^2\ddot{q}_1$$

$$\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{q}_1 = \begin{bmatrix} l_1\cos q_1 & l_1\sin q_1 \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} -l_1\sin q_1\cdot\dot{q}_1 \\ l_1\cos q_1\cdot\dot{q}_1 \end{bmatrix}\dot{q}_1 = 0$$

$$\mathbf{J}^T\mathbf{F}^e = \begin{bmatrix} l_1\cos q_1 & l_1\sin q_1 \end{bmatrix}\begin{bmatrix} 0 \\ -mg \end{bmatrix} = -l_1 mg\sin q_1$$



$$\boxed{ml_1^2}\boxed{\ddot{q}_1} = \boxed{-l_1 mg\sin q_1}$$

Mass moment of inertia

Angular acceleration

Moment induced by gravity

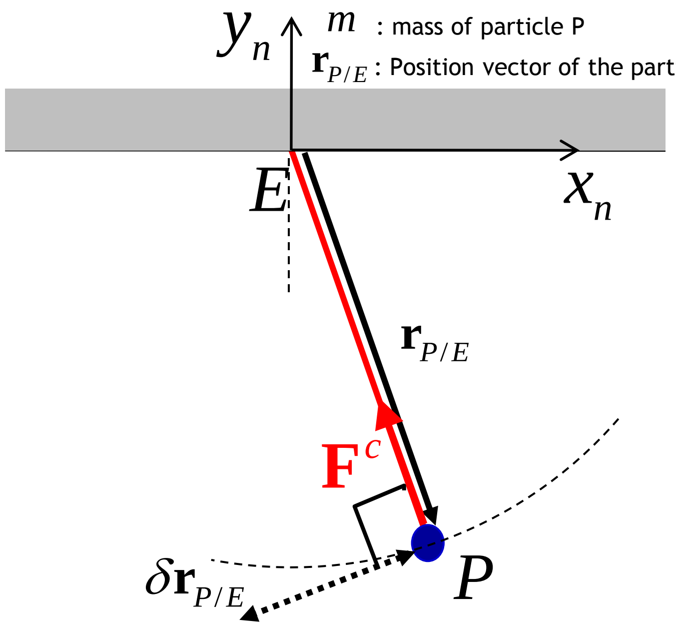
3.5 Absolute coordinate formulation
Example of Pendulum

Example of the pendulum
- Augmented Formulation

$n-frame$: Inertial reference frame

m : mass of particle P

$\mathbf{r}_{P/E}$: Position vector of the particle P



$$\delta\mathbf{r}_{P/E} = \begin{bmatrix} \delta x_{P/E} \\ \delta y_{P/E} \end{bmatrix}$$ **: Virtual displacement**

$$\mathbf{M}\ddot{\mathbf{r}}_{P/E} = \mathbf{F}^e + \mathbf{F}^c \quad , where\ \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$\mathbf{F}^e$: Known external force

$\mathbf{F}^c$: Unknown constraint reaction force

The constraint reaction force $\mathbf{F}^c$ is perpendicular to the vector $\delta\mathbf{r}_{P/E}$

$$\delta\mathbf{r}_{P/E} \cdot \mathbf{M}\ddot{\mathbf{r}}_{P/E} = \delta\mathbf{r}_{P/E} \cdot \mathbf{F}^e + \cancel{\delta\mathbf{r}_{P/E}} \cdot \mathbf{F}^c$$

$$\boxed{\mathbf{A}\cdot\mathbf{B} = \mathbf{A}^T\mathbf{B}}$$

$$\delta\mathbf{r}_{P/E}{}^T\mathbf{M}\ddot{\mathbf{r}}_{P/E} = \delta\mathbf{r}_{P/E}{}^T\mathbf{F}^e \text{ - ①}$$

2 variables $(x_{P/E}, y_{P/E})$ 1 equation

From kinematic relation

$$x_{P/E}^2 + y_{P/E}^2 = l_1^2 \text{ ②}$$

2 variables $(x_{P/E}, y_{P/E})$, 1 equations

2 variables$(x_{P/E}, y_{P/E})$, 2 equations

Example of the pendulum - Augmented Formulation

$$\delta \mathbf{r}_{P/E}{}^{T} \mathbf{M}\ddot{\mathbf{r}}_{P/E} = \delta \mathbf{r}_{P/E}{}^{T} \mathbf{F}^{e}$$ - ①

$$\delta \mathbf{r}_{P/E}{}^{T} \left(\mathbf{M}\ddot{\mathbf{r}}_{P/E} - \mathbf{F}^{e} \right) = 0$$

$$\begin{bmatrix} \delta x_{P/E} & \delta y_{P/E} \end{bmatrix} \begin{bmatrix} m \cdot \ddot{x}_{P/E} - F_x^e \\ m \cdot \ddot{y}_{P/E} - F_y^e \end{bmatrix} = 0$$

$$x_{P/E}^2 + y_{P/E}^2 = l_1^2$$ - ②

$$x_{P/E}^2 + y_{P/E}^2 - l_1^2 = 0$$

$$C\left(x_{P/E}, y_{P/E} \right) = 0$$

Lagrange Multiplier Method

$$\begin{bmatrix}\delta x_{P/E} & \delta y_{P/E}\end{bmatrix}\begin{bmatrix} m\ \ddot{x}_{P/E} - F_x^e \\ m\ \ddot{y}_{P/E} - F_y^e \end{bmatrix} = 0$$

$$C(x_{P/E}, y_{P/E}) = x_{P/E}{}^2 + y_{P/E}{}^2 - l^2 = 0$$

$$\delta x_{P/E}\underline{\left(m\ \ddot{x}_{P/E} - F_x^e\right)}\ \neq 0$$

$$+\delta y_{P/E}\underline{\left(m\ \ddot{y}_{P/E} - F_y^e\right)} = 0$$

$$\neq 0$$

✓If $\delta x_{P/E}, \delta y_{P/E}$ are independent.

✓However, the variables are dependent.

✓ Total derivative of $C\left(x_{P/E},\ y_{P/E}\right)$

$$\delta C\left(x_{P/E},\ y_{P/E}\right) = \frac{\partial C\left(x_{P/E},\ y_{P/E}\right)}{\partial\ x_{P/E}}\delta x_{P/E} + \frac{\partial C\left(x_{P/E},\ y_{P/E}\right)}{\partial\ y_{P/E}}\delta y_{P/E} = 0$$

✓ To eliminate the dependent variable among $\delta x_{P/E}, \delta y_{P/E}$, we will use Lagrange multiplier λ

$$\delta x_{P/E}\left(m\ \ddot{x}_{P/E} - F_x^e\right) + \delta y_{P/E}\left(m\ \ddot{y}_{P/E} - F_y^e\right)$$

$$+\lambda\left(\frac{\partial C\left(x_{P/E},\ y_{P/E}\right)}{\partial\ x_{P/E}}\delta x_{P/E} + \frac{\partial C\left(x_{P/E},\ y_{P/E}\right)}{\partial\ y_{P/E}}\delta y_{P/E}\right) = 0$$

Lagrange Multiplier Method

- ✓ To eliminate the dependent variable among $\delta x_{P/E}, \delta y_{P/E}$, we will use Lagrange multiplier λ

$$\delta x_{P/E}\left(m\,\ddot{x}_{P/E}-F_x^e\right)+\delta y_{P/E}\left(m\,\ddot{y}_{P/E}-F_y^e\right)+\lambda\left(\frac{\partial C(x_{P/E},\ y_{P/E})}{\partial\ x_{P/E}}\delta x_{P/E}+\frac{\partial C(x_{P/E},\ y_{P/E})}{\partial\ y_{P/E}}\delta y_{P/E}\right)=0$$

$$\delta x_{P/E}\left(m\,\ddot{x}_{P/E}-F_x^e+\lambda\frac{\partial C(x_{P/E},\ y_{P/E})}{\partial\ x_{P/E}}\right)+\delta y_{P/E}\underbrace{\left(m\,\ddot{y}_{P/E}-F_y^e+\lambda\frac{\partial C(x_{P/E},\ y_{P/E})}{\partial\ y_{P/E}}\right)}_{=0}=0$$

- ✓If $\delta y_{P/E}$ is dependent variable, we will choose appropriate λ to eliminate $\delta y_{P/E}$.

$$m\,\ddot{y}_{P/E}-F_y^e+\lambda\frac{\partial C(x_{P/E},\ y_{P/E})}{\partial\ y_{P/E}}=0$$

- ✓Because $\delta x_{P/E}$ is independent variable.

$$\delta x_{P/E}\left(m\,\ddot{x}_{P/E}-F_x^e+\lambda\frac{\partial C(x_{P/E},y_{P/E})}{\partial x_{P/E}}\right)=0$$

$$m\,\ddot{x}_{P/E}-F_x^e+\lambda\frac{\partial C(x_{P/E},y_{P/E})}{\partial x_{P/E}}=0$$

$$m\,\ddot{y}_{P/E}-F_y^e+\lambda\frac{\partial C(x_{P/E},\ y_{P/E})}{\partial\ y_{P/E}}=0$$

$$m\,\ddot{x}_{P/E}-F_x^e+\lambda\frac{\partial C(x_{P/E},y_{P/E})}{\partial x_{P/E}}=0$$

- ✓Kinematic constraint

$$C(x_{P/E},y_{P/E})=x_{P/E}{}^2+y_{P/E}{}^2-l_1^2=0$$

Example of the pendulum
- Augmented Formulation

$$\delta\mathbf{r}_{P/E}{}^{T}\mathbf{M}\ddot{\mathbf{r}}_{P/E}=\delta\mathbf{r}_{P/E}{}^{T}\mathbf{F}^{e}$$ - ①

$$\delta\mathbf{r}_{P/E}{}^{T}\left(\mathbf{M}\ddot{\mathbf{r}}_{P/E}-\mathbf{F}^{e}\right)=0$$

$$\begin{bmatrix}\delta x_{P/E} & \delta y_{P/E}\end{bmatrix}\begin{bmatrix} m\cdot\ddot{x}_{P/E}-F_x^e \\ m\cdot\ddot{y}_{P/E}-F_y^e\end{bmatrix}=0$$

$$x_{P/E}^2+y_{P/E}^2=l_1^2$$ - ②

$$x_{P/E}^2+y_{P/E}^2-l_1^2=0$$

$$C\left(x_{P/E},y_{P/E}\right)=0$$

Lagrange multiplier method

$$m\cdot\boxed{\ddot{x}_{P/E}}-F_x^e+\boxed{\lambda}\frac{\partial C\left(x_{P/E},\ y_{P/E}\right)}{\partial\ x_{P/E}}=0$$

$$m\cdot\boxed{\ddot{y}_{P/E}}-F_y^e+\boxed{\lambda}\frac{\partial C\left(x_{P/E},\ y_{P/E}\right)}{\partial\ y_{P/E}}=0$$

$$C\left(x_{P/E},\ y_{P/E}\right)=\boxed{x_{P/E}}^2+\boxed{y_{P/E}}^2-l_1^2=0$$

➡ **3 variables** $x_{P/E},\ y_{P/E},\lambda$, **3 equations**

Example of the pendulum
- Augmented Formulation

$$m\cdot \ddot{x}_{P/E} - F_x^e + \lambda \frac{\partial C\left(x_{P/E},\ y_{P/E} \right)}{\partial\ \ x_{P/E}} = 0$$

$$m\cdot \ddot{y}_{P/E} - F_y^e + \lambda \frac{\partial C\left(x_{P/E},\ y_{P/E} \right)}{\partial\ \ y_{P/E}} = 0$$

$$C\left(x_{P/E},\ y_{P/E} \right) = x_{P/E}^2 + y_{P/E}^2 - l_1^2 = 0$$

Matrix representation

$$\underbrace{\boxed{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}}_{\mathbf{M}} \underbrace{\boxed{\begin{bmatrix} \ddot{x}_{P/E} \\ \ddot{y}_{P/E} \end{bmatrix}}}_{\ddot{\mathbf{r}}_{P/E}} - \underbrace{\boxed{\begin{bmatrix} F_x^e \\ F_y^e \end{bmatrix}}}_{\mathbf{F}^e} + \lambda \underbrace{\boxed{\begin{bmatrix} \dfrac{\partial C(x_{P/E}, y_{P/E})}{\partial x_{P/E}} \\ \dfrac{\partial C(x_{P/E}, y_{P/E})}{\partial x_{P/E}} \end{bmatrix}}}_{\mathbf{C_r}^T} = 0$$

$$\mathbf{M}\ddot{\mathbf{r}}_{P/E} - \mathbf{F}^e + \mathbf{C_r}^T \lambda = 0 \quad ,where\ \mathbf{r}_{P/E} = \begin{bmatrix} x_{P/E} \\ y_{P/E} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} F_x^e \\ F_y^e \end{bmatrix}, \mathbf{C_r} = \begin{bmatrix} \dfrac{\partial C(x_{P/E}, y_{P/E})}{\partial\, x_{P/E}} & \dfrac{\partial C(x_{P/E}, y_{P/E})}{\partial\, y_{P/E}} \end{bmatrix}$$

Example of the pendulum
- Augmented Formulation

$$m\cdot \enclose{circle}{\ddot{x}_{P/E}} - F_x^e + \enclose{circle}{\lambda} \frac{\partial C(\ x_{P/E},\ y_{P/E})}{\partial\ \ x_{P/E}} = 0$$

$$m\cdot \enclose{circle}{\ddot{y}_{P/E}} - F_y^e + \enclose{circle}{\lambda} \frac{\partial C(\ x_{P/E},\ y_{P/E})}{\partial\ \ y_{P/E}} = 0$$

Matrix representation

$$\mathbf{M}\,\ddot{\mathbf{r}}_{P/E} - \mathbf{F}^e + \mathbf{C_r}^T\,\lambda = 0$$

$$,where\ \mathbf{r}_{P/E} = \begin{bmatrix} x_{P/E} \\ y_{P/E} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} F_x^e \\ F_y^e \end{bmatrix}$$

$$,\mathbf{C_r} = \begin{bmatrix} \dfrac{\partial C(x_{P/E}, y_{P/E})}{\partial\, x_{P/E}} & \dfrac{\partial C(x_{P/E}, y_{P/E})}{\partial\, y_{P/E}} \end{bmatrix}$$

$$C(\ x_{P/E},\ \ y_{P/E}) = \enclose{circle}{x_{P/E}}^2 + \enclose{circle}{y_{P/E}}^2 - l_1^2 = 0$$

$$x_{P/E} = x_{P/E}(t), y_{P/E} = y_{P/E}(t)$$

Suppose that the constraint is the function of time t.

$$C\left(x_{P/E}(t), y_{P/E}(t); t\right)$$

Time derivative (Chain rule)

$$\frac{dC\left(x_{P/E}(t), y_{P/E}(t); t\right)}{dt} =$$

$$\frac{\partial C\left(x_{P/E}(t), y_{P/E}(t); t\right)}{\partial x_{P/E}} \frac{dx_{P/E}(t)}{dt} + \frac{\partial C\left(x_{P/E}(t), y_{P/E}(t); t\right)}{\partial y_{P/E}} \frac{dy_{P/E}(t)}{dt} + \frac{\partial C\left(x_{P/E}(t), y_{P/E}(t); t\right)}{\partial t}$$

cf) total derivative of $C\left(x_{P/E}(t), y_{P/E}(t); t\right)$

$$dC\left(x_{P/E}(t), y_{P/E}(t); t\right) = \frac{\partial C\left(x_{P/E}(t), y_{P/E}(t); t\right)}{\partial x_{P/E}} dx_{P/E}(t) + \frac{\partial C\left(x_{P/E}(t), y_{P/E}(t); t\right)}{\partial y_{P/E}} dy_{P/E}(t) + \frac{\partial C\left(x_{P/E}(t), y_{P/E}(t); t\right)}{\partial t} dt$$

Example of the pendulum
- Augmented Formulation

$$m\cdot \ddot{x}_{P/E} - F_x^e + \lambda \frac{\partial C(x_{P/E},\ y_{P/E})}{\partial\ x_{P/E}} = 0$$

$$m\cdot \ddot{y}_{P/E} - F_y^e + \lambda \frac{\partial C(x_{P/E},\ y_{P/E})}{\partial\ y_{P/E}} = 0$$

Matrix representation

$$\boxed{\begin{gathered}\mathbf{M}\ddot{\mathbf{r}}_{P/E} - \mathbf{F}^e + \mathbf{C_r}^T\lambda = 0 \\ ,where\ \mathbf{r}_{P/E} = \begin{bmatrix} x_{P/E} \\ y_{P/E}\end{bmatrix}, \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m\end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} F_x^e \\ F_y^e\end{bmatrix} \\ ,\mathbf{C_r} = \begin{bmatrix} \dfrac{\partial C(x_{P/E}, y_{P/E})}{\partial x_{P/E}} & \dfrac{\partial C(x_{P/E}, y_{P/E})}{\partial y_{P/E}}\end{bmatrix}\end{gathered}}$$

$$C(\ x_{P/E},\ y_{P/E}) = x_{P/E}^2 + y_{P/E}^2 - l_1^2 = 0$$

Time derivative (Chain rule)

$$\frac{dC(x_{P/E}(t), y_{P/E}(t);t)}{dt} = \frac{\partial C(x_{P/E}(t), y_{P/E}(t);t)}{\partial x_{P/E}}\frac{dx_{P/E}(t)}{dt} + \frac{\partial C(x_{P/E}(t), y_{P/E}(t);t)}{\partial y_{P/E}}\frac{dy_{P/E}(t)}{dt} + \frac{\partial C(x_{P/E}(t), y_{P/E}(t);t)}{\partial t}$$

$$\frac{dC(x_{P/E}(t), y_{P/E}(t);t)}{dt} = \underbrace{\boxed{\begin{bmatrix}\dfrac{\partial C(x_{P/E}(t), y_{P/E}(t);t)}{\partial x_{P/E}} & \dfrac{\partial C(x_{P/E}(t), y_{P/E}(t);t)}{\partial y_{P/E}}\end{bmatrix}}}_{\mathbf{C_r}} \underbrace{\boxed{\begin{bmatrix}\dfrac{dx_{P/E}(t)}{dt} \\ \dfrac{dy_{P/E}(t)}{dt}\end{bmatrix}}}_{\dot{\mathbf{r}}_{P/E}} + \underbrace{\boxed{\dfrac{\partial C(x_{P/E}(t), y_{P/E}(t);t)}{\partial t}}}_{\mathbf{C}_t}$$

$$\frac{dC}{dt} = \mathbf{C_r}\dot{\mathbf{r}}_{P/E} + \mathbf{C}_t\ \ ,where\ \mathbf{C_r} = \begin{bmatrix}\dfrac{\partial C}{\partial x_{P/E}} & \dfrac{\partial C}{\partial y_{P/E}}\end{bmatrix}, \dot{\mathbf{r}}_{P/E} = \begin{bmatrix}\dfrac{dx_{P/E}}{dt} \\ \dfrac{dy_{P/E}}{dt}\end{bmatrix}, \mathbf{C}_t = \frac{\partial C}{\partial t}$$

Example of the pendulum
- Augmented Formulation

$$m\cdot \ddot{x}_{P/E} - F_x^e + \lambda \frac{\partial C\left(\ x_{P/E},\ \ y_{P/E}\right)}{\partial\ \ x_{P/E}} = 0$$

$$m\cdot \ddot{y}_{P/E} - F_y^e + \lambda \frac{\partial C\left(\ x_{P/E},\ \ y_{P/E}\right)}{\partial\ \ y_{P/E}} = 0$$

Matrix representation

$$\mathbf{M}\,\ddot{\mathbf{r}}_{P/E} - \mathbf{F}^e + \mathbf{C_r}^T\,\lambda = 0$$

$$,where\ \mathbf{r}_{P/E} = \begin{bmatrix} x_{P/E} \\ y_{P/E} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} F_x^e \\ F_y^e \end{bmatrix}$$

$$,\mathbf{C_r} = \begin{bmatrix} \dfrac{\partial C\left(x_{P/E}, y_{P/E}\right)}{\partial\, x_{P/E}} & \dfrac{\partial C\left(x_{P/E}, y_{P/E}\right)}{\partial\, y_{P/E}} \end{bmatrix}$$

$$C\left(\ x_{P/E},\ \ y_{P/E}\right) = x_{P/E}^{\ 2} + y_{P/E}^{\ 2} - l_1^2 = 0$$

Time derivative (Chain rule)

$$\frac{\boxed{dC}}{dt} = \mathbf{C}_{\mathbf{r}}\,\dot{\mathbf{r}}_{P/E} + \mathbf{C}_t \quad ,where\ \mathbf{C_r} = \begin{bmatrix} \dfrac{\partial C}{\partial x_{P/E}} & \dfrac{\partial C}{\partial y_{P/E}} \end{bmatrix}, \dot{\mathbf{r}}_{P/E} = \begin{bmatrix} \dfrac{dx_{P/E}}{dt} \\ \dfrac{dy_{P/E}}{dt} \end{bmatrix}, \mathbf{C}_t = \frac{\partial C}{\partial t}$$

Time derivative (Chain rule)

$$\frac{d\,\boxed{\left(\mathbf{C_r}\,\dot{\mathbf{r}}_{P/E} + \mathbf{C}_t\right)}}{dt} = \left(\mathbf{C_r}\,\dot{\mathbf{r}}_{P/E} + \mathbf{C}_t\right)_{\mathbf{r}}\dot{\mathbf{r}}_{P/E} + \left(\mathbf{C_r}\,\dot{\mathbf{r}}_{P/E} + \mathbf{C}_t\right)_t$$

$$= \left(\mathbf{C_r}\,\dot{\mathbf{r}}_{P/E}\right)_{\mathbf{r}}\dot{\mathbf{r}}_{P/E} + \mathbf{C}_{t\mathbf{r}}\dot{\mathbf{r}}_{P/E} + \left(\mathbf{C_r}\,\dot{\mathbf{r}}_{P/E}\right)_t + \mathbf{C}_{tt}$$

$$= \left(\mathbf{C_r}\,\dot{\mathbf{r}}_{P/E}\right)_{\mathbf{r}}\dot{\mathbf{r}}_{P/E} + \mathbf{C_{rt}}\dot{\mathbf{r}}_{P/E} + \mathbf{C_{rt}}\,\dot{\mathbf{r}}_{P/E} + \mathbf{C_r}\,\ddot{\mathbf{r}}_{P/E} + \mathbf{C}_{tt}$$

$$= \left(\mathbf{C_r}\,\dot{\mathbf{r}}_{P/E}\right)_{\mathbf{r}}\dot{\mathbf{r}}_{P/E} + 2\mathbf{C_{rt}}\dot{\mathbf{r}}_{P/E} + \mathbf{C_r}\,\ddot{\mathbf{r}}_{P/E} + \mathbf{C}_{tt} \qquad ,where\,\mathbf{C_{rt}} = \frac{\partial \mathbf{C_r}}{\partial t}, \mathbf{C}_{tt} = \frac{\partial \mathbf{C}_t}{\partial t}$$

Example of the pendulum - Augmented Formulation

$$C(x_{P/E}, y_{P/E}) = x_{P/E}^2 + y_{P/E}^2 - l_1^2 = 0$$

Time derivative (Chain rule)

$$\frac{dC}{dt} = \mathbf{C_r}\dot{\mathbf{r}}_{P/E} + \mathbf{C}_t \quad , where\ \mathbf{C_r} = \begin{bmatrix} \frac{\partial C}{\partial x_{P/E}} & \frac{\partial C}{\partial y_{P/E}} \end{bmatrix}, \dot{\mathbf{r}}_{P/E} = \begin{bmatrix} \frac{dx_{P/E}}{dt} \\ \frac{dy_{P/E}}{dt} \end{bmatrix}, \mathbf{C}_t = \frac{\partial C}{\partial t}$$

$$m \cdot \ddot{x}_{P/E} - F_x^e + \lambda \frac{\partial C(x_{P/E}, y_{P/E})}{\partial x_{P/E}} = 0$$

$$m \cdot \ddot{y}_{P/E} - F_y^e + \lambda \frac{\partial C(x_{P/E}, y_{P/E})}{\partial y_{P/E}} = 0$$

Matrix representation

$$\mathbf{M}\ddot{\mathbf{r}}_{P/E} - \mathbf{F}^e + \mathbf{C_r}^T \lambda = 0$$

$$, where\ \mathbf{r}_{P/E} = \begin{bmatrix} x_{P/E} \\ y_{P/E} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} F_x^e \\ F_y^e \end{bmatrix}$$

$$, \mathbf{C_r} = \begin{bmatrix} \frac{\partial C(x_{P/E}, y_{P/E})}{\partial x_{P/E}} & \frac{\partial C(x_{P/E}, y_{P/E})}{\partial y_{P/E}} \end{bmatrix}$$

Time derivative (Chain rule)

$$\frac{d(\mathbf{C_r}\dot{\mathbf{r}}_{P/E} + \mathbf{C}_t)}{dt} = (\mathbf{C_r}\dot{\mathbf{r}}_{P/E})_{\mathbf{r}}\dot{\mathbf{r}}_{P/E} + 2\mathbf{C_{rt}}\dot{\mathbf{r}}_{P/E} + \mathbf{C_r}\ddot{\mathbf{r}}_{P/E} + \mathbf{C}_{tt} \quad , where\ \mathbf{C_{rt}} = \frac{\partial \mathbf{C_r}}{\partial t}, \mathbf{C}_{tt} = \frac{\partial \mathbf{C}_t}{\partial t}$$

$$(\mathbf{C_r}\dot{\mathbf{r}}_{P/E})_{\mathbf{r}}\dot{\mathbf{r}}_{P/E} + 2\mathbf{C_{rt}}\dot{\mathbf{r}}_{P/E} + \mathbf{C_r}\ddot{\mathbf{r}}_{P/E} + \mathbf{C}_{tt} = 0$$

$$\mathbf{C_r}\ddot{\mathbf{r}}_{P/E} = -(\mathbf{C_r}\dot{\mathbf{r}}_{P/E})_{\mathbf{r}}\dot{\mathbf{r}}_{P/E} - 2\mathbf{C_{rt}}\dot{\mathbf{r}}_{P/E} - \mathbf{C}_{tt}$$

$$\mathbf{C_r}\ddot{\mathbf{r}}_{P/E} = \mathbf{F}^d \quad , where\ \mathbf{F}^d = -(\mathbf{C_r}\dot{\mathbf{r}}_{P/E})_{\mathbf{r}}\dot{\mathbf{r}}_{P/E} - 2\mathbf{C_{rt}}\dot{\mathbf{r}}_{P/E} - \mathbf{C}_{tt}$$

Example of the pendulum
- Augmented Formulation

$$\mathbf{M}\;\ddot{\mathbf{r}}_{P/E} - \mathbf{F}^{e} + \mathbf{C}_{r}^{\;T}\lambda = 0$$

$$, where\ \mathbf{r}_{P/E} = \begin{bmatrix} x_{P/E} \\ y_{P/E} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^{e} = \begin{bmatrix} F_x^e \\ F_y^e \end{bmatrix}, \mathbf{C}_{r} = \begin{bmatrix} \dfrac{\partial C(x_{P/E}, y_{P/E})}{\partial x_{P/E}} & \dfrac{\partial C(x_{P/E}, y_{P/E})}{\partial y_{P/E}} \end{bmatrix}$$

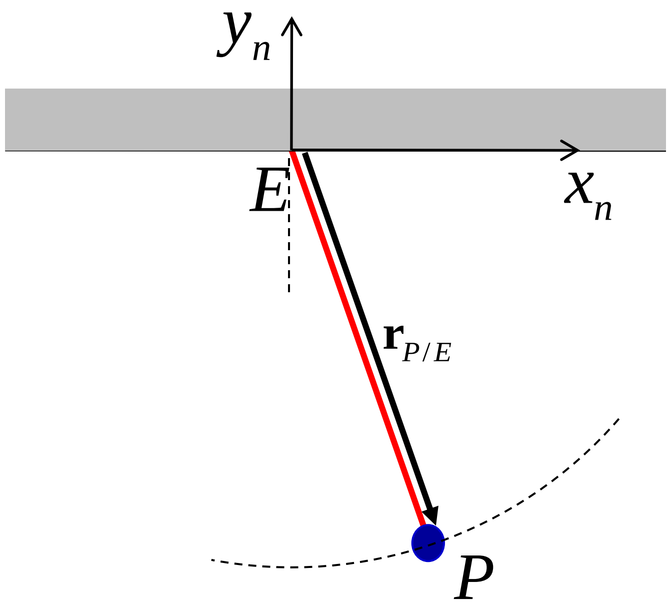
$$\mathbf{C}_{r}\ddot{\mathbf{r}}_{P/E} = \mathbf{F}^{d}$$

$$, where\ \mathbf{F}^{d} = -(\mathbf{C}_{r}\dot{\mathbf{r}}_{P})_{r}\dot{\mathbf{r}}_{P/E} - 2\mathbf{C}_{rt}\dot{\mathbf{r}}_{P/E} - \mathbf{C}_{tt} \quad , \mathbf{C}_{\mathbf{r}t} = \frac{\partial \mathbf{C}_{\mathbf{r}}}{\partial t}, \mathbf{C}_{tt} = \frac{\partial \mathbf{C}_{t}}{\partial t}$$

$$\Downarrow$$

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_{r}^{\;T} \\ \mathbf{C}_{r} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_{P/E} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{e} \\ \mathbf{F}^{d} \end{bmatrix}$$

Example of the pendulum
- Augmented Formulation



$n-frame$: Inertial reference frame
m : mass of particle P
$\mathbf{r}_{P/E}$: Position vector of the particle P

Augmented formulation

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_r{}^T \\ \mathbf{C}_r & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_{P/E} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e \\ \mathbf{F}^d \end{bmatrix}$$

$$\left(\mathbf{r}_{P/E} = \begin{bmatrix} x_{P/E} \\ y_{P/E} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} F_x^e \\ F_y^e \end{bmatrix}, \mathbf{C}_r = \begin{bmatrix} \frac{\partial C(x_{P/E}, y_{P/E})}{\partial x_{P/E}} & \frac{\partial C(x_{P/E}, y_{P/E})}{\partial y_{P/E}} \end{bmatrix}, \right.$$

$$\left. \mathbf{F}^d = -(\mathbf{C}_r \dot{\mathbf{r}}_{P/E})_r \dot{\mathbf{r}}_{P/E} - 2\mathbf{C}_{rt} \dot{\mathbf{r}}_{P/E} - C_{tt} \right)$$

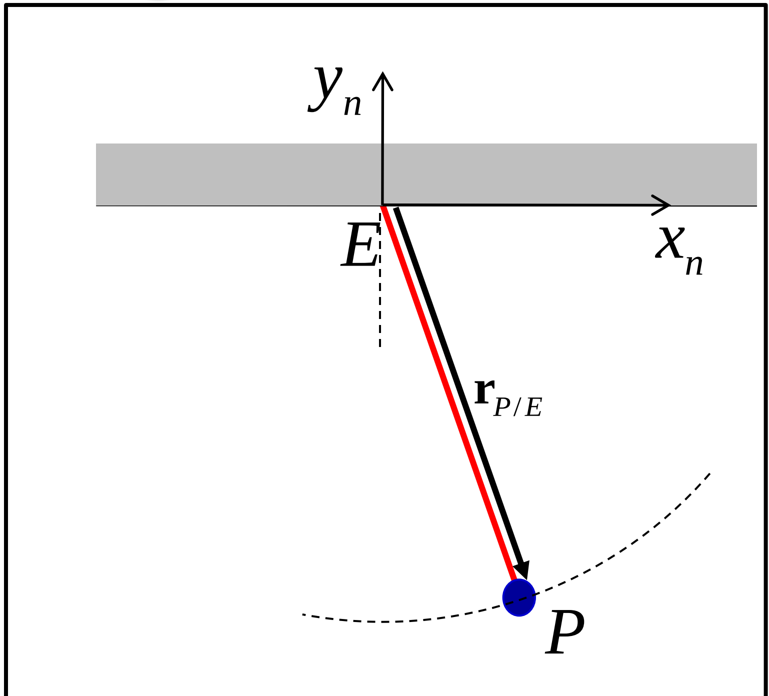
- Constraint $C(x_{P/E}, y_{P/E}) = x_{P/E}{}^2 + y_{P/E}{}^2 - l_1^2 = 0$

- Gravitational force $\mathbf{F}^e = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$

$$\mathbf{C}_r = \begin{bmatrix} 2 \cdot x_{P/E} & 2 \cdot y_{P/E} \end{bmatrix} \quad \mathbf{C}_{rt} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \mathbf{C}_{tt} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{C}_r \dot{\mathbf{r}}_{P/E} = \begin{bmatrix} 2 \cdot x_{P/E} & 2 \cdot y_{P/E} \end{bmatrix} \begin{bmatrix} \dot{x}_{P/E} \\ \dot{y}_{P/E} \end{bmatrix} = 2 \cdot x_{P/E} \cdot \dot{x}_{P/E} + 2 \cdot y_{P/E} \cdot \dot{y}_{P/E}$$

Example of the pendulum - Augmented Formulation



$n-$ *frame* : Inertial reference frame

m : mass of particle P

$\mathbf{r}_{P/E}$: Position vector of the particle P

Augmented formulation

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_r^T \\ \mathbf{C}_r & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_{P/E} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e \\ \mathbf{F}^d \end{bmatrix}$$

$$\left(\mathbf{r}_{P/E} = \begin{bmatrix} x_{P/E} \\ y_{P/E} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} F_x^e \\ F_y^e \end{bmatrix}, \mathbf{C}_r = \left[\frac{\partial C(x_{P/E}, y_{P/E})}{\partial x_{P/E}} \quad \frac{\partial C(x_{P/E}, y_{P/E})}{\partial y_{P/E}} \right], \right.$$

$$\left. \mathbf{F}^d = -(\mathbf{C}_r \dot{\mathbf{r}}_{P/E})_r \dot{\mathbf{r}}_{P/E} - 2\mathbf{C}_{rt} \dot{\mathbf{r}}_{P/E} - C_{tt} \right)$$

$$\mathbf{C}_r = \begin{bmatrix} 2 \cdot x_{P/E} & 2 \cdot y_{P/E} \end{bmatrix} \quad \mathbf{C}_{rt} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \mathbf{C}_{tt} = \begin{bmatrix} 0 \end{bmatrix}$$

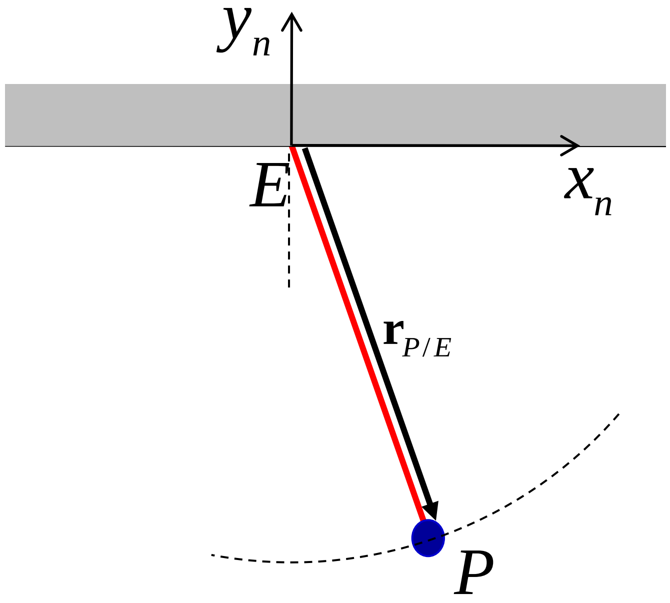
$$\mathbf{C}_r \dot{\mathbf{r}}_{P/E} = \begin{bmatrix} 2 \cdot x_{P/E} & 2 \cdot y_{P/E} \end{bmatrix} \begin{bmatrix} \dot{x}_{P/E} \\ \dot{y}_{P/E} \end{bmatrix} = 2 \cdot x_{P/E} \cdot \dot{x}_{P/E} + 2 \cdot y_{P/E} \cdot \dot{y}_{P/E}$$

$$x_P = l_1 \sin q_1, \ y_P = -l_1 \cos q_1$$

$$\dot{x}_P = l_1 \cos q_1 \cdot \dot{q}_1, \ \dot{y}_P = l_1 \sin q_1 \cdot \dot{q}_1$$

$$\begin{aligned} \mathbf{C}_r \dot{\mathbf{r}}_{P/E} &= 2 \cdot x_{P/E} \cdot \dot{x}_{P/E} + 2 \cdot y_{P/E} \cdot \dot{y}_{P/E} \\ &= 2 \cdot l_1 \sin q_1 \cdot l_1 \cos q_1 \cdot \dot{q}_1 - 2 \cdot l_1 \cos q_1 \cdot l_1 \sin q_1 \cdot \dot{q}_1 \\ &= 0 \end{aligned}$$

Example of the pendulum
- Augmented Formulation



$n-frame$: Inertial reference frame

m : mass of particle P

$\mathbf{r}_{P/E}$: Position vector of the particle P

Augmented formulation

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_r{}^T \\ \mathbf{C}_r & 0 \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{r}}_{P/E} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e \\ \mathbf{F}^d \end{bmatrix}$$

$$\left(\mathbf{r}_{P/E} = \begin{bmatrix} x_{P/E} \\ y_{P/E} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \mathbf{F}^e = \begin{bmatrix} F_x^e \\ F_y^e \end{bmatrix}, \mathbf{C}_r = \begin{bmatrix} \frac{\partial C(x_{P/E}, y_{P/E})}{\partial x_{P/E}} & \frac{\partial C(x_{P/E}, y_{P/E})}{\partial y_{P/E}} \end{bmatrix}, \right.$$

$$\left. \mathbf{F}^d = -(\mathbf{C}_r \dot{\mathbf{r}}_{P/E})_r \dot{\mathbf{r}}_{P/E} - 2\mathbf{C}_{rt} \dot{\mathbf{r}}_{P/E} - C_{tt} \right)$$

$$\mathbf{C}_r = \begin{bmatrix} 2 \cdot x_{P/E} & 2 \cdot y_{P/E} \end{bmatrix} \quad \mathbf{C}_{rt} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \mathbf{C}_{tt} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{C}_r \dot{\mathbf{r}}_{P/E} = \begin{bmatrix} 2 \cdot x_{P/E} & 2 \cdot y_{P/E} \end{bmatrix} \begin{bmatrix} \dot{x}_{P/E} \\ \dot{y}_{P/E} \end{bmatrix} = 0$$

$$(\mathbf{C}_r \dot{\mathbf{r}}_{P/E})_r = 0 \qquad (\mathbf{C}_r \dot{\mathbf{r}}_{P/E})_r \dot{\mathbf{r}}_{P/E} = 0$$

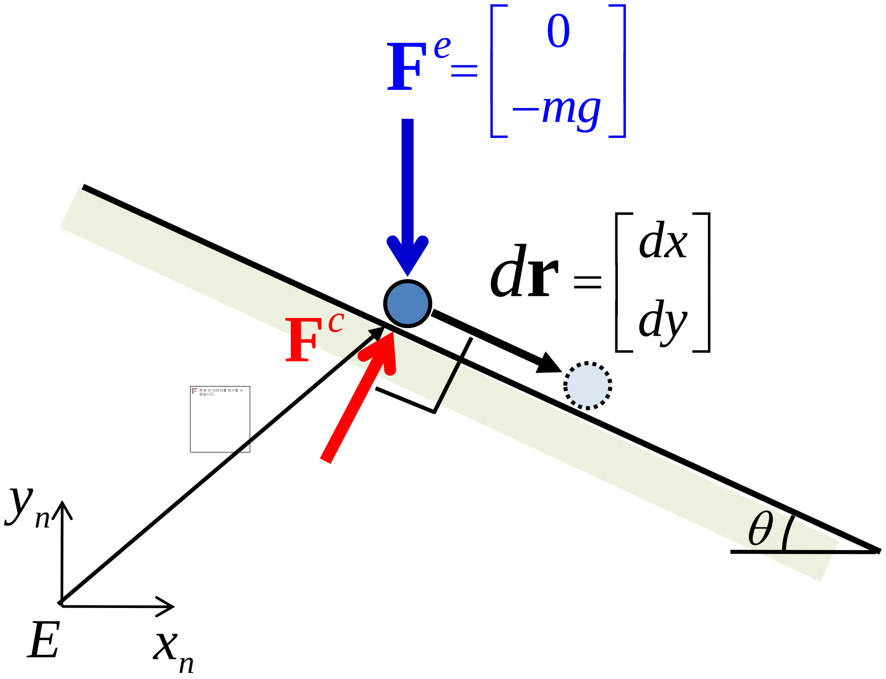
✓**Equations of motion**

$$\begin{bmatrix} m & 0 & 2 \cdot x_{P/E} \\ 0 & m & 2 \cdot y_{P/E} \\ 2 \cdot x_{P/E} & 2 \cdot y_{P/E} & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_{P/E} \\ \ddot{y}_{P/E} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}$$

Reference)
Virtual Displacment

Virtual displacement
- Moving Particle on the Slope

The angle of inclination is constant.



$\mathbf{F}^e$: External force

$\mathbf{F}^c$: Constraint force

$d\mathbf{r}$: Actual displacement

✓**Newton's 2nd law**

$$m\ddot{\mathbf{r}} = \mathbf{F}^e + \mathbf{F}^c$$

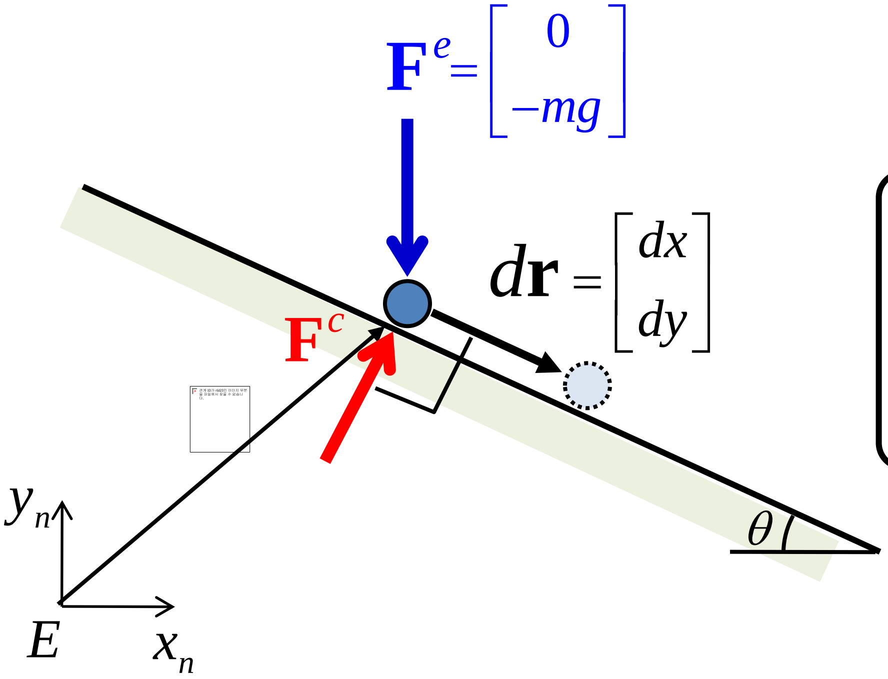
✓**D'Alembert's principle**

$$\mathbf{F}^e + \mathbf{F}^c - m\ddot{\mathbf{r}} = 0$$

$-m\ddot{\mathbf{r}}$: inertial fore

Virtual displacement
- Moving Particle on the Slope

The angle of inclination is constant.



$\mathbf{F}^e$: External force

$\mathbf{F}^c$: Constraint force

$d\mathbf{r}$: Actual displacement

$d\mathbf{r}$ represents the actual movement of the system over an elapsed time dt

Jerry Ginsberg, Engineering Dynamics, Cambridge university press, p. 409

✓D'Alembert's principle

$$\mathbf{F}^e + \mathbf{F}^c - m\ddot{\mathbf{r}} = 0$$

✓Actual work

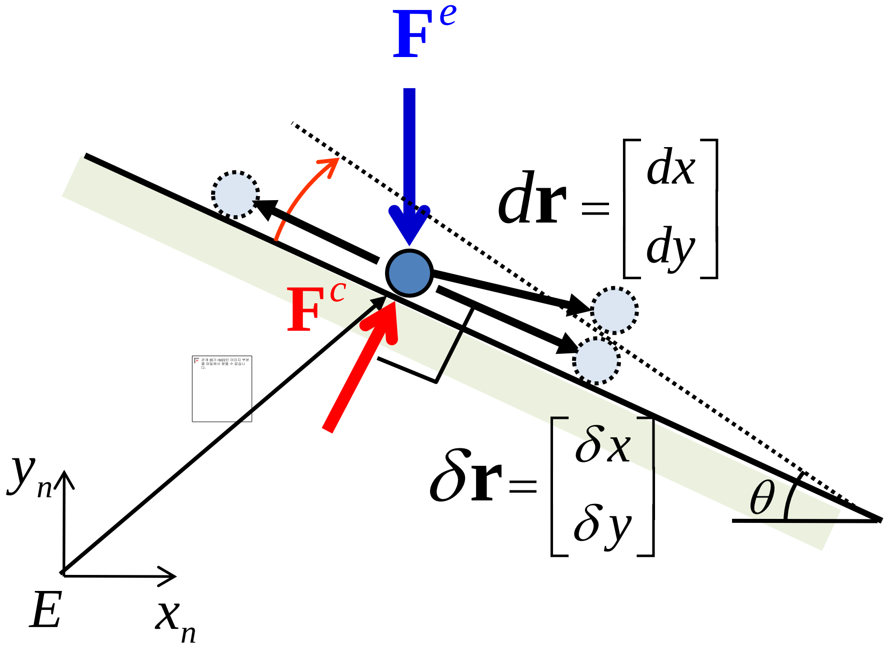
$$dW = d\mathbf{r} \cdot \left(\mathbf{F}^e + \mathbf{F}^c - m\ddot{\mathbf{r}} \right) = 0$$

$$dW = d\mathbf{r} \cdot \left(\mathbf{F}^e - m\ddot{\mathbf{r}} \right) = 0 \leftarrow d\mathbf{r} \cdot \mathbf{F}^c = 0$$

Virtual displacement

- Moving Particle on the Slope

The angle of inclination is not constant.



$\mathbf{F}^e$: External force

$\mathbf{F}^c$: Constraint force

$d\mathbf{r}$: **Actual displacement**

$\delta\mathbf{r}$: **Virtual displacement**

$\delta\mathbf{r}$ represents **the shift in the position** at **instant *t***.
Because the position shift does not represent an actual movement, we say that $\delta\mathbf{r}$ represents a **virtual displacement**.

Jerry Ginsberg, Engineering Dynamics, Cambridge university press, 2008, p. 409

This infinitesimal change is imposed by us on a set of variables as a kind of **mathematical experimental**.

Lanczos, C. The variational principles of mechanics, University of Toronto press, 1970, p.38

✓**Actual work** $d\mathbf{r}\cdot\mathbf{F}^c = 0$ **?**

$$dW = d\mathbf{r}\cdot\left(\mathbf{F}^e + \mathbf{F}^c - m\ddot{\mathbf{r}}\right) = 0$$

✓**Virtual work**

$$\delta W = \delta\mathbf{r}\cdot\left(\mathbf{F}^e + \mathbf{F}^c - m\ddot{\mathbf{r}}\right) = 0$$

$$\delta W = \delta\mathbf{r}\cdot\left(\mathbf{F}^e - m\ddot{\mathbf{r}}\right) = 0 \leftarrow\!\lrcorner\ \delta\mathbf{r}\cdot\mathbf{F}^c = 0 \ \ !!!$$

Jerry Ginsberg, Engineering Dynamics, Cambridge university press, 2008, pp.410 ~ 411

Virtual displacement
- Moving Particle on the Slope

Constraint surface has a shape that is time invariant.

$$C(x,y,z)=0$$

↓ **Total derivative**

$$\frac{\partial C}{\partial x}dx+\frac{\partial C}{\partial y}dy+\frac{\partial C}{\partial z}dz=0$$

↓ **If** $x=x(t), y=y(t), z=z(t)$ **then, we can differentiate with respect to time** t

$$\frac{\partial C}{\partial x}\frac{dx}{dt}+\frac{\partial C}{\partial y}\frac{dy}{dt}+\frac{\partial C}{\partial z}\frac{dz}{dt}=0$$

Constraint surface has a shape that is time variant.

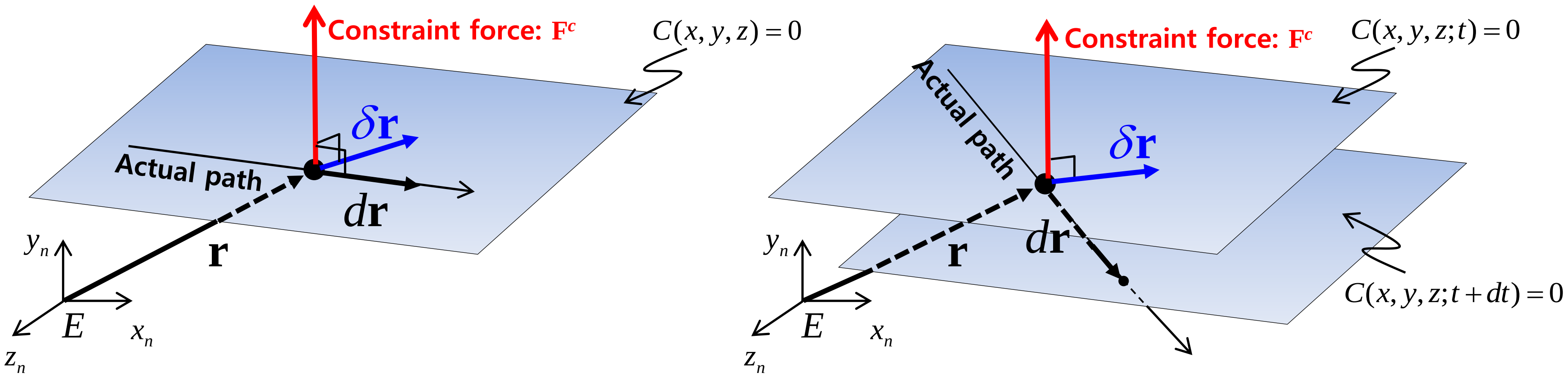
$$C(x,y,z;t)=0$$

↓ **Total derivative**

$$\frac{\partial C}{\partial x}dx+\frac{\partial C}{\partial y}dy+\frac{\partial C}{\partial z}dz+\frac{\partial C}{\partial t}dt=0$$

↓ **If** $x=x(t), y=y(t), z=z(t)$ **then, we can differentiate with respect to time** t

$$\frac{\partial C}{\partial x}\frac{dx}{dt}+\frac{\partial C}{\partial y}\frac{dy}{dt}+\frac{\partial C}{\partial z}\frac{dz}{dt}+\frac{\partial C}{\partial t}=0$$



Direction of constraint force is normal to the surface.

The work performed by the constraint force in any virtual displacement is zero

Naval Architecture & Ocean Engineering

Topics in ship design automation

4. Euler Angle and Euler Parameter

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September, 2010

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4.1 Angular and Linear Velocity

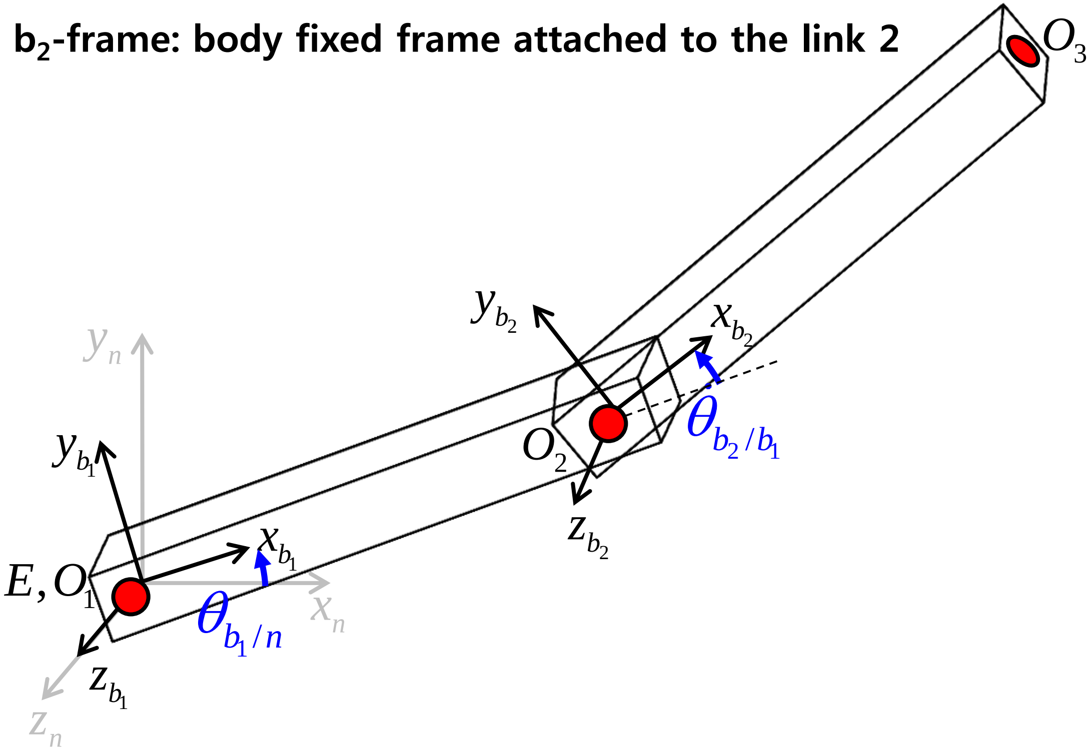
Example of 2-Link Arm

- Configuration

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2



Example of 2-Link Arm
- Configuration

n-frame: Inertial reference frame

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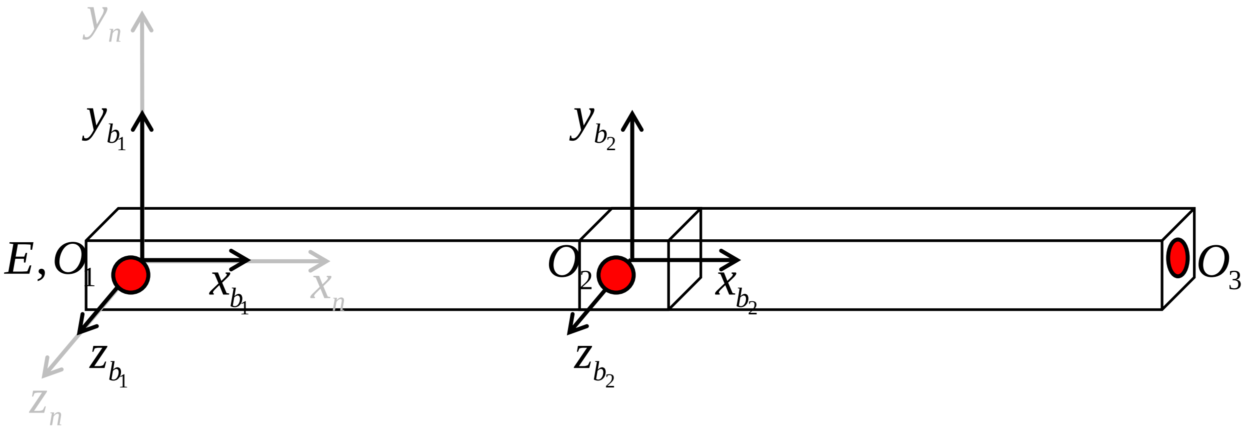

Example of 2-Link Arm

- Configuration

n-frame: Inertial reference frame

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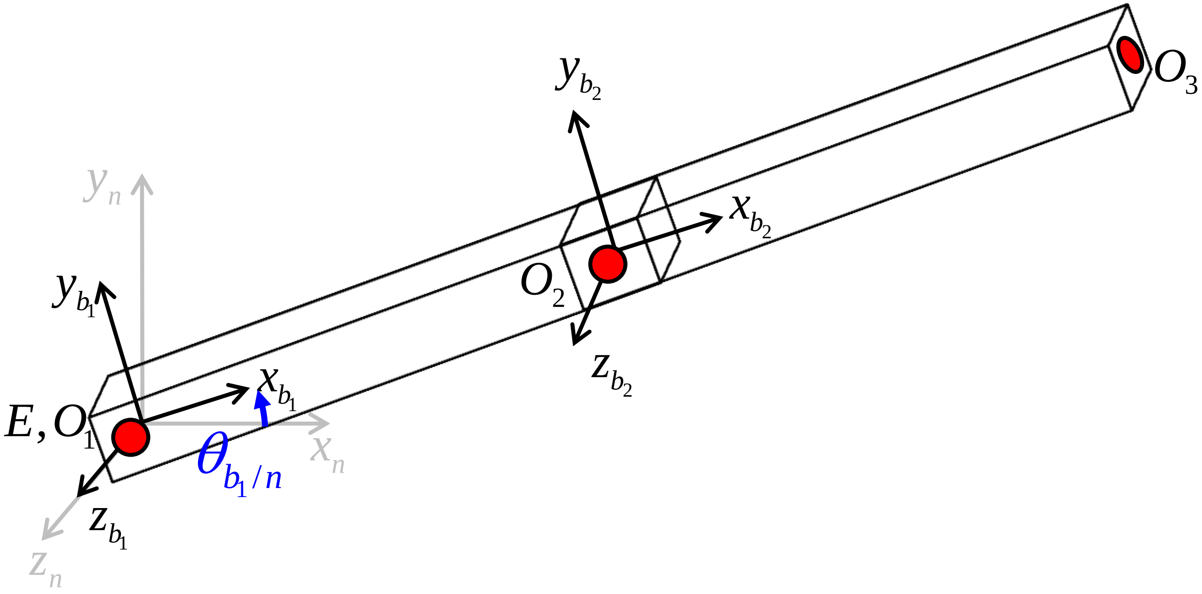


Example of 2-Link Arm

- position vector of end-effector(O_3)

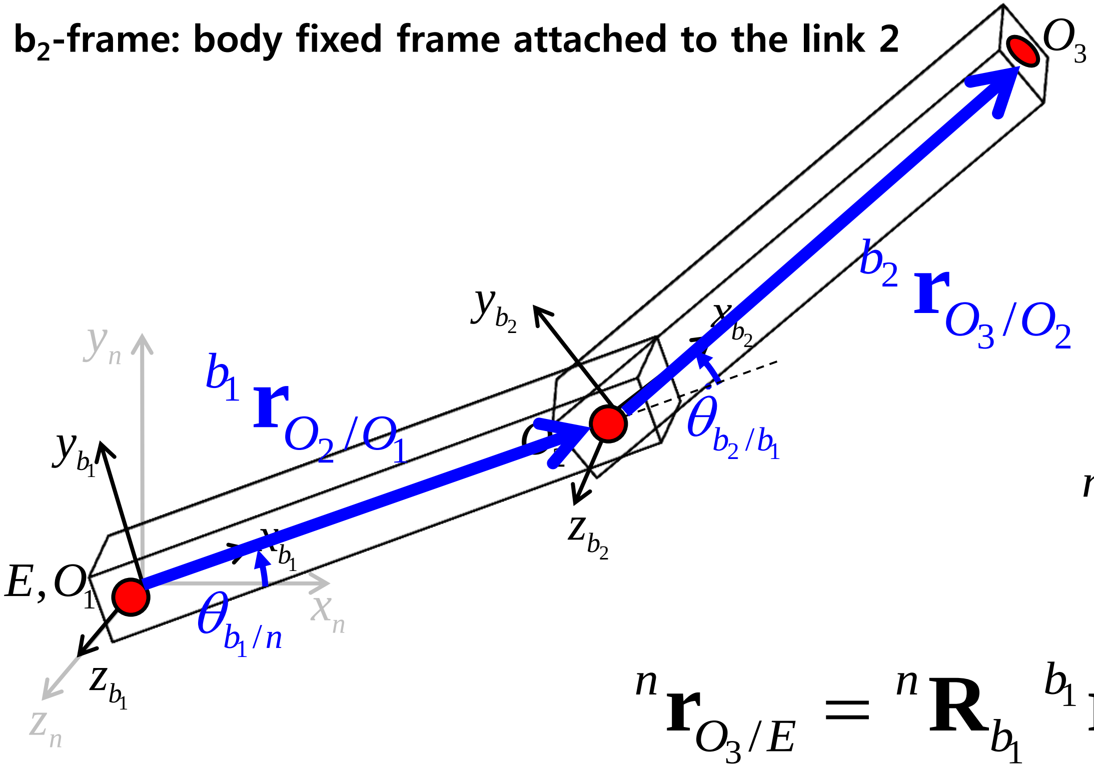
$$ {}^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\theta_{b_1/n} & -\sin\theta_{b_1/n} & 0 \\ \sin\theta_{b_1/n} & \cos\theta_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

$$ {}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & -\sin\theta_{b_2/b_1} & 0 \\ \sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



$$ {}^{n}\mathbf{R}_{b_2} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2} $$

$$ {}^{n}\mathbf{r}_{O_3/E} = {}^{n}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{r}_{O_3/O_2} $$

$$ \downarrow $$

$$ {}^{n}\mathbf{r}_{O_3/E} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} $$

$$ \downarrow $$

$$ {}^{n}\mathbf{r}_{O_3/E} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} $$

Example of 2-Link Arm

- linear velocity vector of end-effector(O_3)

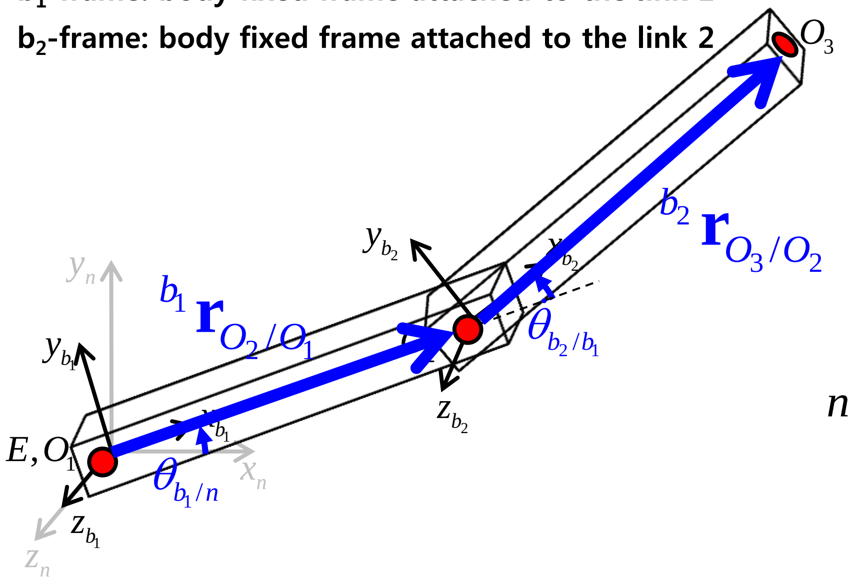
$$
{}^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\theta_{b_1/n} & -\sin\theta_{b_1/n} & 0 \\ \sin\theta_{b_1/n} & \cos\theta_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & -\sin\theta_{b_2/b_1} & 0 \\ \sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2



$$
{}^{n}\mathbf{r}_{O_3/E} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2}
$$

↓ **Time derivative**

$$
\frac{d}{dt}\,{}^{n}\mathbf{r}_{O_3/E} = \frac{d}{dt}\left({}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + \frac{d}{dt}\left({}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2}\right)
$$

↓

$$
\frac{d}{dt}\,{}^{n}\mathbf{r}_{O_3/E} = \frac{d}{dt}\,{}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{R}_{b_1}\underbrace{\cancel{\frac{d}{dt}\,{}^{b_1}\mathbf{r}_{O_2/O_1}}}_{\text{constant}} + \frac{d}{dt}\,{}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} + {}^{n}\mathbf{R}_{b_2}\underbrace{\cancel{\frac{d}{dt}\,{}^{b_2}\mathbf{r}_{O_3/O_2}}}_{\text{constant}}
$$

Because the point O_1, O_2, and O_3 are fixed on the rigid body

Example of 2-Link Arm

- linear velocity vector of end-effector(O_3)

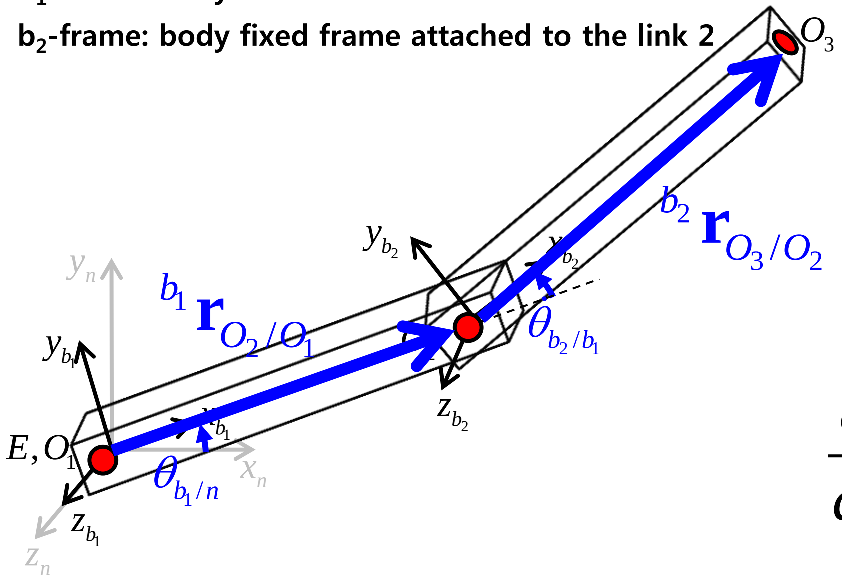
$$ {}^n\mathbf{R}_{b_1} = \begin{bmatrix} \cos\theta_{b_1/n} & -\sin\theta_{b_1/n} & 0 \\ \sin\theta_{b_1/n} & \cos\theta_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & -\sin\theta_{b_2/b_1} & 0 \\ \sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2



$$ \frac{d}{dt}{}^n\mathbf{r}_{O_3/E} = \frac{d}{dt}{}^n\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + \frac{d}{dt}{}^n\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} $$

$$ \downarrow \qquad \frac{d}{dt}{}^n\mathbf{R}_b = {}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{R}_b $$

$$ \frac{d}{dt}{}^n\mathbf{r}_{O_3/E} = \boxed{{}^n\boldsymbol{\omega}_{b_1/n}} \times {}^n\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + \boxed{{}^n\boldsymbol{\omega}_{b_2/n}} \times {}^n\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} $$

We should calculate the angular velocities using $\dot{\theta}_{b_1/n}, \dot{\theta}_{b_2/b_1}$ **.**

Example of 2-Link Arm

- Angular velocities

$${}^{n}\boldsymbol{\omega}_{b_1/n},\ {}^{n}\boldsymbol{\omega}_{b_2/n}$$

$${}^{n}\mathbf{R}_{b_1}=\begin{bmatrix}\cos\theta_{b_1/n} & -\sin\theta_{b_1/n} & 0\\ \sin\theta_{b_1/n} & \cos\theta_{b_1/n} & 0\\ 0 & 0 & 1\end{bmatrix}$$

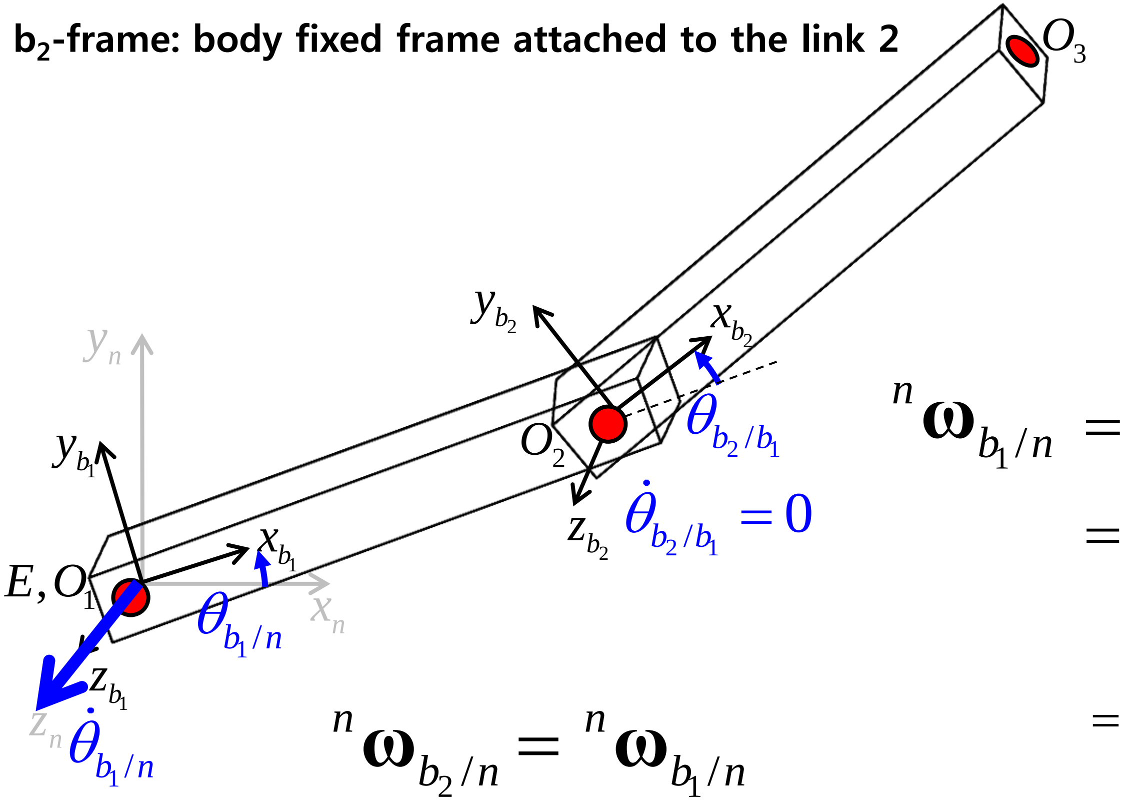
$${}^{b_1}\mathbf{R}_{b_2}=\begin{bmatrix}\cos\theta_{b_2/b_1} & -\sin\theta_{b_2/b_1} & 0\\ \sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1} & 0\\ 0 & 0 & 1\end{bmatrix}$$

$${}^{n}\mathbf{R}_{b_2}={}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}$$

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2



$${}^{n}\boldsymbol{\omega}_{b_2/n}={}^{n}\boldsymbol{\omega}_{b_1/n}$$

$$\begin{aligned}{}^{n}\boldsymbol{\omega}_{b_1/n}&={}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\boldsymbol{\omega}_{b_1/n}\\&={}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{k}_{b_1}\cdot\dot{\theta}_{b_1/n}\\&=\begin{bmatrix}\cos\theta_{b_1/n} & -\sin\theta_{b_1/n} & 0\\ \sin\theta_{b_1/n} & \cos\theta_{b_1/n} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\dot{\theta}_{b_1/n}\\&=\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\dot{\theta}_{b_1/n}\end{aligned}$$

Example of 2-Link Arm

- Angular velocities

$${}^{n}\boldsymbol{\omega}_{b_1/n},\ {}^{n}\boldsymbol{\omega}_{b_2/n}$$

$${}^{n}\mathbf{R}_{b_1}=\begin{bmatrix}\cos\theta_{b_1/n} & -\sin\theta_{b_1/n} & 0\\ \sin\theta_{b_1/n} & \cos\theta_{b_1/n} & 0\\ 0 & 0 & 1\end{bmatrix}$$

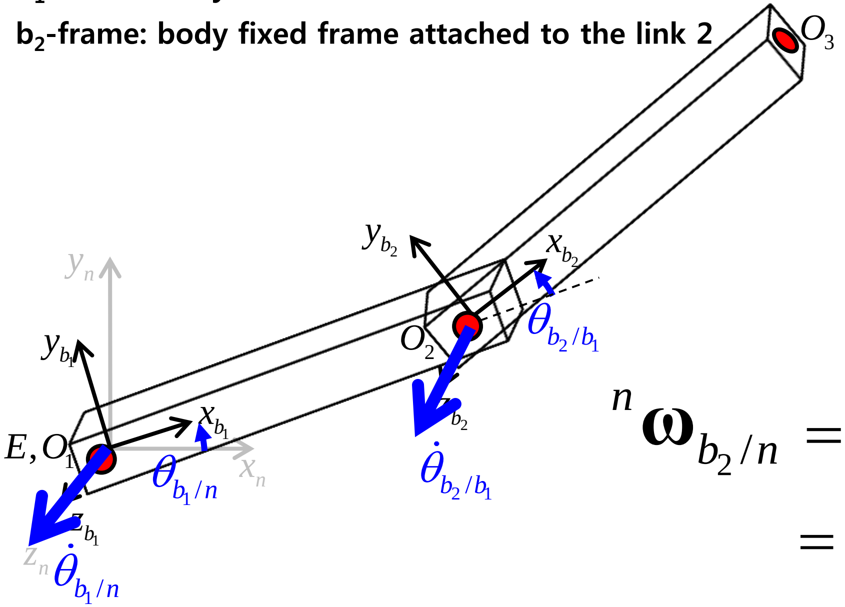
$${}^{b_1}\mathbf{R}_{b_2}=\begin{bmatrix}\cos\theta_{b_2/b_1} & -\sin\theta_{b_2/b_1} & 0\\ \sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1} & 0\\ 0 & 0 & 1\end{bmatrix}$$

$${}^{n}\mathbf{R}_{b_2}={}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}$$

n-frame: Inertial reference frame

b$_1$-frame: body fixed frame attached to the link 1

b$_2$-frame: body fixed frame attached to the link 2



$$\begin{aligned}{}^{n}\boldsymbol{\omega}_{b_2/n}&={}^{n}\boldsymbol{\omega}_{b_1/n}+{}^{n}\boldsymbol{\omega}_{b_2/b_1}\\&={}^{n}\boldsymbol{\omega}_{b_1/n}+\boxed{{}^{n}\mathbf{R}_{b_2}}\,{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_2}\end{aligned}$$

$$={}^{n}\boldsymbol{\omega}_{b_1/n}+\boxed{\begin{bmatrix}\cos\theta_{b_1/n} & -\sin\theta_{b_1/n} & 0\\ \sin\theta_{b_1/n} & \cos\theta_{b_1/n} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta_{b_2/b_1} & -\sin\theta_{b_2/b_1} & 0\\ \sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1} & 0\\ 0 & 0 & 1\end{bmatrix}}\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\dot{\theta}_{b_2/b_1}$$

$$=\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\dot{\theta}_{b_1/n}+\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\dot{\theta}_{b_2/b_1}$$

Example of 2-Link Arm

- linear velocity vector of end-effector(O_3)

$$
{}^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\theta_{b_1/n} & -\sin\theta_{b_1/n} & 0 \\ \sin\theta_{b_1/n} & \cos\theta_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

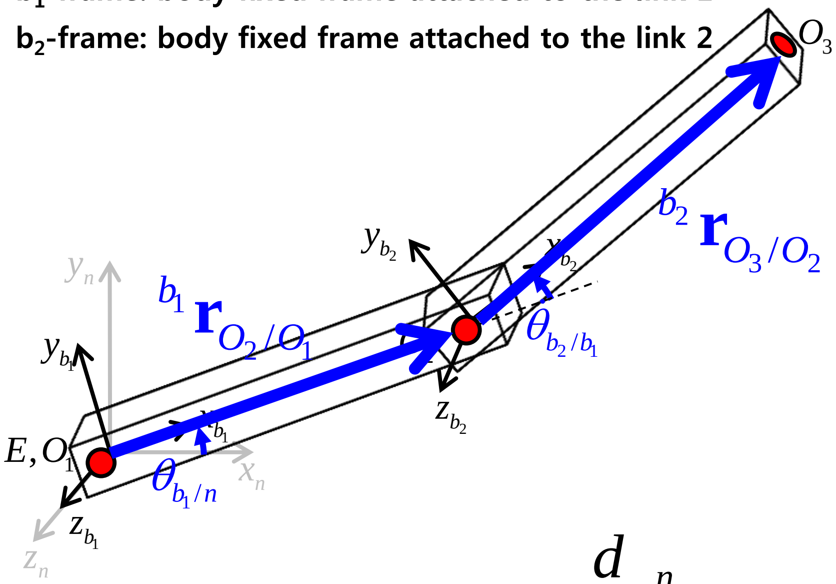
$$
{}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & -\sin\theta_{b_2/b_1} & 0 \\ \sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{n}\mathbf{R}_{b_2} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}
$$

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2



$$
\frac{d}{dt}\,{}^{n}\mathbf{r}_{O_3/E} = \boxed{{}^{n}\boldsymbol{\omega}_{b_1/n}} \times {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + \boxed{{}^{n}\boldsymbol{\omega}_{b_2/n}} \times {}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2}
$$

$$
{}^{n}\boldsymbol{\omega}_{b_1/n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\theta}_{b_1/n} \qquad {}^{n}\boldsymbol{\omega}_{b_2/n} = {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{n}\boldsymbol{\omega}_{b_2/b_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\theta}_{b_1/n} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\theta}_{b_2/b_1}
$$

We can calculate the angular velocities using $\dot{\theta}_{b_1/n}, \dot{\theta}_{b_2/b_1}$.

4.2 Euler Angle

Example of 3-Link Arm

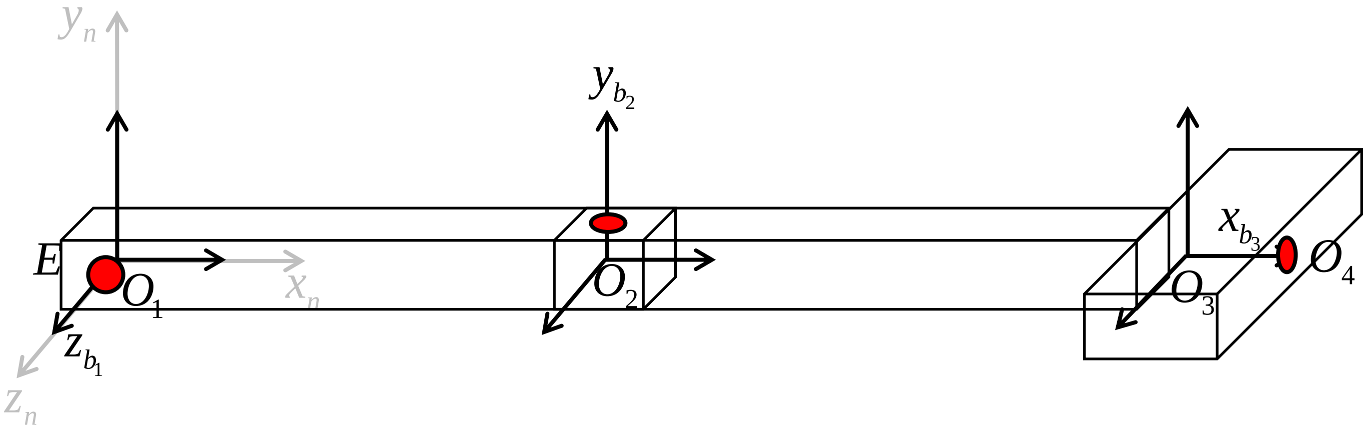
- Configuration

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

b_3-frame: body fixed frame attached to the link 3



Inertial frame

Example of 3-Link Arm

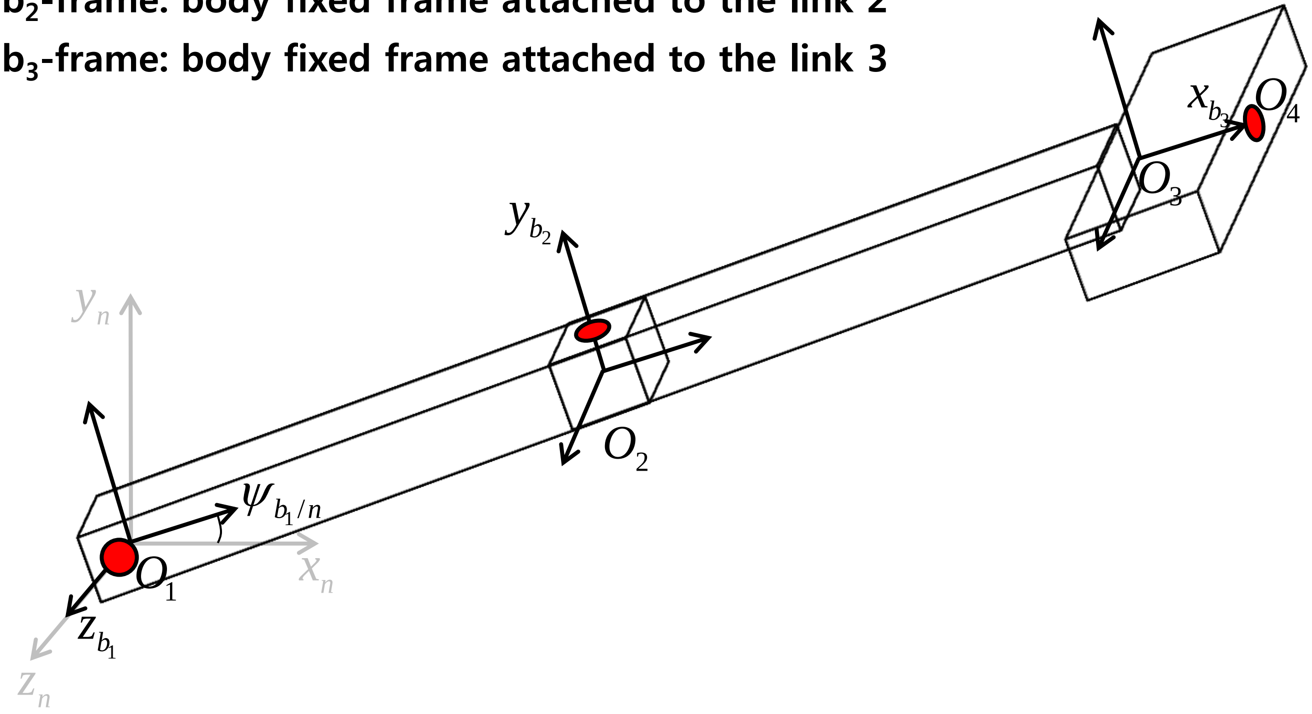
- Configuration

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

b_3-frame: body fixed frame attached to the link 3



Example of 3-Link Arm

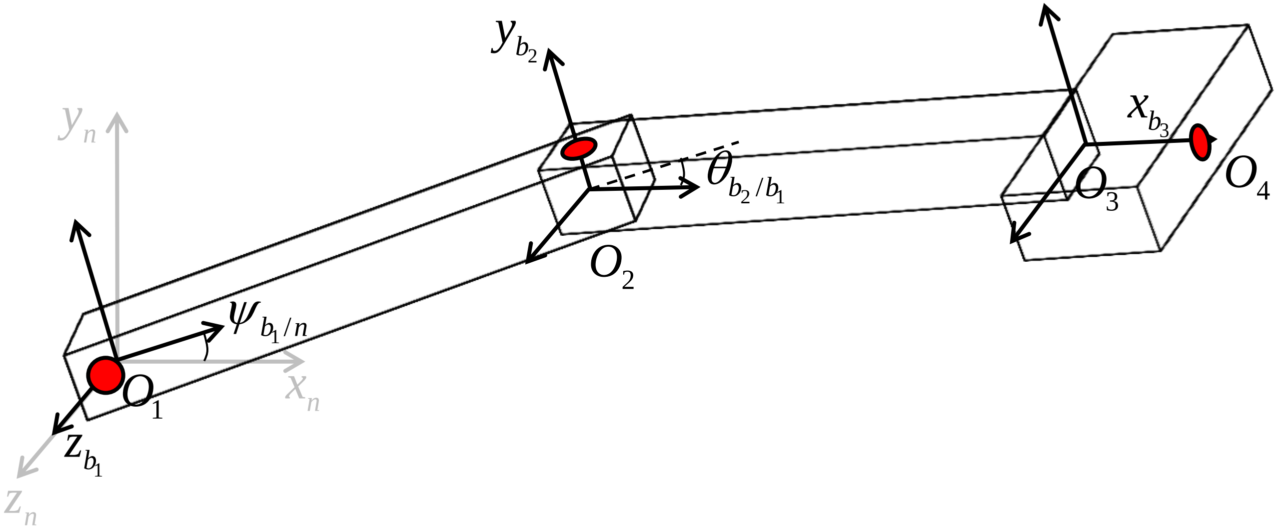
- Configuration

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

b_3-frame: body fixed frame attached to the link 3



Example of 3-Link Arm

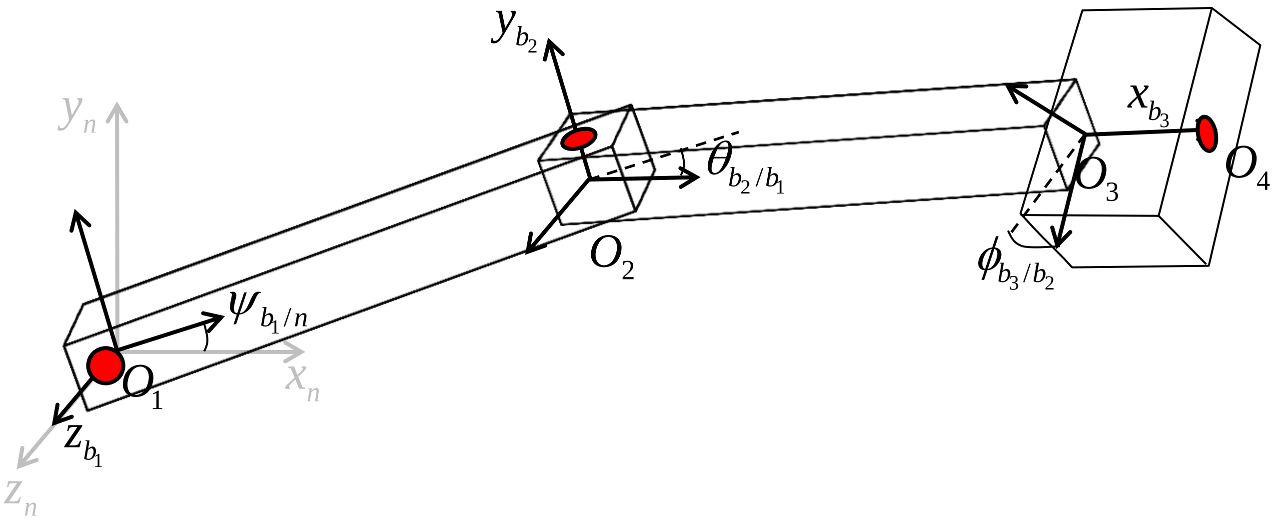
- Configuration

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

b_3-frame: body fixed frame attached to the link 3



Example of 3-Link Arm

- position vector of end-effector(O_4)

$$^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix}$$

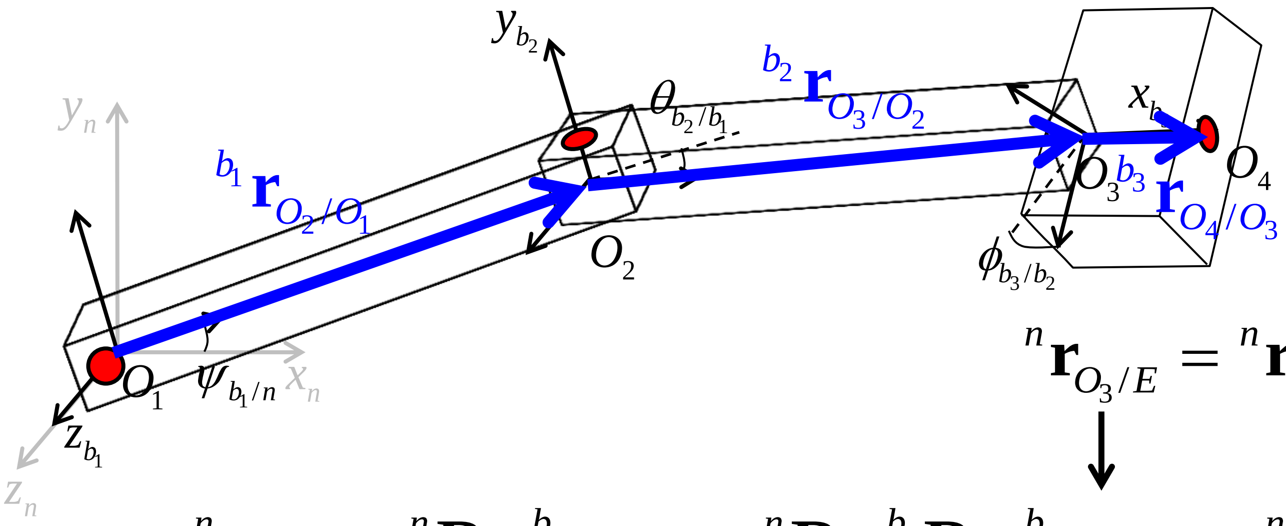
$$^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix}$$

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

b_3-frame: body fixed frame attached to the link 3



$$^{n}\mathbf{r}_{O_3/E} = {}^{n}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{r}_{O_3/O_2} + {}^{n}\mathbf{r}_{O_4/O_3}$$

$$\downarrow$$

$$^{n}\mathbf{r}_{O_3/E} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} + {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{R}_{b_3}\,{}^{b_3}\mathbf{r}_{O_4/O_3}$$

$$\downarrow$$

$$^{n}\mathbf{r}_{O_3/E} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} + {}^{n}\mathbf{R}_{b_3}\,{}^{b_3}\mathbf{r}_{O_4/O_3}$$

Example of 3-Link Arm

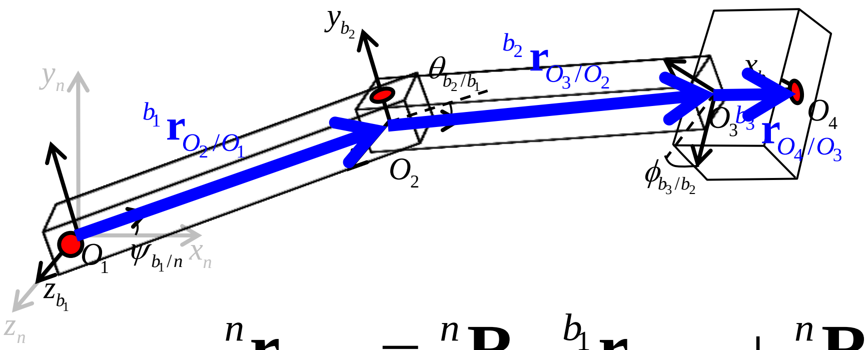
- position vector of end-effector(O_4)

n-frame: Inertial reference frame
b_1-frame: body fixed frame attached to the link 1
b_2-frame: body fixed frame attached to the link 2
b_3-frame: body fixed frame attached to the link 3

$${}^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix}$$

$${}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix}$$



$${}^{n}\mathbf{r}_{O_3/E} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} + \boxed{{}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{R}_{b_3}}\,{}^{b_3}\mathbf{r}_{O_4/O_3}$$

Calculation of rotational transformation matrix using ZYX Euler angle $\phi_{b_3/b_2}, \theta_{b_2/b_1}, \psi_{b_1/n}$

$${}^{n}\mathbf{R}_{b_3} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix}$$

$${}^{n}\mathbf{R}_{b_3} = \begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} & \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} - \sin\psi_{b_1/n}\cos\phi_{b_3/b_2} & \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} + \sin\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} & \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} + \cos\psi_{b_1/n}\cos\phi_{b_3/b_2} & \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} - \cos\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ -\sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1}\sin\phi_{b_3/b_2} & \cos\theta_{b_2/b_1}\cos\phi_{b_3/b_2} \end{bmatrix}$$

Example of 3-Link Arm

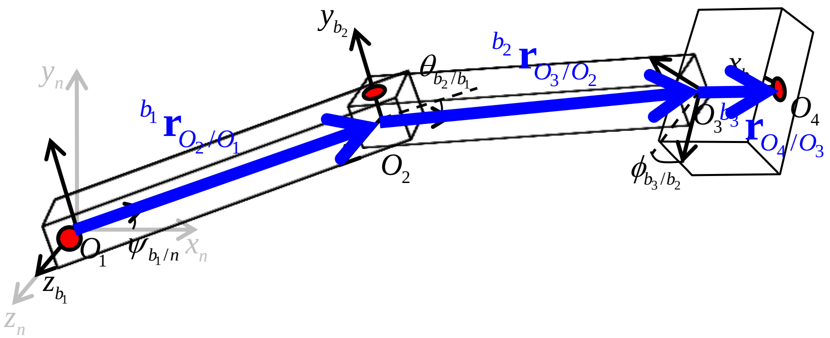
- linear velocity vector of end-effector(O_4)

$$
{}^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix}
$$

$$
{}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix}
$$

n-frame: Inertial reference frame
b_1-frame: body fixed frame attached to the link 1
b_2-frame: body fixed frame attached to the link 2
b_3-frame: body fixed frame attached to the link 3



$$
{}^{n}\mathbf{r}_{O_3/E} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} + {}^{n}\mathbf{R}_{b_3}\,{}^{b_3}\mathbf{r}_{O_4/O_3}
$$

↓**Time derivative**

$$
\frac{d}{dt}\,{}^{n}\mathbf{r}_{O_3/E} = \frac{d}{dt}\,{}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{R}_{b_1}\,\cancel{\frac{d}{dt}\,\underline{{}^{b_1}\mathbf{r}_{O_2/O_1}}} + \frac{d}{dt}\,{}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} + {}^{n}\mathbf{R}_{b_2}\,\cancel{\frac{d}{dt}\,\underline{{}^{b_2}\mathbf{r}_{O_3/O_2}}} + \frac{d}{dt}\,{}^{n}\mathbf{R}_{b_3}\,{}^{b_3}\mathbf{r}_{O_4/O_3} + {}^{n}\mathbf{R}_{b_3}\,\cancel{\frac{d}{dt}\,\underline{{}^{b_3}\mathbf{r}_{O_4/O_3}}}
$$

constant **constant** **constant**

Because the point O_1, O_2, O_3, and O_4 are fixed on the rigid body

Example of 3-Link Arm

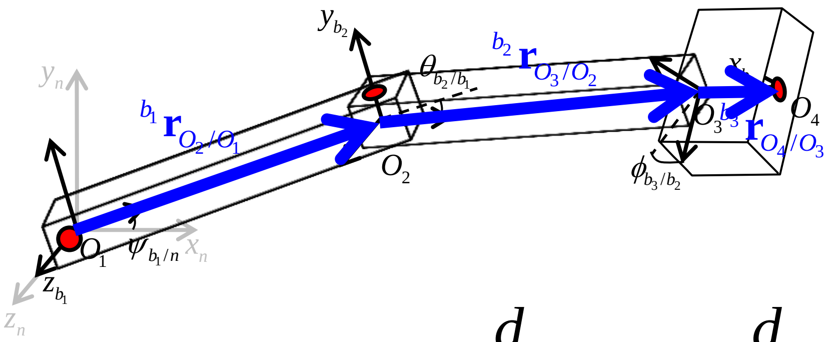
- linear velocity vector of end-effector(O_4)

$$ {}^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix} $$

$$ {}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix} $$

n-frame: Inertial reference frame
b_1-frame: body fixed frame attached to the link 1
b_2-frame: body fixed frame attached to the link 2
b_3-frame: body fixed frame attached to the link 3



$$ \frac{d}{dt}\,{}^{n}\mathbf{r}_{O_3/E} = \frac{d}{dt}\,{}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + \frac{d}{dt}\,{}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} + \frac{d}{dt}\,{}^{n}\mathbf{R}_{b_3}\,{}^{b_3}\mathbf{r}_{O_4/O_3} $$

$$ \Big\downarrow \quad \frac{d}{dt}\,{}^{n}\mathbf{R}_{b} = {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{R}_{b} $$

$$ \frac{d}{dt}\,{}^{n}\mathbf{r}_{O_3/E} = \boxed{{}^{n}\boldsymbol{\omega}_{b_1/n}} \times {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + \boxed{{}^{n}\boldsymbol{\omega}_{b_2/n}} \times {}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} + \boxed{{}^{n}\boldsymbol{\omega}_{b_3/n}} \times {}^{n}\mathbf{R}_{b_3}\,{}^{b_3}\mathbf{r}_{O_4/O_3} $$

We should calculate the angular velocities using time derivative of Euler angle $\dot{\phi}_{b_3/b_2},\ \dot{\theta}_{b_2/b_1},\ \dot{\psi}_{b_1/n}$.

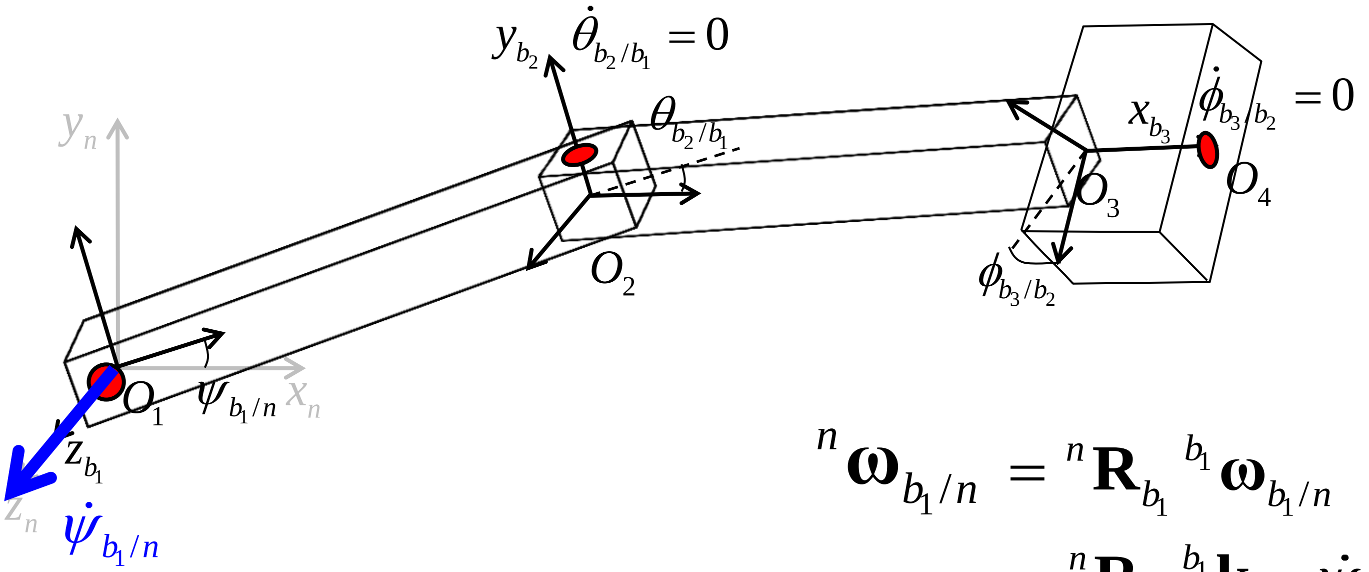
Example of 3-Link Arm

- angular velocities

$${}^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix}$$

$${}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix}$$



$$\begin{aligned} {}^{n}\boldsymbol{\omega}_{b_1/n} &= {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\boldsymbol{\omega}_{b_1/n} \\ &= {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{k}_{b_1}\cdot\dot{\psi}_{b_1/n} \\ &= \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot\dot{\psi}_{b_1/n} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot\dot{\psi}_{b_1/n} \end{aligned}$$

$${}^{n}\boldsymbol{\omega}_{b_3/n} = {}^{n}\boldsymbol{\omega}_{b_2/n} = {}^{n}\boldsymbol{\omega}_{b_1/n}$$

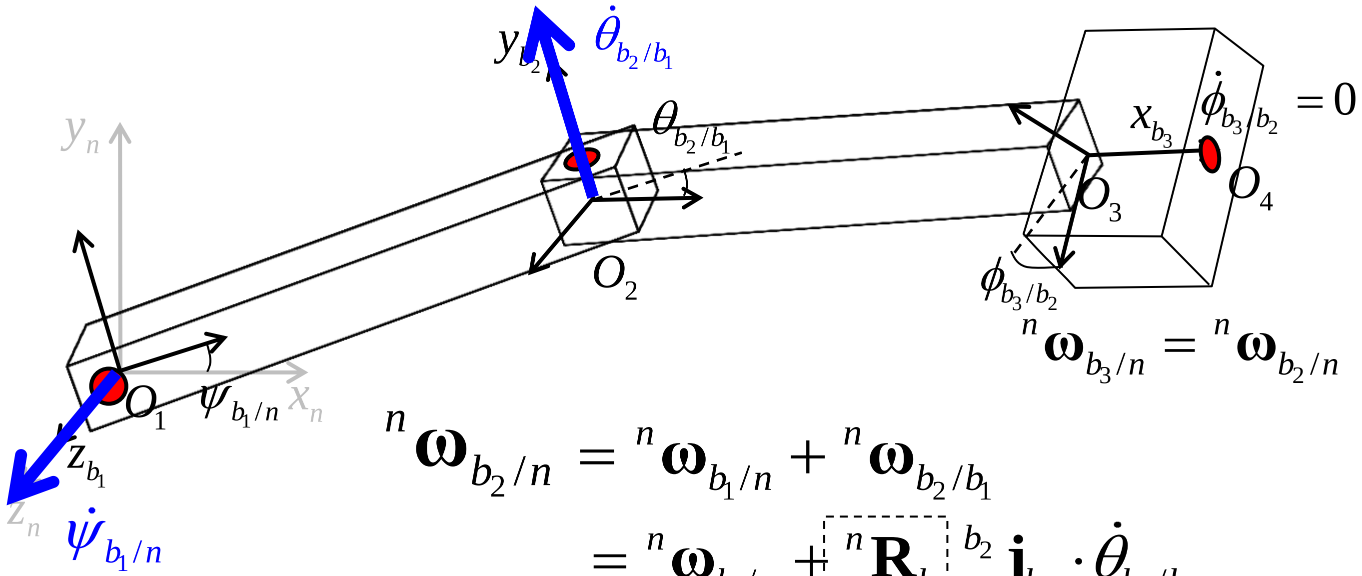
Example of 3-Link Arm

- angular velocities

$${}^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix}$$

$${}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix}$$



$${}^{n}\boldsymbol{\omega}_{b_3/n} = {}^{n}\boldsymbol{\omega}_{b_2/n}$$

$${}^{n}\boldsymbol{\omega}_{b_1/n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\psi}_{b_1/n}$$

$${}^{n}\boldsymbol{\omega}_{b_2/b_1} = \begin{bmatrix} -\sin\psi_{b_1/n} \\ \cos\psi_{b_1/n} \\ 0 \end{bmatrix} \cdot \dot{\theta}_{b_2/b_1}$$

$$\begin{aligned} {}^{n}\boldsymbol{\omega}_{b_2/n} &= {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{n}\boldsymbol{\omega}_{b_2/b_1} \\ &= {}^{n}\boldsymbol{\omega}_{b_1/n} + \boxed{{}^{n}\mathbf{R}_{b_2}} \, {}^{b_2}\mathbf{j}_{b_2} \cdot \dot{\theta}_{b_2/b_1} \end{aligned}$$

$$= {}^{n}\boldsymbol{\omega}_{b_1/n} + \boxed{\begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \dot{\theta}_{b_2/b_1}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\psi}_{b_1/n} + \begin{bmatrix} -\sin\psi_{b_1/n} \\ \cos\psi_{b_1/n} \\ 0 \end{bmatrix} \cdot \dot{\theta}_{b_2/b_1}$$

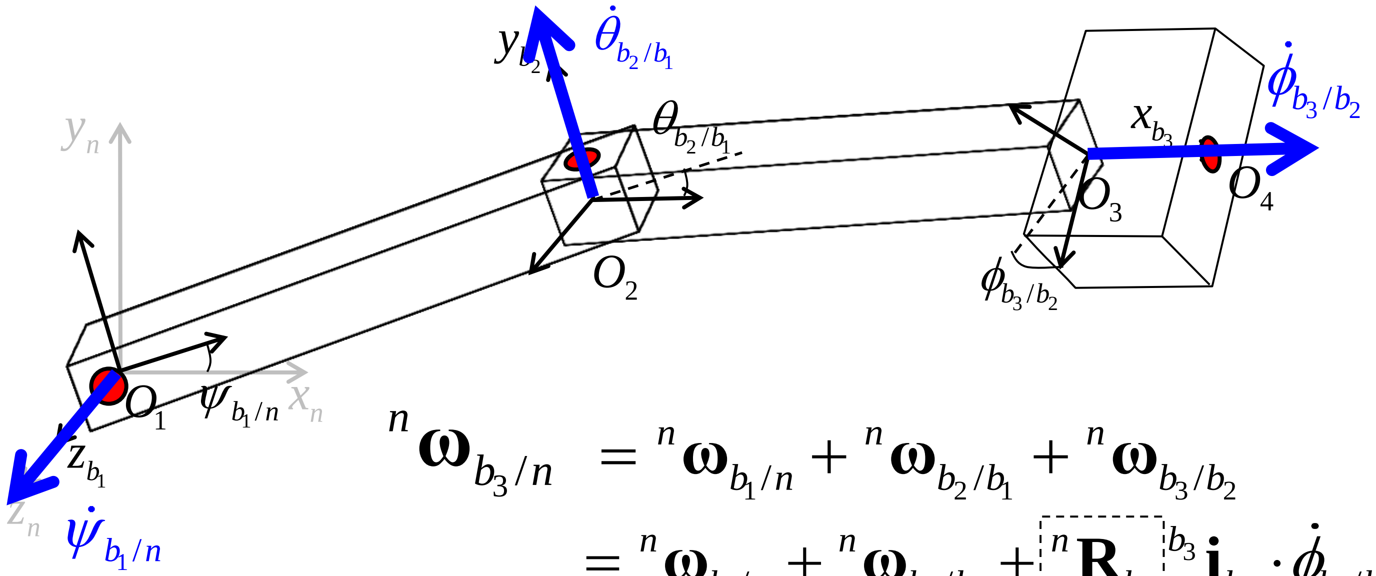
Example of 3-Link Arm

- angular velocities

$$^n\mathbf{R}_{b_1} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix}$$

$$^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix}$$



$$^n\boldsymbol{\omega}_{b_1/n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\psi}_{b_1/n}$$

$$^n\boldsymbol{\omega}_{b_2/b_1} = \begin{bmatrix} -\sin\psi_{b_1/n} \\ \cos\psi_{b_1/n} \\ 0 \end{bmatrix} \cdot \dot{\theta}_{b_2/b_1}$$

$$^n\boldsymbol{\omega}_{b_3/n} = {}^n\boldsymbol{\omega}_{b_1/n} + {}^n\boldsymbol{\omega}_{b_2/b_1} + {}^n\boldsymbol{\omega}_{b_3/b_2}$$

$$= {}^n\boldsymbol{\omega}_{b_1/n} + {}^n\boldsymbol{\omega}_{b_2/b_1} + \boxed{{}^n\mathbf{R}_{b_3}}\, {}^{b_3}\mathbf{i}_{b_3} \cdot \dot{\phi}_{b_3/b_2}$$

$$= {}^n\boldsymbol{\omega}_{b_1/n} + {}^n\boldsymbol{\omega}_{b_2/b_1} + \boxed{\begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \dot{\phi}_{b_3/b_2}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\psi}_{b_1/n} + \begin{bmatrix} -\sin\psi_{b_1/n} \\ \cos\psi_{b_1/n} \\ 0 \end{bmatrix} \cdot \dot{\theta}_{b_2/b_1} + \boxed{\begin{bmatrix} \cos\psi_{b_1/n} \cos\theta_{b_2/b_1} \\ \sin\psi_{b_1/n} \cos\theta_{b_2/b_1} \\ -\sin\theta_{b_2/b_1} \end{bmatrix}} \cdot \dot{\phi}_{b_3/b_2}$$

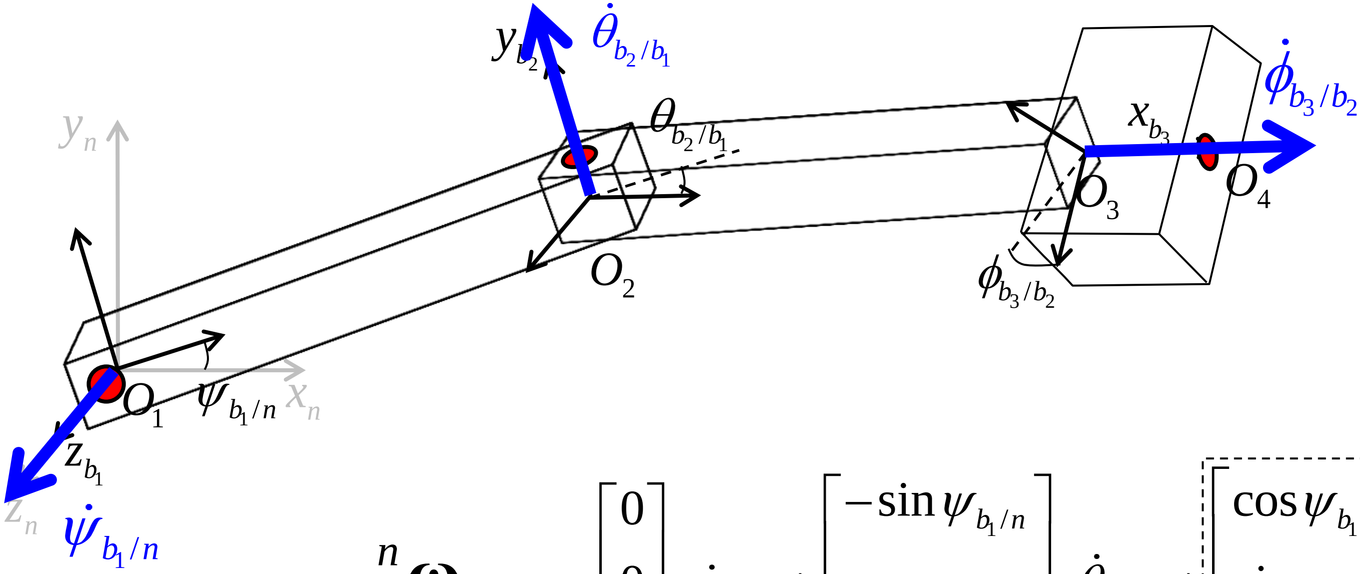
Example of 3-Link Arm

- angular velocities

$$ {}^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix} $$

$$ {}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix} $$



$$ {}^{n}\boldsymbol{\omega}_{b_3/n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\psi}_{b_1/n} + \begin{bmatrix} -\sin\psi_{b_1/n} \\ \cos\psi_{b_1/n} \\ 0 \end{bmatrix} \cdot \dot{\theta}_{b_2/b_1} + \boxed{\begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} \\ -\sin\theta_{b_2/b_1} \end{bmatrix}} \cdot \dot{\phi}_{b_3/b_2} $$

$$ = \underbrace{\begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} & \cos\psi_{b_1/n} & 0 \\ -\sin\theta_{b_2/b_1} & 0 & 1 \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} \dot{\phi}_{b_3/b_2} \\ \dot{\theta}_{b_2/b_1} \\ \dot{\psi}_{b_1/n} \end{bmatrix}}_{\dot{\boldsymbol{\gamma}}} $$

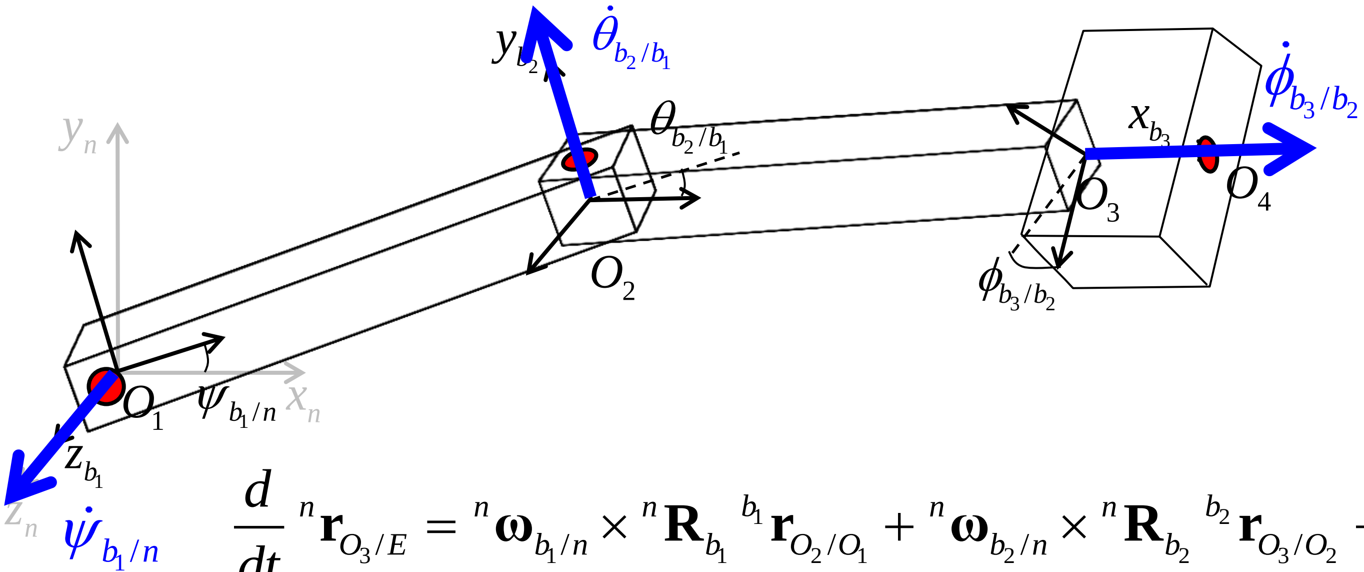
Example of 3-Link Arm

- linear velocity vector of end-effector(O_4)

$$ {}^{n}\mathbf{R}_{b_1} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{b_1}\mathbf{R}_{b_2} = \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix} $$

$$ {}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix} $$



$$ \frac{d}{dt}\,{}^{n}\mathbf{r}_{O_3/E} = {}^{n}\boldsymbol{\omega}_{b_1/n} \times {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{n}\boldsymbol{\omega}_{b_2/n} \times {}^{n}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{r}_{O_3/O_2} + {}^{n}\boldsymbol{\omega}_{b_3/n} \times {}^{n}\mathbf{R}_{b_3}\,{}^{b_3}\mathbf{r}_{O_4/O_3} $$

$$ {}^{n}\boldsymbol{\omega}_{b_1/n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\psi}_{b_1/n},\ {}^{n}\boldsymbol{\omega}_{b_2/n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\psi}_{b_1/n} + \begin{bmatrix} -\sin\psi_{b_1/n} \\ \cos\psi_{b_1/n} \\ 0 \end{bmatrix} \cdot \dot{\theta}_{b_2/b_1},\ {}^{n}\boldsymbol{\omega}_{b_3/n} = \underbrace{\begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} & \cos\psi_{b_1/n} & 0 \\ -\sin\theta_{b_2/b_1} & 0 & 1 \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} \dot{\phi}_{b_3/b_2} \\ \dot{\theta}_{b_2/b_1} \\ \dot{\psi}_{b_1/n} \end{bmatrix}}_{\dot{\boldsymbol{\gamma}}} $$

We can calculate the angular velocities using time derivative of Euler angle $\dot{\phi}_{b_3/b_2},\ \dot{\theta}_{b_2/b_1},\ \dot{\psi}_{b_1/n}$.

Summary of Euler angles

$$ {}^{n}\mathbf{R}_{b_3} = \begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} & \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} - \sin\psi_{b_1/n}\cos\phi_{b_3/b_2} & \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} + \sin\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} & \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} + \cos\psi_{b_1/n}\cos\phi_{b_3/b_2} & \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} - \cos\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ -\sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1}\sin\phi_{b_3/b_2} & \cos\theta_{b_2/b_1}\cos\phi_{b_3/b_2} \end{bmatrix} $$

Calculation of rotational transformation matrix using ZYX Euler angle $\phi_{b_3/b_2}, \theta_{b_2/b_1}, \psi_{b_1/n}$

$$ {}^{n}\boldsymbol{\omega}_{b_3/n} = \underbrace{\begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} & \cos\psi_{b_1/n} & 0 \\ -\sin\theta_{b_2/b_1} & 0 & 1 \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} \dot{\phi}_{b_3/b_2} \\ \dot{\theta}_{b_2/b_1} \\ \dot{\psi}_{b_1/n} \end{bmatrix}}_{\dot{\boldsymbol{\gamma}}} $$

We can calculate the angular velocities using time derivative of Euler angle $\dot{\phi}_{b_3/b_2}, \dot{\theta}_{b_2/b_1}, \dot{\psi}_{b_1/n}$.

4.3 Gimbal lock of the Euler angle

Inverse Dynamics of 3-Link Arm

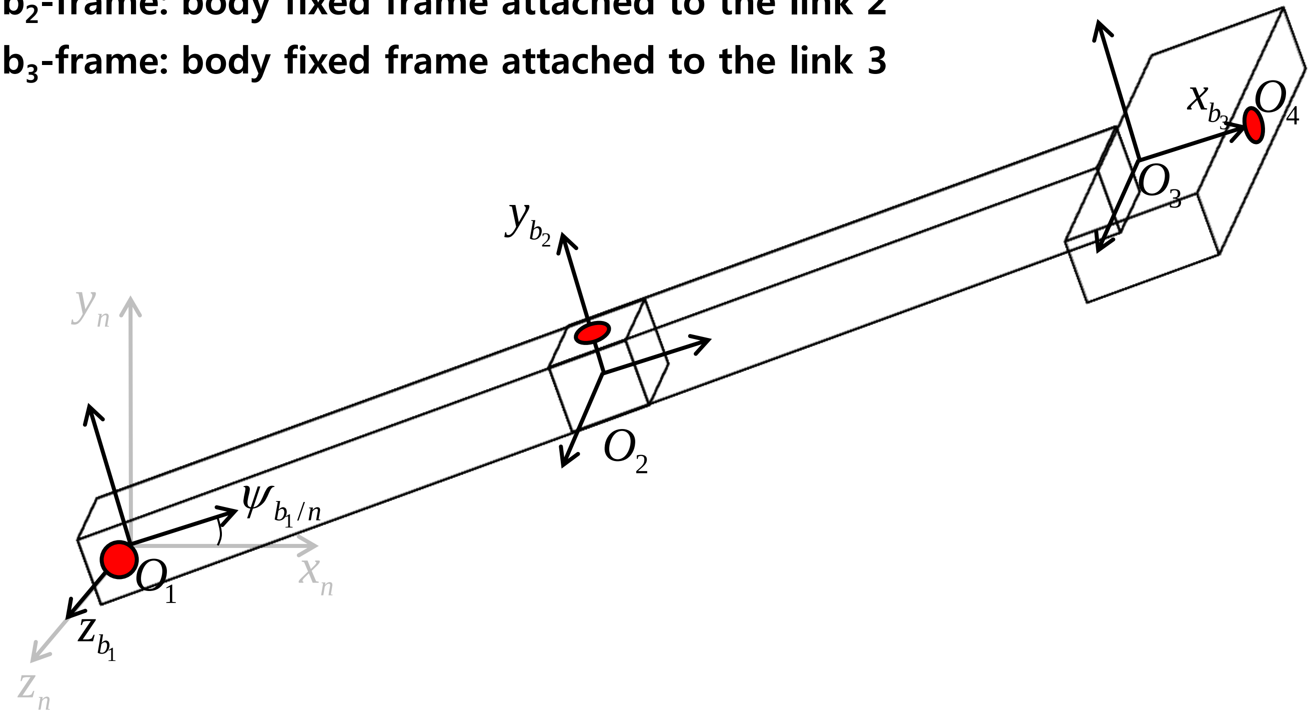
- Gimbal lock of the Euler angle(1).

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

b_3-frame: body fixed frame attached to the link 3



Example of 3-Link Arm

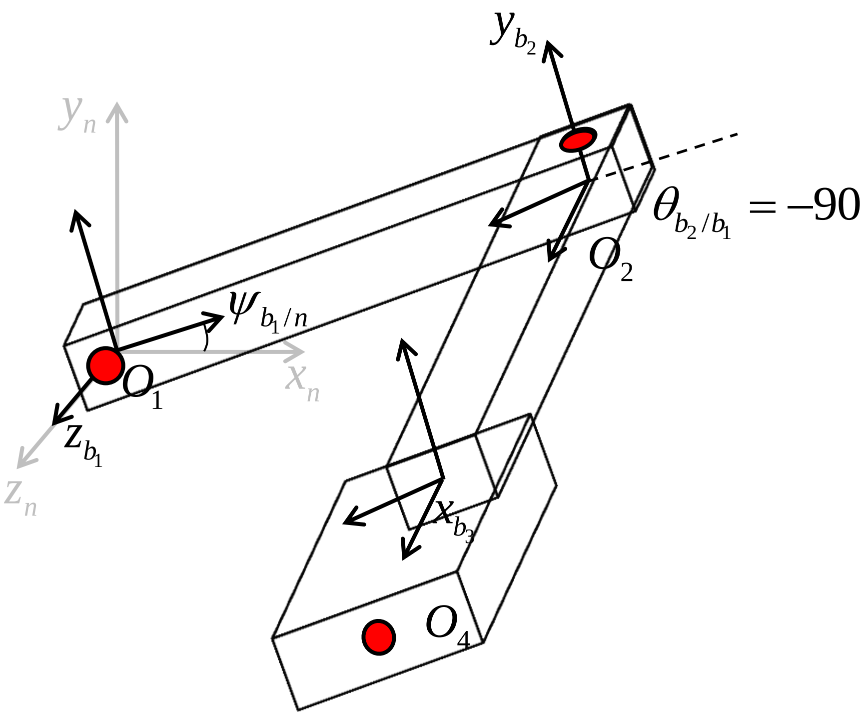
- Gimbal lock of the Euler angle(1).

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

b_3-frame: body fixed frame attached to the link 3



Example of 3-Link Arm

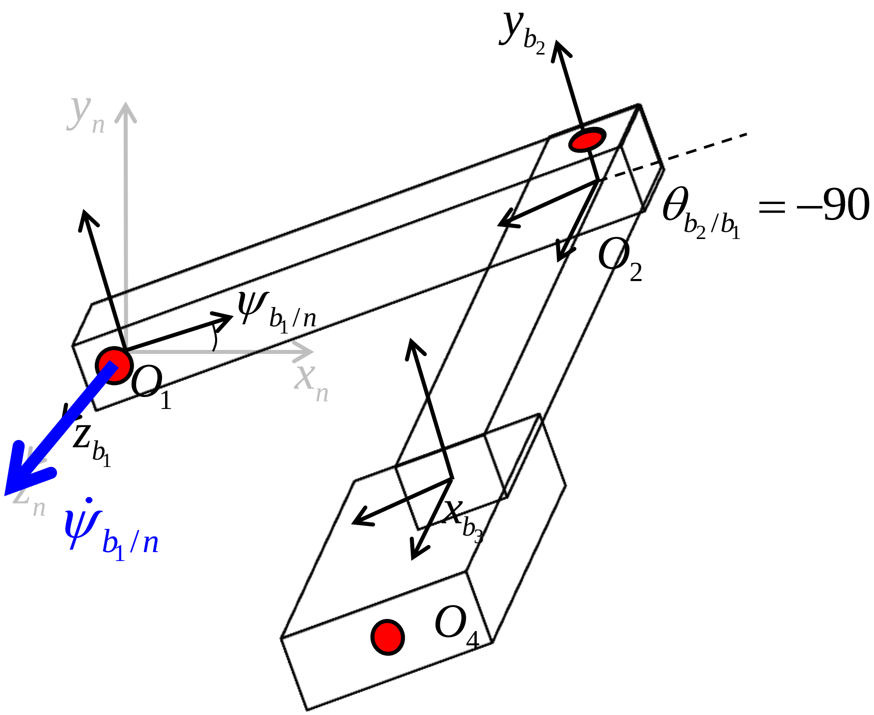
- Gimbal lock of the Euler angle(1).

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

b_3-frame: body fixed frame attached to the link 3



$${}^{n}\boldsymbol{\omega}_{b_3/n} = {}^{n}\boldsymbol{\omega}_{b_1/n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\psi}_{b_1/n}$$

Example of 3-Link Arm

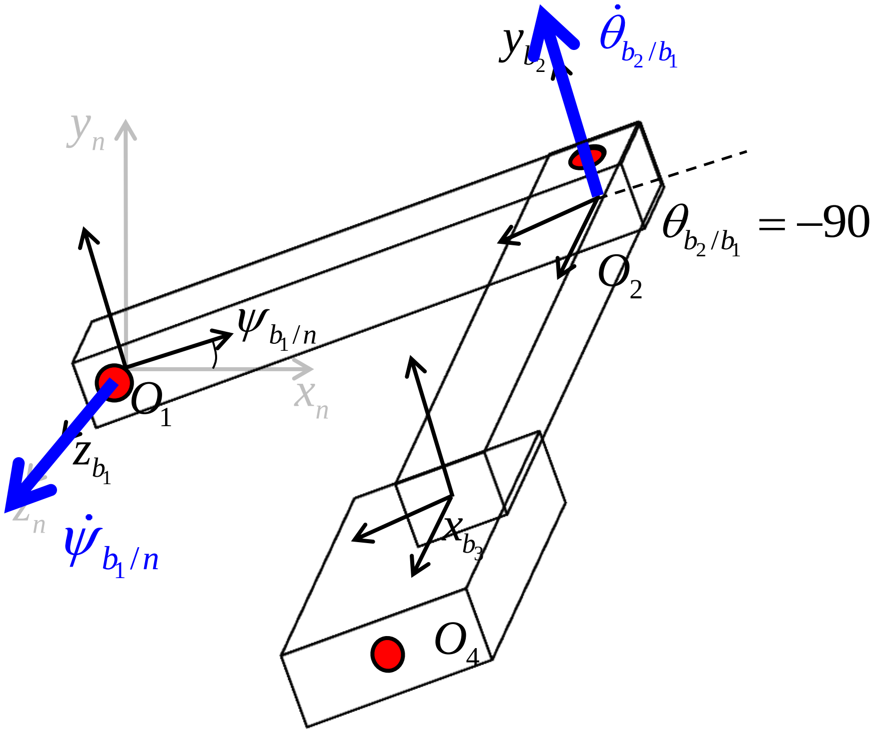
- Gimbal lock of the Euler angle(1).

n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

b_3-frame: body fixed frame attached to the link 3



$$ {}^{n}\boldsymbol{\omega}_{b_3/n} = {}^{n}\boldsymbol{\omega}_{b_2/n} $$

$$ = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\psi}_{b_1/n} + \begin{bmatrix} -\sin\psi_{b_1/n} \\ \cos\psi_{b_1/n} \\ 0 \end{bmatrix} \cdot \dot{\theta}_{b_2/b_1} $$

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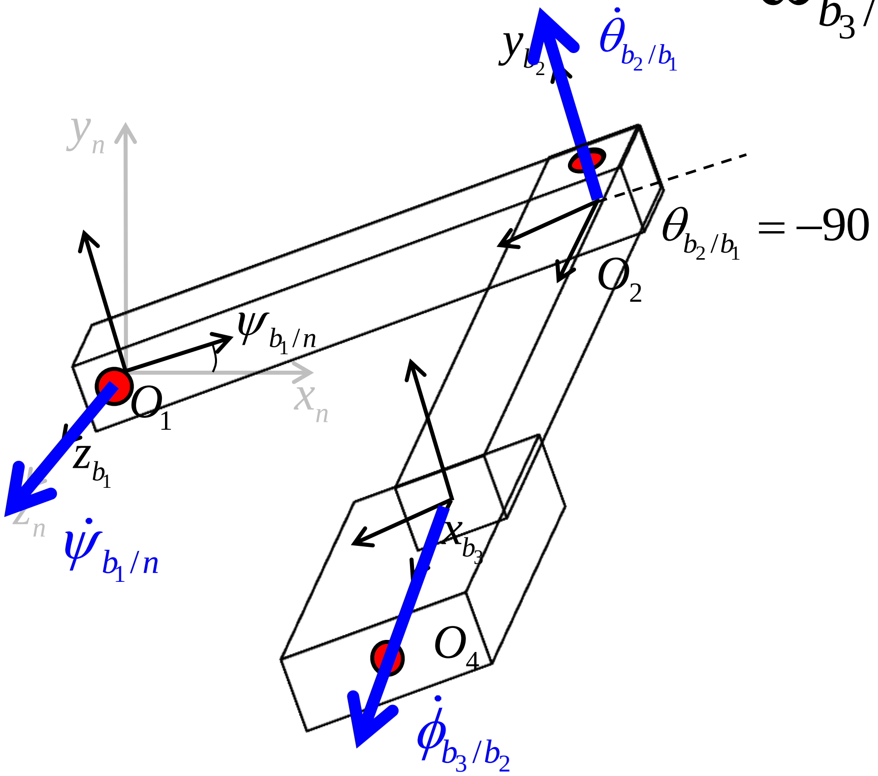

$$
{}^{n}\boldsymbol{\omega}_{b_3/n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \dot{\psi}_{b_1/n} + \begin{bmatrix} -\sin\psi_{b_1/n} \\ \cos\psi_{b_1/n} \\ 0 \end{bmatrix} \cdot \dot{\theta}_{b_2/b_1} + \begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} \\ -\sin\theta_{b_2/b_1} \end{bmatrix} \cdot \dot{\phi}_{b_3/b_2}
$$

$$
= \begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} & \cos\psi_{b_1/n} & 0 \\ -\sin\theta_{b_2/b_1} & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{b_3/b_2} \\ \dot{\theta}_{b_2/b_1} \\ \dot{\psi}_{b_1/n} \end{bmatrix}
$$

$$
{}^{n}\boldsymbol{\omega}_{b_3/n} = \mathbf{G}\dot{\boldsymbol{\gamma}}
$$

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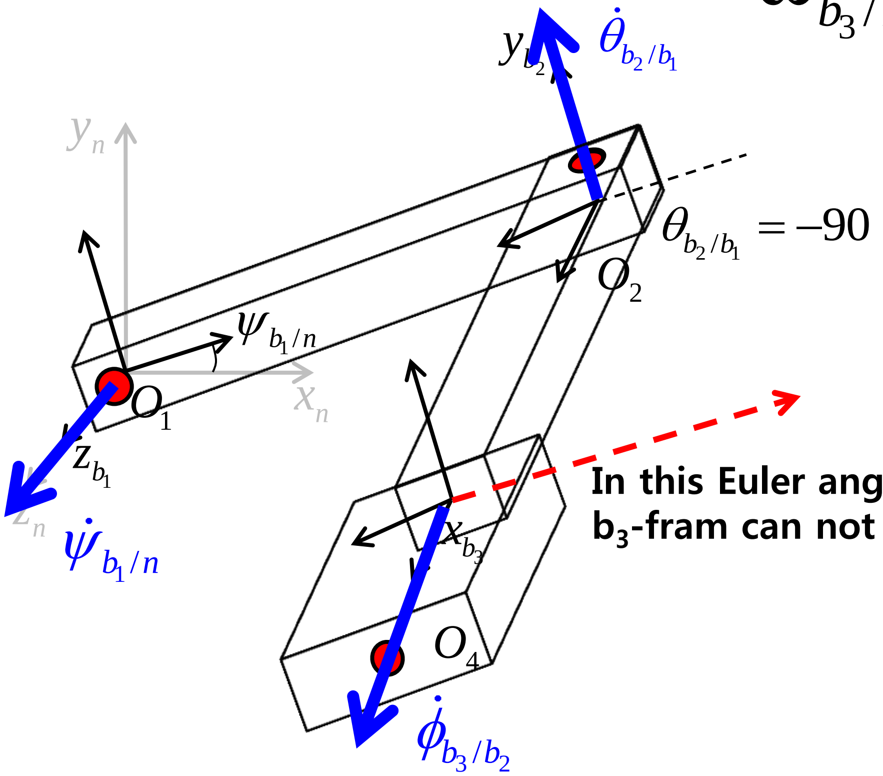
$$ {}^{n}\boldsymbol{\omega}_{b_3/n} = \begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} & \cos\psi_{b_1/n} & 0 \\ -\sin\theta_{b_2/b_1} & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{b_3/b_2} \\ \dot{\theta}_{b_2/b_1} \\ \dot{\psi}_{b_1/n} \end{bmatrix} $$



when $\theta_{b_2/b_1} = 90$

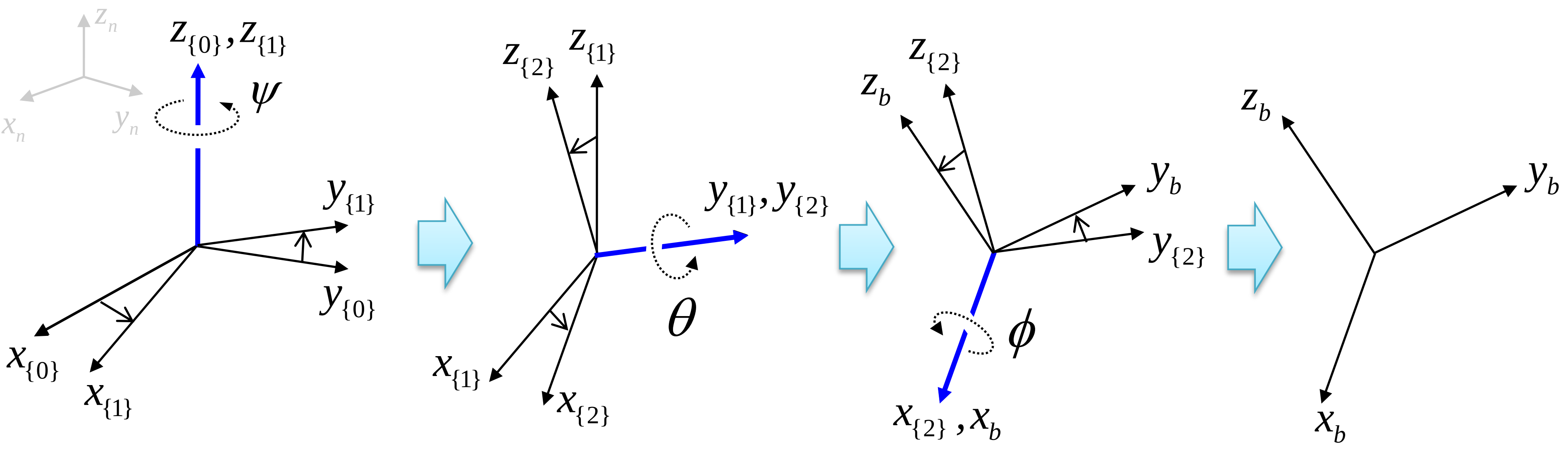
$$ {}^{n}\boldsymbol{\omega}_{b_3/n} = \begin{bmatrix} 0 & -\sin\psi_{b_1/n} & 0 \\ 0 & \cos\psi_{b_1/n} & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{b_3/b_2} \\ \dot{\theta}_{b_2/b_1} \\ \dot{\psi}_{b_1/n} \end{bmatrix} $$

In this Euler angle $(\theta_{b_2/b_1} = 90)$ **,**
b_3-fram can not rotate about this axis.

Finite rotation vs. infinitesimal rotation

1) Ginsberg, J., Engineering Dynamics, Cambridge University Press, pp.105, 2008.

✓ Example of 3-D rotation: ZYX Euler angles



Rotation of an angle ψ about the $z_{\{0\}}$ axis

$$\mathbf{R}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}$$

Rotation of an angle θ about the $y_{\{1\}}$ axis

$$\mathbf{R}_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Rotation of an angle ϕ about the $x_{\{2\}}$ axis

$$\mathbf{R}_x = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation(transformation) matrix

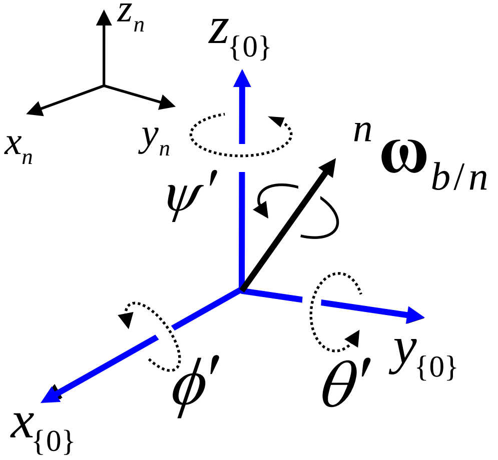
$$\mathbf{R} = \mathbf{R}_z \ \mathbf{R}_y \ \mathbf{R}_x$$

Finite rotation

The rotation transformation depends on the sequence in which the rotations occur.[1)]

Finite rotation vs. infinitesimal rotation

- ✓ **If the angles are very small, the rotation matrix is**



$$\mathbf{R} = \mathbf{R}_z\ \mathbf{R}_y\ \mathbf{R}_x,$$

$$\mathbf{R}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi' & \sin\psi' \\ 0 & -\sin\psi' & \cos\psi' \end{bmatrix}, \mathbf{R}_y = \begin{bmatrix} \cos\theta' & 0 & -\sin\theta' \\ 0 & 1 & 0 \\ \sin\theta' & 0 & \cos\theta' \end{bmatrix}, \mathbf{R}_x = \begin{bmatrix} \cos\phi' & \sin\phi' & 0 \\ -\sin\phi' & \cos\phi' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cos\psi' \approx 1$
$\sin\psi' \approx \psi'$

$\cos\theta' \approx 1$
$\sin\theta' \approx \theta'$

$\cos\phi' \approx 1$
$\sin\phi' \approx \phi'$

$$\mathbf{R}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \psi' \\ 0 & -\psi' & 1 \end{bmatrix}, \qquad \mathbf{R}_y = \begin{bmatrix} 1 & 0 & -\theta' \\ 0 & 1 & 0 \\ \theta' & 0 & 1 \end{bmatrix}, \qquad \mathbf{R}_x = \begin{bmatrix} 1 & \phi' & 0 \\ -\phi' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \psi' + \cancel{\theta'\phi'} & -\theta' + \cancel{\psi'\phi'} \\ -\psi' & 1 - \cancel{\psi'\theta'\phi'} & \phi' + \cancel{\psi'\theta'} \\ \theta & -\phi' & 1 \end{bmatrix} = \begin{bmatrix} 1 & \psi' & -\theta' \\ -\psi' & 1 & \phi' \\ \theta & -\phi' & 1 \end{bmatrix}$$

Infinitesimal rotation[1)]

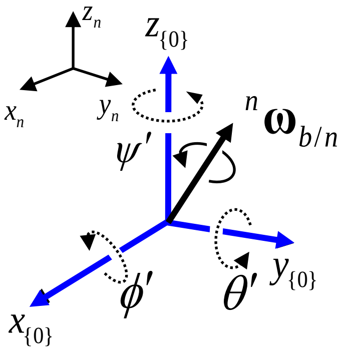
$${}^{n}\boldsymbol{\omega}_{b/n} = \begin{bmatrix} \dfrac{d\phi'}{dt} & \dfrac{d\theta'}{dt} & \dfrac{d\psi'}{dt} \end{bmatrix}^T$$

[1)] Ginsberg, J., Engineering Dynamics, Cambridge University Press, pp.120, 2008.

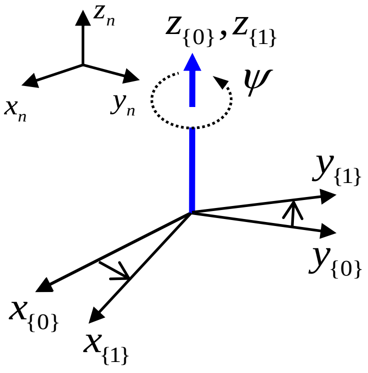
Finite rotation vs. infinitesimal rotation

1) Ginsberg, J., Engineering Dynamics, Cambridge University Press, pp.120, 2008.

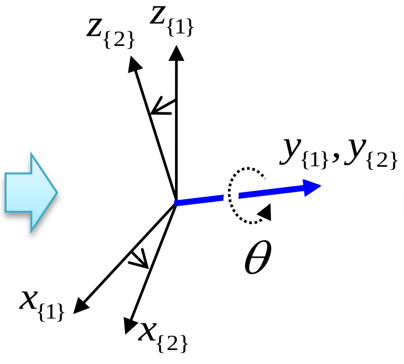
✓ Angular velocity (infinitesimal rotation)



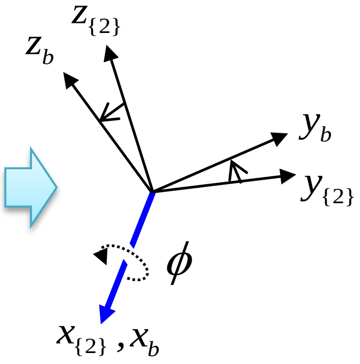
✓ ZYX Euler angles (finite rotation)



Rotation of an angle ψ about the z{0} axis



Rotation of an angle θ about the y{1} axis



Rotation of an angle ϕ about the x{2} axis

$$^n\boldsymbol{\omega}_{b/n} = \begin{bmatrix} \frac{d\phi'}{dt} & \frac{d\theta'}{dt} & \frac{d\psi'}{dt} \end{bmatrix}^T \neq \dot{\boldsymbol{\gamma}} = \begin{bmatrix} \frac{d\phi}{dt} & \frac{d\theta}{dt} & \frac{d\psi}{dt} \end{bmatrix}^T$$

$$\boxed{^n\boldsymbol{\omega}_{b/n} = {}^n\mathbf{G}\dot{\boldsymbol{\gamma}}}$$

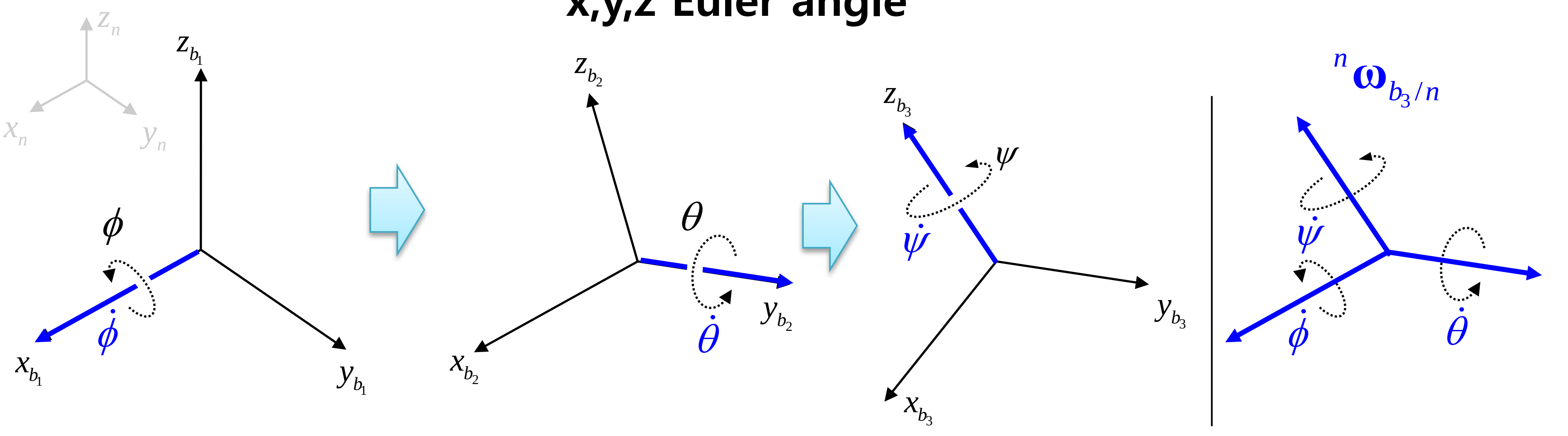
$$where\ {}^n\mathbf{G} = \begin{bmatrix} 1 & 0 & \sin\theta \\ 0 & \cos\phi & -\sin\phi\cos\theta \\ 0 & \sin\phi & \cos\phi\cos\theta \end{bmatrix}$$

Relationship between the time derivative of the Euler angle and angular velocity

x,y,z Euler angle at time: $\begin{bmatrix}\phi & \theta & \psi\end{bmatrix}^T$

Time derivative of Euler angle: $\begin{bmatrix}\dot\phi & \dot\theta & \dot\psi\end{bmatrix}^T$

x,y,z Euler angle



$$ {}^n\boldsymbol{\omega}_{b_3/n} = {}^n\mathbf{i}_{b_1}\dot\phi + {}^n\mathbf{j}_{b_2}\dot\theta + {}^n\mathbf{k}_{b_3}\dot\psi $$

$$ = \begin{bmatrix}1\\0\\0\end{bmatrix}\dot\phi + \begin{bmatrix}0\\\cos\phi\\\sin\phi\end{bmatrix}\dot\theta + \begin{bmatrix}\sin\theta\\-\sin\phi\cos\theta\\\cos\phi\cos\theta\end{bmatrix}\dot\psi = \underset{\mathbf{G}}{\begin{bmatrix}1 & 0 & \sin\theta\\0 & \cos\phi & -\sin\phi\cos\theta\\0 & \sin\phi & \cos\phi\cos\theta\end{bmatrix}}\underset{\dot{\boldsymbol{\gamma}}}{\begin{bmatrix}\dot\phi\\\dot\theta\\\dot\psi\end{bmatrix}} \Rightarrow {}^n\boldsymbol{\omega}_{b_3/n} = \mathbf{G}\dot{\boldsymbol{\gamma}} $$

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 68

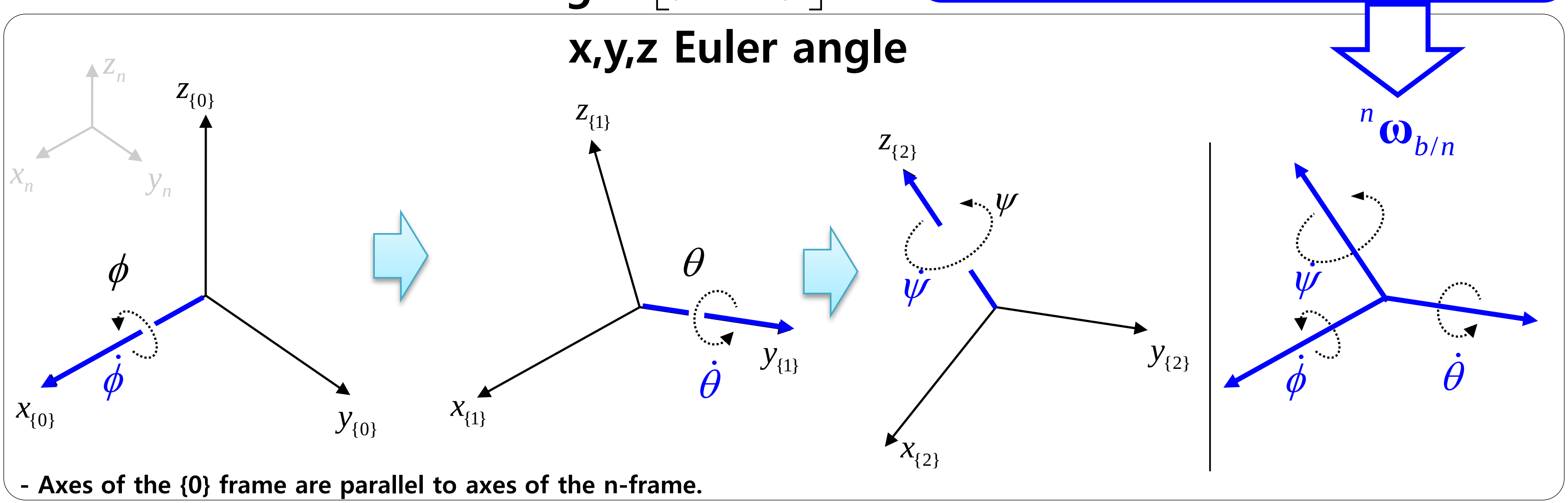
Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Relation between the time derivative of the Euler angle and angular velocity in 3 dimensional motion

x,y,z Euler angle at time: $\begin{bmatrix}\phi & \theta & \psi\end{bmatrix}^T$

Time derivative of Euler angle: $\begin{bmatrix}\dot{\phi} & \dot{\theta} & \dot{\psi}\end{bmatrix}^T$

An angular velocity $^n\boldsymbol{\omega}_{b/n}$ is the sum of simple rotations.



x,y,z Euler angle

- Axes of the {0} frame are parallel to axes of the n-frame.

The time derivative of Euler angle $\begin{bmatrix}\dot{\phi} & \dot{\theta} & \dot{\psi}\end{bmatrix}^T$ is the angular velocity components directed along $^n\mathbf{x}_{\{0\}}, {}^n\mathbf{y}_{\{1\}}, {}^n\mathbf{z}_{\{2\}}$ axes.

$$^n\boldsymbol{\omega}_{b/n} = {}^n\mathbf{x}_{\{0\}}\dot{\phi} + {}^n\mathbf{y}_{\{1\}}\dot{\theta} + {}^n\mathbf{z}_{\{2\}}\dot{\psi}$$

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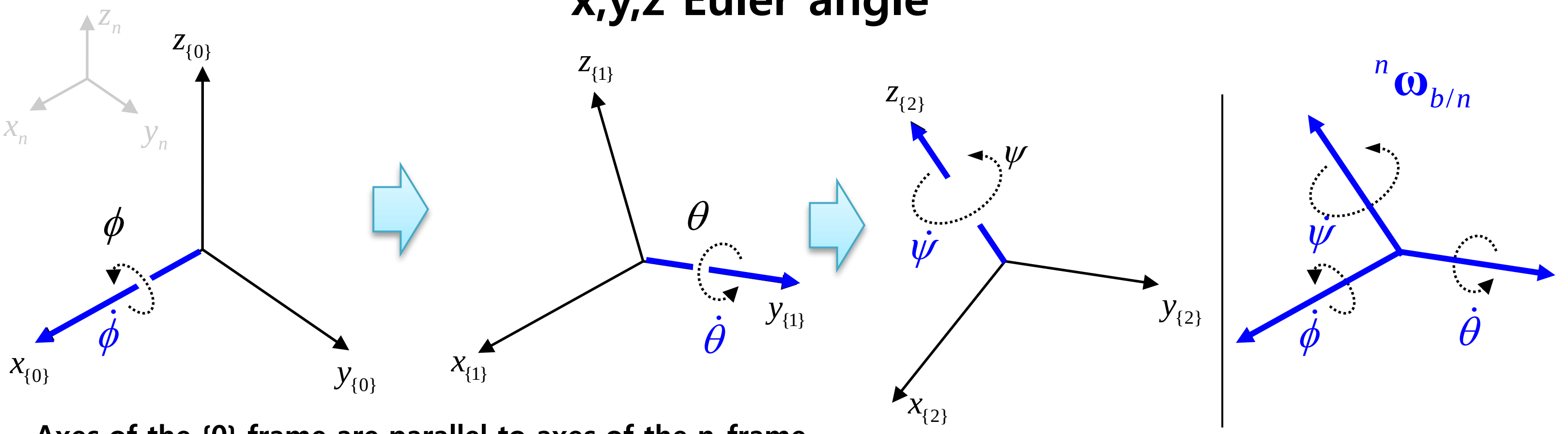
Relation between the time derivative of the Euler angle and angular velocity in 3 dimensional motion

x,y,z Euler angle at time: $[\phi \quad \theta \quad \psi]^T$ **Time derivative of Euler angle:** $[\dot\phi \quad \dot\theta \quad \dot\psi]^T$

x,y,z Euler angle



- Axes of the {0} frame are parallel to axes of the n-frame.

$${}^{n}\boldsymbol{\omega}_{b/n} = {}^{n}\mathbf{x}_{\{0\}}\dot\phi + {}^{n}\mathbf{y}_{\{1\}}\dot\theta + {}^{n}\mathbf{z}_{\{2\}}\dot\psi$$

$$= {}^{n}\mathbf{R}_{\{0\}}\,{}^{\{0\}}\mathbf{x}_{\{0\}}\dot\phi + {}^{n}\mathbf{R}_{\{0\}}\,{}^{\{0\}}\mathbf{R}_{\{1\}}\,{}^{\{1\}}\mathbf{y}_{\{1\}}\dot\theta + {}^{n}\mathbf{R}_{\{0\}}\,{}^{\{0\}}\mathbf{R}_{\{1\}}\,{}^{\{1\}}\mathbf{R}_{\{2\}}\,{}^{\{2\}}\mathbf{z}_{\{2\}}\dot\psi$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\dot\phi + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\dot\theta + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\dot\psi$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\dot\phi + \begin{bmatrix} 0 \\ \cos\phi \\ \sin\phi \end{bmatrix}\dot\theta + \begin{bmatrix} \sin\theta \\ -\sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix}\dot\psi = \underbrace{\begin{bmatrix} 1 & 0 & \sin\theta \\ 0 & \cos\phi & -\sin\phi\cos\theta \\ 0 & \sin\phi & \cos\phi\cos\theta \end{bmatrix}}_{{}^{n}\mathbf{G}}\underbrace{\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}}_{\dot{\boldsymbol{\gamma}}} \Rightarrow \boxed{{}^{n}\boldsymbol{\omega}_{b/n} = {}^{n}\mathbf{G}\dot{\boldsymbol{\gamma}}}$$

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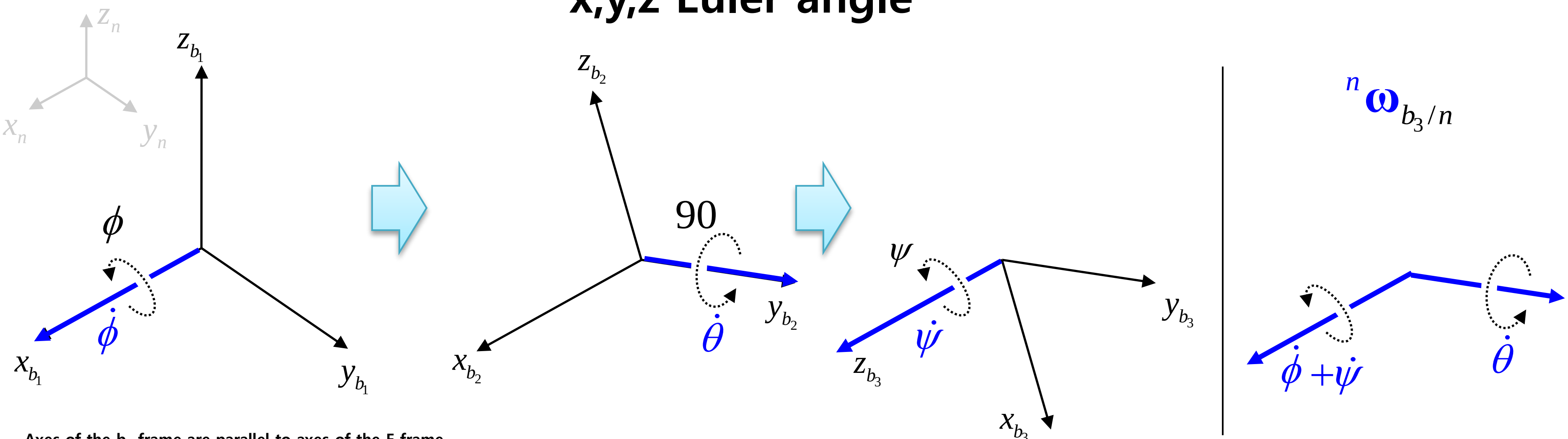
Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Gimbal lock of the Euler angle(1).

x,y,z Euler angle at time: $\begin{bmatrix}\phi & 90 & \psi\end{bmatrix}^T$

Time derivative of Euler angle: $\begin{bmatrix}\dot\phi & \dot\theta & \dot\psi\end{bmatrix}^T$

x,y,z Euler angle



- Axes of the b_1 frame are parallel to axes of the E frame.

$$
{}^{n}\boldsymbol{\omega}_{b_3/n} = {}^{n}\mathbf{i}_{b_1}\dot\phi + {}^{n}\mathbf{j}_{b_2}\dot\theta + {}^{n}\mathbf{k}_{b_3}\dot\psi
$$

$$
= \begin{bmatrix}1\\0\\0\end{bmatrix}\dot\phi + \begin{bmatrix}0\\\cos\phi\\\sin\phi\end{bmatrix}\dot\theta + \begin{bmatrix}1\\0\\0\end{bmatrix}\dot\psi = \overset{\mathbf{G}}{\boxed{\begin{bmatrix}1 & 0 & 1\\0 & \cos\phi & 0\\0 & \sin\phi & 0\end{bmatrix}}}\overset{\dot{\boldsymbol{\gamma}}}{\begin{bmatrix}\dot\phi\\\dot\theta\\\dot\psi\end{bmatrix}}
$$

$\mathbf{G}$ does not have inverse matrix.

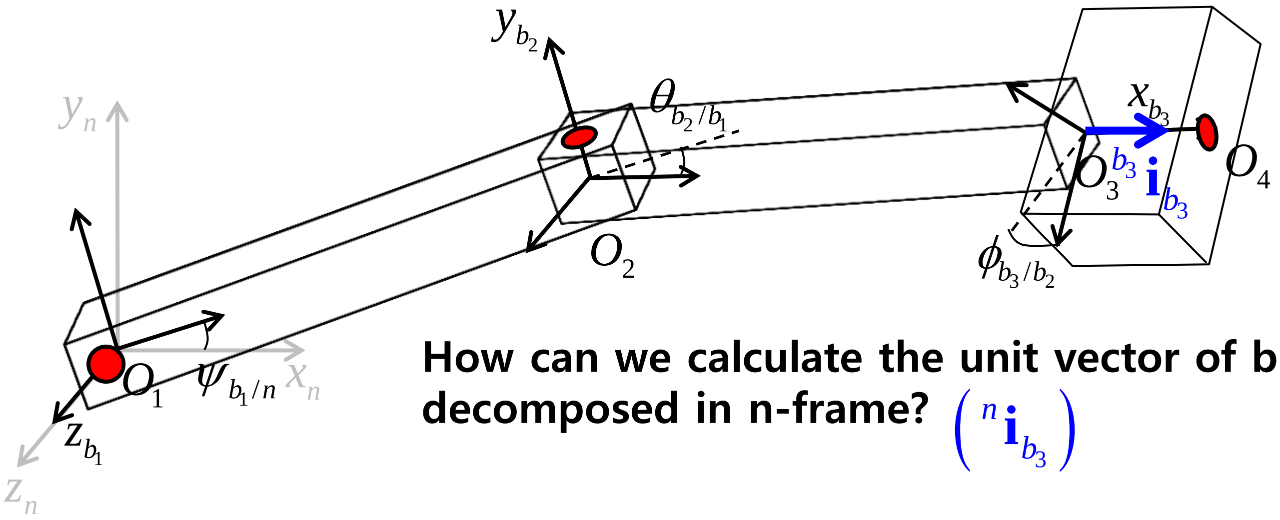
We cannot determine $\dot{\boldsymbol{\gamma}}$ from ${}^{n}\boldsymbol{\omega}_{b_3/n}$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 210 ~ 213

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

Example of 3-Link Arm

- the relationship between the unit vector of body fixed frame and rotational transformation matrix



How can we calculate the unit vector of b_3-frame in x direction $\left({}^{b_3}\mathbf{i}_{b_3}\right)$ decomposed in n-frame? $\left({}^{n}\mathbf{i}_{b_3}\right)$

$${}^{n}\mathbf{i}_{b_3} = {}^{n}\mathbf{R}_{b_3}\,{}^{b_3}\mathbf{i}_{b_3}$$

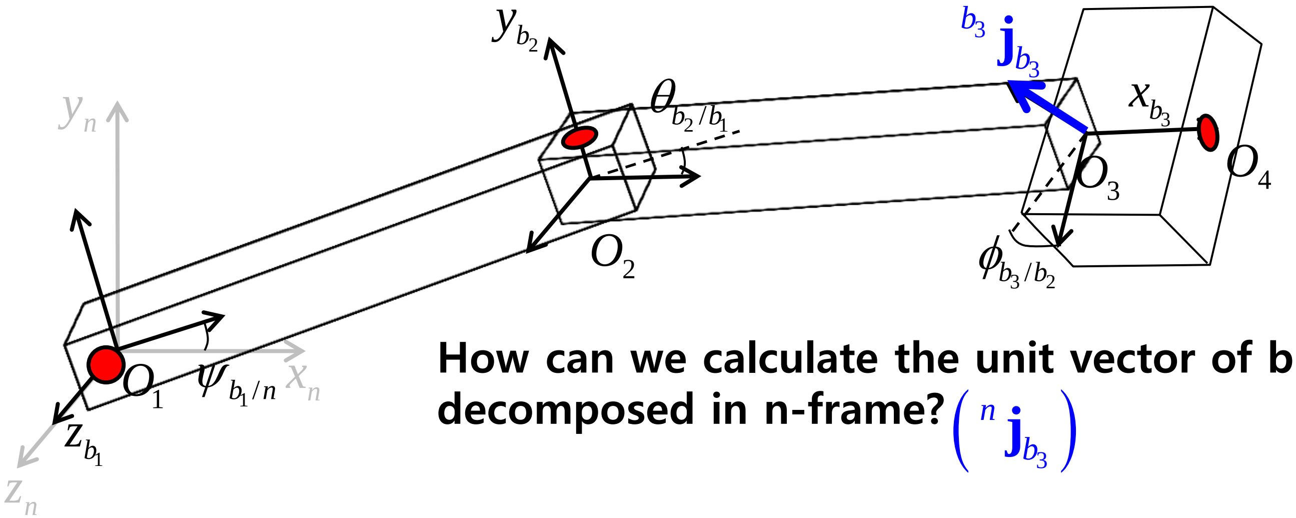
$$= \left[\boxed{\begin{matrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} \\ -\sin\theta_{b_2/b_1} \end{matrix}}\begin{matrix} \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} - \sin\psi_{b_1/n}\cos\phi_{b_3/b_2} & \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} + \sin\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} + \cos\psi_{b_1/n}\cos\phi_{b_3/b_2} & \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} - \cos\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ \cos\theta_{b_2/b_1}\sin\phi_{b_3/b_2} & \cos\theta_{b_2/b_1}\cos\phi_{b_3/b_2} \end{matrix}\right]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\begin{matrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} \\ -\sin\theta_{b_2/b_1} \end{matrix}} \end{bmatrix}$$

The first column of ${}^{n}\mathbf{R}_{b_3}$ is the unit vector of b_3-frame in x direction decomposed in n-frame.

Example of 3-Link Arm

- the relationship between the unit vector of body fixed frame and rotational transformation matrix



How can we calculate the unit vector of b$_3$-frame in y direction $\left({}^{b_3}\mathbf{j}_{b_3}\right)$ decomposed in n-frame? $\left({}^{n}\mathbf{j}_{b_3}\right)$

$$ {}^{n}\mathbf{j}_{b_3} = {}^{n}\mathbf{R}_{b_3}\, {}^{b_3}\mathbf{j}_{b_3} $$

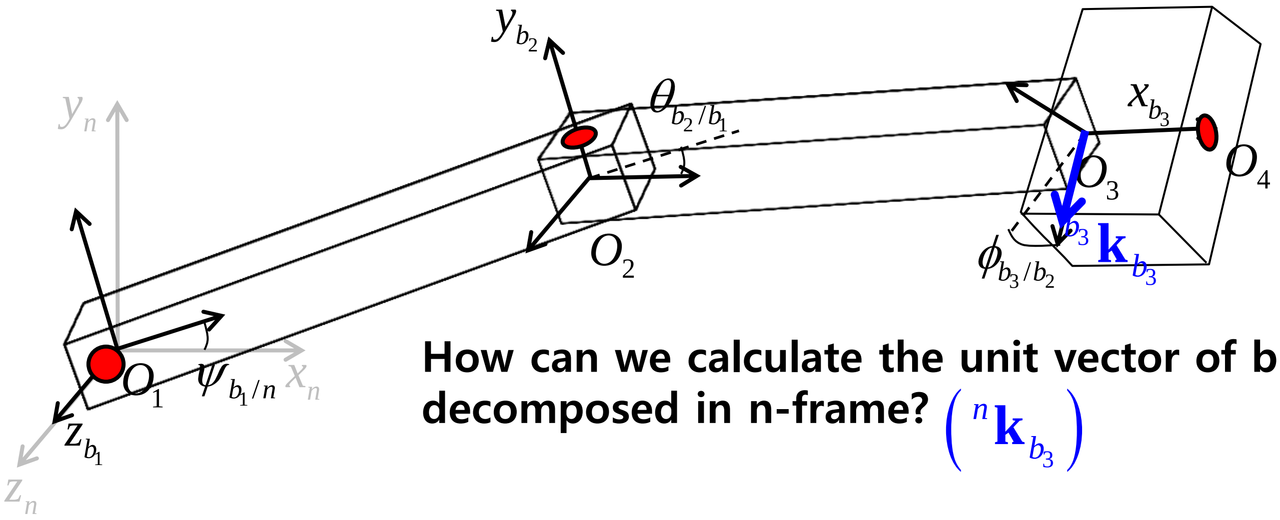
$$ = \begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} & \boxed{\begin{matrix}\cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} - \sin\psi_{b_1/n}\cos\phi_{b_3/b_2} \\ \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} + \cos\psi_{b_1/n}\cos\phi_{b_3/b_2} \\ \cos\theta_{b_2/b_1}\sin\phi_{b_3/b_2}\end{matrix}} & \begin{matrix}\cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} + \sin\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} - \cos\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ \cos\theta_{b_2/b_1}\cos\phi_{b_3/b_2}\end{matrix} \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} & & \\ -\sin\theta_{b_2/b_1} & & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} $$

$$ = \boxed{\begin{bmatrix} \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} - \sin\psi_{b_1/n}\cos\phi_{b_3/b_2} \\ \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} + \cos\psi_{b_1/n}\cos\phi_{b_3/b_2} \\ \cos\theta_{b_2/b_1}\sin\phi_{b_3/b_2} \end{bmatrix}} $$

The second column of ${}^{n}\mathbf{R}_{b_3}$ is the unit vector of b$_3$-frame in y direction decomposed in n-frame.

Example of 3-Link Arm

- the relationship between the unit vector of body fixed frame and rotational transformation matrix



How can we calculate the unit vector of b$_3$-frame in z direction $\left({}^{b_3}\mathbf{k}_{b_3}\right)$ decomposed in n-frame? $\left({}^{n}\mathbf{k}_{b_3}\right)$

$${}^{n}\mathbf{k}_{b_3} = {}^{n}\mathbf{R}_{b_3}\,{}^{b_3}\mathbf{k}_{b_3}$$

$$= \begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} & \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} - \sin\psi_{b_1/n}\cos\phi_{b_3/b_2} & \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} + \sin\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} & \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} + \cos\psi_{b_1/n}\cos\phi_{b_3/b_2} & \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} - \cos\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ -\sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1}\sin\phi_{b_3/b_2} & \cos\theta_{b_2/b_1}\cos\phi_{b_3/b_2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} + \sin\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} - \cos\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ \cos\theta_{b_2/b_1}\cos\phi_{b_3/b_2} \end{bmatrix}}$$

The third column of ${}^{n}\mathbf{R}_{b_3}$ is the unit vector of b$_3$-frame in z direction decomposed in n-frame.

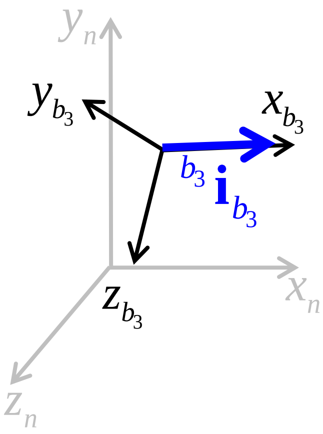
the relationship between the unit vector of body fixed frame and rotational transformation matrix

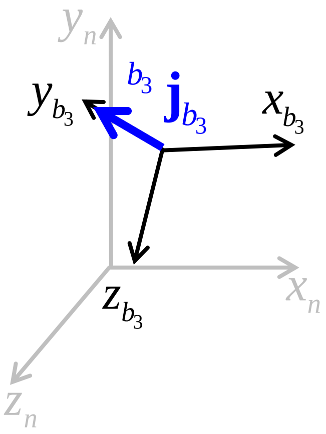
$$
{}^{n}\mathbf{R}_{b_3} = \begin{bmatrix} \cos\psi_{b_1/n}\cos\theta_{b_2/b_1} & \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} - \sin\psi_{b_1/n}\cos\phi_{b_3/b_2} & \cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} + \sin\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1} & \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} + \cos\psi_{b_1/n}\cos\phi_{b_3/b_2} & \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} - \cos\psi_{b_1/n}\sin\phi_{b_3/b_2} \\ -\sin\theta_{b_2/b_1} & \cos\theta_{b_2/b_1}\sin\phi_{b_3/b_2} & \cos\theta_{b_2/b_1}\cos\phi_{b_3/b_2} \end{bmatrix}
$$

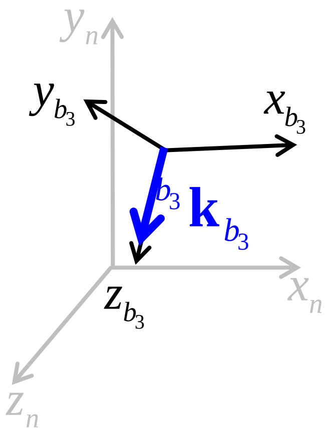
The unit vector in x direction of b_3-frame decomposed in n-frame.

The unit vector in y direction of b_3-frame decomposed in n-frame.

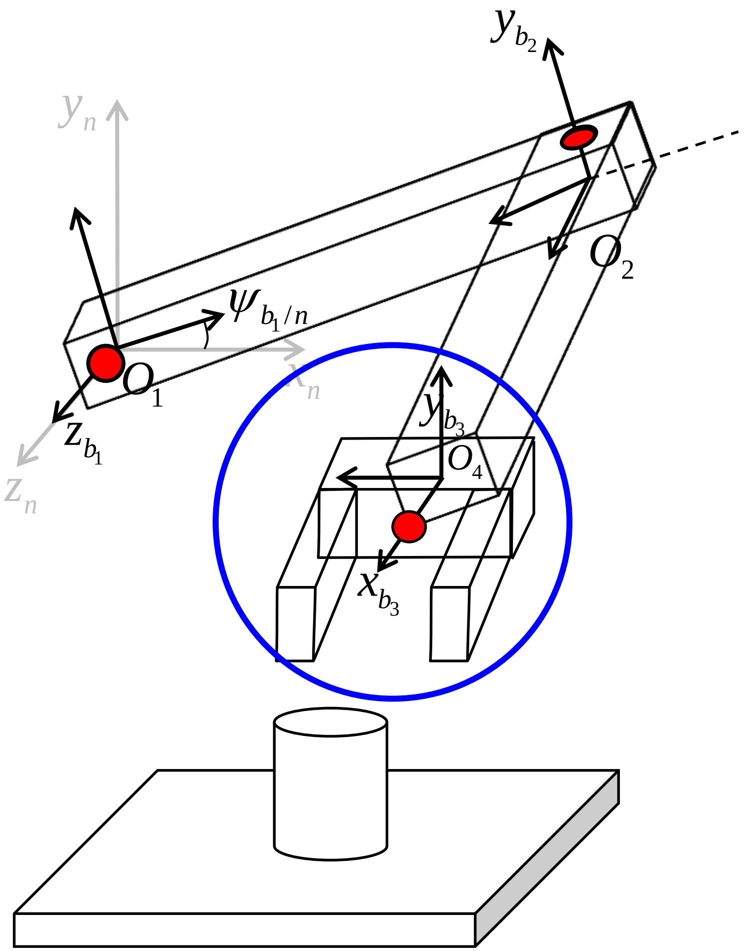
The unit vector in z direction of b_3-frame decomposed in n-frame.







Example of 3-Link Arm
- determination of Euler angles



For determination of required orientation of end-effector(b_3-frame), how can we calculate Euler angles?

Suppose that the direction of the grip should be $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$

Because the direction of the grip and direction of ${}^n\mathbf{i}_{b_3}$ is same,

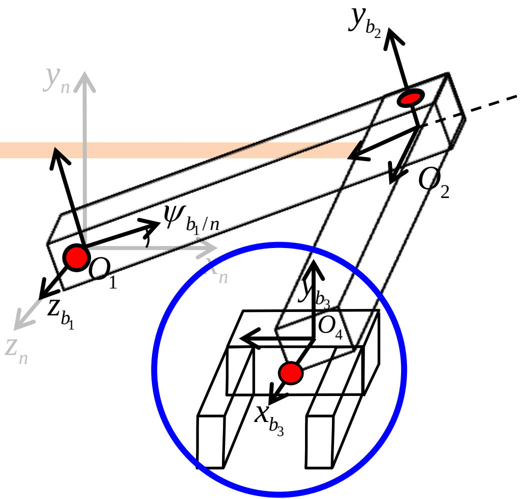
$$ {}^n\mathbf{i}_{b_3} = {}^n\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

For the upright position of the grip

$$ {}^n\mathbf{j}_{b_3} = {}^n\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} $$

Example of 3-Link Arm

- determination of Euler angles



Given: Required orientation of the end-effector

Find: Euler angles

Since $${}^{n}\mathbf{i}_{b_3} = {}^{n}\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T, \quad {}^{n}\mathbf{j}_{b_3} = {}^{n}\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T, \quad {}^{n}\mathbf{R}_{b_3} = \begin{bmatrix} \boxed{\begin{matrix}0\\0\\1\end{matrix}} & \begin{matrix}\boxed{0}\\1\\0\end{matrix} & \begin{matrix}R_{13}\\R_{23}\\R_{33}\end{matrix}\end{bmatrix}$$

Rotational transformation matrix with Euler angle ϕ_{b_3/b_2}, θ_{b_2/b_1}, $\psi_{b_1/n}$ is

$${}^{n}\mathbf{R}_{b_3} = \begin{bmatrix} \boxed{\begin{matrix}\cos\psi_{b_1/n}\cos\theta_{b_2/b_1}\\ \sin\psi_{b_1/n}\cos\theta_{b_2/b_1}\\ -\sin\theta_{b_2/b_1}\end{matrix}} & \begin{matrix}\cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} - \sin\psi_{b_1/n}\cos\phi_{b_3/b_2}\\ \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\sin\phi_{b_3/b_2} + \cos\psi_{b_1/n}\cos\phi_{b_3/b_2}\\ \cos\theta_{b_2/b_1}\sin\phi_{b_3/b_2}\end{matrix} & \begin{matrix}\cos\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} + \sin\psi_{b_1/n}\sin\phi_{b_3/b_2}\\ \sin\psi_{b_1/n}\sin\theta_{b_2/b_1}\cos\phi_{b_3/b_2} - \cos\psi_{b_1/n}\sin\phi_{b_3/b_2}\\ \cos\theta_{b_2/b_1}\cos\phi_{b_3/b_2}\end{matrix}\end{bmatrix}$$

$$\Big\downarrow \; \theta_{b_2/b_1} = -\frac{n}{2}\pi$$

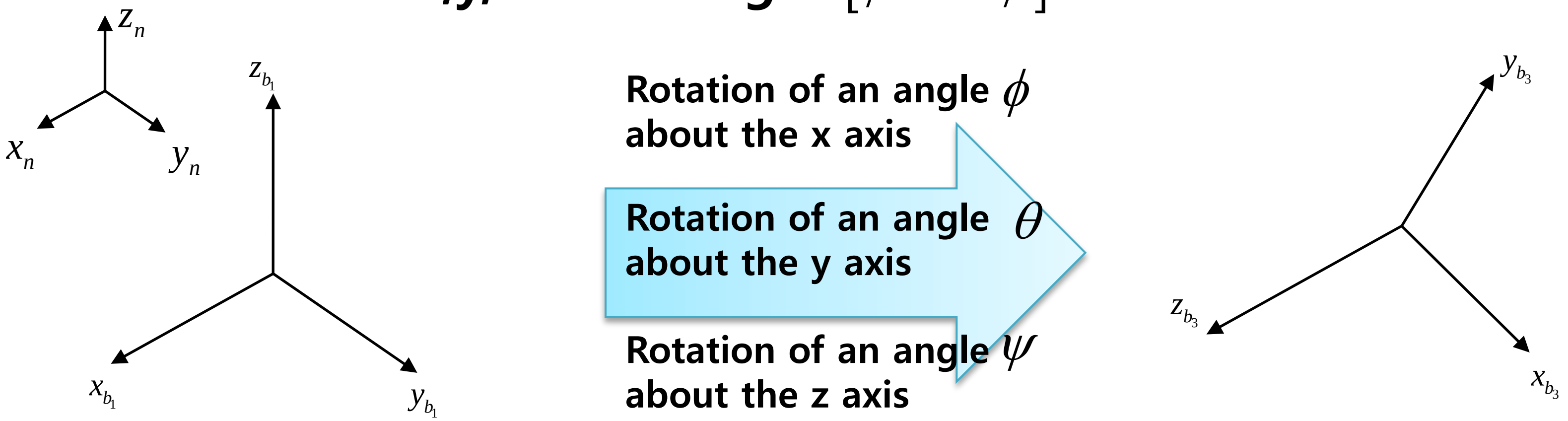
$${}^{n}\mathbf{R}_{b_3} = \begin{bmatrix} 0 & \boxed{-\sin\left(\psi_{b_1/n} + \phi_{b_3/b_2}\right)} & -\cos\left(\psi_{b_1/n} + \phi_{b_3/b_2}\right)\\ 0 & \cos\left(\psi_{b_1/n} + \phi_{b_3/b_2}\right) & -\sin\left(\psi_{b_1/n} + \phi_{b_3/b_2}\right)\\ 1 & 0 & 0\end{bmatrix} \longrightarrow -\sin\left(\psi_{b_1/n} + \phi_{b_3/b_2}\right) = 0$$

$$\psi_{b_1/n} + \phi_{b_3/b_2} = n\pi$$

This is an indeterminate equation.

Gimbal lock of the Euler angle(2)

x,y,z Euler angle $[\phi \quad \theta \quad \psi]^T$



Rotation of an angle ϕ about the x axis

Rotation of an angle θ about the y axis

Rotation of an angle ψ about the z axis

- Axes of the b_1 frame are parallel to axes of the E-frame.

$$
{}^{n}\mathbf{R}_{b_3} = \begin{bmatrix} \cos\theta\cos\psi & -\cos\theta\sin\psi & \sin\theta \\ \sin\phi\sin\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & -\sin\phi\cos\theta \\ -\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi + \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}
$$

For given $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

1. θ = 90°. $\Longrightarrow$ $${}^{n}\mathbf{R}_{b_3} = \begin{bmatrix} 0 & 0 & 1 \\ \sin(\phi+\psi) & \cos(\phi+\psi) & 0 \\ -\cos(\phi+\psi) & \sin(\phi+\psi) & 0 \end{bmatrix}$$
2. $\phi+\psi$ = 90°.
3. This is an indeterminate equation.

When the rotation transformation matrix ${}^{n}\mathbf{R}_{b_3}$ is given, the Euler angle $[\phi \quad \theta \quad \psi]^T$ are not uniquely determined.

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 210 ~ 213

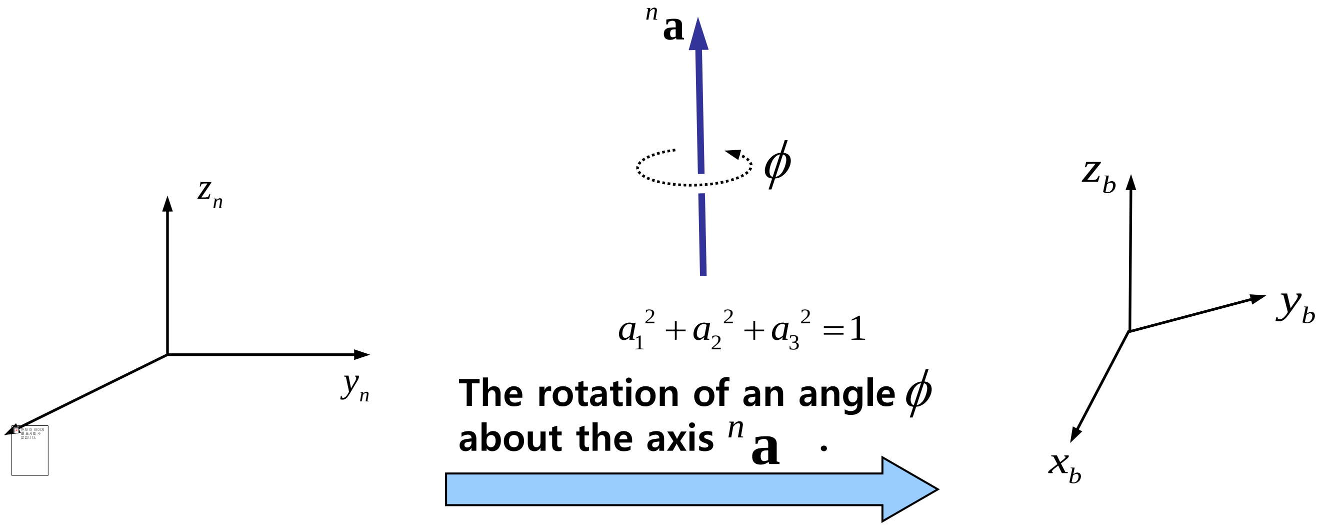
Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.229~p.231

4.4 Euler's Theorem

Euler's Theorem on Rotation

Euler's Theorem on Rotation

Every change in the relative orientation of two rigid bodies or n-frame and b-frame can be produced by means of a simple rotation about the unit vector ⁿa with angle ϕ



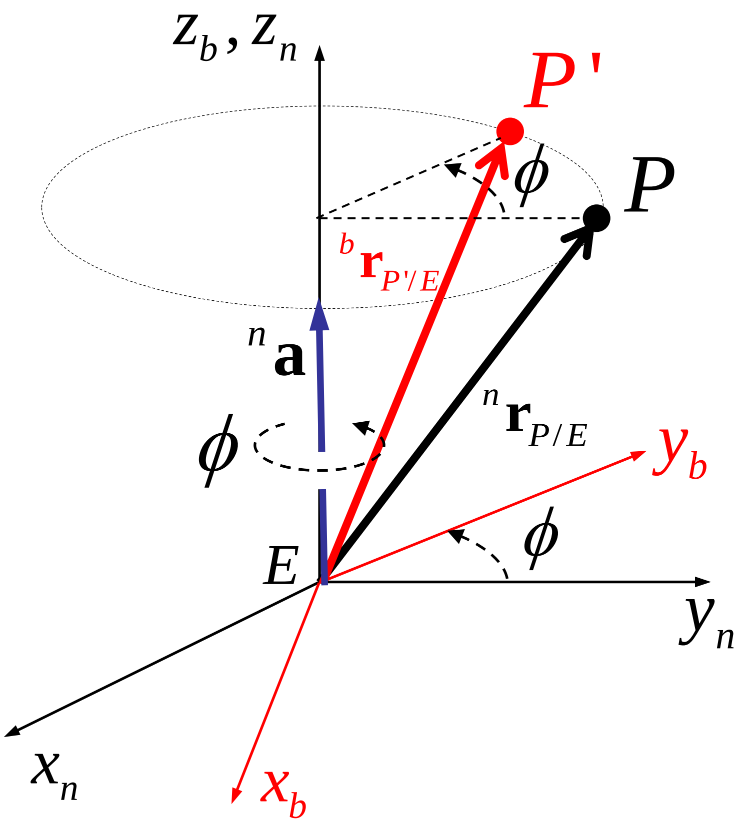
$$a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2} = 1$$

The rotation of an angle ϕ about the axis $^n\mathbf{a}$.

Euler’s Theorem on Rotation

$${}^{n}\mathbf{R}_{b_3} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2}\end{bmatrix}$$

Calculation of rotational transformation matrix using ZYX Euler angle $\phi_{b_3/b_2}, \theta_{b_2/b_1}, \psi_{b_1/n}$



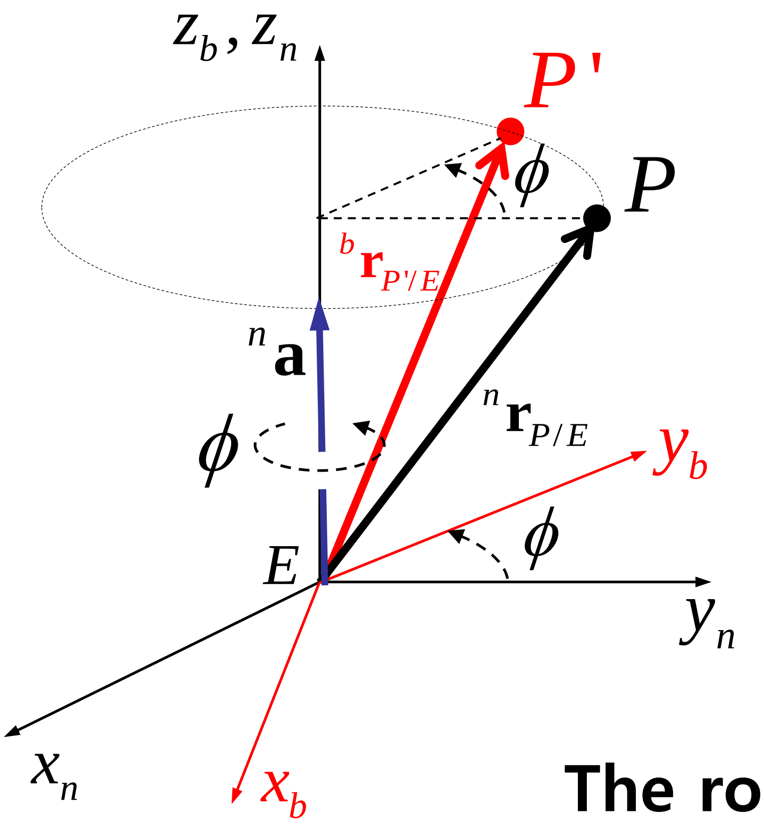
Point P is fixed on the n-frame.

The point P and n-frame rotate about axis ⁿa with angle ϕ , and become point P’ and b-frame.

Euler's Theorem on Rotation

$$^{n}\mathbf{R}_{b_3} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix}$$

Calculation of rotational transformation matrix using ZYX Euler angle $\phi_{b_3/b_2}, \theta_{b_2/b_1}, \psi_{b_1/n}$



1. Given:
The position vector of point P' in b-frame $\boxed{b}\mathbf{r}_{P'/E}$

2. Find:
The position vector of point P' in n-frame $\boxed{n}\mathbf{r}_{P'/E}$

$$^{n}\mathbf{r}_{P'/E} = \boxed{^{n}\mathbf{R}_{b}}\,{}^{b}\mathbf{r}_{P'/E}$$

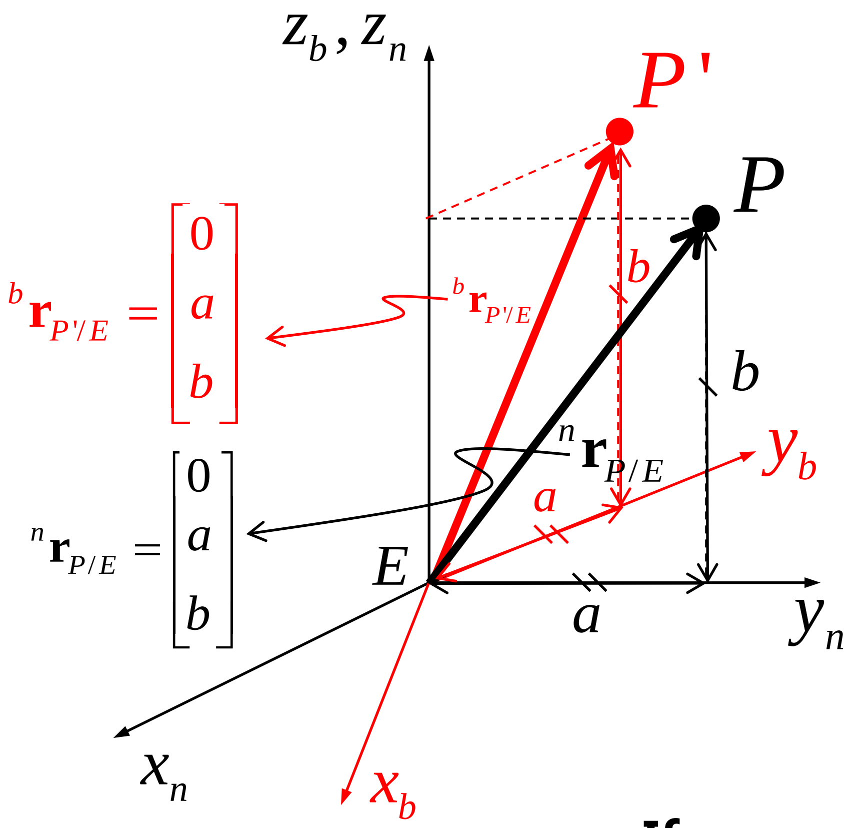
We need to know $^{n}\mathbf{R}_{b}$

The rotational transformation matrix $^{n}\mathbf{R}_{b}$ can be represented as a function of rotation axis $^{n}\mathbf{a}$ and rotation angle ϕ

Euler's Theorem on Rotation

$$ {}^{n}\mathbf{R}_{b_3} = {}^{n}\mathbf{R}_{b_1} {}^{b_1}\mathbf{R}_{b_2} {}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix} $$

Calculation of rotational transformation matrix using ZYX Euler angle $\phi_{b_3/b_2}, \theta_{b_2/b_1}, \psi_{b_1/n}$



$$ {}^{n}\mathbf{r}_{P'/E} = \boxed{{}^{n}\mathbf{R}_{b}}\, {}^{b}\mathbf{r}_{P'/E} $$

Coordinate transformation matrix

(b-frame → n-frame)

The components of ${}^{n}\mathbf{r}_{P/E}$ are same with the components of ${}^{b}\mathbf{r}_{P'/E}$

$$ {}^{n}\mathbf{r}_{P'/E} = \boxed{\mathbf{R}_{{}^{n}\mathbf{a},\phi}}\, {}^{n}\mathbf{r}_{P/E} $$

Vector rotation matrix

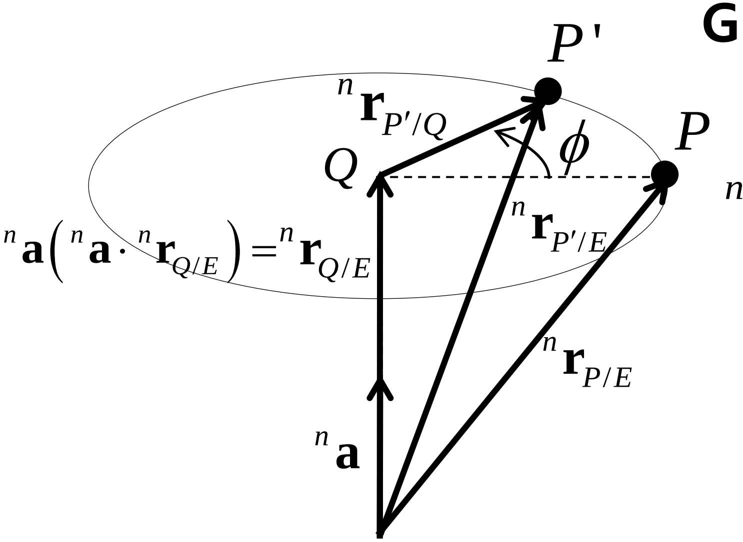
(about ${}^{n}\mathbf{a}$ with angle ϕ ,Point P → Point P')

If we can derive the vector rotation matrix $\mathbf{R}_{{}^{n}\mathbf{a},\phi}$, then we can also derive the coordinate transformation matrix ${}^{n}\mathbf{R}_{b}$

Euler's Theorem on Rotation

$$ {}^{n}\mathbf{R}_{b_3} = {}^{n}\mathbf{R}_{b_1}\,{}^{b_1}\mathbf{R}_{b_2}\,{}^{b_2}\mathbf{R}_{b_3} = \begin{bmatrix} \cos\psi_{b_1/n} & -\sin\psi_{b_1/n} & 0 \\ \sin\psi_{b_1/n} & \cos\psi_{b_1/n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_{b_2/b_1} & 0 & \sin\theta_{b_2/b_1} \\ 0 & 1 & 0 \\ -\sin\theta_{b_2/b_1} & 0 & \cos\theta_{b_2/b_1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{b_3/b_2} & -\sin\phi_{b_3/b_2} \\ 0 & \sin\phi_{b_3/b_2} & \cos\phi_{b_3/b_2} \end{bmatrix} $$

Calculation of rotational transformation matrix using ZYX Euler angle $\phi_{b_3/b_2}, \theta_{b_2/b_1}, \psi_{b_1/n}$

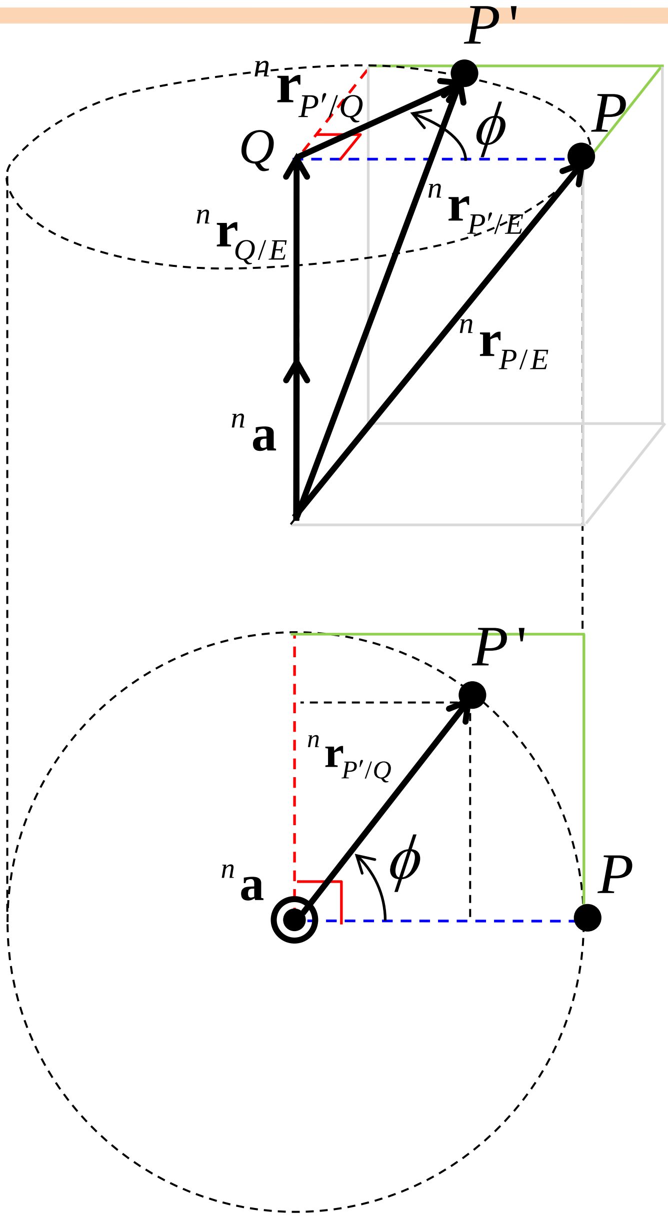


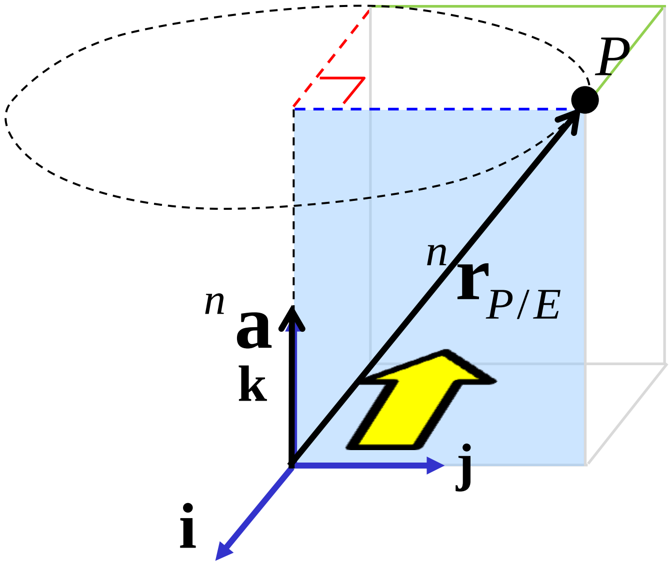
Given: ${}^{n}\mathbf{r}_{P/E}, {}^{n}\mathbf{a}, \phi$ **Find: ${}^{n}\mathbf{r}_{P'/E}$**

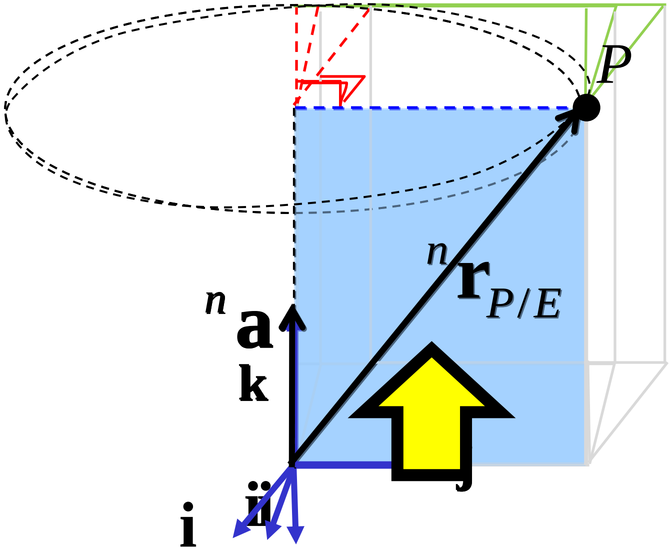
${}^{n}\mathbf{a}$: Euler axis **ϕ: Euler angle**

$$ {}^{n}\mathbf{r}_{P'/E} = {}^{n}\mathbf{r}_{Q/E} + {}^{n}\mathbf{r}_{P'/Q} $$

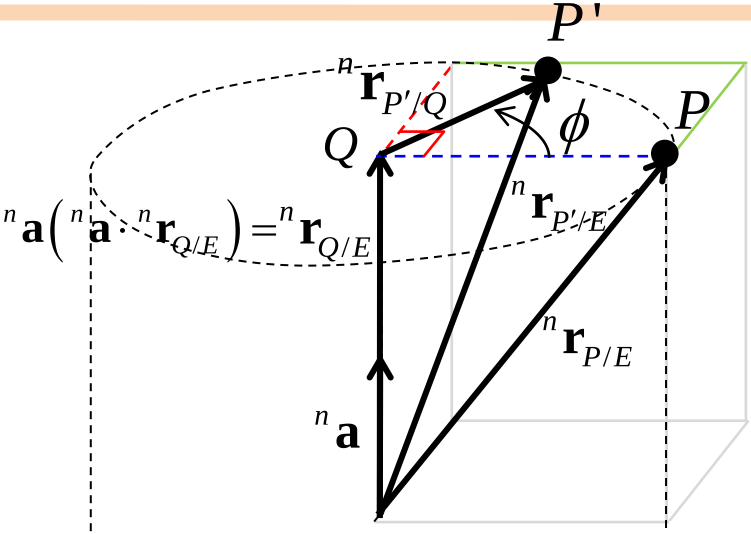
Euler's Theorem on Rotation

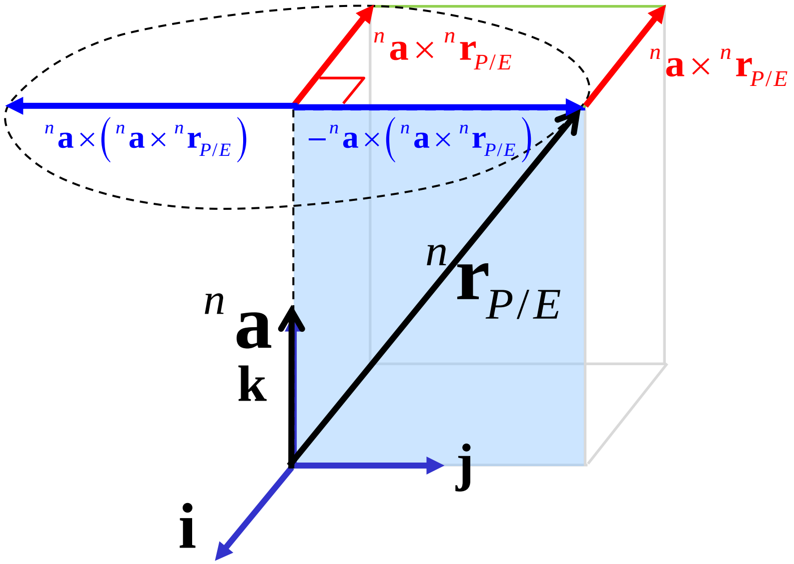


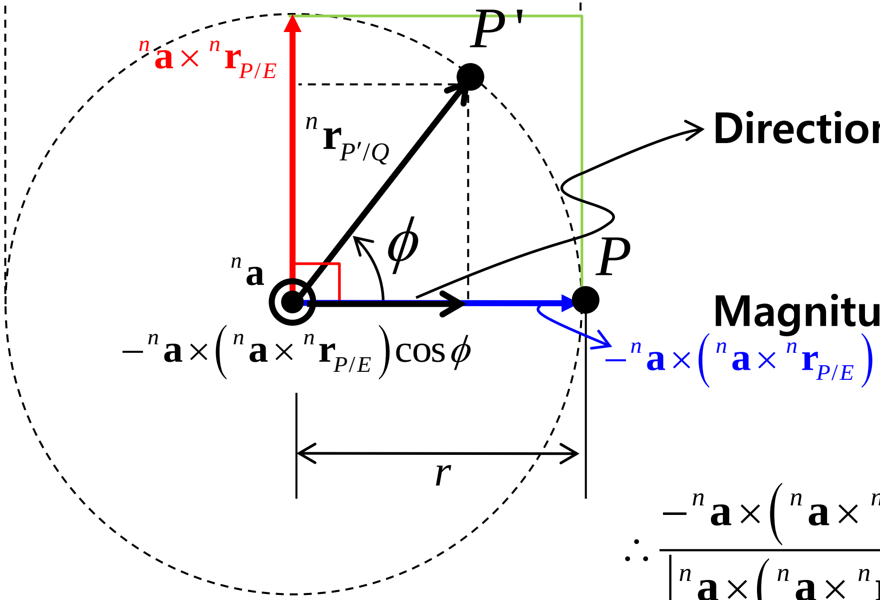




Euler's Theorem on Rotation





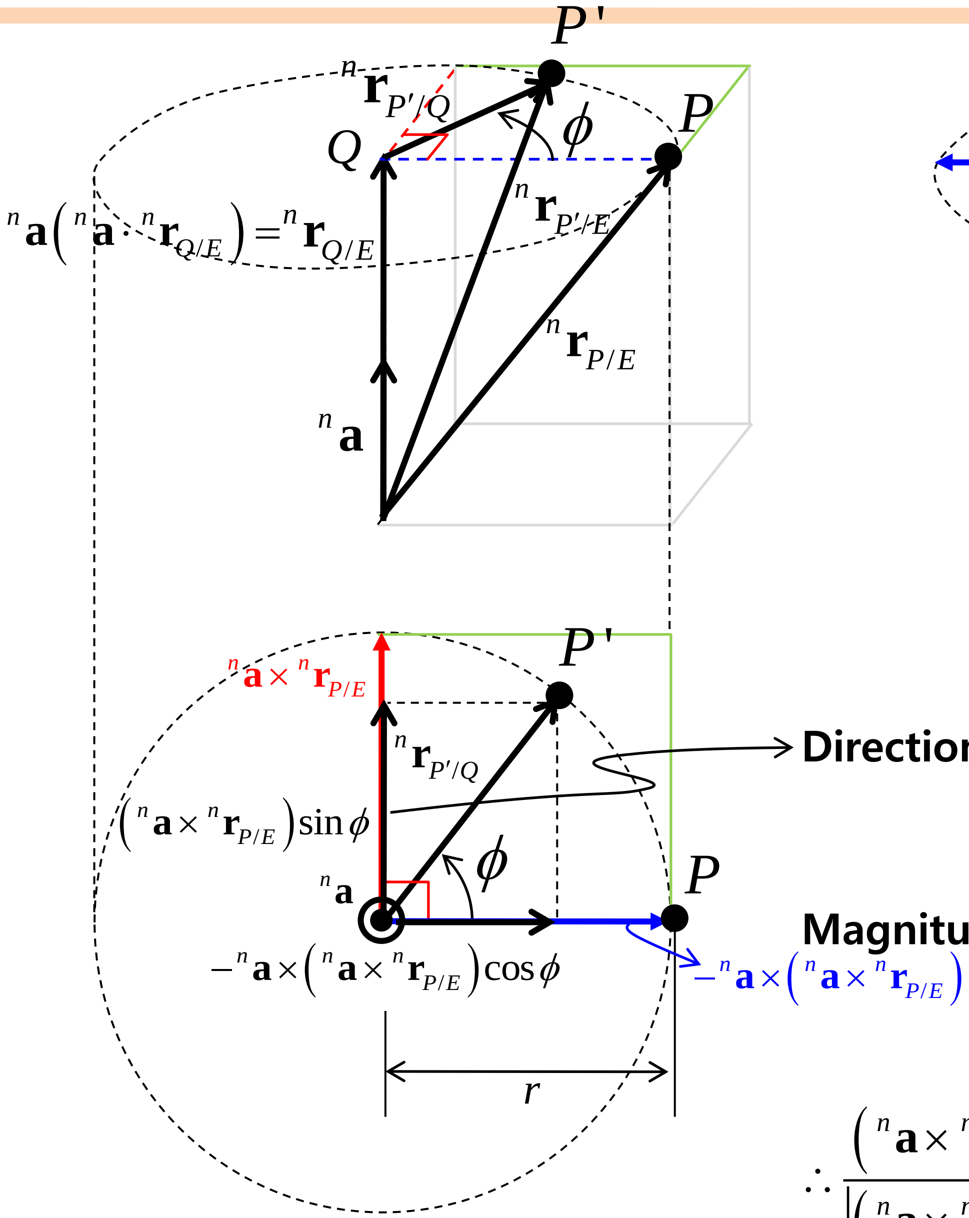


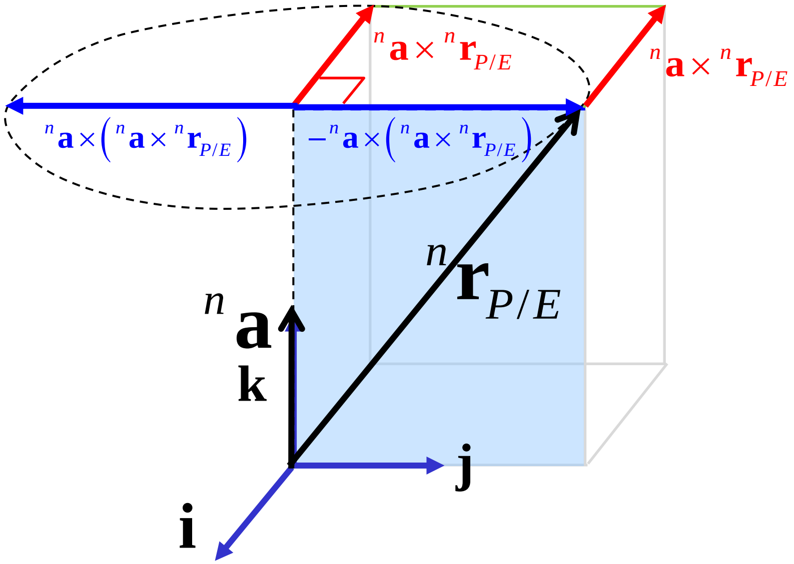
Direction:
$$\frac{-{}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)}{\left|{}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\right|}$$

Magnitude:
$$\left|{}^n\mathbf{r}_{P'/Q}\right|\cos\phi = r\cos\phi$$
$$=\left|{}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\right|\cos\phi$$

$$\therefore\ \frac{-{}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)}{\left|{}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\right|}\left|{}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\right|\cos\phi=-{}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\cos\phi$$

Euler's Theorem on Rotation





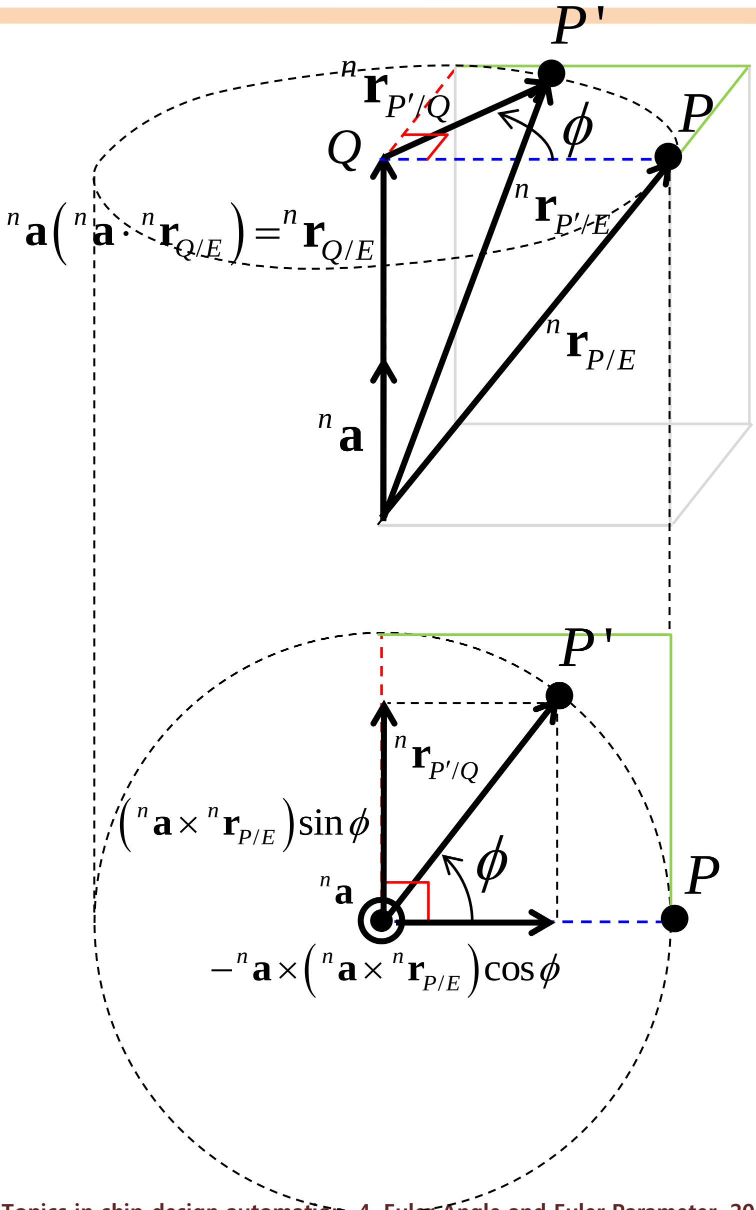
Direction: $\dfrac{\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)}{\left|\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\right|}$

Magnitude: $\left|{}^n\mathbf{r}_{P'/Q}\right|\sin\phi = r\sin\phi$

$$= \left|\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\right|\sin\phi$$

$$\therefore \frac{\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)}{\left|\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\right|}\left|\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\right|\sin\phi=\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\sin\phi$$

Euler's Theorem on Rotation



Given: $^n\mathbf{r}_{P/E}$, $^n\mathbf{a}, \phi$ **Find:** $^n\mathbf{r}_{P'/E}$

$^n\mathbf{a}$: Euler axis ϕ: Euler angle

$$^n\mathbf{r}_{P'/E} = {}^n\mathbf{r}_{Q/E} + {}^n\mathbf{r}_{P'/Q}$$

$$= {}^n\mathbf{a}\left({}^n\mathbf{a}\cdot{}^n\mathbf{r}_{P/E}\right) - {}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\cos\phi + \left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\sin\phi$$

Euler's Theorem on Rotation

$$\mathbf{a}\cdot\mathbf{b} \triangleq \mathbf{a}^T\mathbf{b}$$

$$\mathbf{a}\times\mathbf{b} \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_1 \\ b_1 \end{bmatrix}$$

$$= \mathbf{a}^{\times}\mathbf{b}$$

$${}^{n}\mathbf{a}\left({}^{n}\mathbf{a}\cdot{}^{n}\mathbf{r}_{P/E}\right) - {}^{n}\mathbf{a}\times\left({}^{n}\mathbf{a}\times{}^{n}\mathbf{r}_{P/E}\right)\cos\phi + \left({}^{n}\mathbf{a}\times{}^{n}\mathbf{r}_{P/E}\right)\sin\phi$$

Matrix representation

$${}^{n}\mathbf{a}\left({}^{n}\mathbf{a}^{T}\,{}^{n}\mathbf{r}_{P/E}\right) - {}^{n}\mathbf{a}^{\times}\left({}^{n}\mathbf{a}^{\times}\,{}^{n}\mathbf{r}_{P/E}\right)\cos\phi + \left({}^{n}\mathbf{a}^{\times}\,{}^{n}\mathbf{r}_{P/E}\right)\sin\phi$$

$${}^{n}\mathbf{a}\left({}^{n}\mathbf{a}^{T}\,{}^{n}\mathbf{r}_{P/E}\right) = \begin{bmatrix} {}^{n}a_x \\ {}^{n}a_y \\ {}^{n}a_z \end{bmatrix}\left[{}^{n}a_x\,{}^{n}r_{P/E,x} + {}^{n}a_y\,{}^{n}r_{P/E,y} + {}^{n}a_z\,{}^{n}r_{P/E,z}\right]$$

$$= \begin{bmatrix} {}^{n}a_x{}^{2}\,{}^{n}r_{P/E,x} + {}^{n}a_x\,{}^{n}a_y\,{}^{n}r_{P/E,y} + {}^{n}a_x\,{}^{n}a_z\,{}^{n}r_{P/E,z} \\ {}^{n}a_x\,{}^{n}a_y\,{}^{n}r_{P/E,x} + {}^{n}a_y{}^{2}\,{}^{n}r_{P/E,y} + {}^{n}a_y\,{}^{n}a_z\,{}^{n}r_{P/E,z} \\ {}^{n}a_x\,{}^{n}a_z\,{}^{n}r_{P/E,x} + {}^{n}a_y\,{}^{n}a_z\,{}^{n}r_{P/E,y} + {}^{n}a_z{}^{2}\,{}^{n}r_{P/E,z} \end{bmatrix}$$

$$= \begin{bmatrix} {}^{n}a_x{}^{2} & {}^{n}a_x\,{}^{n}a_y & {}^{n}a_x\,{}^{n}a_z \\ {}^{n}a_x\,{}^{n}a_y & {}^{n}a_y{}^{2} & {}^{n}a_y\,{}^{n}a_z \\ {}^{n}a_x\,{}^{n}a_z & {}^{n}a_y\,{}^{n}a_z & {}^{n}a_z{}^{2} \end{bmatrix}\begin{bmatrix} {}^{n}r_{P/E,x} \\ {}^{n}r_{P/E,y} \\ {}^{n}r_{P/E,z} \end{bmatrix} = {}^{n}\mathbf{a}\,{}^{n}\mathbf{a}^{T}\,{}^{n}\mathbf{r}_{P/E}$$

$${}^{n}\mathbf{a}\,{}^{n}\mathbf{a}^{T}\,{}^{n}\mathbf{r}_{P/E} - {}^{n}\mathbf{a}^{\times}\left({}^{n}\mathbf{a}^{\times}\,{}^{n}\mathbf{r}_{P/E}\right)\cos\phi + \left({}^{n}\mathbf{a}^{\times}\,{}^{n}\mathbf{r}_{P/E}\right)\sin\phi$$

Euler's Theorem on Rotation

$$\mathbf{a}\cdot\mathbf{b}\triangleq\mathbf{a}^T\mathbf{b}$$

$$\mathbf{a}\times\mathbf{b}\triangleq\begin{bmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{bmatrix}\begin{bmatrix}b_1\\ b_1\\ b_1\end{bmatrix}$$

$$=\mathbf{a}^{\times}\mathbf{b}$$

$${}^n\mathbf{a}\left({}^n\mathbf{a}\cdot{}^n\mathbf{r}_{P/E}\right)-{}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\cos\phi+\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\sin\phi$$

Matrix representation

$${}^n\mathbf{a}\left({}^n\mathbf{a}^T\,{}^n\mathbf{r}_{P/E}\right)-{}^n\mathbf{a}^{\times}\left({}^n\mathbf{a}^{\times}\,{}^n\mathbf{r}_{P/E}\right)\cos\phi+\left({}^n\mathbf{a}^{\times}\,{}^n\mathbf{r}_{P/E}\right)\sin\phi$$

$${}^n\mathbf{a}\,{}^n\mathbf{a}^T\,{}^n\mathbf{r}_{P/E}-{}^n\mathbf{a}^{\times}\left({}^n\mathbf{a}^{\times}\,{}^n\mathbf{r}_{P/E}\right)\cos\phi+\left({}^n\mathbf{a}^{\times}\,{}^n\mathbf{r}_{P/E}\right)\sin\phi$$

$${}^n\mathbf{a}^{\times}\left({}^n\mathbf{a}^{\times}\,{}^n\mathbf{r}_{P/E}\right)=\begin{bmatrix}0 & -{}^na_z & {}^na_y\\ {}^na_z & 0 & -{}^na_x\\ -{}^na_y & {}^na_x & 0\end{bmatrix}\begin{bmatrix}0 & -{}^na_z & {}^na_y\\ {}^na_z & 0 & -{}^na_x\\ -{}^na_y & {}^na_x & 0\end{bmatrix}\begin{bmatrix}{}^nr_{P/E,x}\\ {}^nr_{P/E,y}\\ {}^nr_{P/E,z}\end{bmatrix}$$

$$=\begin{bmatrix}-{}^na_y{}^2-{}^na_z{}^2 & {}^na_x\,{}^na_y & {}^na_x\,{}^na_z\\ {}^na_x\,{}^na_y & -{}^na_x{}^2-{}^na_z{}^2 & {}^na_y\,{}^na_z\\ {}^na_x\,{}^na_z & {}^na_y\,{}^na_z & -{}^na_x{}^2-{}^na_y{}^2\end{bmatrix}\begin{bmatrix}{}^nr_{P/E,x}\\ {}^nr_{P/E,y}\\ {}^nr_{P/E,z}\end{bmatrix}$$

$$=\left(\begin{bmatrix}{}^na_x{}^2 & {}^na_x\,{}^na_y & {}^na_x\,{}^na_z\\ {}^na_x\,{}^na_y & {}^na_y{}^2 & {}^na_y\,{}^na_z\\ {}^na_x\,{}^na_z & {}^na_y\,{}^na_z & {}^na_z{}^2\end{bmatrix}-\begin{bmatrix}{}^na_x{}^2+{}^na_y{}^2+{}^na_z{}^2 & 0 & 0\\ 0 & {}^na_x{}^2+{}^na_y{}^2+{}^na_z{}^2 & 0\\ 0 & 0 & {}^na_x{}^2+{}^na_y{}^2+{}^na_z{}^2\end{bmatrix}\right)\begin{bmatrix}{}^nr_{P/E,x}\\ {}^nr_{P/E,y}\\ {}^nr_{P/E,z}\end{bmatrix}$$

$$=\left({}^n\mathbf{a}\,{}^n\mathbf{a}^T-\mathbf{I}\right){}^n\mathbf{r}_{P/E}$$

$${}^n\mathbf{a}\,{}^n\mathbf{a}^T\,{}^n\mathbf{r}_{P/E}-\left({}^n\mathbf{a}\,{}^n\mathbf{a}^T-\mathbf{I}\right){}^n\mathbf{r}_{P/E}\cos\phi+{}^n\mathbf{a}^{\times}\,{}^n\mathbf{r}_{P/E}\sin\phi$$

$$=\left[{}^n\mathbf{a}\,{}^n\mathbf{a}^T-\left({}^n\mathbf{a}\,{}^n\mathbf{a}^T-\mathbf{I}\right)\cos\phi+{}^n\mathbf{a}^{\times}\sin\phi\right]{}^n\mathbf{r}_{P/E}$$

Euler's Theorem on Rotation

$$\mathbf{a}\cdot\mathbf{b} \triangleq \mathbf{a}^T\mathbf{b}$$

$$\mathbf{a}\times\mathbf{b} \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_1 \\ b_1 \end{bmatrix}$$

$$= \mathbf{a}^{\times}\mathbf{b}$$

$$= \mathbf{S}(\mathbf{a})\mathbf{b}$$

$${}^n\mathbf{a}\left({}^n\mathbf{a}\cdot{}^n\mathbf{r}_{P/E}\right) - {}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\cos\phi + \left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\sin\phi$$

Matrix representation

$${}^n\mathbf{a}\left({}^n\mathbf{a}^T\,{}^n\mathbf{r}_{P/E}\right) - {}^n\mathbf{a}^{\times}\left({}^n\mathbf{a}^{\times}\,{}^n\mathbf{r}_{P/E}\right)\cos\phi + \left({}^n\mathbf{a}^{\times}\,{}^n\mathbf{r}_{P/E}\right)\sin\phi$$

$$= \left[{}^n\mathbf{a}\,{}^n\mathbf{a}^T - \left({}^n\mathbf{a}\,{}^n\mathbf{a}^T - \mathbf{I}\right)\cos\phi + {}^n\mathbf{a}^{\times}\sin\phi\right]{}^n\mathbf{r}_{P/E}$$

$$= \left[{}^n\mathbf{a}\,{}^n\mathbf{a}^T - \cos\phi\left({}^n\mathbf{a}\,{}^n\mathbf{a}^T - \mathbf{I}\right) + \sin\phi\,{}^n\mathbf{a}^{\times}\right]{}^n\mathbf{r}_{P/E}$$

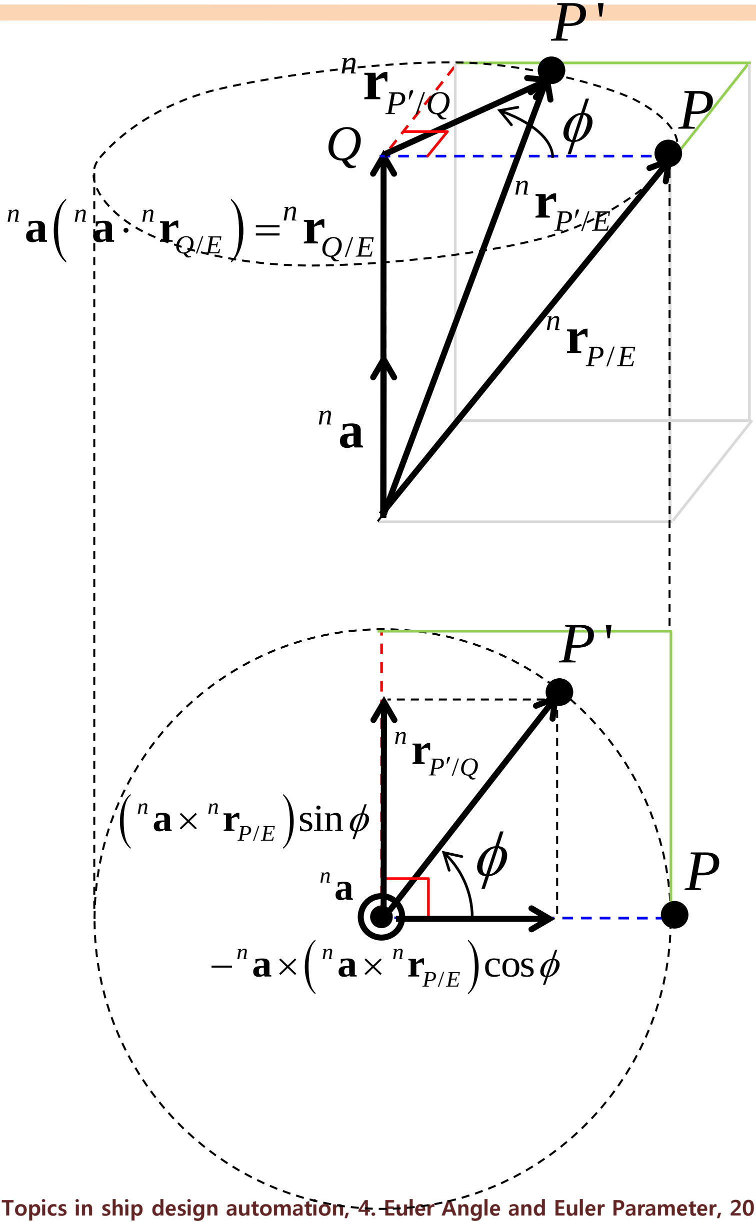
$$= \left[{}^n\mathbf{a}\,{}^n\mathbf{a}^T - \cos\phi\,{}^n\mathbf{a}\,{}^n\mathbf{a}^T + \cos\phi\mathbf{I} + \sin\phi\,{}^n\mathbf{a}^{\times}\right]{}^n\mathbf{r}_{P/E}$$

$$= \left[\left(1-\cos\phi\right){}^n\mathbf{a}\,{}^n\mathbf{a}^T + \cos\phi\mathbf{I} + \sin\phi\,{}^n\mathbf{a}^{\times}\right]{}^n\mathbf{r}_{P/E}$$

$$= \left[\cos\phi\mathbf{I} + \left(1-\cos\phi\right){}^n\mathbf{a}\,{}^n\mathbf{a}^T + \sin\phi\,{}^n\mathbf{a}^{\times}\right]{}^n\mathbf{r}_{P/E}$$

⇨ **Hughes, P. C, Spacecraft attitude dynamics, Dover publication, 2004, pp.13**

Euler's Theorem on Rotation



Given: $^n\mathbf{r}_{P/E},\ {}^n\mathbf{a},\phi$ **Find:** $^n\mathbf{r}_{P'/E}$

$^n\mathbf{a}$: Euler axis ϕ: Euler angle

$$^n\mathbf{r}_{P'/E} = {}^n\mathbf{r}_{Q/E} + {}^n\mathbf{r}_{P'/Q}$$

$$= {}^n\mathbf{a}\left({}^n\mathbf{a}\cdot{}^n\mathbf{r}_{P/E}\right) - {}^n\mathbf{a}\times\left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\cos\phi + \left({}^n\mathbf{a}\times{}^n\mathbf{r}_{P/E}\right)\sin\phi$$

$$= \left[\cos\phi\mathbf{I} + (1-\cos\phi)\,{}^n\mathbf{a}\,{}^n\mathbf{a}^T + \sin\phi\,{}^n\mathbf{a}^\times\right]{}^n\mathbf{r}_{P/E}$$

$$= \underbrace{\left[\mathbf{I} + \sin\phi\mathbf{S}\left({}^n\mathbf{a}\right) + (1-\cos\phi)\mathbf{S}^2\left({}^n\mathbf{a}\right)\right]{}^n\mathbf{r}_{P/E}}_{\displaystyle\downarrow\ \mathbf{R}_{{}^n\mathbf{a},\phi}}$$

$$,\textit{where}\ \ \mathbf{S}\left({}^n\mathbf{a}\right) = {}^n\mathbf{a}^\times$$

$$= \begin{bmatrix} 0 & -{}^na_z & {}^na_y \\ {}^na_z & 0 & -{}^na_x \\ -{}^na_y & {}^na_x & 0 \end{bmatrix}$$

Euler's Theorem on Rotation and Principle rotations

${}^n\mathbf{a}$: **Euler axis** ϕ: **Euler angle**

$$ {}^n\mathbf{r}_{P'/E} = \underbrace{\left[\mathbf{I}+\sin\phi\,\mathbf{S}\left({}^n\mathbf{a}\right)+\left(1-\cos\phi\right)\mathbf{S}^2\left({}^n\mathbf{a}\right)\right]}_{\downarrow\ \mathbf{R}_{{}^n\mathbf{a},\phi}}\,{}^n\mathbf{r}_{P/E} \quad ,where\ \mathbf{S}\left({}^n\mathbf{a}\right)=\begin{bmatrix} 0 & -{}^na_z & {}^na_y \\ {}^na_z & 0 & -{}^na_x \\ -{}^na_y & {}^na_x & 0 \end{bmatrix} $$

$$ \mathbf{R}_{{}^n\mathbf{a},\phi} = \begin{bmatrix} (1-\cos\phi)\,{}^na_x^2+\cos\phi & (1-\cos\phi)\,{}^na_x\,{}^na_y - a_z\sin\phi & (1-\cos\phi)\,{}^na_z\,{}^na_x + {}^na_y\sin\phi \\ (1-\cos\phi)\,{}^na_x\,{}^na_y + {}^na_z\sin\phi & (1-\cos\phi)\,{}^na_y^2+\cos\phi & (1-\cos\phi)\,{}^na_y\,{}^na_z - {}^na_x\sin\phi \\ (1-\cos\phi)\,{}^na_z\,{}^na_x - {}^na_y\sin\phi & (1-\cos\phi)\,{}^na_y\,{}^na_z + a_x\sin\phi & (1-\cos\phi)\,{}^na_z^2+\cos\phi \end{bmatrix} $$

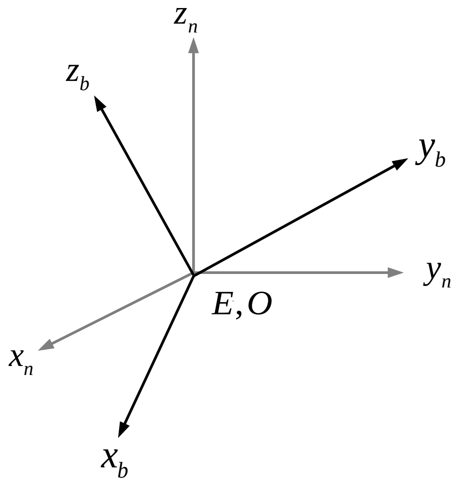
Principle rotations

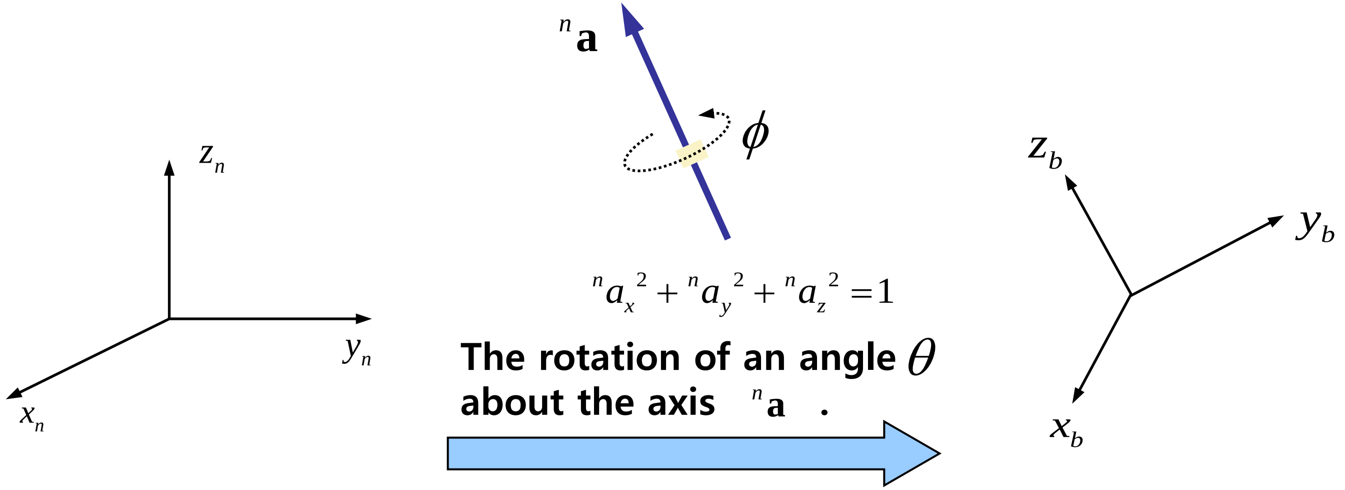
$$ {}^na_x=1, {}^na_y=0, {}^na_z=0 \qquad {}^na_x=0, {}^na_y=1, {}^na_z=0 \qquad {}^na_x=0, {}^na_y=0, {}^na_z=1 $$

$$ \mathbf{R}_{x,\phi}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \qquad \mathbf{R}_{y,\theta}=\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \mathbf{R}_{z,\psi}=\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

4.5 Euler parameter

Euler's Theorem on Rotation





$$ {}^n a_x{}^2 + {}^n a_y{}^2 + {}^n a_z{}^2 = 1 $$

The rotation of an angle θ about the axis ${}^n\mathbf{a}$.

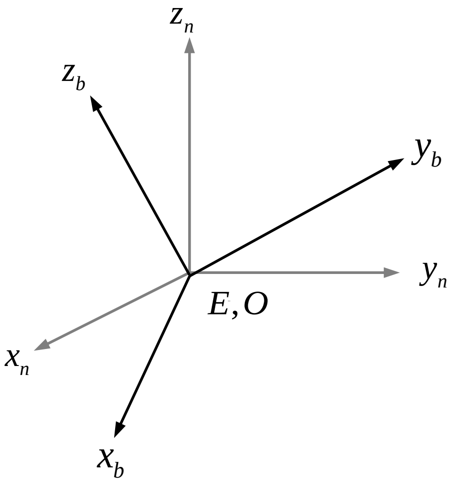
- This rotation is expressed in terms of the angle of rotation ϕ and a unit vector ${}^n\mathbf{a}$ along the axis of rotation.

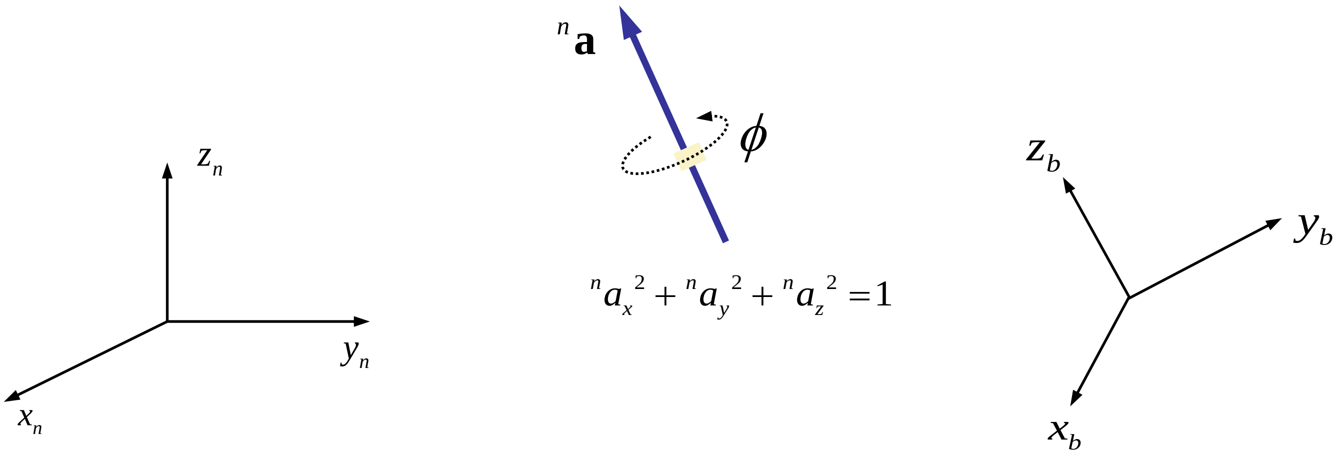
${}^n\mathbf{R}_b$: Rotation transformation matrix, ${}^n\mathbf{r} = {}^n\mathbf{R}_b \, {}^b\mathbf{r}$

- The rotation transformation matrix is expressed in terms of the angle of rotation ϕ and a unit vector ${}^n\mathbf{a}$.

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

Euler parameter - rotation transformation matrix





$${}^n a_x{}^2 + {}^n a_y{}^2 + {}^n a_z{}^2 = 1$$

$$\mathbf{R}_{\mathbf{a},\phi} = \begin{bmatrix} (1-\cos\phi)\,{}^n a_x^2 + \cos\phi & (1-\cos\phi)\,{}^n a_x\,{}^n a_y - {}^n a_z \sin\phi & (1-\cos\phi)\,{}^n a_z\,{}^n a_x + {}^n a_y \sin\phi \\ (1-\cos\phi)\,{}^n a_x\,{}^n a_y + {}^n a_z \sin\phi & (1-\cos\phi)\,{}^n a_y^2 + \cos\phi & (1-\cos\phi)\,{}^n a_y\,{}^n a_z - {}^n a_x \sin\phi \\ (1-\cos\phi)\,{}^n a_z\,{}^n a_x - {}^n a_y \sin\phi & (1-\cos\phi)\,{}^n a_y\,{}^n a_z + {}^n a_x \sin\phi & (1-\cos\phi)\,{}^n a_z^2 + \cos\phi \end{bmatrix}$$

$${}^n\mathbf{R}_b = \begin{bmatrix} 2(\theta_0{}^2 - \theta_1{}^2) - 1 & 2(\theta_1\theta_2 - \theta_0\theta_3) & 2(\theta_1\theta_3 + \theta_0\theta_2) \\ 2(\theta_1\theta_2 + \theta_0\theta_3) & 2(\theta_0{}^2 + \theta_2{}^2) - 1 & 2(\theta_2\theta_3 - \theta_0\theta_1) \\ 2(\theta_1\theta_3 - \theta_0\theta_2) & 2(\theta_2\theta_3 + \theta_0\theta_1) & 2(\theta_0{}^2 + \theta_3{}^2) - 1 \end{bmatrix}$$

, where $\theta_0 = \cos\frac{\phi}{2}$ $, \theta_1 = {}^n a_x \sin\frac{\phi}{2}$

$, \theta_2 = {}^n a_y \sin\frac{\phi}{2}$ $, \theta_3 = {}^n a_z \sin\frac{\phi}{2}$

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

(derivation)

$$\mathbf{R}_{\mathbf{a},\phi}=\begin{bmatrix}\boxed{(1-\cos\phi)^n a_x^2+\cos\phi} & \boxed{(1-\cos\phi)^n a_x{}^n a_y-{}^n a_z\sin\phi} & (1-\cos\phi)^n a_z{}^n a_x+{}^n a_y\sin\phi\\ (1-\cos\phi)^n a_x{}^n a_y+{}^n a_z\sin\phi & (1-\cos\phi)^n a_y^2+\cos\phi & (1-\cos\phi)^n a_y{}^n a_z-{}^n a_x\sin\phi\\ (1-\cos\phi)^n a_z{}^n a_x-{}^n a_y\sin\phi & (1-\cos\phi)^n a_y{}^n a_z+{}^n a_x\sin\phi & (1-\cos\phi)^n a_z^2+\cos\phi\end{bmatrix}$$

$$^n\mathbf{R}_b=\begin{bmatrix}\boxed{2(\theta_0{}^2+\theta_1{}^2)-1} & \boxed{2(\theta_1\theta_2-\theta_0\theta_3)} & 2(\theta_1\theta_3+\theta_0\theta_2)\\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0{}^2+\theta_2{}^2)-1 & 2(\theta_2\theta_3-\theta_0\theta_1)\\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0{}^2+\theta_3{}^2)-1\end{bmatrix}$$

, where $\theta_0=\cos\frac{\phi}{2}$ $,\theta_1={}^n a_x\sin\frac{\phi}{2}$

$,\theta_2={}^n a_y\sin\frac{\phi}{2}$ $,\theta_3={}^n a_z\sin\frac{\phi}{2}$

$$^n\mathbf{R}_{b\_(1,1)}=2(\theta_0{}^2+\theta_1{}^2)-1$$

$$\Downarrow\quad \theta_0=\cos\frac{\phi}{2}\ ,\theta_1={}^n a_x\sin\frac{\phi}{2}$$

$$=2\left(\cos^2\frac{\phi}{2}+{}^n a_x{}^2\ \sin^2\frac{\phi}{2}\right)-1$$

$$\Downarrow\quad \cos^2\frac{\phi}{2}=\frac{1+\cos\phi}{2}\ ,\sin^2\frac{\phi}{2}=\frac{1-\cos\phi}{2}$$

$$=2\left(\frac{1+\cos\phi}{2}+{}^n a_x{}^2\ \frac{1-\cos\phi}{2}\right)-1$$

$$=1+\cos\phi+{}^n a_x{}^2(1-\cos\phi)-1$$

$$=(1-\cos\phi)\,{}^n a_x{}^2+\cos\phi$$

$$=\mathbf{R}_{\mathbf{a},\phi\_(1,1)}$$

$$^n\mathbf{R}_{b\_(1,2)}=2(\theta_1\theta_2-\theta_0\theta_3)$$

$$\Downarrow\quad \theta_0=\cos\frac{\phi}{2}\ ,\theta_1={}^n a_x\sin\frac{\phi}{2}\quad \theta_2={}^n a_y\sin\frac{\phi}{2}\ ,\theta_3={}^n a_z\sin\frac{\phi}{2}$$

$$=2\left({}^n a_x\sin\frac{\phi}{2}\cdot{}^n a_y\sin\frac{\phi}{2}-\cos\frac{\phi}{2}\cdot{}^n a_z\sin\frac{\phi}{2}\right)$$

$$\Downarrow\quad \sin^2\frac{\phi}{2}=\frac{1-\cos\phi}{2}\ ,\cos\frac{\phi}{2}\sin\frac{\phi}{2}=\frac{\sin\phi}{2}$$

$$=2\left({}^n a_x{}^n a_y\frac{1-\cos\phi}{2}-{}^n a_z\frac{\sin\phi}{2}\right)$$

$$=(1-\cos\phi)\,{}^n a_x{}^n a_y-{}^n a_z\sin\phi$$

$$=\mathbf{R}_{\mathbf{a},\phi\_(1,2)}$$

(derivation)

$$\mathbf{R}_{\mathbf{a},\phi}=\begin{bmatrix}(1-\cos\phi)\,{}^{n}a_x^2+\cos\phi & (1-\cos\phi)\,{}^{n}a_x\,{}^{n}a_y-{}^{n}a_z\sin\phi & \boxed{(1-\cos\phi)\,{}^{n}a_z\,{}^{n}a_x+{}^{n}a_y\sin\phi}\\ \boxed{(1-\cos\phi)\,{}^{n}a_x\,{}^{n}a_y+{}^{n}a_z\sin\phi} & (1-\cos\phi)\,{}^{n}a_y^2+\cos\phi & (1-\cos\phi)\,{}^{n}a_y\,{}^{n}a_z-{}^{n}a_x\sin\phi\\ (1-\cos\phi)\,{}^{n}a_z\,{}^{n}a_x-{}^{n}a_y\sin\phi & (1-\cos\phi)\,{}^{n}a_y\,{}^{n}a_z+{}^{n}a_x\sin\phi & (1-\cos\phi)\,{}^{n}a_z^2+\cos\phi\end{bmatrix}$$

$${}^{n}\mathbf{R}_b=\begin{bmatrix}2({\theta_0}^2+{\theta_1}^2)-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & \boxed{2(\theta_1\theta_3+\theta_0\theta_2)}\\ \boxed{2(\theta_1\theta_2+\theta_0\theta_3)} & 2({\theta_0}^2+{\theta_2}^2)-1 & 2(\theta_2\theta_3-\theta_0\theta_1)\\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2({\theta_0}^2+{\theta_3}^2)-1\end{bmatrix}$$

, where $\theta_0=\cos\frac{\phi}{2}\quad,\theta_1={}^{n}a_x\sin\frac{\phi}{2}$

$,\theta_2={}^{n}a_y\sin\frac{\phi}{2}\quad,\theta_3={}^{n}a_z\sin\frac{\phi}{2}$

$${}^{n}\mathbf{R}_{b\_(1,3)}=2(\theta_1\theta_3+\theta_0\theta_2)$$

$$\Big\Downarrow\quad \theta_0=\cos\frac{\phi}{2}\ ,\theta_1={}^{n}a_x\sin\frac{\phi}{2}\quad \theta_2={}^{n}a_y\sin\frac{\phi}{2}\ ,\theta_3={}^{n}a_z\sin\frac{\phi}{2}$$

$$=2\left({}^{n}a_x\sin\frac{\phi}{2}\cdot{}^{n}a_z\sin\frac{\phi}{2}+\cos\frac{\phi}{2}\cdot{}^{n}a_y\sin\frac{\phi}{2}\right)$$

$$\Big\Downarrow\quad \sin^2\frac{\phi}{2}=\frac{1-\cos\phi}{2}\quad,\cos\frac{\phi}{2}\sin\frac{\phi}{2}=\frac{\sin\phi}{2}$$

$$=2\left({}^{n}a_x\,{}^{n}a_z\frac{1-\cos\phi}{2}+{}^{n}a_y\frac{\sin\phi}{2}\right)$$

$$=(1-\cos\phi)\,{}^{n}a_z\,{}^{n}a_x+{}^{n}a_y\sin\phi$$

$$=\mathbf{R}_{\mathbf{a},\phi\_(1,3)}$$

$${}^{n}\mathbf{R}_{b\_(2,1)}=2(\theta_1\theta_2+\theta_0\theta_3)$$

$$\Big\Downarrow\quad \theta_0=\cos\frac{\phi}{2}\ ,\theta_1={}^{n}a_x\sin\frac{\phi}{2}\quad \theta_2={}^{n}a_y\sin\frac{\phi}{2}\ ,\theta_3={}^{n}a_z\sin\frac{\phi}{2}$$

$$=2\left({}^{n}a_x\sin\frac{\phi}{2}\cdot{}^{n}a_y\sin\frac{\phi}{2}+\cos\frac{\phi}{2}\cdot{}^{n}a_z\sin\frac{\phi}{2}\right)$$

$$\Big\Downarrow\quad \sin^2\frac{\phi}{2}=\frac{1-\cos\phi}{2}\quad,\cos\frac{\phi}{2}\sin\frac{\phi}{2}=\frac{\sin\phi}{2}$$

$$=2\left({}^{n}a_x\,{}^{n}a_y\frac{1-\cos\phi}{2}+{}^{n}a_z\frac{\sin\phi}{2}\right)$$

$$=(1-\cos\phi)\,{}^{n}a_x\,{}^{n}a_y+{}^{n}a_z\sin\phi$$

$$=\mathbf{R}_{\mathbf{a},\phi\_(2,1)}$$

(derivation)

$$\mathbf{R}_{\mathbf{a},\phi}=\begin{bmatrix}(1-\cos\phi)\,{}^n a_x^2+\cos\phi & (1-\cos\phi)\,{}^n a_x\,{}^n a_y-{}^n a_z\sin\phi & (1-\cos\phi)\,{}^n a_z\,{}^n a_x+{}^n a_y\sin\phi\\ (1-\cos\phi)\,{}^n a_x\,{}^n a_y+{}^n a_z\sin\phi & \boxed{(1-\cos\phi)\,{}^n a_y^2+\cos\phi} & \boxed{(1-\cos\phi)\,{}^n a_y\,{}^n a_z-{}^n a_x\sin\phi}\\ (1-\cos\phi)\,{}^n a_z\,{}^n a_x-{}^n a_y\sin\phi & (1-\cos\phi)\,{}^n a_y\,{}^n a_z+{}^n a_x\sin\phi & (1-\cos\phi)\,{}^n a_z^2+\cos\phi\end{bmatrix}$$

$${}^n\mathbf{R}_b=\begin{bmatrix}2(\theta_0{}^2+\theta_1{}^2)-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2)\\ 2(\theta_1\theta_2+\theta_0\theta_3) & \boxed{2(\theta_0{}^2+\theta_2{}^2)-1} & \boxed{2(\theta_2\theta_3-\theta_0\theta_1)}\\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0{}^2+\theta_3{}^2)-1\end{bmatrix}$$

, where $\theta_0=\cos\frac{\phi}{2}$ $,\theta_1={}^n a_x\sin\frac{\phi}{2}$

$,\theta_2={}^n a_y\sin\frac{\phi}{2}$ $,\theta_3={}^n a_z\sin\frac{\phi}{2}$

$${}^n\mathbf{R}_{b\_(2,2)}=2(\theta_0{}^2+\theta_2{}^2)-1$$

$$\Downarrow\quad \theta_0=\cos\frac{\phi}{2}\ ,\theta_2={}^n a_y\sin\frac{\phi}{2}$$

$$=2\left(\cos^2\frac{\phi}{2}+{}^n a_y{}^2\ \sin^2\frac{\phi}{2}\right)-1$$

$$\Downarrow\quad \cos^2\frac{\phi}{2}=\frac{1+\cos\phi}{2}\ ,\sin^2\frac{\phi}{2}=\frac{1-\cos\phi}{2}$$

$$=2\left(\frac{1+\cos\phi}{2}+{}^n a_y{}^2\ \frac{1-\cos\phi}{2}\right)-1$$

$$=1+\cos\phi+{}^n a_y{}^2(1-\cos\phi)-1$$

$$=(1-\cos\phi)\,{}^n a_y{}^2+\cos\phi$$

$$=\mathbf{R}_{\mathbf{a},\phi\_(2,2)}$$

$${}^n\mathbf{R}_{b\_(2,3)}=2(\theta_2\theta_3-\theta_0\theta_1)$$

$$\Downarrow\quad \theta_0=\cos\frac{\phi}{2}\ ,\theta_1={}^n a_x\sin\frac{\phi}{2}\quad \theta_2={}^n a_y\sin\frac{\phi}{2}\ ,\theta_3={}^n a_z\sin\frac{\phi}{2}$$

$$=2\left({}^n a_y\sin\frac{\phi}{2}\cdot{}^n a_z\sin\frac{\phi}{2}-\cos\frac{\phi}{2}\cdot{}^n a_x\sin\frac{\phi}{2}\right)$$

$$\Downarrow\quad \sin^2\frac{\phi}{2}=\frac{1-\cos\phi}{2}\ ,\cos\frac{\phi}{2}\sin\frac{\phi}{2}=\frac{\sin\phi}{2}$$

$$=2\left({}^n a_y\,{}^n a_z\frac{1-\cos\phi}{2}-{}^n a_x\frac{\sin\phi}{2}\right)$$

$$=(1-\cos\phi)\,{}^n a_y\,{}^n a_z-{}^n a_x\sin\phi$$

$$=\mathbf{R}_{\mathbf{a},\phi\_(2,3)}$$

(derivation)

$$\mathbf{R}_{\mathbf{a},\phi}=\begin{bmatrix}(1-\cos\phi)^n a_x^2+\cos\phi & (1-\cos\phi)^n a_x{}^n a_y-{}^n a_z\sin\phi & (1-\cos\phi)^n a_z{}^n a_x+{}^n a_y\sin\phi\\ (1-\cos\phi)^n a_x{}^n a_y+{}^n a_z\sin\phi & (1-\cos\phi)^n a_y^2+\cos\phi & (1-\cos\phi)^n a_y{}^n a_z-{}^n a_x\sin\phi\\ \boxed{(1-\cos\phi)^n a_z{}^n a_x-{}^n a_y\sin\phi} & \boxed{(1-\cos\phi)^n a_y{}^n a_z+{}^n a_x\sin\phi} & (1-\cos\phi)^n a_z^2+\cos\phi\end{bmatrix}$$

$$^n\mathbf{R}_b=\begin{bmatrix}2(\theta_0{}^2+\theta_1{}^2)-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2)\\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0{}^2+\theta_2{}^2)-1 & 2(\theta_2\theta_3-\theta_0\theta_1)\\ \boxed{2(\theta_1\theta_3-\theta_0\theta_2)} & \boxed{2(\theta_2\theta_3+\theta_0\theta_1)} & 2(\theta_0{}^2+\theta_3{}^2)-1\end{bmatrix}$$

, where $\theta_0=\cos\frac{\phi}{2}$ $,\theta_1={}^n a_x\sin\frac{\phi}{2}$ $,\theta_2={}^n a_y\sin\frac{\phi}{2}$ $,\theta_3={}^n a_z\sin\frac{\phi}{2}$

$$^n\mathbf{R}_{b\_(3,1)}=2(\theta_1\theta_3-\theta_0\theta_2)$$

$$\Big\Downarrow \quad \theta_0=\cos\frac{\phi}{2}\ ,\theta_1={}^n a_x\sin\frac{\phi}{2}\quad \theta_2={}^n a_y\sin\frac{\phi}{2}\ ,\theta_3={}^n a_z\sin\frac{\phi}{2}$$

$$=2\left({}^n a_x\sin\frac{\phi}{2}\cdot{}^n a_z\sin\frac{\phi}{2}-\cos\frac{\phi}{2}\cdot{}^n a_y\sin\frac{\phi}{2}\right)$$

$$\Big\Downarrow \quad \sin^2\frac{\phi}{2}=\frac{1-\cos\phi}{2}\ ,\cos\frac{\phi}{2}\sin\frac{\phi}{2}=\frac{\sin\phi}{2}$$

$$=2\left({}^n a_x{}^n a_z\frac{1-\cos\phi}{2}-{}^n a_y\frac{\sin\phi}{2}\right)$$

$$=(1-\cos\phi)\,{}^n a_z{}^n a_x-{}^n a_y\sin\phi$$

$$=\mathbf{R}_{\mathbf{a},\phi\_(3,1)}$$

$$^n\mathbf{R}_{b\_(3,2)}=2(\theta_2\theta_3+\theta_0\theta_1)$$

$$\Big\Downarrow \quad \theta_0=\cos\frac{\phi}{2}\ ,\theta_1={}^n a_x\sin\frac{\phi}{2}\quad \theta_2={}^n a_y\sin\frac{\phi}{2}\ ,\theta_3={}^n a_z\sin\frac{\phi}{2}$$

$$=2\left({}^n a_y\sin\frac{\phi}{2}\cdot{}^n a_z\sin\frac{\phi}{2}+\cos\frac{\phi}{2}\cdot{}^n a_x\sin\frac{\phi}{2}\right)$$

$$\Big\Downarrow \quad \sin^2\frac{\phi}{2}=\frac{1-\cos\phi}{2}\ ,\cos\frac{\phi}{2}\sin\frac{\phi}{2}=\frac{\sin\phi}{2}$$

$$=2\left({}^n a_y{}^n a_z\frac{1-\cos\phi}{2}+{}^n a_x\frac{\sin\phi}{2}\right)$$

$$=(1-\cos\phi)\,{}^n a_y{}^n a_z+{}^n a_x\sin\phi$$

$$=\mathbf{R}_{\mathbf{a},\phi\_(3,2)}$$

(derivation)

$$\mathbf{R}_{\mathbf{a},\phi}=\begin{bmatrix}(1-\cos\phi)^n a_x^2+\cos\phi & (1-\cos\phi)^n a_x\,{}^n a_y-{}^n a_z\sin\phi & (1-\cos\phi)^n a_z\,{}^n a_x+{}^n a_y\sin\phi\\ (1-\cos\phi)^n a_x\,{}^n a_y+{}^n a_z\sin\phi & (1-\cos\phi)^n a_y^2+\cos\phi & (1-\cos\phi)^n a_y\,{}^n a_z-{}^n a_x\sin\phi\\ (1-\cos\phi)^n a_z\,{}^n a_x-{}^n a_y\sin\phi & (1-\cos\phi)^n a_y\,{}^n a_z+{}^n a_x\sin\phi & \boxed{(1-\cos\phi)^n a_z^2+\cos\phi}\end{bmatrix}$$

$${}^n\mathbf{R}_b=\begin{bmatrix}2(\theta_0{}^2+\theta_1{}^2)-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2)\\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0{}^2+\theta_2{}^2)-1 & 2(\theta_2\theta_3-\theta_0\theta_1)\\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & \boxed{2(\theta_0{}^2+\theta_3{}^2)-1}\end{bmatrix}$$

, where $\theta_0=\cos\frac{\phi}{2}$ $,\theta_1={}^n a_x\sin\frac{\phi}{2}$

$,\theta_2={}^n a_y\sin\frac{\phi}{2}$ $,\theta_3={}^n a_z\sin\frac{\phi}{2}$

$${}^n\mathbf{R}_{b\_(3,3)}=2(\theta_0{}^2+\theta_3{}^2)-1$$

$$\Downarrow\quad \theta_0=\cos\frac{\phi}{2}\ ,\theta_3={}^n a_z\sin\frac{\phi}{2}$$

$$=2\left(\cos^2\frac{\phi}{2}+{}^n a_z{}^2\ \sin^2\frac{\phi}{2}\right)-1$$

$$\Downarrow\quad \cos^2\frac{\phi}{2}=\frac{1+\cos\phi}{2}\ ,\sin^2\frac{\phi}{2}=\frac{1-\cos\phi}{2}$$

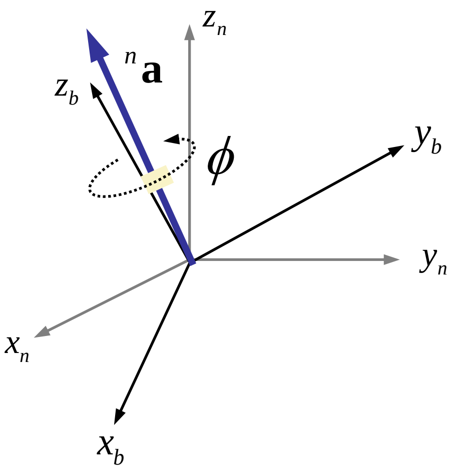
$$=2\left(\frac{1+\cos\phi}{2}+{}^n a_z{}^2\ \frac{1-\cos\phi}{2}\right)-1$$

$$=1+\cos\phi+{}^n a_z{}^2\left(1-\cos\phi\right)-1$$

$$=\left(1-\cos\phi\right){}^n a_z{}^2+\cos\phi$$

$$=\mathbf{R}_{\mathbf{a},\phi\_(3,3)}$$

Euler parameter - rotation transformation matrix



- The rotation transformation matrix is expressed in terms of the angle of rotation ϕ and a unit vector $^n\mathbf{a}$.

$$^n\mathbf{R}_b = \begin{bmatrix} 2(\theta_0{}^2+\theta_1{}^2)-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0{}^2+\theta_2{}^2)-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0{}^2+\theta_3{}^2)-1 \end{bmatrix}$$

, where $\theta_0 = \cos\frac{\phi}{2}$ $, \theta_1 = {}^n a_x \sin\frac{\phi}{2}$ $, \theta_2 = {}^n a_y \sin\frac{\phi}{2}$ $, \theta_3 = {}^n a_z \sin\frac{\phi}{2}$

$\begin{bmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3\end{bmatrix}^T$: Euler parameter $\mathbf{p}$

- $^n\mathbf{R}_b$ can be expressed in terms of components of $\mathbf{p}$.
- $\mathbf{p}$ can be expressed in terms of components of $^n\mathbf{R}_b$.

$$\theta_0{}^2 = \frac{tr\,{}^n\mathbf{R}_b+1}{4} \quad ,\theta_1{}^2 = \frac{1+2\,{}^nR_{b,11}-tr\,{}^n\mathbf{R}_b}{4} \quad ,\theta_2{}^2 = \frac{1+2\,{}^nR_{b,22}-tr\,{}^n\mathbf{R}_b}{4} \quad ,\theta_3{}^2 = \frac{1+2\,{}^nR_{b,33}-tr\,{}^n\mathbf{R}_b}{4}$$

$, where\ tr\,{}^n\mathbf{R}_b = {}^nR_{b,11} + {}^nR_{b,22} + {}^nR_{b,33}$

- By using Euler parameter one can avoid the problem of singularities of the rotation matrix. (Shabana, pp. 81)

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

Euler parameter - rotation transformation matrix

$$
{}^{n}\mathbf{R}_{b} = \begin{bmatrix} 2(\theta_0^{\ 2}+\theta_1^{\ 2})-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0^{\ 2}+\theta_2^{\ 2})-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0^{\ 2}+\theta_3^{\ 2})-1 \end{bmatrix}
$$

$$
\begin{aligned}
\frac{tr\,{}^{n}\mathbf{R}_{b}+1}{4} &= \frac{2(\theta_0^{\ 2}+\theta_1^{\ 2})-1+2(\theta_0^{\ 2}+\theta_2^{\ 2})-1+2(\theta_0^{\ 2}+\theta_3^{\ 2})-1+1}{4} \\
&= \frac{6\theta_0^{\ 2}+2\theta_1^{\ 2}+2\theta_2^{\ 2}+2\theta_3^{\ 2}-2}{4} \\
&= \frac{6\theta_0^{\ 2}+2\theta_1^{\ 2}+2\theta_2^{\ 2}+2\theta_3^{\ 2}-2(\theta_0^{\ 2}+\theta_1^{\ 2}+\theta_2^{\ 2}+\theta_3^{\ 2}}{4} \\
&= \frac{4\theta_0^{\ 2}}{4} \\
&= \theta_0^{\ 2}
\end{aligned}
$$

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

Euler parameter - rotation transformation matrix

$${}^{n}\mathbf{R}_{b}=\begin{bmatrix} 2(\theta_0{}^2+\theta_1{}^2)-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0{}^2+\theta_2{}^2)-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0{}^2+\theta_3{}^2)-1 \end{bmatrix}$$

$$\frac{1+2\,{}^{n}R_{b,11}-tr\,{}^{n}\mathbf{R}_{b}}{4}=\frac{1+4(\theta_0{}^2+\theta_1{}^2)-2-(2(\theta_0{}^2+\theta_1{}^2)-1+2(\theta_0{}^2+\theta_2{}^2)-1+2(\theta_0{}^2+\theta_3{}^2)-1)}{4}$$

$$=\frac{1+4(\theta_0{}^2+\theta_1{}^2)-2-2(\theta_0{}^2+\theta_1{}^2)+1-2(\theta_0{}^2+\theta_2{}^2)+1-2(\theta_0{}^2+\theta_3{}^2)+1}{4}$$

$$=\frac{2+4(\theta_0{}^2+\theta_1{}^2)-2(\theta_0{}^2+\theta_1{}^2)-2(\theta_0{}^2+\theta_2{}^2)-2(\theta_0{}^2+\theta_3{}^2)}{4}$$

$$=\frac{2+2(\theta_0{}^2+\theta_1{}^2)-2(\theta_0{}^2+\theta_2{}^2)-2(\theta_0{}^2+\theta_3{}^2)}{4}$$

$$=\frac{2(\theta_0{}^2+\theta_1{}^2+\theta_2{}^2+\theta_3{}^2)+2(\theta_0{}^2+\theta_1{}^2)-2(\theta_0{}^2+\theta_2{}^2)-2(\theta_0{}^2+\theta_3{}^2)}{4}$$

$$=\frac{4\theta_1{}^2}{4}$$

$$=\theta_1{}^2$$

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

Euler parameter - rotation transformation matrix

$$
{}^{n}\mathbf{R}_{b}=\begin{bmatrix} 2(\theta_0{}^2+\theta_1{}^2)-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0{}^2+\theta_2{}^2)-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0{}^2+\theta_3{}^2)-1 \end{bmatrix}
$$

$$
\begin{aligned}
\frac{1+2\,{}^{n}R_{b,22}-tr\,{}^{n}\mathbf{R}_{b}}{4} &= \frac{1+4(\theta_0{}^2+\theta_2{}^2)-2-(2(\theta_0{}^2+\theta_1{}^2)-1+2(\theta_0{}^2+\theta_2{}^2)-1+2(\theta_0{}^2+\theta_3{}^2)-1)}{4} \\
&= \frac{1+4(\theta_0{}^2+\theta_2{}^2)-2-2(\theta_0{}^2+\theta_1{}^2)+1-2(\theta_0{}^2+\theta_2{}^2)+1-2(\theta_0{}^2+\theta_3{}^2)+1}{4} \\
&= \frac{2+4(\theta_0{}^2+\theta_2{}^2)-2(\theta_0{}^2+\theta_1{}^2)-2(\theta_0{}^2+\theta_2{}^2)-2(\theta_0{}^2+\theta_3{}^2)}{4} \\
&= \frac{2+2(\theta_0{}^2+\theta_2{}^2)-2(\theta_0{}^2+\theta_1{}^2)-2(\theta_0{}^2+\theta_3{}^2)}{4} \\
&= \frac{2(\theta_0{}^2+\theta_1{}^2+\theta_2{}^2+\theta_3{}^2)+2(\theta_0{}^2+\theta_2{}^2)-2(\theta_0{}^2+\theta_1{}^2)-2(\theta_0{}^2+\theta_3{}^2)}{4} \\
&= \frac{4\theta_2{}^2}{4} \\
&= \theta_2{}^2
\end{aligned}
$$

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

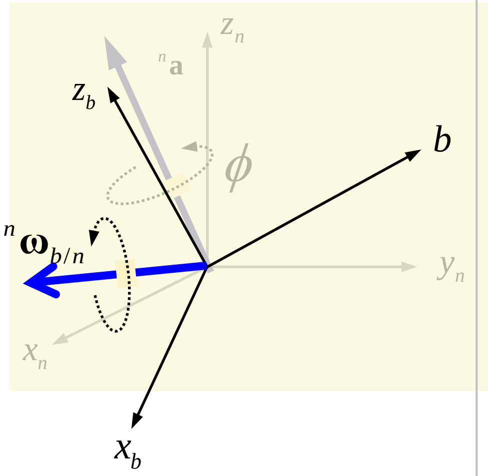
Euler parameter - rotation transformation matrix

$$
{}^{n}\mathbf{R}_{b} = \begin{bmatrix} 2(\theta_0^{2}+\theta_1^{2})-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0^{2}+\theta_2^{2})-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0^{2}+\theta_3^{2})-1 \end{bmatrix}
$$

$$
\begin{aligned}
\frac{1+2\,{}^{n}R_{b,33}-tr\,{}^{n}\mathbf{R}_{b}}{4} &= \frac{1+4(\theta_0^{2}+\theta_3^{2})-2-(2(\theta_0^{2}+\theta_1^{2})-1+2(\theta_0^{2}+\theta_2^{2})-1+2(\theta_0^{2}+\theta_3^{2})-1)}{4} \\
&= \frac{1+4(\theta_0^{2}+\theta_3^{2})-2-2(\theta_0^{2}+\theta_1^{2})+1-2(\theta_0^{2}+\theta_2^{2})+1-2(\theta_0^{2}+\theta_3^{2})+1}{4} \\
&= \frac{2+4(\theta_0^{2}+\theta_3^{2})-2(\theta_0^{2}+\theta_1^{2})-2(\theta_0^{2}+\theta_2^{2})-2(\theta_0^{2}+\theta_3^{2})}{4} \\
&= \frac{2+2(\theta_0^{2}+\theta_3^{2})-2(\theta_0^{2}+\theta_1^{2})-2(\theta_0^{2}+\theta_2^{2})}{4} \\
&= \frac{2(\theta_0^{2}+\theta_1^{2}+\theta_2^{2}+\theta_3^{2})+2(\theta_0^{2}+\theta_3^{2})-2(\theta_0^{2}+\theta_1^{2})-2(\theta_0^{2}+\theta_2^{2})}{4} \\
&= \frac{4\theta_3^{2}}{4} \\
&= \theta_3^{2}
\end{aligned}
$$

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

Euler parameter - angular velocity



- The rotation transformation matrix is expressed in terms of the angle of rotation ϕ and a unit vector $^n\mathbf{a}$.

$$^n\mathbf{R}_b = \begin{bmatrix} 2(\theta_0^{\,2}+\theta_1^{\,2})-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0^{\,2}+\theta_2^{\,2})-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0^{\,2}+\theta_3^{\,2})-1 \end{bmatrix}$$

, where $\theta_0 = \cos\frac{\phi}{2}$ $,\theta_1 = {}^n a_x \sin\frac{\phi}{2}$ $,\theta_2 = {}^n a_y \sin\frac{\phi}{2}$ $,\theta_3 = {}^n a_z \sin\frac{\phi}{2}$

$\begin{bmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3\end{bmatrix}^T$: Euler parameter $\mathbf{p}$

$^n\boldsymbol{\omega}_{b/n}$: Angular velocity vector

cf) Euler angle

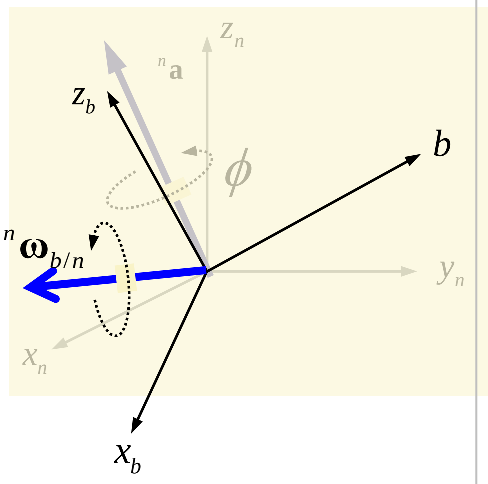
$$^n\boldsymbol{\omega}_{b/n} = \mathbf{G}\dot{\boldsymbol{\gamma}}$$

$$^n\boldsymbol{\omega}_{b/n} = 2\mathbf{E}\dot{\mathbf{p}} \quad , where\ \mathbf{E} = \begin{bmatrix} -\theta_1 & \theta_0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0 \end{bmatrix}$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 206

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

Euler parameter - angular velocity



- The rotation transformation matrix is expressed in terms of the angle of rotation ϕ and a unit vector $^n\mathbf{a}$.

$$^n\mathbf{R}_b = \begin{bmatrix} 2(\theta_0^{\,2}+\theta_1^{\,2})-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0^{\,2}+\theta_2^{\,2})-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0^{\,2}+\theta_3^{\,2})-1 \end{bmatrix}$$

, where $\theta_0 = \cos\frac{\phi}{2}$ $,\theta_1 = {}^na_x\sin\frac{\phi}{2}$ $,\theta_2 = {}^na_y\sin\frac{\phi}{2}$ $,\theta_3 = {}^na_z\sin\frac{\phi}{2}$

$[\theta_0 \quad \theta_1 \quad \theta_2 \quad \theta_3]^T$: Euler parameter $\mathbf{p}$

$^n\boldsymbol{\omega}_{b/n}$: Angular velocity vector

$$^n\boldsymbol{\omega}_{b/n} = 2\mathbf{E}\dot{\mathbf{p}} \quad , where\ \mathbf{E} = \begin{bmatrix} -\theta_1 & \theta_0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0 \end{bmatrix}$$

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{E}^T\,{}^n\boldsymbol{\omega}_{b/n}$$

cf) Euler angle

$$^n\boldsymbol{\omega}_{b/n} = \mathbf{G}\dot{\boldsymbol{\gamma}}$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 206

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

Euler parameter - angular velocity

$$\boxed{{}^{n}\boldsymbol{\omega}_{b/n} = 2\mathbf{E}\dot{\mathbf{p}}}$$

$$,\,where\ \mathbf{E} = \begin{bmatrix} -\theta_1 & \theta_0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0 \end{bmatrix}$$

$\begin{bmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3\end{bmatrix}^T$: **Euler parameter $\mathbf{p}$**

$$\theta_0 = \cos\frac{\phi}{2}, \theta_1 = {}^{n}a_x \sin\frac{\phi}{2}, \theta_2 = {}^{n}a_y \sin\frac{\phi}{2}, \theta_3 = {}^{n}a_z \sin\frac{\phi}{2}$$

$${}^{n}\mathbf{R}_b = \mathbf{E}\bar{\mathbf{E}}^T \quad \text{---(1)}$$

$${}^{n}\tilde{\boldsymbol{\omega}}_{b/n} = {}^{n}\dot{\mathbf{R}}_b\, {}^{n}\mathbf{R}_b^T \quad \text{---(2)}$$

$${}^{n}\mathbf{R}_b = \begin{bmatrix} 2(\theta_0^{\,2}+\theta_1^{\,2})-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0^{\,2}+\theta_2^{\,2})-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0^{\,2}+\theta_3^{\,2})-1 \end{bmatrix}$$

Substituting (1) into (2)

$$\bar{\mathbf{E}} = \begin{bmatrix} -\theta_1 & \theta_0 & \theta_3 & -\theta_2 \\ -\theta_2 & -\theta_3 & \theta_0 & \theta_1 \\ -\theta_3 & \theta_2 & -\theta_1 & \theta_0 \end{bmatrix}$$

$${}^{n}\tilde{\boldsymbol{\omega}}_{b/n} = 2\dot{\mathbf{E}}\mathbf{E}^T \quad \text{---(3)}$$

$$\longrightarrow {}^{n}\boldsymbol{\omega}_{b/n} = 2\mathbf{E}\dot{\mathbf{p}}$$

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

(Derivation)

$$^{n}\mathbf{R}_b = \mathbf{E}\bar{\mathbf{E}}^T$$

Proof)

L.H.S

$$^{n}\mathbf{R}_b = \begin{bmatrix} 2(\theta_0{}^2+\theta_1{}^2)-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0{}^2+\theta_2{}^2)-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0{}^2+\theta_3{}^2)-1 \end{bmatrix}$$

R.H.S

$$\mathbf{E}\bar{\mathbf{E}}^T = \begin{bmatrix} -\theta_1 & \theta_0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0 \end{bmatrix} \begin{bmatrix} -\theta_1 & -\theta_2 & -\theta_3 \\ \theta_0 & -\theta_3 & \theta_2 \\ \theta_3 & \theta_0 & -\theta_1 \\ -\theta_2 & \theta_1 & \theta_0 \end{bmatrix}$$

$$, where\ \mathbf{E} = \begin{bmatrix} -\theta_1 & \theta_0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0 \end{bmatrix}, \bar{\mathbf{E}} = \begin{bmatrix} -\theta_1 & \theta_0 & \theta_3 & -\theta_2 \\ -\theta_2 & -\theta_3 & \theta_0 & \theta_1 \\ -\theta_3 & \theta_2 & -\theta_1 & \theta_0 \end{bmatrix}$$

$$= \begin{bmatrix} \theta_1{}^2+\theta_0{}^2-\theta_3{}^2-\theta_2{}^2 & \theta_1\theta_2-\theta_0\theta_3-\theta_3\theta_0+\theta_2\theta_1 & \theta_1\theta_3+\theta_0\theta_2+\theta_3\theta_1+\theta_2\theta_0 \\ \theta_2\theta_1+\theta_3\theta_0+\theta_0\theta_3+\theta_1\theta_2 & \theta_2{}^2-\theta_3{}^2+\theta_0{}^2-\theta_1{}^2 & \theta_2\theta_3+\theta_3\theta_2-\theta_0\theta_1-\theta_1\theta_0 \\ \theta_3\theta_1-\theta_2\theta_0+\theta_1\theta_3-\theta_0\theta_2 & \theta_3\theta_2+\theta_2\theta_3+\theta_1\theta_0+\theta_0\theta_1 & \theta_3{}^2-\theta_2{}^2-\theta_1{}^2+\theta_0{}^2 \end{bmatrix}$$

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(Derivation)

$[\theta_0 \quad \theta_1 \quad \theta_2 \quad \theta_3]^T$: **Euler parameter** $\mathbf{p}$

$$\theta_0 = \cos\frac{\phi}{2}, \theta_1 = {}^n a_x \sin\frac{\phi}{2}, \theta_2 = {}^n a_y \sin\frac{\phi}{2}, \theta_3 = {}^n a_z \sin\frac{\phi}{2}$$

$$\theta_0{}^2 + \theta_2{}^2 + \theta_3{}^2 + \theta_4{}^2 = \cos^2\frac{\phi}{2} + {}^n a_x^2 \sin^2\frac{\phi}{2} + {}^n a_y^2 \sin^2\frac{\phi}{2} + {}^n a_z^2 \sin^2\frac{\phi}{2}$$

$$= \cos^2\frac{\phi}{2} + \left({}^n a_x^2 + {}^n a_y^2 + {}^n a_z^2\right)\sin^2\frac{\phi}{2}$$

$$= \cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}$$

$$= 1$$

$$ {}^n\mathbf{R}_b = \mathbf{E}\overline{\mathbf{E}}^T $$

Proof)

L.H.S

$$ {}^n\mathbf{R}_b = \begin{bmatrix} 2(\theta_0{}^2+\theta_1{}^2)-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0{}^2+\theta_2{}^2)-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0{}^2+\theta_3{}^2)-1 \end{bmatrix} $$

R.H.S

$$ \mathbf{E}\overline{\mathbf{E}}^T = \begin{bmatrix} \theta_1{}^2+\theta_0{}^2-\theta_3{}^2-\theta_2{}^2 & \theta_1\theta_2-\theta_0\theta_3-\theta_3\theta_0+\theta_2\theta_1 & \theta_1\theta_3+\theta_0\theta_2+\theta_3\theta_1+\theta_2\theta_0 \\ \theta_2\theta_1+\theta_3\theta_0+\theta_0\theta_3+\theta_1\theta_2 & \theta_2{}^2-\theta_3{}^2+\theta_0{}^2-\theta_1{}^2 & \theta_2\theta_3+\theta_3\theta_2-\theta_0\theta_1-\theta_1\theta_0 \\ \theta_3\theta_1-\theta_2\theta_0+\theta_1\theta_3-\theta_0\theta_2 & \theta_3\theta_2+\theta_2\theta_3+\theta_1\theta_0+\theta_0\theta_1 & \theta_3{}^2-\theta_2{}^2-\theta_1{}^2+\theta_0{}^2 \end{bmatrix} $$

$$ = \begin{bmatrix} 2\left(\theta_0{}^2+\theta_1{}^2\right)-\left(\theta_1{}^2+\theta_0{}^2+\theta_3{}^2+\theta_2{}^2\right) & 2\left(\theta_1\theta_2-\theta_0\theta_3\right) & 2\left(\theta_1\theta_3+\theta_0\theta_2\right) \\ 2\left(\theta_1\theta_2+\theta_0\theta_3\right) & 2\left(\theta_0{}^2+\theta_2{}^2\right)-\left(\theta_0{}^2+\theta_2{}^2+\theta_3{}^2+\theta_1{}^2\right) & 2\left(\theta_2\theta_3-\theta_0\theta_1\right) \\ 2\left(\theta_1\theta_3-\theta_0\theta_2\right) & 2\left(\theta_2\theta_3+\theta_0\theta_1\right) & 2\left(\theta_0{}^2+\theta_3{}^2\right)-\left(\theta_3{}^2+\theta_0{}^2+\theta_2{}^2+\theta_1{}^2\right) \end{bmatrix} $$

$$ = \begin{bmatrix} 2\left(\theta_0{}^2+\theta_1{}^2\right)-1 & 2\left(\theta_1\theta_2-\theta_0\theta_3\right) & 2\left(\theta_1\theta_3+\theta_0\theta_2\right) \\ 2\left(\theta_1\theta_2+\theta_0\theta_3\right) & 2\left(\theta_0{}^2+\theta_2{}^2\right)-1 & 2\left(\theta_2\theta_3-\theta_0\theta_1\right) \\ 2\left(\theta_1\theta_3-\theta_0\theta_2\right) & 2\left(\theta_2\theta_3+\theta_0\theta_1\right) & 2\left(\theta_0{}^2+\theta_3{}^2\right)-1 \end{bmatrix} $$

$$ \boxed{\therefore\ {}^n\mathbf{R}_b = \mathbf{E}\overline{\mathbf{E}}^T} $$

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(Derivation)

$$ {}^{n}\tilde{\boldsymbol{\omega}}_{b/n} = {}^{n}\dot{\mathbf{R}}_{b}\,{}^{n}\mathbf{R}_{b}^{T} $$

Proof)

$$ {}^{n}\dot{\mathbf{R}}_{b} = {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{R}_{b} $$

$$ = {}^{n}\tilde{\boldsymbol{\omega}}_{b/n}\,{}^{n}\mathbf{R}_{b} $$

$$ \boldsymbol{\omega}\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} $$

$$ \tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}\times $$

↓

$$ {}^{n}\dot{\mathbf{R}}_{b} = {}^{n}\tilde{\boldsymbol{\omega}}_{b/n}\,{}^{n}\mathbf{R}_{b} $$

$$ {}^{n}\dot{\mathbf{R}}_{b}\,{}^{n}\mathbf{R}_{b}^{T} = {}^{n}\tilde{\boldsymbol{\omega}}_{b/n}\,{}^{n}\mathbf{R}_{b}\,{}^{n}\mathbf{R}_{b}^{T} $$

$$ {}^{n}\dot{\mathbf{R}}_{b}\,{}^{n}\mathbf{R}_{b}^{T} = {}^{n}\tilde{\boldsymbol{\omega}}_{b/n}\,{}^{n}\mathbf{R}_{b}\,{}^{b}\mathbf{R}_{n} $$

$$ \boxed{{}^{n}\dot{\mathbf{R}}_{b}\,{}^{n}\mathbf{R}_{b}^{T} = {}^{n}\tilde{\boldsymbol{\omega}}_{b/n}} $$

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(Derivation)

$${}^{n}\tilde{\boldsymbol{\omega}}_{b/n} = 2\dot{\mathbf{E}}\mathbf{E}^{T}$$

$$, where\ \mathbf{E} = \begin{bmatrix} -\theta_1 & \theta_0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0 \end{bmatrix}, \bar{\mathbf{E}} = \begin{bmatrix} -\theta_1 & \theta_0 & \theta_3 & -\theta_2 \\ -\theta_2 & -\theta_3 & \theta_0 & \theta_1 \\ -\theta_3 & \theta_2 & -\theta_1 & \theta_0 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^{T}$$

Proof)

$${}^{n}\mathbf{R}_{b} = \mathbf{E}\bar{\mathbf{E}}^{T} \qquad \text{---(1)}$$

$${}^{n}\tilde{\boldsymbol{\omega}}_{b/n} = {}^{n}\dot{\mathbf{R}}_{b}\,{}^{n}\mathbf{R}_{b}^{T} \qquad \text{---(2)}$$

Substituting (1) into (2)

R.H.S of eq. (2)

$${}^{n}\dot{\mathbf{R}}_{b}\,{}^{n}\mathbf{R}_{b}^{T}$$

$${}^{n}\mathbf{R}_{b} = \mathbf{E}\bar{\mathbf{E}}^{T}$$

$${}^{n}\dot{\mathbf{R}}_{b} = \dot{\mathbf{E}}\bar{\mathbf{E}}^{T} + \mathbf{E}\dot{\bar{\mathbf{E}}}^{T}$$

$${}^{n}\mathbf{R}_{b}^{T} = \left(\mathbf{E}\bar{\mathbf{E}}^{T}\right)^{T} = \left(\bar{\mathbf{E}}^{T}\right)^{T}\left(\mathbf{E}\right)^{T} = \bar{\mathbf{E}}\mathbf{E}^{T}$$

$$= \left(\dot{\mathbf{E}}\bar{\mathbf{E}}^{T} + \mathbf{E}\dot{\bar{\mathbf{E}}}^{T}\right)\bar{\mathbf{E}}\mathbf{E}^{T}$$

$$\mathbf{E}\dot{\bar{\mathbf{E}}}^{T} = \dot{\mathbf{E}}\bar{\mathbf{E}}^{T}$$

$$= \left(\dot{\mathbf{E}}\bar{\mathbf{E}}^{T} + \dot{\mathbf{E}}\bar{\mathbf{E}}^{T}\right)\bar{\mathbf{E}}\mathbf{E}^{T}$$

$$= 2\dot{\mathbf{E}}\bar{\mathbf{E}}^{T}\bar{\mathbf{E}}\mathbf{E}^{T}$$

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(Derivation)

$${}^{n}\tilde{\boldsymbol{\omega}}_{b/n} = 2\dot{\mathbf{E}}\mathbf{E}^{T}$$

$$, where\ \mathbf{E} = \begin{bmatrix} -\theta_1 & \theta_0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0 \end{bmatrix}, \bar{\mathbf{E}} = \begin{bmatrix} -\theta_1 & \theta_0 & \theta_3 & -\theta_2 \\ -\theta_2 & -\theta_3 & \theta_0 & \theta_1 \\ -\theta_3 & \theta_2 & -\theta_1 & \theta_0 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^{T}$$

Proof)

R.H.S of eq. (2)

$${}^{n}\dot{\mathbf{R}}_{b}\,{}^{n}\mathbf{R}_{b}^{T}$$

$$= 2\dot{\mathbf{E}}\bar{\mathbf{E}}^{T}\bar{\mathbf{E}}\mathbf{E}^{T}$$

$$\Big\Downarrow \ \bar{\mathbf{E}}^{T}\bar{\mathbf{E}} = \mathbf{I}_4 + \mathbf{p}\mathbf{p}^{T}$$

$$= 2\dot{\mathbf{E}}\left(\mathbf{I}_4 + \mathbf{p}\mathbf{p}^{T}\right)\mathbf{E}^{T}$$

$$= 2\dot{\mathbf{E}}\mathbf{I}_4\mathbf{E}^{T} + 2\dot{\mathbf{E}}\mathbf{p}\mathbf{p}^{T}\mathbf{E}^{T}$$

$$= 2\dot{\mathbf{E}}\mathbf{E}^{T} + 2\dot{\mathbf{E}}\mathbf{p}\mathbf{p}^{T}\mathbf{E}^{T}$$

$$= 2\dot{\mathbf{E}}\mathbf{E}^{T} + 2\dot{\mathbf{E}}\mathbf{p}\left(\mathbf{E}\mathbf{p}\right)^{T}$$

$$\Big\Downarrow \ \mathbf{E}\mathbf{p} = 0$$

$$= 2\dot{\mathbf{E}}\mathbf{E}^{T}$$

$${}^{n}\mathbf{R}_{b} = \mathbf{E}\bar{\mathbf{E}}^{T} \qquad \text{---(1)}$$

$${}^{n}\tilde{\boldsymbol{\omega}}_{b/n} = {}^{n}\dot{\mathbf{R}}_{b}\,{}^{n}\mathbf{R}_{b}^{T} \qquad \text{---(2)}$$

Substituting (1) into (2)

$$\boxed{\therefore\ {}^{n}\tilde{\boldsymbol{\omega}}_{b/n} = 2\dot{\mathbf{E}}\mathbf{E}^{T}}$$

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(Derivation)

$$^{n}\boldsymbol{\omega}_{b/n} = 2\mathbf{E}\dot{\mathbf{p}}$$

Proof)

$$^{n}\tilde{\boldsymbol{\omega}}_{b/n} = 2\dot{\mathbf{E}}\mathbf{E}^{T}$$ **---(3)**

L.H.S of eq. (3)

$$^{n}\tilde{\boldsymbol{\omega}}_{b/n} = {}^{n}\boldsymbol{\omega}_{b/n}\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

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(Derivation)

$$\dot{\mathbf{E}} = \begin{bmatrix} -\dot{\theta}_1 & \dot{\theta}_0 & -\dot{\theta}_3 & \dot{\theta}_2 \\ -\dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_0 & -\dot{\theta}_1 \\ -\dot{\theta}_3 & -\dot{\theta}_2 & \dot{\theta}_1 & \dot{\theta}_0 \end{bmatrix} \quad \mathbf{E}^T = \begin{bmatrix} -\theta_1 & -\theta_2 & -\theta_3 \\ \theta_0 & \theta_3 & -\theta_2 \\ -\theta_3 & \theta_0 & \theta_1 \\ \theta_2 & -\theta_1 & \theta_0 \end{bmatrix}$$

$$^n\boldsymbol{\omega}_{b/n} = 2\mathbf{E}\dot{\mathbf{p}}$$

Proof)

$$^n\tilde{\boldsymbol{\omega}}_{b/n} = 2\dot{\mathbf{E}}\mathbf{E}^T \text{ ---(3)}$$

L.H.S of eq. (3)

$$^n\tilde{\boldsymbol{\omega}}_{b/n} = {^n\boldsymbol{\omega}_{b/n}}\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

R.H.S of eq. (3)

$$2\dot{\mathbf{E}}\mathbf{E}^T = 2\begin{bmatrix} -\dot{\theta}_1 & \dot{\theta}_0 & -\dot{\theta}_3 & \dot{\theta}_2 \\ -\dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_0 & -\dot{\theta}_1 \\ -\dot{\theta}_3 & -\dot{\theta}_2 & \dot{\theta}_1 & \dot{\theta}_0 \end{bmatrix}\begin{bmatrix} -\theta_1 & -\theta_2 & -\theta_3 \\ \theta_0 & \theta_3 & -\theta_2 \\ -\theta_3 & \theta_0 & \theta_1 \\ \theta_2 & -\theta_1 & \theta_0 \end{bmatrix}$$

$$= 2\begin{bmatrix} \dot{\theta}_1\theta_1 + \dot{\theta}_0\theta_0 + \dot{\theta}_3\theta_3 + \dot{\theta}_2\theta_2 & \dot{\theta}_1\theta_2 + \dot{\theta}_0\theta_3 - \dot{\theta}_3\theta_0 - \dot{\theta}_2\theta_1 & \dot{\theta}_1\theta_3 - \dot{\theta}_0\theta_2 - \dot{\theta}_3\theta_1 + \dot{\theta}_2\theta_0 \\ \dot{\theta}_2\theta_1 + \dot{\theta}_3\theta_0 - \dot{\theta}_0\theta_3 - \dot{\theta}_1\theta_2 & \dot{\theta}_2\theta_2 + \dot{\theta}_3\theta_3 + \dot{\theta}_0\theta_0 + \dot{\theta}_1\theta_1 & \dot{\theta}_2\theta_3 - \dot{\theta}_3\theta_2 + \dot{\theta}_0\theta_1 - \dot{\theta}_1\theta_0 \\ \dot{\theta}_3\theta_1 - \dot{\theta}_2\theta_0 - \dot{\theta}_1\theta_3 + \dot{\theta}_0\theta_2 & \dot{\theta}_3\theta_2 - \dot{\theta}_2\theta_3 + \dot{\theta}_1\theta_0 - \dot{\theta}_0\theta_1 & \dot{\theta}_3\theta_3 + \dot{\theta}_2\theta_2 + \dot{\theta}_1\theta_1 + \dot{\theta}_0\theta_0 \end{bmatrix}$$

L.H.S of eq. (3) = R.H.S of eq. (3)

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = 2\begin{bmatrix} \dot{\theta}_1\theta_1 + \dot{\theta}_0\theta_0 + \dot{\theta}_3\theta_3 + \dot{\theta}_2\theta_2 & \dot{\theta}_1\theta_2 + \dot{\theta}_0\theta_3 - \dot{\theta}_3\theta_0 - \dot{\theta}_2\theta_1 & \dot{\theta}_1\theta_3 - \dot{\theta}_0\theta_2 - \dot{\theta}_3\theta_1 + \dot{\theta}_2\theta_0 \\ \dot{\theta}_2\theta_1 + \dot{\theta}_3\theta_0 - \dot{\theta}_0\theta_3 - \dot{\theta}_1\theta_2 & \dot{\theta}_2\theta_2 + \dot{\theta}_3\theta_3 + \dot{\theta}_0\theta_0 + \dot{\theta}_1\theta_1 & \dot{\theta}_2\theta_3 - \dot{\theta}_3\theta_2 + \dot{\theta}_0\theta_1 - \dot{\theta}_1\theta_0 \\ \dot{\theta}_3\theta_1 - \dot{\theta}_2\theta_0 - \dot{\theta}_1\theta_3 + \dot{\theta}_0\theta_2 & \dot{\theta}_3\theta_2 - \dot{\theta}_2\theta_3 + \dot{\theta}_1\theta_0 - \dot{\theta}_0\theta_1 & \dot{\theta}_3\theta_3 + \dot{\theta}_2\theta_2 + \dot{\theta}_1\theta_1 + \dot{\theta}_0\theta_0 \end{bmatrix}$$

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(Derivation)

$$ {}^{n}\boldsymbol{\omega}_{b/n} = 2\mathbf{E}\dot{\mathbf{p}} $$

$$ \dot{\mathbf{E}} = \begin{bmatrix} -\dot{\theta}_1 & \dot{\theta}_0 & -\dot{\theta}_3 & \dot{\theta}_2 \\ -\dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_0 & -\dot{\theta}_1 \\ -\dot{\theta}_3 & -\dot{\theta}_2 & \dot{\theta}_1 & \dot{\theta}_0 \end{bmatrix} \quad \mathbf{E}^T = \begin{bmatrix} -\theta_1 & -\theta_2 & -\theta_3 \\ \theta_0 & \theta_3 & -\theta_2 \\ -\theta_3 & \theta_0 & \theta_1 \\ \theta_2 & -\theta_1 & \theta_0 \end{bmatrix} $$

Proof)

L.H.S of eq. (3) = R.H.S of eq. (3)

$$ {}^{n}\tilde{\boldsymbol{\omega}}_{b/n} = 2\dot{\mathbf{E}}\mathbf{E}^T \text{ ---(3)} $$

$$ \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = 2\begin{bmatrix} \dot{\theta}_1\theta_1 + \dot{\theta}_0\theta_0 + \dot{\theta}_3\theta_3 + \dot{\theta}_2\theta_2 & \dot{\theta}_1\theta_2 + \dot{\theta}_0\theta_3 - \dot{\theta}_3\theta_0 - \dot{\theta}_2\theta_1 & \boxed{\dot{\theta}_1\theta_3 - \dot{\theta}_0\theta_2 - \dot{\theta}_3\theta_1 + \dot{\theta}_2\theta_0} \\ \boxed{\dot{\theta}_2\theta_1 + \dot{\theta}_3\theta_0 - \dot{\theta}_0\theta_3 - \dot{\theta}_1\theta_2} & \dot{\theta}_2\theta_2 + \dot{\theta}_3\theta_3 + \dot{\theta}_0\theta_0 + \dot{\theta}_1\theta_1 & \dot{\theta}_2\theta_3 - \dot{\theta}_3\theta_2 + \dot{\theta}_0\theta_1 - \dot{\theta}_1\theta_0 \\ \dot{\theta}_3\theta_1 - \dot{\theta}_2\theta_0 - \dot{\theta}_1\theta_3 + \dot{\theta}_0\theta_2 & \boxed{\dot{\theta}_3\theta_2 - \dot{\theta}_2\theta_3 + \dot{\theta}_1\theta_0 - \dot{\theta}_0\theta_1} & \dot{\theta}_3\theta_3 + \dot{\theta}_2\theta_2 + \dot{\theta}_1\theta_1 + \dot{\theta}_0\theta_0 \end{bmatrix} $$

$$ \boxed{\begin{aligned} \omega_1 &= 2\left(\dot{\theta}_3\theta_2 - \dot{\theta}_2\theta_3 + \dot{\theta}_1\theta_0 - \dot{\theta}_0\theta_1\right) \\ \therefore\ \omega_2 &= 2\left(\dot{\theta}_1\theta_3 - \dot{\theta}_0\theta_2 - \dot{\theta}_3\theta_1 + \dot{\theta}_2\theta_0\right) \\ \omega_3 &= 2\left(\dot{\theta}_2\theta_1 + \dot{\theta}_3\theta_0 - \dot{\theta}_0\theta_3 - \dot{\theta}_1\theta_2\right) \end{aligned}} $$

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(Derivation)

$$\mathbf{E}=\begin{bmatrix}-\theta_1 & \theta_0 & -\theta_3 & \theta_2\\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1\\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0\end{bmatrix},$$

$$\dot{\mathbf{E}}=\begin{bmatrix}-\dot\theta_1 & \dot\theta_0 & -\dot\theta_3 & \dot\theta_2\\ -\dot\theta_2 & \dot\theta_3 & \dot\theta_0 & -\dot\theta_1\\ -\dot\theta_3 & -\dot\theta_2 & \dot\theta_1 & \dot\theta_0\end{bmatrix} \quad \mathbf{E}^T=\begin{bmatrix}-\theta_1 & -\theta_2 & -\theta_3\\ \theta_0 & \theta_3 & -\theta_2\\ -\theta_3 & \theta_0 & \theta_1\\ \theta_2 & -\theta_1 & \theta_0\end{bmatrix}$$

$$\mathbf{p}=\begin{bmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3\end{bmatrix}^T$$

$$^{n}\tilde{\boldsymbol{\omega}}_{b/n}=2\dot{\mathbf{E}}\mathbf{E}^T \quad \text{---(3)}$$

$$^{n}\boldsymbol{\omega}_{b/n}=2\mathbf{E}\dot{\mathbf{p}}$$

Proof)

L.H.S of eq. (3) = R.H.S of eq. (3)

$$\omega_1=2\left(\dot\theta_3\theta_2-\dot\theta_2\theta_3+\dot\theta_1\theta_0-\dot\theta_0\theta_1\right)$$

$$\omega_2=2\left(\dot\theta_1\theta_3-\dot\theta_0\theta_2-\dot\theta_3\theta_1+\dot\theta_2\theta_0\right)$$

$$\omega_3=2\left(\dot\theta_2\theta_1+\dot\theta_3\theta_0-\dot\theta_0\theta_3-\dot\theta_1\theta_2\right)$$

$$^{n}\boldsymbol{\omega}_{b/n}=\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}=\begin{bmatrix}2\left(\dot\theta_3\theta_2-\dot\theta_2\theta_3+\dot\theta_1\theta_0-\dot\theta_0\theta_1\right)\\ 2\left(\dot\theta_1\theta_3-\dot\theta_0\theta_2-\dot\theta_3\theta_1+\dot\theta_2\theta_0\right)\\ 2\left(\dot\theta_2\theta_1+\dot\theta_3\theta_0-\dot\theta_0\theta_3-\dot\theta_1\theta_2\right)\end{bmatrix}=2\underbrace{\begin{bmatrix}-\theta_1 & \theta_0 & -\theta_3 & \theta_2\\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1\\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0\end{bmatrix}}_{\mathbf{E}}\underbrace{\begin{bmatrix}\dot\theta_0\\ \dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}}_{\dot{\mathbf{p}}}$$

$$\therefore\ ^{n}\boldsymbol{\omega}_{b/n}=2\mathbf{E}\dot{\mathbf{p}}$$

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(Derivation)

$${}^{n}\boldsymbol{\omega}_{b/n} = 2\mathbf{E}\dot{\mathbf{p}}$$

$$\mathbf{E}^{T}\,{}^{n}\boldsymbol{\omega}_{b/n} = 2\mathbf{E}^{T}\mathbf{E}\dot{\mathbf{p}}$$

$$\Big\Downarrow\ \bar{\mathbf{E}}^{T}\bar{\mathbf{E}} = \mathbf{I}_4 + \mathbf{p}\mathbf{p}^{T}$$

$$\mathbf{E}^{T}\,{}^{n}\boldsymbol{\omega}_{b/n} = 2\left(\mathbf{I}_4 + \mathbf{p}\mathbf{p}^{T}\right)\dot{\mathbf{p}}$$

$$\mathbf{E}^{T}\,{}^{n}\boldsymbol{\omega}_{b/n} = 2\mathbf{I}_4\dot{\mathbf{p}} + 2\mathbf{p}\mathbf{p}^{T}\dot{\mathbf{p}}$$

$$\mathbf{E}^{T}\,{}^{n}\boldsymbol{\omega}_{b/n} = 2\mathbf{I}_4\dot{\mathbf{p}} + 2\mathbf{p}\left(\dot{\mathbf{p}}^{T}\mathbf{p}\right)^{T}$$

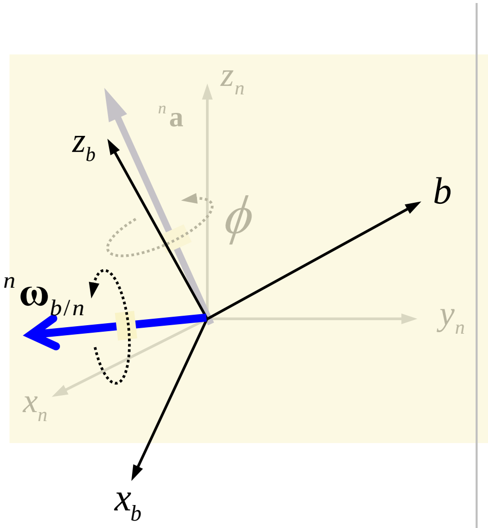
$$\Big\Downarrow\ \dot{\mathbf{p}}^{T}\mathbf{p} = 0$$

$$\mathbf{E}^{T}\,{}^{n}\boldsymbol{\omega}_{b/n} = 2\mathbf{I}_4\dot{\mathbf{p}}$$

$$\mathbf{E}^{T}\,{}^{n}\boldsymbol{\omega}_{b/n} = 2\dot{\mathbf{p}}$$

$$\boxed{\therefore \dot{\mathbf{p}} = \frac{1}{2}\mathbf{E}^{T}\,{}^{n}\boldsymbol{\omega}_{b/n}}$$

Orientation of the rigid body in spatial motion
- Euler parameter - angular velocity



- The rotation transformation matrix is expressed in terms of the angle of rotation ϕ and a unit vector $^n\mathbf{a}$.

$$^n\mathbf{R}_b = \begin{bmatrix} 2(\theta_0{}^2+\theta_1{}^2)-1 & 2(\theta_1\theta_2-\theta_0\theta_3) & 2(\theta_1\theta_3+\theta_0\theta_2) \\ 2(\theta_1\theta_2+\theta_0\theta_3) & 2(\theta_0{}^2+\theta_2{}^2)-1 & 2(\theta_2\theta_3-\theta_0\theta_1) \\ 2(\theta_1\theta_3-\theta_0\theta_2) & 2(\theta_2\theta_3+\theta_0\theta_1) & 2(\theta_0{}^2+\theta_3{}^2)-1 \end{bmatrix}$$

, where $\theta_0 = \cos\frac{\phi}{2}$ $,\theta_1 = {}^na_x\sin\frac{\phi}{2}$ $,\theta_2 = {}^na_y\sin\frac{\phi}{2}$ $,\theta_3 = {}^na_z\sin\frac{\phi}{2}$

$[\theta_0 \quad \theta_1 \quad \theta_2 \quad \theta_3]^T$: Euler parameter $\mathbf{p}$

$^n\boldsymbol{\omega}_{b/n}$: Angular velocity vector

cf) Euler angle

$$^n\boldsymbol{\omega}_{b/n} = 2\mathbf{E}\dot{\mathbf{p}}$$

$$, where\ \mathbf{E} = \begin{bmatrix} -\theta_1 & \theta_0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0 \end{bmatrix}$$

$$^n\boldsymbol{\omega}_{b/n} = \mathbf{G}\dot{\boldsymbol{\gamma}}$$

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{E}^T\,{}^n\boldsymbol{\omega}_{b/n}$$

Haug, E. J., Intermediate Dynamics, Prentice-Hall, 1992, pp. 206

Ahmed A. Shabana, Dynamics of multibody systems, third edition, Cambridge University Press, 2005, pp. 31

Topics in ship design automation

5. Recursive Formulation

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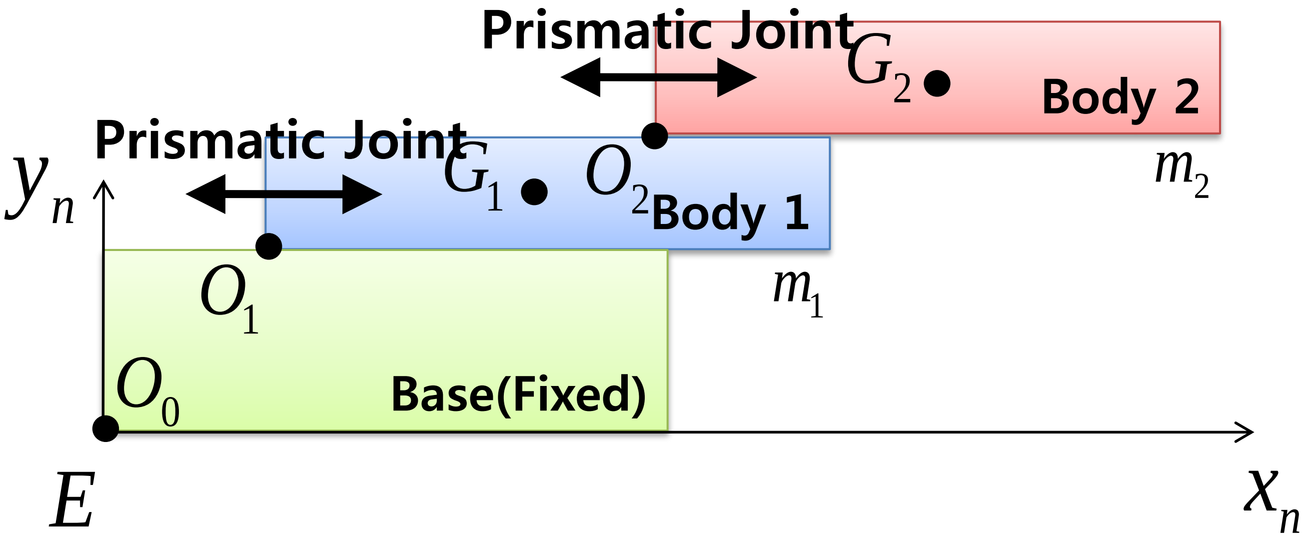
5.1 Inverse and Forward Dynamics

Equations of Motion for Multibody that has two Prismatic Joints - Problem Definition



$n-\mathrm{frame}$: Inertial Frame

G_1 : Center of Mass of Body 1

G_2 : Center of Mass of Body 2

O_0: Origin of the Base

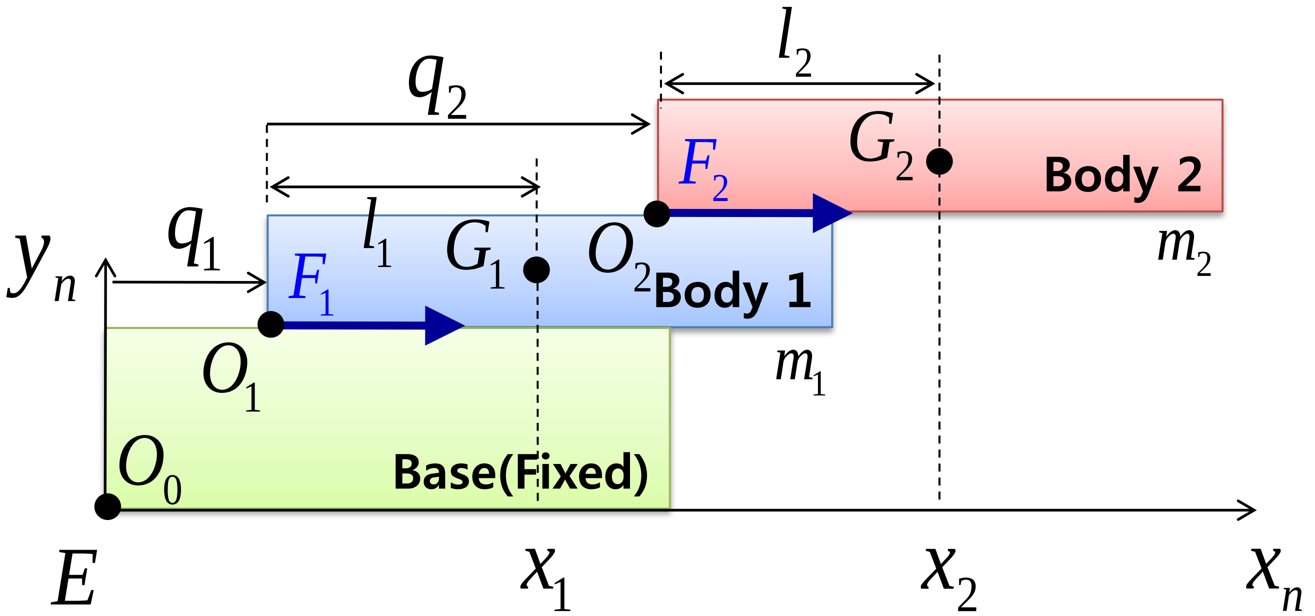
O_1: Origin of the Body 1

O_2: Origin of the Body 2

m_1: Mass of the Body 1

m_2: Mass of the Body 2

Equations of Motion for Multibody that has two Prismatic Joints - Problem Definition



$n-\text{frame}$: Inertial Frame

G_1 : Center of Mass of Body 1

G_2 : Center of Mass of Body 2

O_0: Origin of the Base

O_1: Origin of the Body 1

O_2: Origin of the Body 2

m_1: Mass of the Body 1

m_2: Mass of the Body 2

x_1 : x coordinate of G_1 with respect to the n-frame

x_2 : x coordinate of G_2 with respect to the n-frame

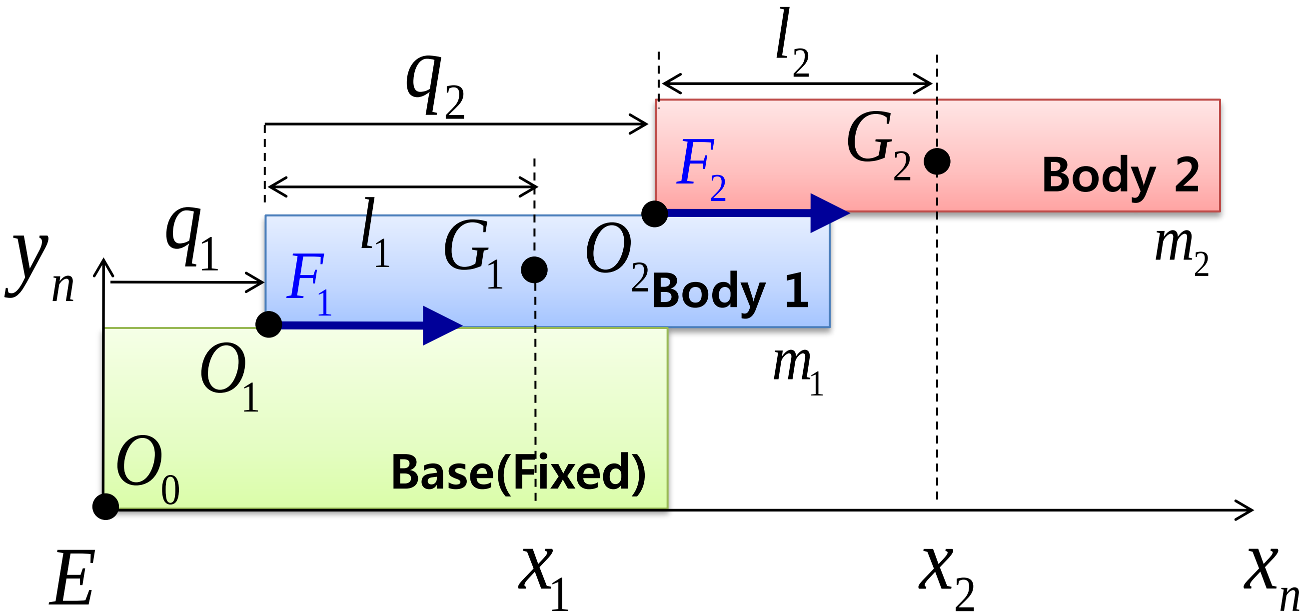
q_1 : Displacement of O_1 with respect to O_0 in x direction

q_2 : Displacement of O_2 with respect to O_1 in x direction

F_1 : Force acting on the body 1 from the base

F_2 : Force acting on the body 2 from the body 1

Equations of Motion for Multibody that has two Prismatic Joints - Derivation of the Equations of Motion



According to Newton's 2nd law

- **Resultant Force acting on the Body 2:** F_2 ⇨ $F_2 = m_2\ddot{x}_2$
- **Resultant Force acting on the Body 1:** $F_1 - F_2$ ⇨ $F_1 - F_2 = m_1\ddot{x}_1$

→ **Substituting** x **with** q

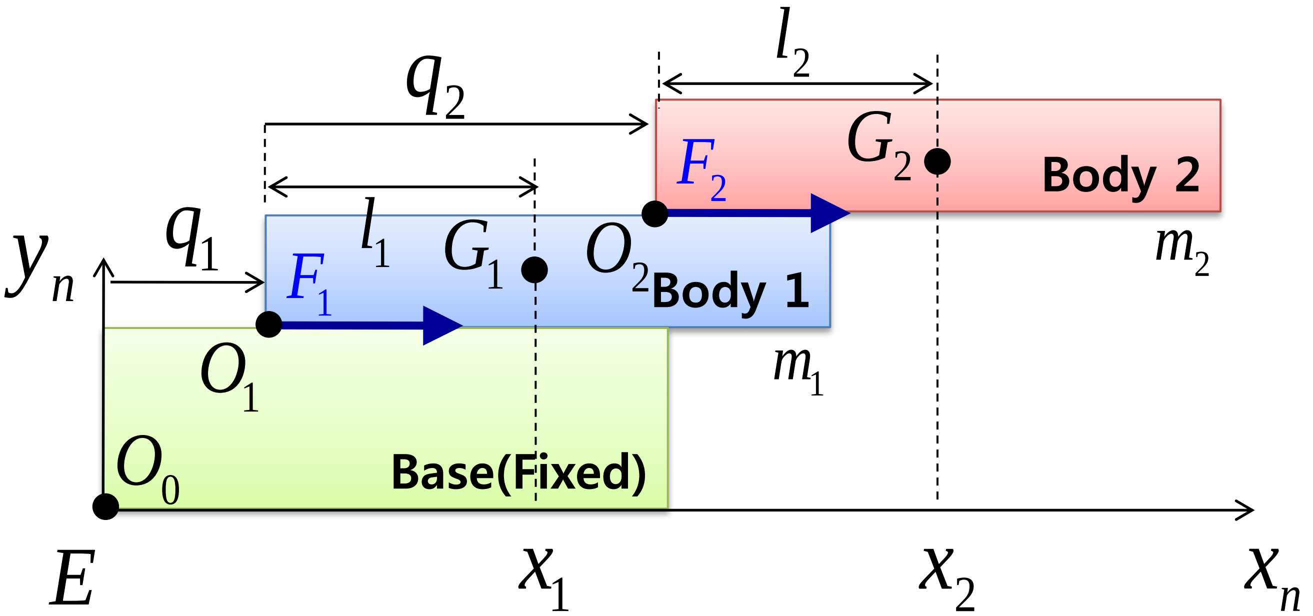
$$\begin{matrix} x_1 = q_1 + l_1 \\ x_2 = q_1 + q_2 + l_2 \end{matrix} \xrightarrow[\text{derivative}]{\text{Time}} \begin{matrix} \dot{x}_1 = \dot{q}_1 \\ \dot{x}_2 = \dot{q}_1 + \dot{q}_2 \end{matrix} \xrightarrow[\text{derivative}]{\text{Time}} \begin{matrix} \ddot{x}_1 = \ddot{q}_1 \\ \ddot{x}_2 = \ddot{q}_1 + \ddot{q}_2 \end{matrix}$$

Substituting x with q

⇨

$$F_1 - F_2 = m_1\ddot{q}_1$$
$$F_2 = m_2\ddot{q}_2 + m_2\ddot{q}_1$$

Equations of Motion for Multibody that has two Prismatic Joints - Derivation of the Equations of Motion



$$F_1 - F_2 = m_1\ddot{q}_1 \quad \text{-------} \; ①$$

$$F_2 = m_2\ddot{q}_2 + m_2\ddot{q}_1 \quad \text{---} \; ②$$

Eq. ②+① $F_1 = \left(m_1 + m_2\right)\ddot{q}_1 + m_2\ddot{q}_2$

Eq. ② $F_2 = m_2\ddot{q}_2 + m_2\ddot{q}_1$

"Inverse Dynamics"

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

Given: $\ddot{q}_1, \ddot{q}_2$ **Find:** F_1, F_2

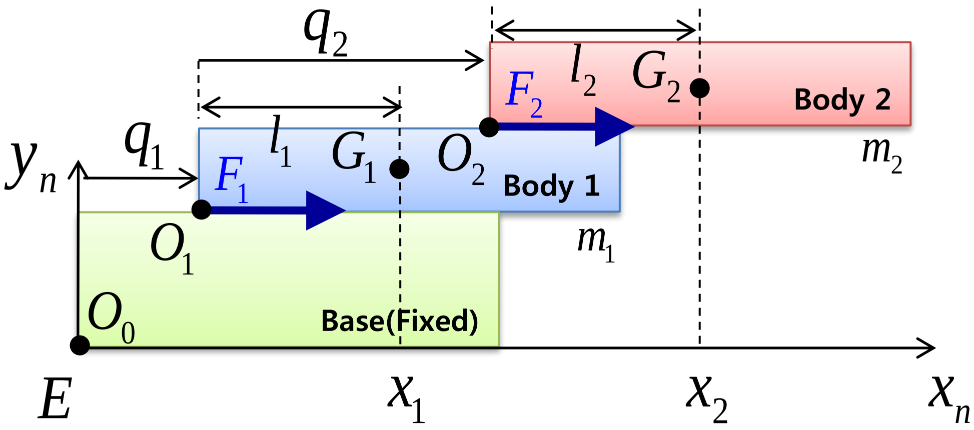
"Forward Dynamics"

$$\begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

Given: F_1, F_2 **Find:** $\ddot{q}_1, \ddot{q}_2$

5.2 Derivation of the Equations of Motion by using "Embedding Technique"

Equations of Motion for Multibody that has two Prismatic Joints
- Derivation of the Equations of Motion by using "Embedding Technique"



Equations of motion

$$\boxed{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}=\begin{bmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}}$$

Relative coordinate formulation (Embedding technique)

$$\tilde{\mathbf{M}}\ddot{\mathbf{q}}+\tilde{\mathbf{k}}=\tilde{\mathbf{F}}^e \quad ,where\ \tilde{\mathbf{M}}=\mathbf{J}^T\mathbf{M}\mathbf{J},\ \tilde{\mathbf{k}}=\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}},\ \tilde{\mathbf{F}}^e=\mathbf{J}^T\mathbf{F}^e$$

$\mathbf{J}$: velocity transformation matrix, $\dot{\mathbf{r}}=\mathbf{J}\dot{\mathbf{q}}$

$$\begin{aligned} x_1&=q_1+l_1 \\ x_2&=q_1+q_2+l_2 \end{aligned} \xrightarrow[\text{derivative}]{\text{Time}} \begin{aligned} \dot{x}_1&=\dot{q}_1 \\ \dot{x}_2&=\dot{q}_1+\dot{q}_2 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\mathbf{r}}}=\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\mathbf{J}}\underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{\mathbf{q}}}$$

From newton's 2nd law $\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\mathbf{M}}\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{\mathbf{r}}}=\underbrace{\begin{bmatrix} F_1-F_2 \\ F_2 \end{bmatrix}}_{\mathbf{F}^e}$

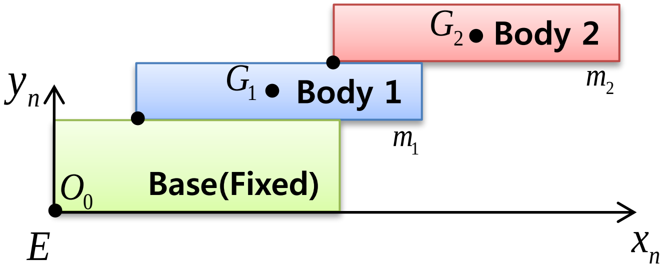
$$\tilde{\mathbf{M}}=\mathbf{J}^T\mathbf{M}\mathbf{J}=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} m_1 & m_2 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}$$

$$\tilde{\mathbf{k}}=\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}=0$$

$$\tilde{\mathbf{F}}^e=\mathbf{J}^T\mathbf{F}^e=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} F_1-F_2 \\ F_2 \end{bmatrix}=\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\tilde{\mathbf{M}}\ddot{\mathbf{q}}+\tilde{\mathbf{k}}=\tilde{\mathbf{F}}^e \Rightarrow \boxed{\begin{bmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}=\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}}$$

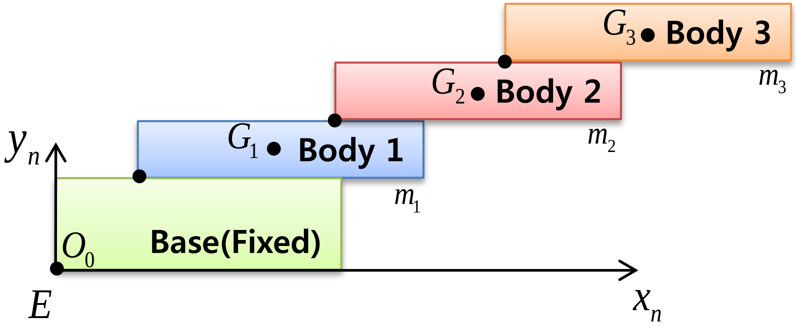
Equations of Motion for Multibody that has two Prismatic Joints
- Derivation of the Equations of Motion by using "Embedding Technique"



2 Prismatic joints

⇨
$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

2 X 2 Matrix



3 Prismatic joints

⇨
$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} m_1 + m_2 + m_3 & m_2 + m_3 & m_3 \\ m_2 + m_3 & m_2 + m_3 & m_3 \\ m_3 & m_3 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix}$$

3 X 3 Matrix

---

n Prismatic joints ⇨ **Inverse of n X n matrix should be calculated**

Computational time is proportional to n^3

Chapra, S. C., Canale R. P. Numerical Methods for Engineers, 5th edition, McGRAW-HILL, 2006

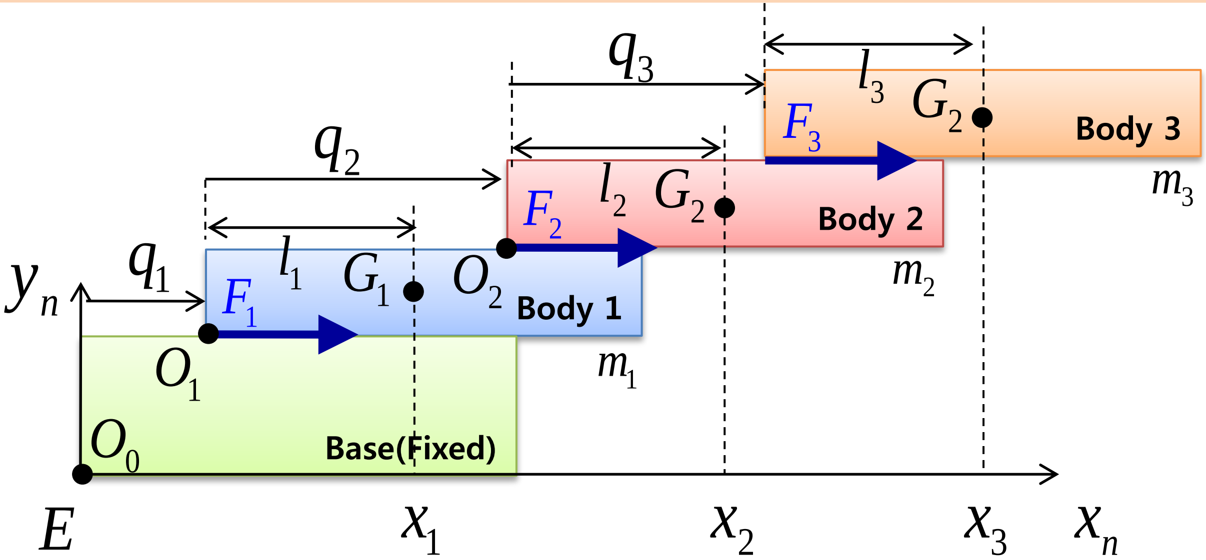
The computational time is by using Gauss elimination. (p.244)

The computational time is $\frac{4n^3}{3} - \frac{n}{3}$ by using LU decomposition. (p.275)

5.3 Solving Inverse Dynamics Problem by using “Recursive Newton-Euler Formulation”

Equations of Motion for Multibody that has two Prismatic Joints

- Recursive Newton-Euler Formulation – Inverse Dynamics



Forward Dynamics

Given: F_1, F_2, F_3

Find: $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3$

Inverse Dynamics

Given: $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3$

Find: F_1, F_2, F_3

① **Since** $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3$ **are known,** $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3$ **can be computed.**

$$\ddot{x}_1 = \ddot{q}_1$$
$$\ddot{x}_2 = \ddot{x}_1 + \ddot{q}_2$$
$$\ddot{x}_3 = \ddot{x}_2 + \ddot{q}_3$$

$$x_1 = q_1 + l_1 \qquad \dot{x}_1 = \dot{q}_1 \qquad \boxed{\ddot{x}_1} = \ddot{q}_1 \quad \cdots\cdot \text{(1)}$$

$$x_2 = q_1 + q_2 + l_2 \xrightarrow[\text{derivative}]{\text{Time}} \dot{x}_2 = \dot{q}_1 + \dot{q}_2 \xrightarrow[\text{derivative}]{\text{Time}} \boxed{\ddot{x}_2} = \boxed{\ddot{q}_1} + \ddot{q}_2 \quad \cdots\cdot \text{(2)}$$

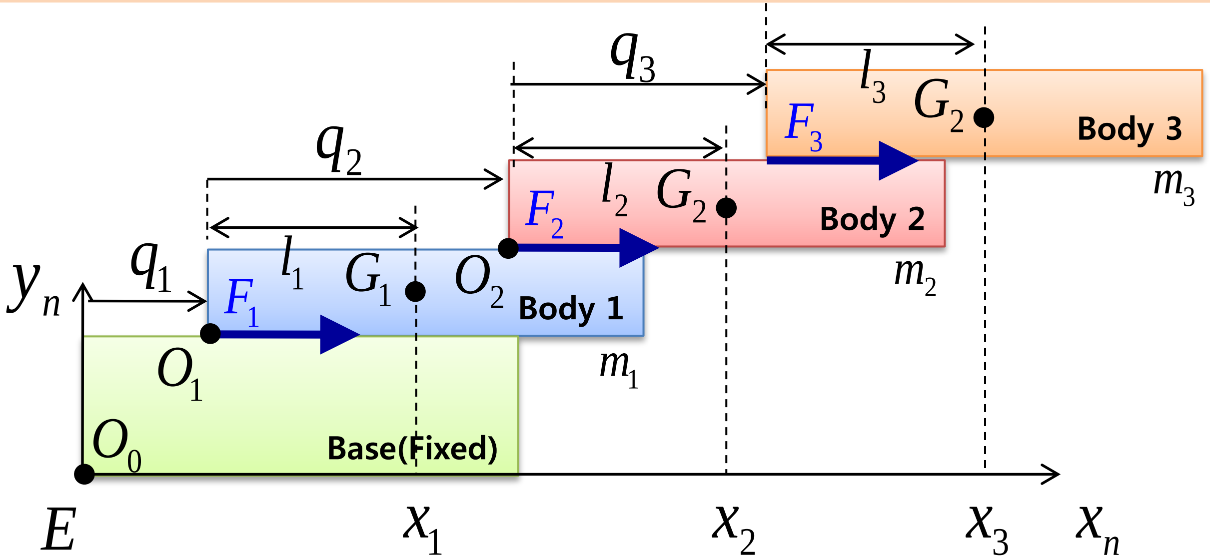
$$x_3 = q_1 + q_2 + q_3 + l_3 \qquad \dot{x}_3 = \dot{q}_1 + \dot{q}_2 + \dot{q}_3 \qquad \ddot{x}_3 = \boxed{\ddot{q}_1 + \ddot{q}_2} + \ddot{q}_3 \cdot\cdot \text{(3)}$$

In eq.(3), $\ddot{q}_1 + \ddot{q}_2$ **should be calculated.**
However, since $\ddot{q}_1 + \ddot{q}_2$ **is done in eq.(2),** $\ddot{q}_1 + \ddot{q}_2$ **can be substituted by** $\ddot{x}_2$ **in eq.(3).**
→ Recursive formulation

➡ $$\ddot{x}_n = \ddot{x}_{n-1} + \ddot{q}_n \left(n : 1 \sim 3, \ddot{x}_0 = 0 \right)$$

Equations of Motion for Multibody that has two Prismatic Joints

- Recursive Newton-Euler Formulation – Inverse Dynamics



Forward Dynamics

Given: F_1, F_2, F_3

Find: $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3$

Inverse Dynamics

Given: $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3$

Find: F_1, F_2, F_3

① **Since** $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3$ **are known,** $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3$ **can be computed.**

$$\ddot{x}_1 = \ddot{q}_1$$
$$\ddot{x}_2 = \ddot{x}_1 + \ddot{q}_2$$
$$\ddot{x}_3 = \ddot{x}_2 + \ddot{q}_3$$

$$\ddot{x}_n = \ddot{x}_{n-1} + \ddot{q}_n \left(n: 1 \sim 3, \ddot{x}_0 = 0 \right)$$

② **Since** $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3$ **are known, the resultant forces exerted on the each bodies can be computed.**

$$\boxed{F_3} = m_3 \ddot{x}_3$$
$$\boxed{F_2} - \boxed{F_3} = m_2 \ddot{x}_2$$
$$F_1 - \boxed{F_2} = m_1 \ddot{x}_1$$

→ Recursive formulation

$$F_n - F_{n+1} = m_n \ddot{x}_n \left(n: 1 \sim 3, F_4 = 0 \right)$$

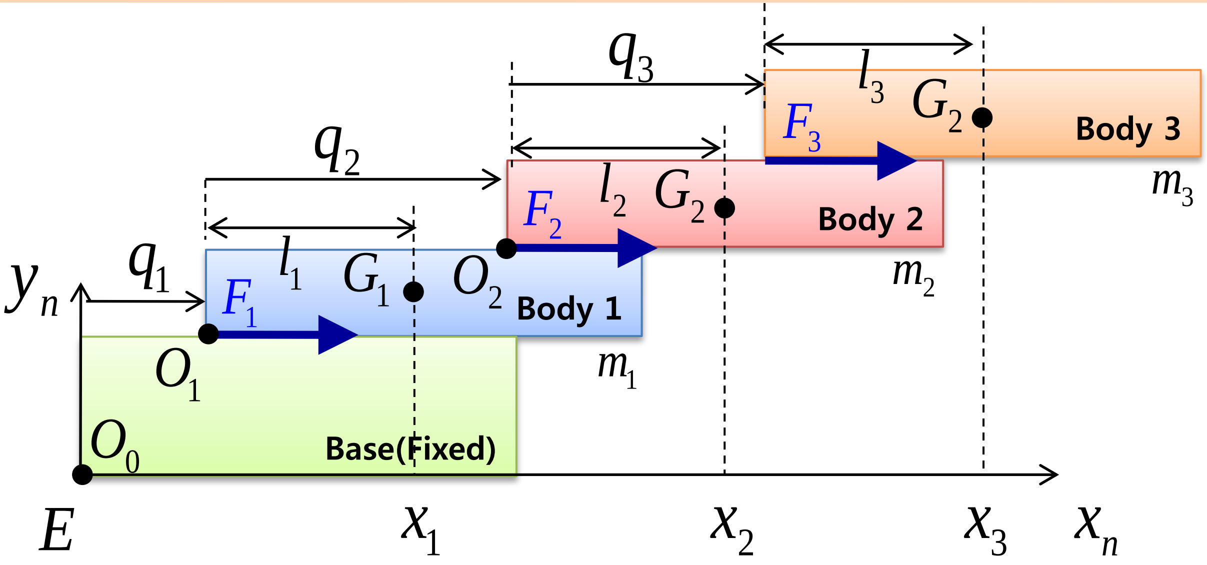
Computational time is proportional to n

Featherstone, R., Rigid Body Dynamics Algorithm, Springer, 2008, p.89
"Recursive Newton-Euler algorithm has a computational complexity of O(n)"

5.4 Solving Forward Dynamics Problem by using "Recursive Newton-Euler Formulation"

Equations of Motion for Multibody that has two Prismatic Joints

- Recursive Newton-Euler Formulation – Forward Dynamics



Forward Dynamics

Given: F_1, F_2, F_3

Find: $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3$

Inverse Dynamics

Given: $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3$

Find: F_1, F_2, F_3

① **Since** F_1, F_2, F_3 **are known,** $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3$ **can be computed.**

$$F_3 = m_3\ddot{x}_3 \qquad \ddot{x}_3 = F_3 / m_3$$
$$F_2 - F_3 = m_2\ddot{x}_2 \quad \Rightarrow \quad \ddot{x}_2 = \left(F_2 - F_3\right) / m_2$$
$$F_1 - F_2 = m_1\ddot{x}_1 \qquad \ddot{x}_1 = \left(F_1 - F_2\right) / m_1$$

$$\Rightarrow \ddot{x}_n = \left(F_n - F_{n+1}\right) / m_n ,\left(n:1 \sim 3, F_4 = 0\right)$$

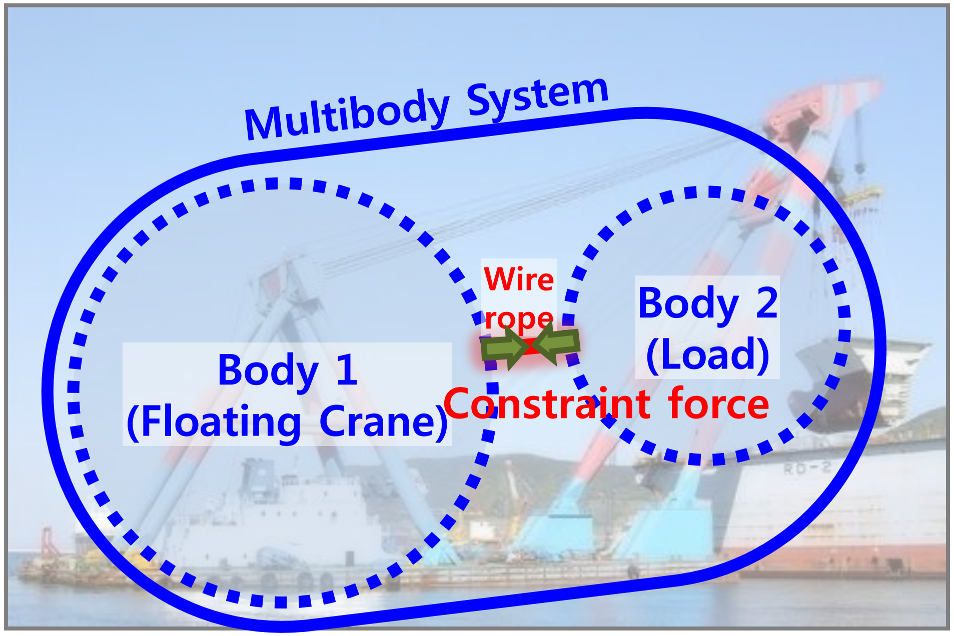
② **Since** $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3$ **are known,** $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3$ **can be computed**

$$\ddot{x}_1 = \ddot{q}_1 \qquad \ddot{q}_1 = \ddot{x}_1$$
$$\ddot{x}_2 = \ddot{x}_1 + \ddot{q}_2 \quad \Rightarrow \quad \ddot{q}_2 = \ddot{x}_2 - \ddot{x}_1$$
$$\ddot{x}_3 = \ddot{x}_2 + \ddot{q}_3 \qquad \ddot{q}_3 = \ddot{x}_3 - \ddot{x}_2$$

$$\Rightarrow \ddot{q}_n = \ddot{x}_n - \ddot{x}_{n-1},\left(n:1 \sim 3, x_0 = 0\right)$$

Computational time is proportional to n

Derivation of Equations of motion
- Multibody Dynamics



Forward Dynamics

Given: External Force
Find: Motion of the Bodies
(Example: Simulation)

Inverse Dynamics

Given: Motion of the Bodies
Find: Required External Force
(Example: Robotics, Control)

Differential-Algebraic Equation(DAE)

[Embedding technique]

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

Computational time is proportional to n³

Recursive Newton-Euler Formulation

Inverse Dynamics

$$\ddot{x}_n = \ddot{x}_{n-1} + \ddot{q}_n \left(n:1\sim 3, \ddot{x}_0 = 0\right)$$

$$F_n - F_{n+1} = m_n \ddot{x}_n \left(n:1\sim 3, F_4 = 0\right)$$

Forward Dynamics

$$\ddot{x}_n = \left(F_n - F_{n+1}\right)/m_n, \left(n:1\sim 3, F_4 = 0\right)$$

$$\ddot{q}_n = \ddot{x}_n - \ddot{x}_{n-1}, \left(n:1\sim 3, x_0 = 0\right)$$

Computational time is proportional to n

5.5 Application of Recursive Newton-Euler Equation
- 2 link robot arm

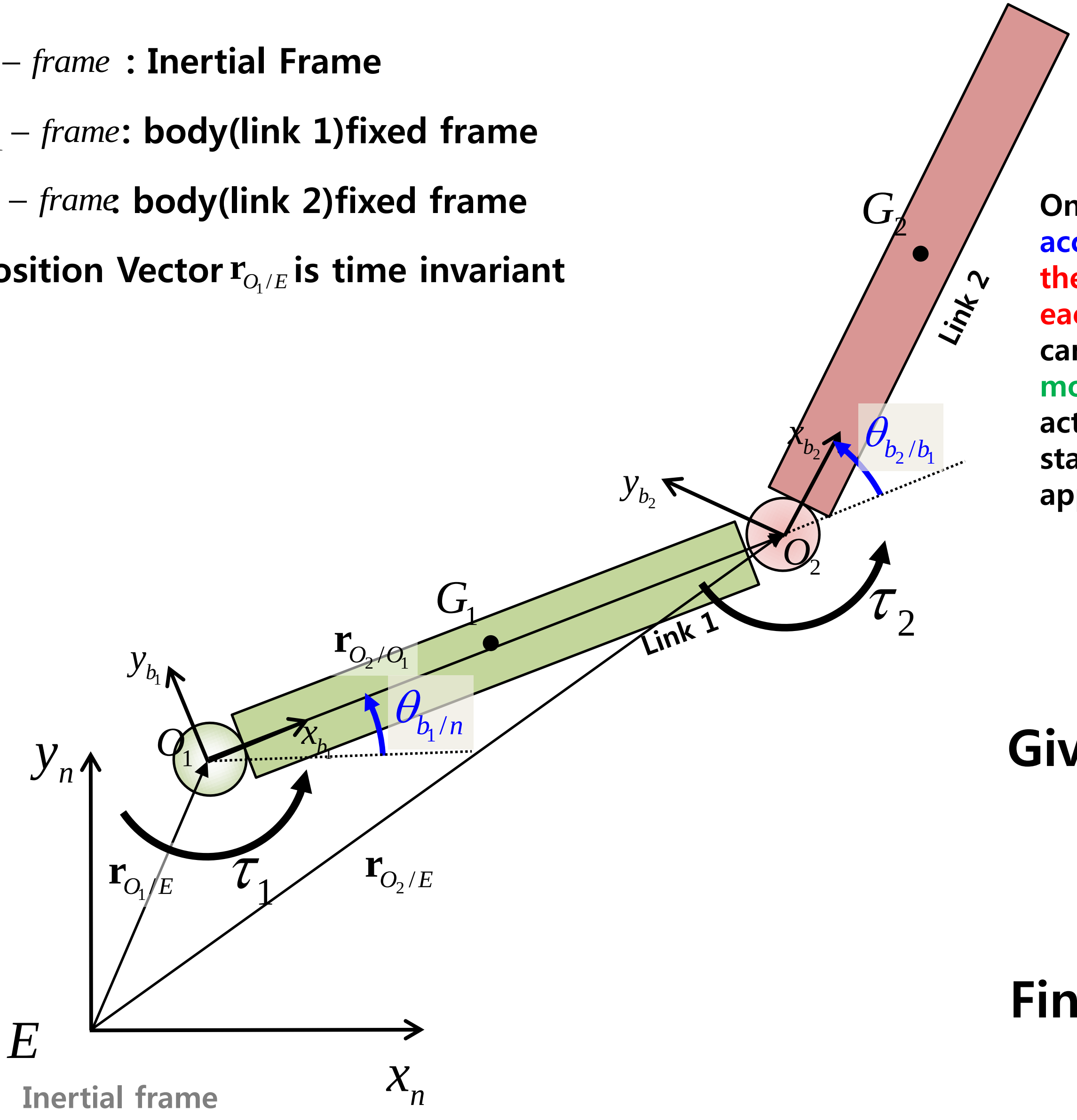
Application of Recursive Newton-Euler Equation - 2 link robot arm - Problem definition

$n-frame$ **: Inertial Frame**

$b_1-frame$**: body(link 1)fixed frame**

$b_2-frame$**: body(link 2)fixed frame**

Position Vector $\mathbf{r}_{O_1/E}$ **is time invariant**



Once the Joint positions(θ), velocities($\dot{\theta}$) and acceleration($\ddot{\theta}$) are known, one can compute the accelerations ($\ddot{\mathbf{r}}$) of the center of mass for each link, and the Newton-Euler Formulation can be utilized to find the forces and moments (τ_1, τ_2) about each axis of the joint acting on each link in a recursive fashion, starting from the force and moment applied to the end effector

"L. Sciavicco and B. Siciliano, Modelling and control of robot manipulators, 2nd edition, Springer, 2000, p. 170"

Inverse Dynamics

Given: **Kinematic Model**

$$\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \ddot{\theta}_{b_1/n}$$

$$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \ddot{\theta}_{b_2/b_1}$$

Generalized Coordinates

Generalized Force

Find: **input torque for link 1 -** τ_1

input torque for link 2 - τ_2

Application of Recursive Newton-Euler Equation

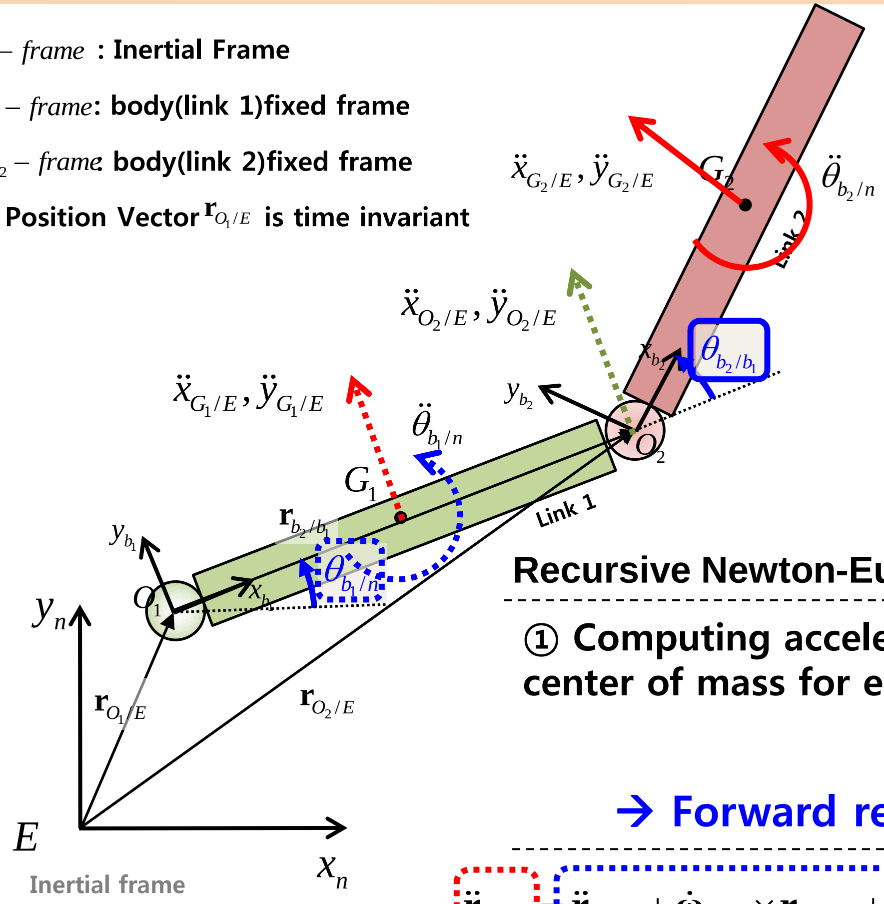
- Inverse Dynamics - Computing acceleration of center of mass for each link

$n-frame$: Inertial Frame

$b_1-frame$: body(link 1)fixed frame

$b_2-frame$: body(link 2)fixed frame

Position Vector $\mathbf{r}_{O_1/E}$ is time invariant



Inverse Dynamics

Given: Kinematic Model

$$\boxed{\begin{matrix}\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \ddot{\theta}_{b_1/n} \\ \theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \ddot{\theta}_{b_2/b_1}\end{matrix}}$$

Find: input torque for link 1 - τ_1

input torque for link 2 - τ_2

Recursive Newton-Euler Formulation	Given	Find	
① Computing acceleration of center of mass for each link	$\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \ddot{\theta}_{b_1/n}$	$\ddot{x}_{G_1/E}, \ddot{y}_{G_1/E}$	From Eq. (1-1)
		$\ddot{x}_{O_2/E}, \ddot{y}_{O_2/E}$	From Eq. (1-2)
→ Forward recursive	$\theta_{b_1/b_2}, \dot{\theta}_{b_1/b_2}, \ddot{\theta}_{b_1/b_2}$	$\ddot{x}_{G_2/E}, \ddot{y}_{G_2/E}, \ddot{\theta}_{b_2/n}$	From Eq. (2)

$$\ddot{\mathbf{r}}_{G_1/E} = \ddot{\mathbf{r}}_{O_1/E} + \dot{\boldsymbol{\omega}}_{b_1/n} \times \mathbf{r}_{G_1/O_1} + \boldsymbol{\omega}_{b_1/n} \times \left(\boldsymbol{\omega}_{b_1/n} \times \mathbf{r}_{G_1/O_1}\right) \quad \text{where} \quad \mathbf{r}_{G_1/E} = \begin{bmatrix} x_{G_1/E} & y_{G_1/E} & 0 \end{bmatrix}^T,\ \boldsymbol{\omega}_{b_1/n} = \begin{bmatrix} 0 & 0 & \theta_{b_1/n} \end{bmatrix}^T \quad \cdots (1\text{-}1)$$

$$\ddot{\mathbf{r}}_{O_2/E} = \ddot{\mathbf{r}}_{O_1/E} + \dot{\boldsymbol{\omega}}_{b_1/n} \times \mathbf{r}_{O_2/O_1} + \boldsymbol{\omega}_{b_1/n} \times \left(\boldsymbol{\omega}_{b_1/n} \times \mathbf{r}_{O_2/O_1}\right) \quad \text{where} \quad \mathbf{r}_{G_1/E} = \begin{bmatrix} x_{G_1/E} & y_{G_1/E} & 0 \end{bmatrix}^T,\ \boldsymbol{\omega}_{b_1/n} = \begin{bmatrix} 0 & 0 & \theta_{b_1/n} \end{bmatrix}^T \quad \cdots (1\text{-}2)$$

$$\ddot{\mathbf{r}}_{G_2/E} = \ddot{\mathbf{r}}_{O_2/E} + \dot{\boldsymbol{\omega}}_{b_2/b_1} \times \mathbf{r}_{G_2/O_2} + \boldsymbol{\omega}_{b_2/b_1} \times \left(\boldsymbol{\omega}_{b_2/b_1} \times \mathbf{r}_{G_2/O_2}\right) \quad \text{where} \quad \mathbf{r}_{G_2/E} = \begin{bmatrix} x_{G_2/E} & y_{G_2/E} & 0 \end{bmatrix}^T,\ \boldsymbol{\omega}_{b_2/b_1} = \begin{bmatrix} 0 & 0 & \theta_{b_2/b_1} \end{bmatrix}^T \quad \cdots (2)$$

$$\ddot{\boldsymbol{\omega}}_{b_2/n} = \ddot{\boldsymbol{\omega}}_{b_1/n} + \ddot{\boldsymbol{\omega}}_{b_2/b_1}$$

Application of Recursive Newton-Euler Equation

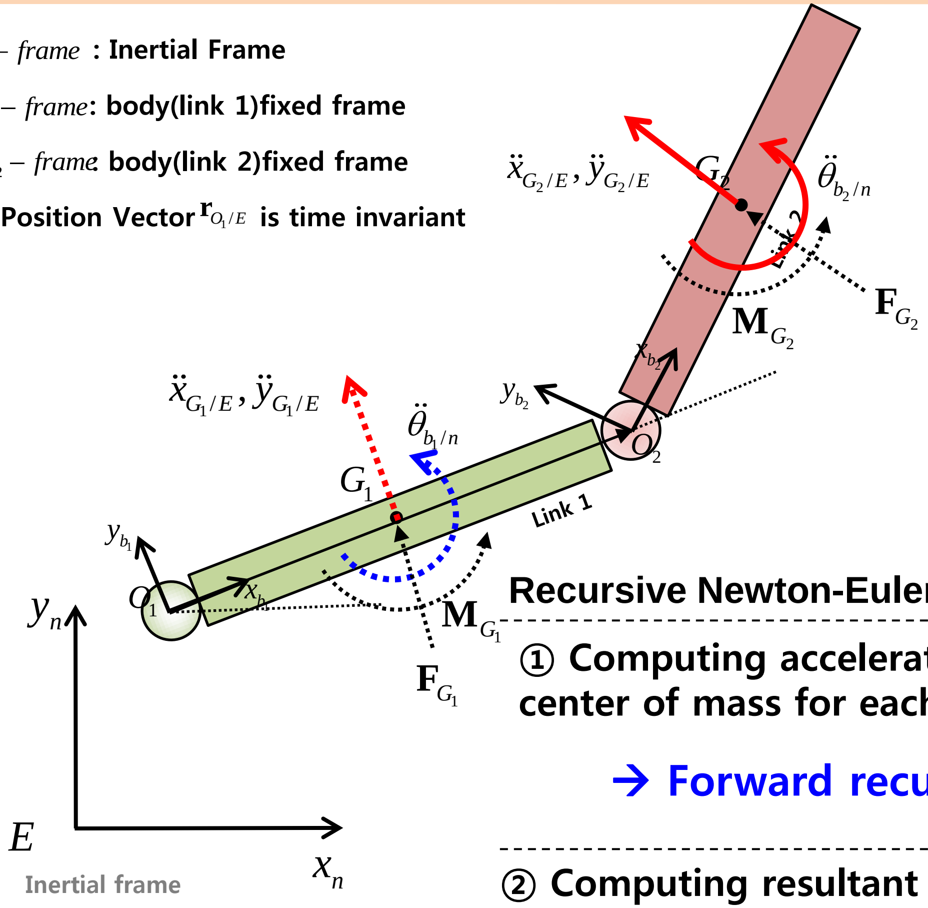
- Inverse Dynamics - Computing resultant force exerted on center of mass

$n-frame$: **Inertial Frame**

$b_1-frame$: **body(link 1)fixed frame**

$b_2-frame$: **body(link 2)fixed frame**

Position Vector $\mathbf{r}_{O_1/E}$ **is time invariant**



Inverse Dynamics

Given: **Kinematic Model**

$$\boxed{\begin{matrix}\theta_{b_1/n},\dot{\theta}_{b_1/n},\ddot{\theta}_{b_1/n}\\ \theta_{b_2/b_1},\dot{\theta}_{b_2/b_1},\ddot{\theta}_{b_2/b_1}\end{matrix}}$$

Find: **input torque for link 1** - τ_1

input torque for link 2 - τ_2

Recursive Newton-Euler Formulation	Given	Find
① Computing acceleration of center of mass for each link → **Forward recursive**	$\theta_{b_1/n},\dot{\theta}_{b_1/n},\ddot{\theta}_{b_1/n}$ $\theta_{b_1/b_2},\dot{\theta}_{b_1/b_2},\ddot{\theta}_{b_1/b_2}$	$\ddot{x}_{G_1/E},\ddot{y}_{G_1/E}$ $\ddot{x}_{G_2/E},\ddot{y}_{G_2/E},\ddot{\theta}_{b_2/n}$
② Computing resultant force exerted on center of mass by using **Newton-Euler equation**.	$\boxed{\ddot{x}_{G_1/E},\ddot{y}_{G_1/E}}\boxed{\ddot{\theta}_{b_1/n}}$	$\boxed{\mathbf{F}_{G_1},\mathbf{M}_{G_1}}$ From Eq. (1)
	$\boxed{\ddot{x}_{G_2/E},\ddot{y}_{G_2/E},\ddot{\theta}_{b_2/n}}$	$\boxed{\mathbf{F}_{G_2},\mathbf{M}_{G_2}}$ From Eq. (2)

$$\boxed{\mathbf{F}_{G_1}}=m_1\boxed{\ddot{\mathbf{r}}_{G_1/E}},\boxed{\mathbf{M}_{G_1}}=\mathbf{I}_{G_1}\boxed{\dot{\boldsymbol{\omega}}_{b_1/n}}\text{ where }\mathbf{r}_{G_1/E}=\begin{bmatrix}x_{G_1/E} & y_{G_1/E} & 0\end{bmatrix}^T,\boldsymbol{\omega}_{b_1/n}=\begin{bmatrix}0 & 0 & \theta_{b_1/n}\end{bmatrix}^T\cdots\quad(1)$$

$$\boxed{\mathbf{F}_{G_2}}=m_2\boxed{\ddot{\mathbf{r}}_{G_2/E}},\boxed{\mathbf{M}_{G_2}}=\mathbf{I}_{G_2}\boxed{\dot{\boldsymbol{\omega}}_{b_2/n}}\text{ where }\mathbf{r}_{G_2/E}=\begin{bmatrix}x_{G_2/E} & y_{G_2/E} & 0\end{bmatrix}^T,\boldsymbol{\omega}_{b_2/n}=\begin{bmatrix}0 & 0 & \theta_{b_2/n}\end{bmatrix}^T\cdots\quad(2)$$

cf) Moment Equation in 3D motion

$$\mathbf{M}_{G_1}=\mathbf{I}_{G_1}\dot{\boldsymbol{\omega}}_{b_1/n}+\boldsymbol{\omega}_{b_1/n}\times\mathbf{I}_{G_1}\boldsymbol{\omega}_{b_1/n}$$

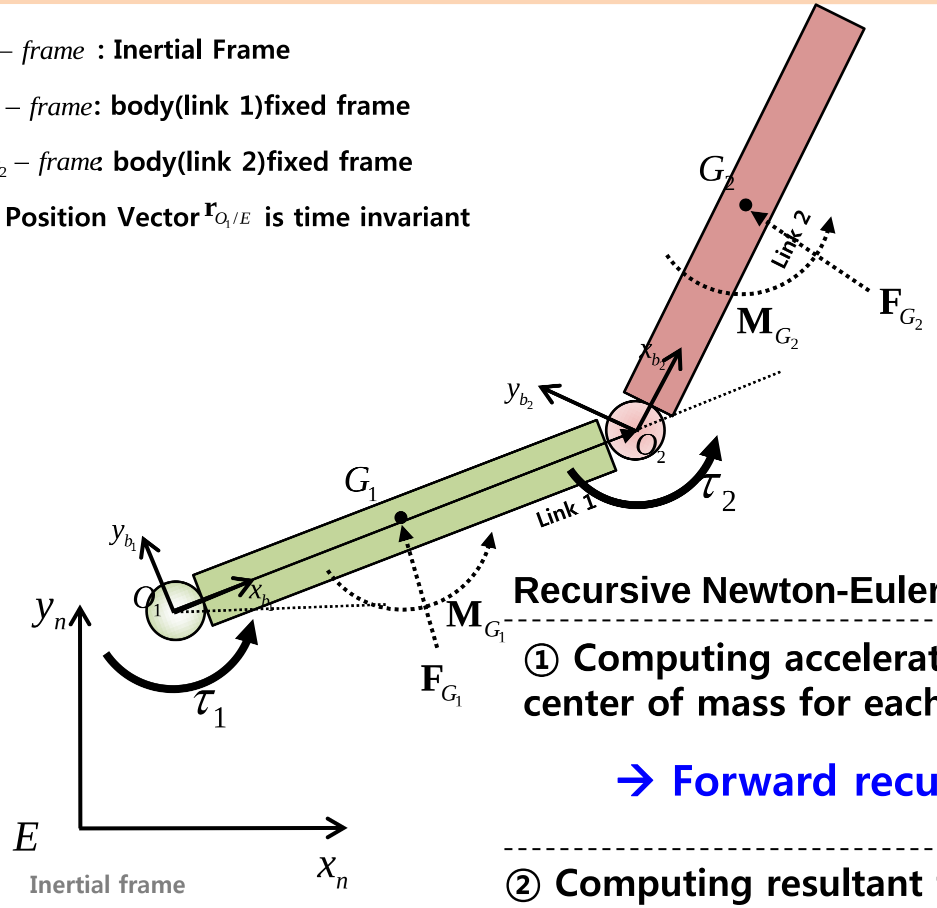
Application of Recursive Newton-Euler Equation
- Inverse Dynamics - Computing input torque for each link

$n-frame$: **Inertial Frame**

$b_1-frame$: **body(link 1)fixed frame**

$b_2-frame$: **body(link 2)fixed frame**

Position Vector $\mathbf{r}_{O_1/E}$ **is time invariant**



Inverse Dynamics

Given: **Kinematic Model**

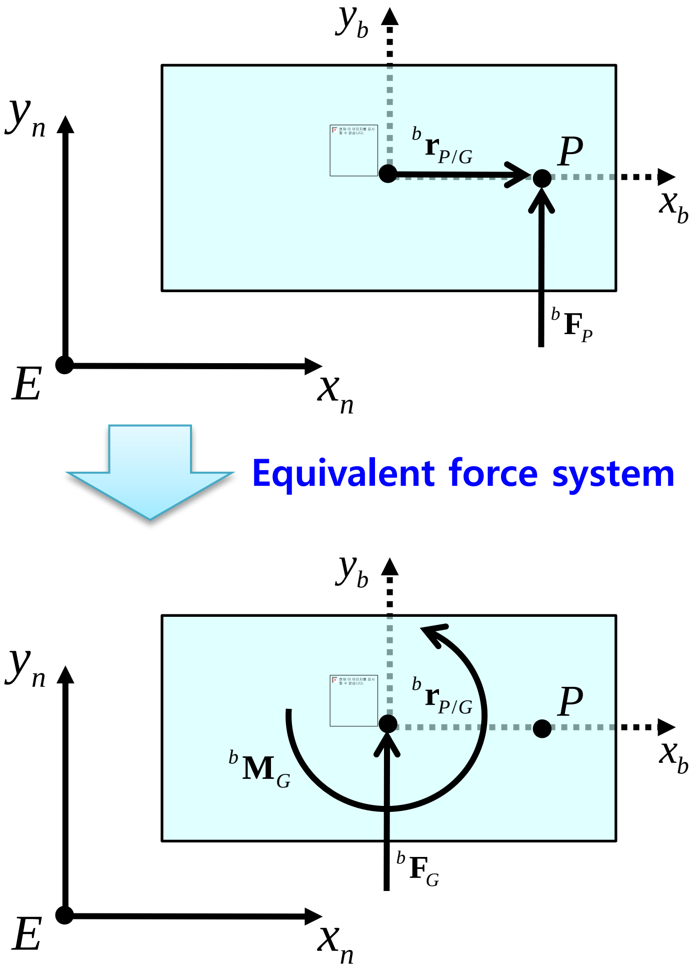
$$\boxed{\begin{matrix}\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \ddot{\theta}_{b_1/n} \\ \theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \ddot{\theta}_{b_2/b_1}\end{matrix}}$$

Find: **input torque for link 1 -** $\boxed{\tau_1}$

input torque for link 2 - $\boxed{\tau_2}$

Recursive Newton-Euler Formulation	Given	Find
① Computing acceleration of center of mass for each link → Forward recursive	$\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \ddot{\theta}_{b_1/n}$ $\theta_{b_1/b_2}, \dot{\theta}_{b_1/b_2}, \ddot{\theta}_{b_1/b_2}$	$\ddot{x}_{G_1/E}, \ddot{y}_{G_1/E}$ $\ddot{x}_{G_2/E}, \ddot{y}_{G_2/E}, \ddot{\theta}_{b_2/n}$
② Computing resultant force exerted on center of mass by using Newton-Euler equation	$\ddot{x}_{G_1/E}, \ddot{y}_{G_1/E}, \ddot{\theta}_{b_1/n}$ $\ddot{x}_{G_2/E}, \ddot{y}_{G_2/E}, \ddot{\theta}_{b_2/n}$	$\mathbf{F}_{G_1}, \mathbf{M}_{G_1}$ $\mathbf{F}_{G_2}, \mathbf{M}_{G_2}$
③ Computing input torque for each link	$\mathbf{F}_{G_2}, \mathbf{M}_{G_2}$ $\mathbf{F}_{G_1}, \mathbf{M}_{G_1}$	$\boxed{\tau_2}$ $\boxed{\tau_1}$

Force and moment exerted on a rigid body



$${}^b\mathbf{I}_G\,{}^b\dot{\boldsymbol{\omega}}_{b/n} = \boxed{{}^b\mathbf{M}_G}$$

${}^b\mathbf{F}_P$: Force acting on the point P decomposed in the b-frame

---

${}^b\mathbf{F}_G$: Force acting on the point G decomposed in the b-frame

$${}^b\mathbf{F}_G = {}^b\mathbf{F}_P$$

- The translational motion is independent of the point where the external force is exerted. (Fossen, 2002, pp. 54)

${}^b\mathbf{M}_G$: Moment about z_b-axis decomposed in the b-frame

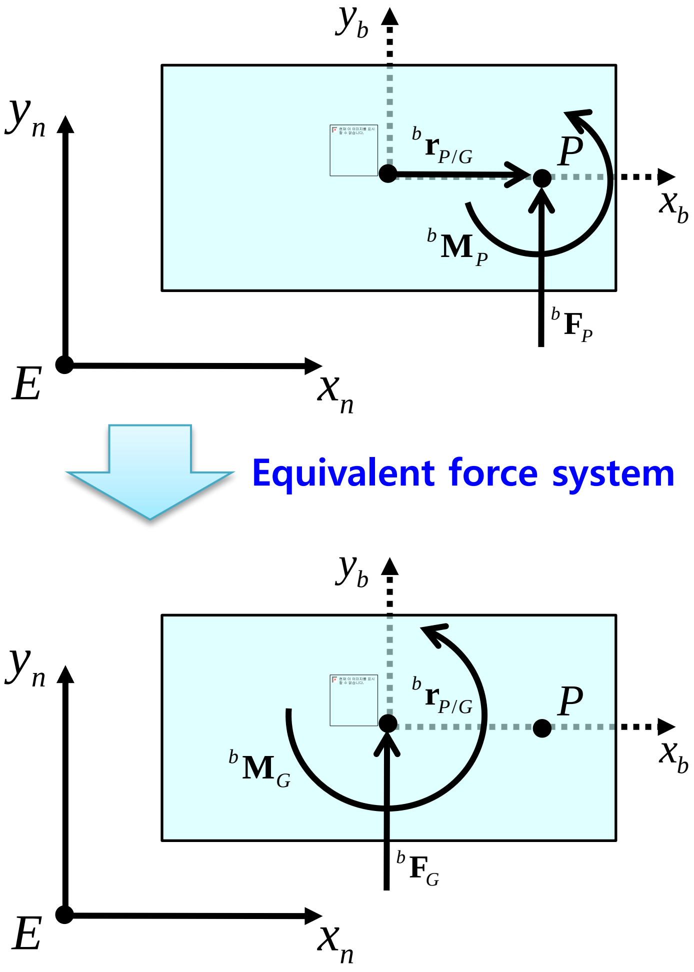
$${}^b\mathbf{M}_G = {}^b\mathbf{r}_{P/G} \times {}^b\mathbf{F}_P$$

The moment is generated by the force exerted on the point P

n-frame: Inertial frame
Point E: Origin of the inertial frame(n-frame)
b-frame: Body fixed frame
Point G: Center of mass,
Origin of the body-fixed frame(b-frame)

- we consider the moment exerted by each interaction force.
- it is reasonable to expect that the resultant moment of a set of forces represents the rotational influence

Force and moment exerted on a rigid body



n-frame: Inertial frame
Point E: Origin of the inertial frame(n-frame)
b-frame: Body fixed frame
Point G: Center of mass,
Origin of the body-fixed frame(b-frame)

$^b\mathbf{F}_P$: Force acting on the point P decomposed in the b-frame

$^b\mathbf{M}_P$: Moment about z-axis through the point P decomposed in the b-frame

---

$^b\mathbf{F}_G$: Force acting on the point G decomposed in the b-frame

$$^b\mathbf{F}_G = {}^b\mathbf{F}_P$$

- The translational motion is independent of the point where the external force is exerted. (Fossen, 2002, pp. 54)

$^b\mathbf{M}_G$: Moment about z_b-axis decomposed in the b-frame

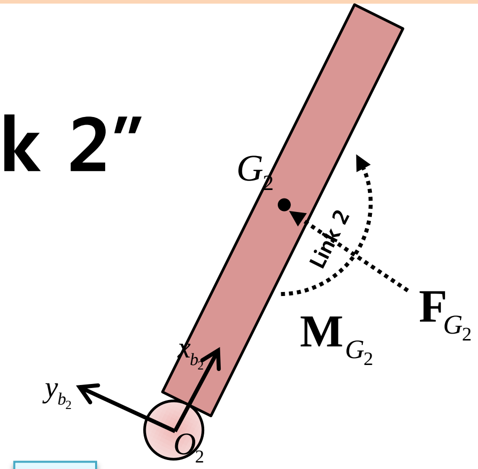
$$^b\mathbf{M}_G = {}^b\mathbf{M}_O + {}^b\mathbf{r}_{P/G} \times {}^b\mathbf{F}_P$$

The moment $^b\mathbf{M}_G$ is the sum of $^b\mathbf{M}_P$ and $^b\mathbf{r}_{P/G} \times {}^b\mathbf{F}_P$

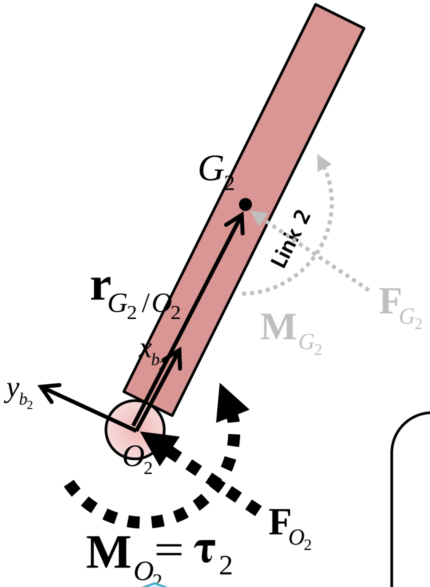
- we consider the moment exerted by each interaction force.
- it is reasonable to expect that the resultant moment of a set of forces represents the rotational influence

Force and moment exerted on a rigid body

"Link 2"



Equivalent force system



input torque for link 2 - τ_2

$\mathbf{F}_{G_2}$: Force acting on the point G_2

$\mathbf{M}_{G_2}$: Moment about z-axis through the point G_2

$\mathbf{F}_{O_2}$: Force acting on the point O_2

$$\mathbf{F}_{O_2} = \mathbf{F}_{G_2}$$

- The translational motion is independent of the point where the external force is exerted. (Fossen, 2002, pp. 54)

$\mathbf{M}_{O_2}$: Moment about z_b-axis

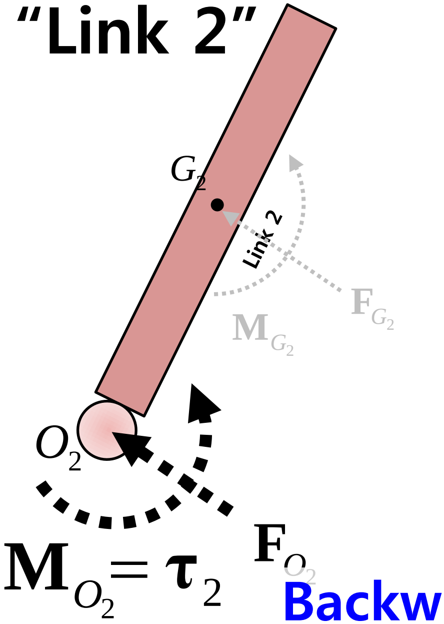
$$\mathbf{M}_{O_2} = \mathbf{M}_{G_2} + \mathbf{r}_{G_2/O_2} \times \mathbf{F}_{G_2}$$

The moment $\mathbf{M}_{O_2}$ is the sum of $\mathbf{M}_{G_2}$ and $\mathbf{r}_{G_2/O_2} \times \mathbf{F}_{G_2}$

Since the joint is revolute type,
the actuator can generate only moment $\mathbf{M}_{O_2}$.
In this case, $\mathbf{F}_{O_2}$ is the constraint force.

Force and moment exerted on a rigid body

Force and Moment exerted on the "Link 2"

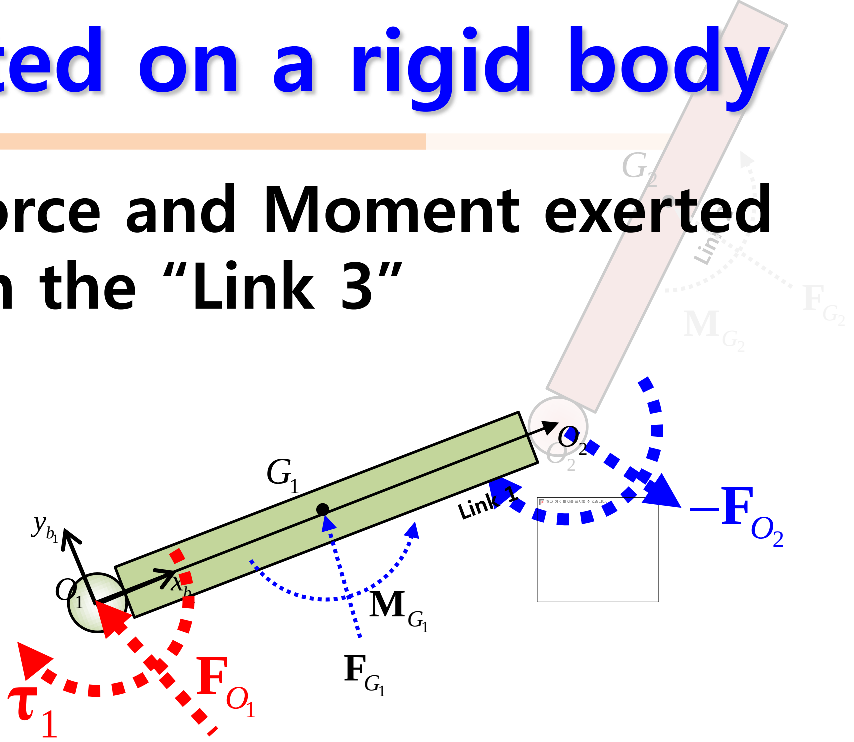


Backward recursive

Torque $\boldsymbol{\tau}_2$ and Force $\mathbf{F}_{O_2}$ should be exerted on the point O_2 of the link 2.

($\boldsymbol{\tau}_2$ and $\mathbf{F}_{O_2}$ are calculated in previous page)

Force and Moment exerted on the "Link 3"



(1) According to Newton's 3rd law, Torque $-\boldsymbol{\tau}_2$ and Force $-\mathbf{F}_{O_2}$ should be exerted on the point O_2 of the link 1.(Known)

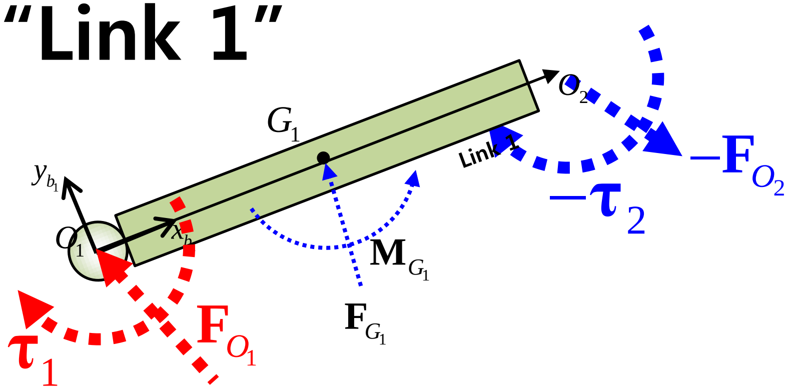
(2) $\boldsymbol{\tau}_1$ and $\mathbf{F}_{O_1}$ will be exerted on the point O_1 .(Unknown)

(3) Resultant force exerted on G_1 should be $\mathbf{M}_{G_1}$ and $\mathbf{F}_{G_1}$.(Known)

→ (1) + (2) should be equal to (3)

Force and moment exerted on a rigid body

"Link 1"



(1) **According to Newton's 3rd law, Torque $-\boldsymbol{\tau}_2$ and Force $-\mathbf{F}_{O_2}$ should be exerted on the point O_2 of the link 1.(Known)**

(2) $\boldsymbol{\tau}_1$ **and** $\mathbf{F}_{O_1}$ **will be exerted on the point** O_1 **.(Unknown)**

(3) **Resultant force exerted on** G_1 **should be** $\mathbf{M}_{G_1}$ **and** $\mathbf{F}_{G_1}$ **.(Known)**

→ **(1) + (2) should be equal to (3)**

Force exerted on O_1

(1) + (2) = (3)

$$(-\mathbf{F}_{O_2}) + \mathbf{F}_{O_1} = \mathbf{F}_{G_1}$$

$$\mathbf{F}_{O_1} = \mathbf{F}_{G_1} + \mathbf{F}_{O_2}$$

Moment(Torque)

about z-axis through point O_1

(1) + (2) = (3)

$$-\boldsymbol{\tau}_2 + \mathbf{r}_{O_2/O_1} \times \left(-\mathbf{F}_{O_2}\right) + \boldsymbol{\tau}_1 = \mathbf{M}_{G_1} + \mathbf{r}_{G_1/O_1} \times \mathbf{F}_{G_1}$$

$$\boldsymbol{\tau}_1 = \mathbf{M}_{G_1} + \mathbf{r}_{G_1/O_1} \times \mathbf{F}_{G_1} + \boldsymbol{\tau}_2 + \mathbf{r}_{O_2/O_1} \times \mathbf{F}_{O_2}$$

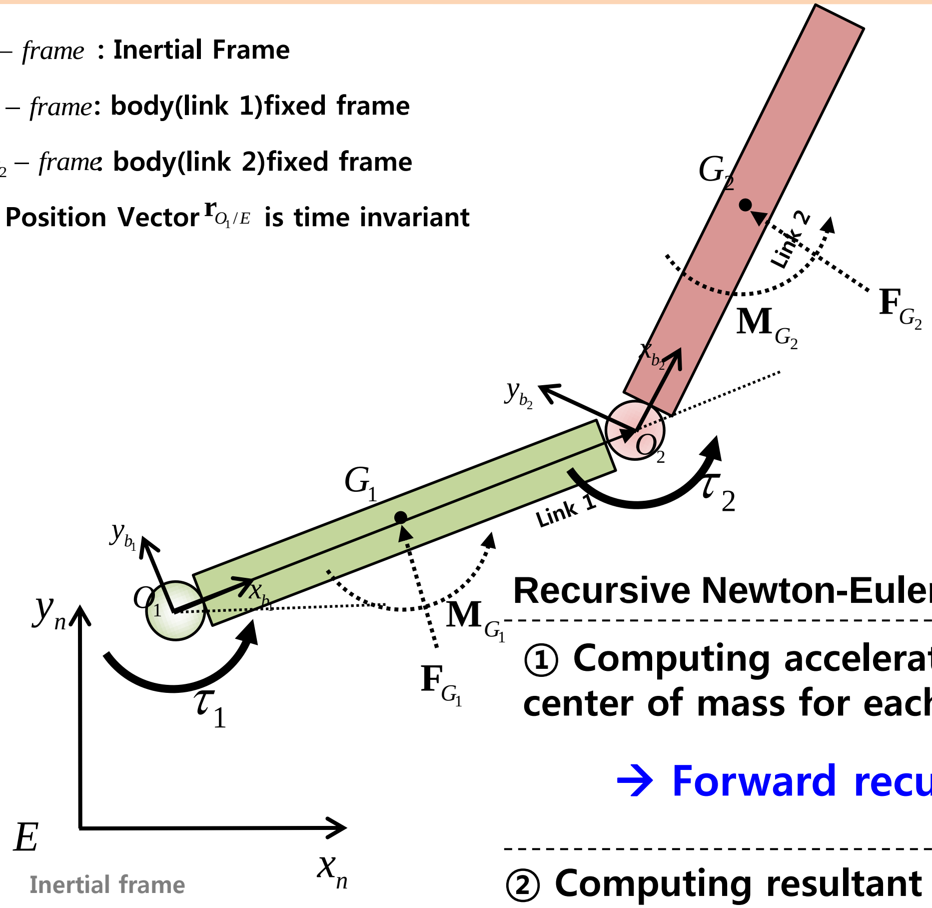
Application of Recursive Newton-Euler Equation
- Inverse Dynamics - Computing input torque for each link

$n-frame$: Inertial Frame

$b_1-frame$: body(link 1)fixed frame

$b_2-frame$: body(link 2)fixed frame

Position Vector $\mathbf{r}_{O_1/E}$ is time invariant



Inverse Dynamics

Given: Kinematic Model

$$\boxed{\begin{matrix}\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \ddot{\theta}_{b_1/n} \\ \theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \ddot{\theta}_{b_2/b_1}\end{matrix}}$$

Find: input torque for link 1 - $\boxed{\tau_1}$

input torque for link 2 - $\boxed{\tau_2}$

Recursive Newton-Euler Formulation	Given	Find
① Computing acceleration of center of mass for each link → Forward recursive	$\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \ddot{\theta}_{b_1/n}$ $\theta_{b_1/b_2}, \dot{\theta}_{b_1/b_2}, \ddot{\theta}_{b_1/b_2}$	$\ddot{x}_{G_1/E}, \ddot{y}_{G_1/E}$ $\ddot{x}_{G_2/E}, \ddot{y}_{G_2/E}, \ddot{\theta}_{b_2/n}$
② Computing resultant force exerted on center of mass by using Newton-Euler equation	$\ddot{x}_{G_1/E}, \ddot{y}_{G_1/E}, \ddot{\theta}_{b_1/n}$ $\ddot{x}_{G_2/E}, \ddot{y}_{G_2/E}, \ddot{\theta}_{b_2/n}$	$\mathbf{F}_{G_1}, \mathbf{M}_{G_1}$ $\mathbf{F}_{G_2}, \mathbf{M}_{G_2}$
③ Computing input torque for each link → Backward recursive	$\boxed{\mathbf{F}_{G_2}, \mathbf{M}_{G_2}}$ $\boxed{\mathbf{F}_{G_1}, \mathbf{M}_{G_1}}$	$\boxed{\tau_2}$ $\boxed{\tau_1}$

Application of Recursive Newton-Euler Equation

Given: **Kinematic Model**

$$\boxed{\begin{matrix}\theta_{b_1/n},\dot{\theta}_{b_1/n},\ddot{\theta}_{b_1/n}\\ \theta_{b_2/b_1},\dot{\theta}_{b_2/b_1},\ddot{\theta}_{b_2/b_1}\end{matrix}}$$

Find: **input torque for link 1 -** $\boxed{\begin{matrix}\tau_1\\ \tau_2\end{matrix}}$

input torque for link 2 -

$$\boxed{\ddot{\mathbf{r}}_{G_1/E}}=\boxed{\ddot{\mathbf{r}}_{O_1/E}}+\boxed{\dot{\boldsymbol{\omega}}_{b_1/n}}\times\mathbf{r}_{G_1/O_1}+\boldsymbol{\omega}_{b_1/n}\times\left(\boldsymbol{\omega}_{b_1/n}\times\mathbf{r}_{G_1/O_1}\right)$$

where $\mathbf{r}_{G_1/E}=\begin{bmatrix}x_{G_1/E} & y_{G_1/E} & 0\end{bmatrix}^T$

$\boldsymbol{\omega}_{b_1/n}=\begin{bmatrix}0 & 0 & \theta_{b_1/n}\end{bmatrix}^T$

$$\boxed{\ddot{\mathbf{r}}_{O_2/E}}=\boxed{\ddot{\mathbf{r}}_{O_1/E}}+\boxed{\dot{\boldsymbol{\omega}}_{b_1/n}}\times\mathbf{r}_{O_2/O_1}+\boldsymbol{\omega}_{b_1/n}\times\left(\boldsymbol{\omega}_{b_1/n}\times\mathbf{r}_{O_2/O_1}\right)$$

where $\mathbf{r}_{G_1/E}=\begin{bmatrix}x_{G_1/E} & y_{G_1/E} & 0\end{bmatrix}^T$

$\boldsymbol{\omega}_{b_1/n}=\begin{bmatrix}0 & 0 & \theta_{b_1/n}\end{bmatrix}^T$

$$\boxed{\ddot{\mathbf{r}}_{G_2/E}}=\boxed{\ddot{\mathbf{r}}_{O_2/E}}+\boxed{\dot{\boldsymbol{\omega}}_{b_2/b_1}}\times\mathbf{r}_{G_2/O_2}+\boldsymbol{\omega}_{b_2/b_1}\times\left(\boldsymbol{\omega}_{b_2/b_1}\times\mathbf{r}_{G_2/O_2}\right)$$

where $\mathbf{r}_{G_2/E}=\begin{bmatrix}x_{G_2/E} & y_{G_2/E} & 0\end{bmatrix}^T$

$\boldsymbol{\omega}_{b_2/b_1}=\begin{bmatrix}0 & 0 & \theta_{b_2/b_1}\end{bmatrix}^T$

$$\boxed{\ddot{\boldsymbol{\omega}}_{b_2/n}}=\boxed{\ddot{\boldsymbol{\omega}}_{b_1/n}}+\boxed{\ddot{\boldsymbol{\omega}}_{b_2/b_1}}$$

$\boxed{\mathbf{F}_{G_1}}=m_1\boxed{\ddot{\mathbf{r}}_{G_1/E}}$, $\boxed{\mathbf{M}_{G_1}}=\mathbf{I}_{G_1}\boxed{\dot{\boldsymbol{\omega}}_{b_1/n}}$ **where** $\mathbf{r}_{G_1/E}=\begin{bmatrix}x_{G_1/E} & y_{G_1/E} & 0\end{bmatrix}^T,\boldsymbol{\omega}_{b_1/n}=\begin{bmatrix}0 & 0 & \theta_{b_1/n}\end{bmatrix}^T$

$\boxed{\mathbf{F}_{G_2}}=m_2\boxed{\ddot{\mathbf{r}}_{G_2/E}}$, $\boxed{\mathbf{M}_{G_2}}=\mathbf{I}_{G_2}\boxed{\dot{\boldsymbol{\omega}}_{b_2/n}}$ **where** $\mathbf{r}_{G_2/E}=\begin{bmatrix}x_{G_2/E} & y_{G_2/E} & 0\end{bmatrix}^T,\boldsymbol{\omega}_{b_2/n}=\begin{bmatrix}0 & 0 & \theta_{b_2/n}\end{bmatrix}^T$

$$\boxed{\mathbf{F}_{O_2}}=\boxed{\mathbf{F}_{G_2}} \qquad \boxed{\mathbf{F}_{O_1}}=\boxed{\mathbf{F}_{G_1}}+\boxed{\mathbf{F}_{O_2}}$$

$$\boxed{\boldsymbol{\tau}_2}=\boxed{\mathbf{M}_{G_2}} \qquad \boxed{\boldsymbol{\tau}_1}=\boxed{\mathbf{M}_{G_1}}+\mathbf{r}_{G_1/O_1}\times\boxed{\mathbf{F}_{G_1}}+\boxed{\boldsymbol{\tau}_2}+\mathbf{r}_{O_2/O_1}\times\boxed{\mathbf{F}_{O_2}}$$

5.6 Recursive Newton-Euler Formulation using Spatial Vector (Inverse Dynamics)

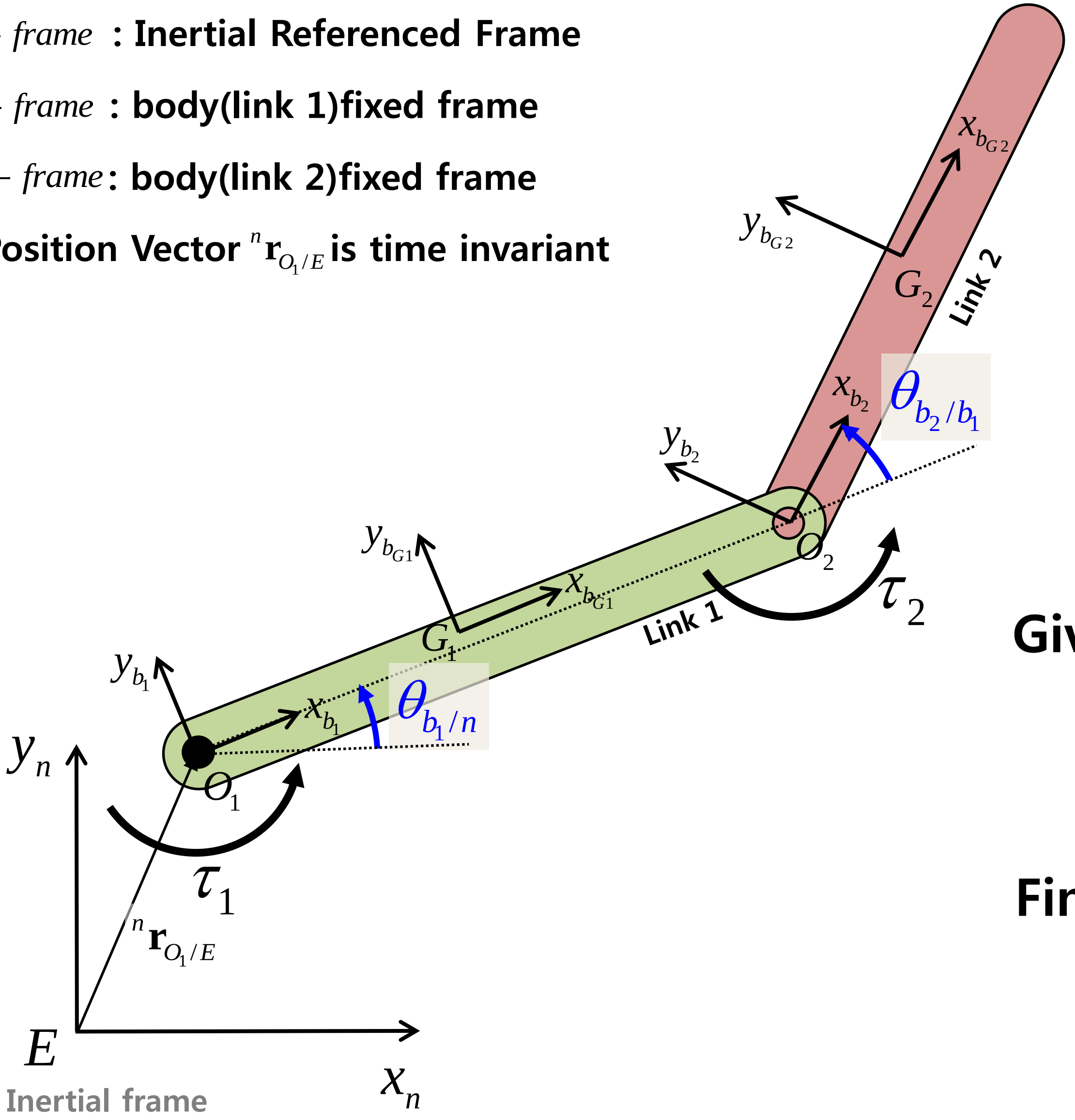
Inverse Dynamics of 2-Link Arm

$n-frame$ **: Inertial Referenced Frame**

$b_1-frame$ **: body(link 1)fixed frame**

$b_2-frame$**: body(link 2)fixed frame**

Position Vector $^n\mathbf{r}_{O_1/E}$ **is time invariant**



$$\theta_{b_1/n} = q_1$$
$$\theta_{b_2/b_1} = q_2$$

Generalized Coordinates

Given: **Kinematic Model**

$$\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \ddot{\theta}_{b_1/n}$$
$$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \ddot{\theta}_{b_2/b_1}$$

Find: **input torque for link 1 -** τ_1

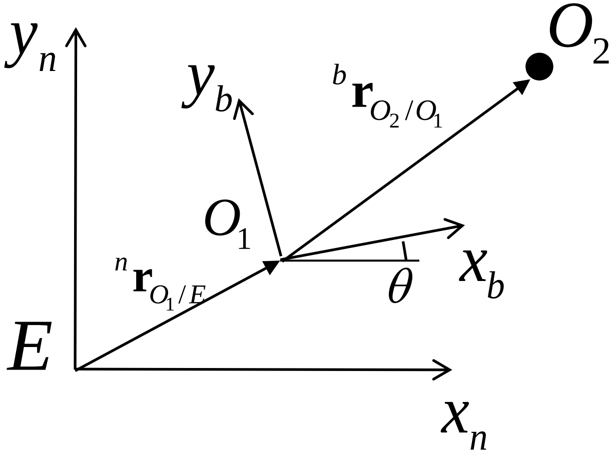
input torque for link 2 - τ_2

Relative Motion
- Rotating reference frame

Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.15

E : Origin of Inertial reference frame

O_1 : Origin of translating and rotating reference frame



$$ {}^{\textcircled{n}}\mathbf{r}_{O_2/E} = {}^{\textcircled{n}}\mathbf{r}_{O_1/E} + {}^{\textcircled{b}}\mathbf{r}_{O_2/O_1} $$

➔These vectors can not be added because they are defined using difference unit vector

---

$$ {}^{n}\mathbf{r}_{O_2/E} = {}^{n}\mathbf{r}_{O_1/E} + {}^{n}\mathbf{r}_{O_2/O_1} $$

➔ These vectors can be added because they are defined using the same unit vector.

${}^{n}\mathbf{R}_b(\theta)$:Rotation matrix that transforms 3D vectors from b to n coordinates.

$$ {}^{n}\mathbf{r}_{O_2/E} = {}^{n}\mathbf{r}_{O_1/E} + {}^{n}\mathbf{R}_b \cdot {}^{b}\mathbf{r}_{O_2/O_1} $$

$$ {}^{n}\mathbf{R}_b = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} $$

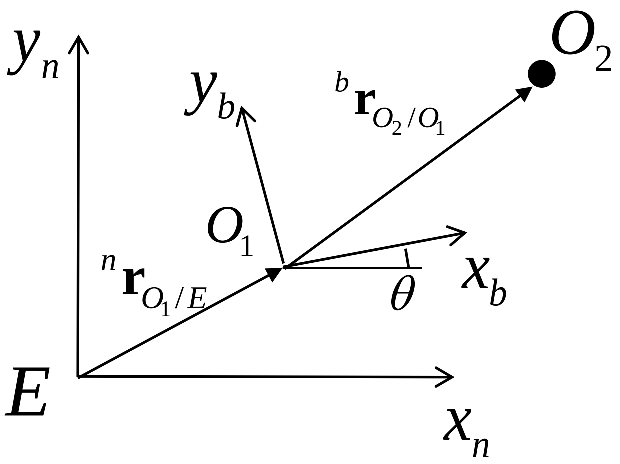
Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.15

Inverse Dynamics of 2-Link Arm

- Linear velocity of b_2-frame

E : Origin of <u>Inertial</u> reference frame

***O*₁** : Origin of <u>translating and rotating</u> reference frame



$$ {}^n\mathbf{r}_{O_2/E} = {}^n\mathbf{r}_{O_1/E} + {}^n\mathbf{R}_b \cdot {}^b\mathbf{r}_{O_2/O_1} $$

the time derivative

$$ \frac{d}{dt}\,{}^n\mathbf{r}_{O_2/E} = \frac{d}{dt}\,{}^n\mathbf{r}_{O_1/E} + \frac{d}{dt}\,{}^n\mathbf{R}_b \cdot {}^b\mathbf{r}_{O_2/O_1} + {}^n\mathbf{R}_b \cdot \frac{d}{dt}\,{}^b\mathbf{r}_{O_2/O_1} $$

$$ \frac{d}{dt}\,{}^n\mathbf{R}_b = {}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{R}_b $$

$$ \frac{d}{dt}\,{}^n\mathbf{r}_{O_2/E} = \frac{d}{dt}\,{}^n\mathbf{r}_{O_1/E} + {}^n\boldsymbol{\omega}_{b/n} \times {}^n\mathbf{R}_b \cdot {}^b\mathbf{r}_{O_2/O_1} + {}^n\mathbf{R}_b \cdot \frac{d}{dt}\,{}^b\mathbf{r}_{O_2/O_1} $$

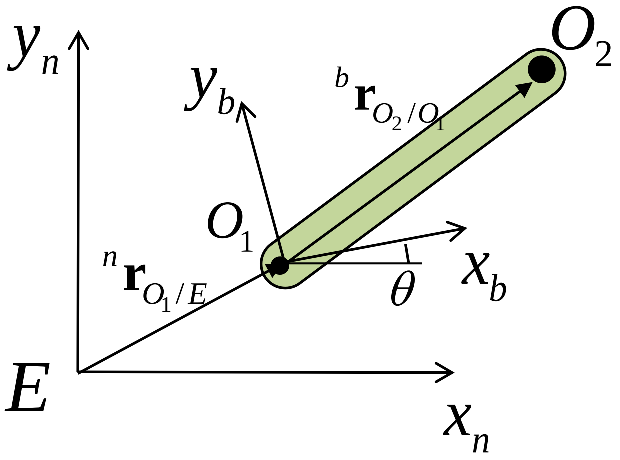
Jerry ginsberg, Engineering Dynamics, Georgia Institute of Technology, 2008, p.15

Inverse Dynamics of 2-Link Arm

- Linear velocity of b_2-frame

E : Origin of Inertial reference frame

***O*₁** : Origin of translating and rotating reference frame



$$ {}^{n}\mathbf{r}_{O_2/E} = {}^{n}\mathbf{r}_{O_1/E} + {}^{n}\mathbf{R}_{b} \cdot {}^{b}\mathbf{r}_{O_2/O_1} $$

$$\Downarrow$$ **the time derivative**

$$ \frac{d}{dt}\,{}^{n}\mathbf{r}_{O_2/E} = \frac{d}{dt}\,{}^{n}\mathbf{r}_{O_1/E} $$

$$ + {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{R}_{b} \cdot {}^{b}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{R}_{b} \cdot \underbrace{\cancel{\frac{d}{dt}\,{}^{b}\mathbf{r}_{O_2/O_1}}}_{\text{constant}} $$

$$\Downarrow$$

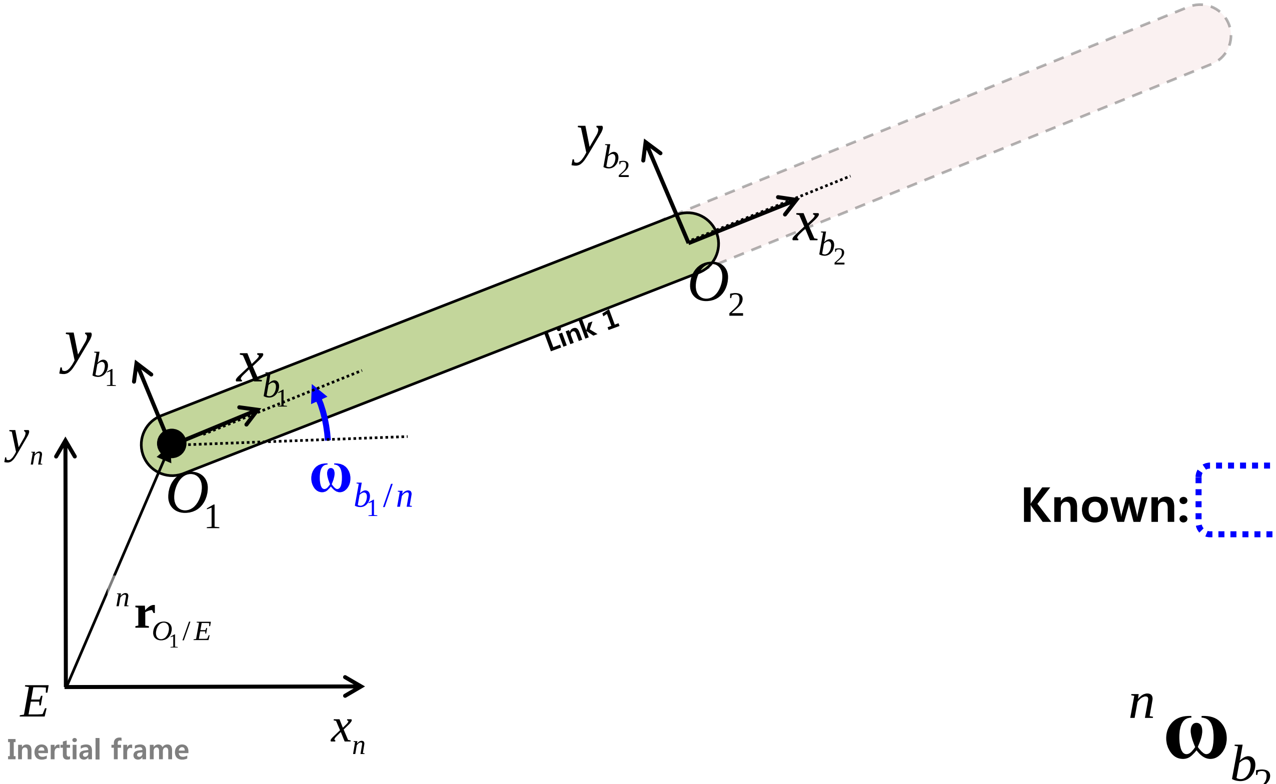
$$ \frac{d}{dt}\,{}^{n}\mathbf{r}_{O_2/E} = \frac{d}{dt}\,{}^{n}\mathbf{r}_{O_1/E} + {}^{n}\boldsymbol{\omega}_{b/n} \times {}^{n}\mathbf{R}_{b} \cdot {}^{b}\mathbf{r}_{O_2/O_1} $$

$$\Downarrow$$

$$ {}^{n}\mathbf{v}_{O_2/E} = {}^{n}\mathbf{v}_{O_1/E} + {}^{n}\boldsymbol{\omega}_{b_1/n} \times {}^{n}\mathbf{r}_{O_2/O_1} $$

Inverse Dynamics of 2-Link Arm

- Angluar velocity of b$_2$-frame

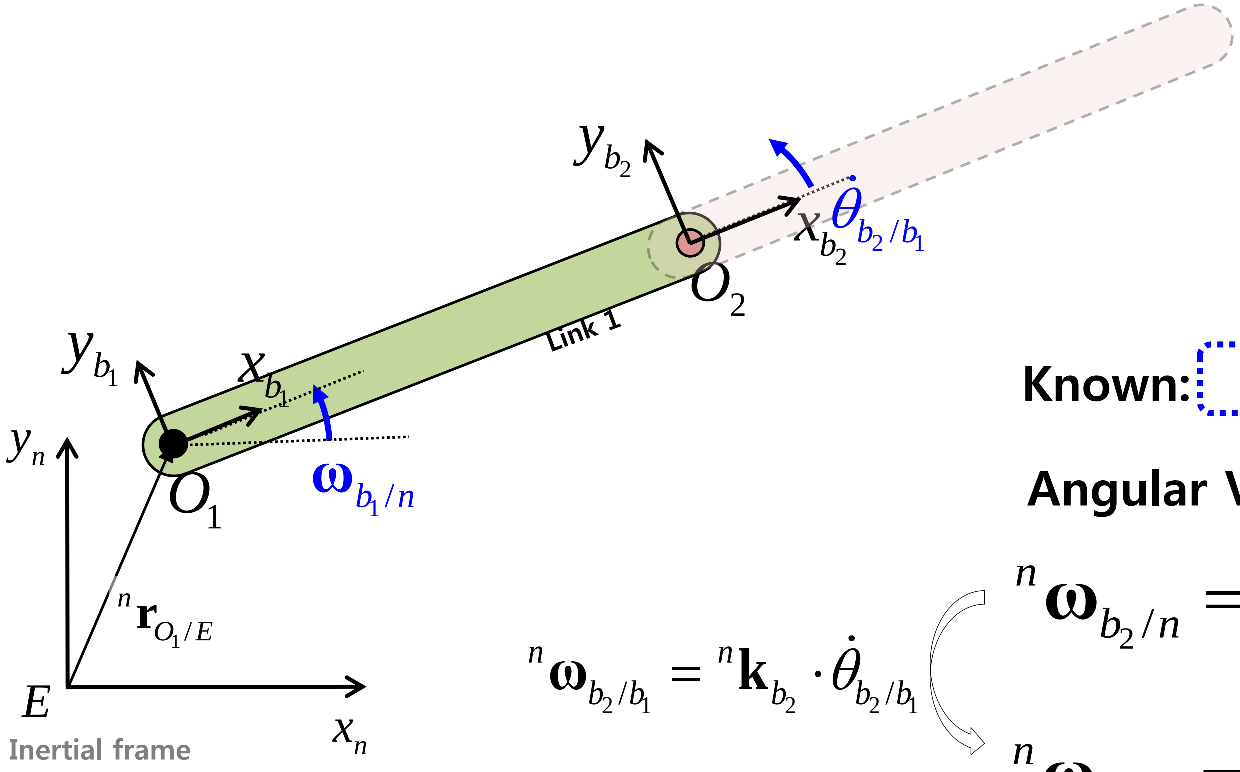


Known: ⬚ Given: ☐

$$ {}^n\boldsymbol{\omega}_{b_2/n} = \boxed{{}^n\boldsymbol{\omega}_{b_1/n}} $$

Inverse Dynamics of 2-Link Arm

- Angluar velocity of b$_2$-frame



Known: ⬚ **Given:** ▢

Angular Velocity of {b$_2$}

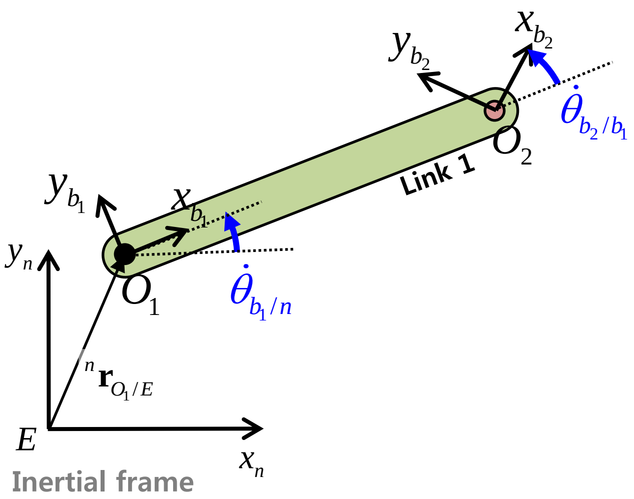
$$^{n}\boldsymbol{\omega}_{b_2/n} = \boxed{^{n}\boldsymbol{\omega}_{b_1/n}} + {}^{n}\boldsymbol{\omega}_{b_2/b_1}$$

$$^{n}\boldsymbol{\omega}_{b_2/b_1} = {}^{n}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1}$$

$$^{n}\boldsymbol{\omega}_{b_2/n} = \boxed{^{n}\boldsymbol{\omega}_{b_1/n}} + {}^{n}\mathbf{k}_{b_2} \boxed{\dot{\theta}_{b_2/b_1}}$$

Inverse Dynamics of 2-Link Arm

- Velocity of b_2-frame



Velocity of {b_2}

Angular Vel.

$$ {}^n\boldsymbol{\omega}_{b_2/n} = \boxed{{}^n\boldsymbol{\omega}_{b_1/n}} + {}^n\mathbf{k}_{b_2}\boxed{\dot{\theta}_{b_2/b_1}} $$

Linear Vel.

$$ {}^n\mathbf{v}_{O_2/E} = \boxed{{}^n\boldsymbol{\omega}_{b_1/n}} \times {}^n\mathbf{r}_{O_2/O_1} + \boxed{{}^n\mathbf{v}_{O_1/E}} $$

${}^n\mathbf{k}_{b_2}$ **Rotation axis of O_2 joint decomposed in inertial frame (unit vector of z-axis of b_2-frame)**

$\boxed{{}^n\boldsymbol{\omega}_{b_1/n}}$ $\boxed{{}^n\mathbf{v}_{O_1/E}}$ ⇨ **This vectors can be calculated using same equation.**

Velocity of {b_1}

$$ \boxed{\begin{aligned} {}^n\boldsymbol{\omega}_{b_1/n} \\ {}^n\mathbf{v}_{O_1/E} \end{aligned}} \begin{aligned} &= {}^n\boldsymbol{\omega}_{n/n} + {}^n\mathbf{k}_{b_1}\boxed{\dot{\theta}_{b_1/n}} \\ &= {}^n\boldsymbol{\omega}_{n/n} \times {}^n\mathbf{r}_{O_1/E} + {}^n\mathbf{v}_{E/E} \end{aligned} $$

$$ {}^n\boldsymbol{\omega}_{n/n} = 0,\ {}^n\mathbf{v}_{E/E} = 0 $$

⇨

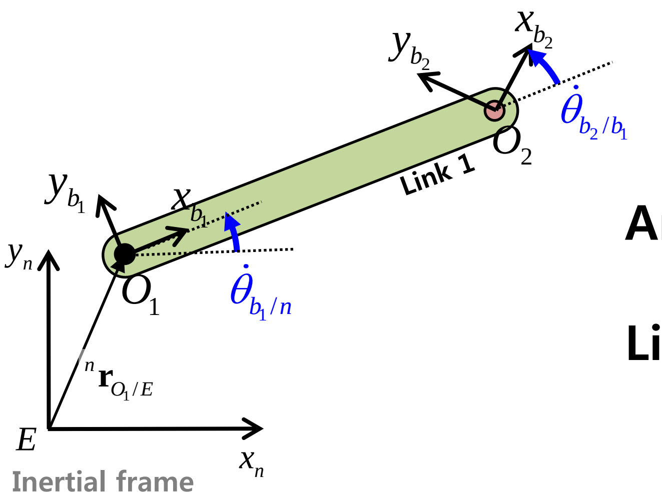
Velocity of {b_2}

$$ \begin{aligned} {}^n\boldsymbol{\omega}_{b_2/n} &= \boxed{{}^n\boldsymbol{\omega}_{b_1/n}} + {}^n\mathbf{k}_{b_2}\boxed{\dot{\theta}_{b_2/b_1}} \\ {}^n\mathbf{v}_{O_2/E} &= \boxed{{}^n\boldsymbol{\omega}_{b_1/n}} \times {}^n\mathbf{r}_{O_2/O_1} + \boxed{{}^n\mathbf{v}_{O_1/E}} \end{aligned} $$

Forward Recursive!!

Inverse Dynamics of 2-Link Arm

- Velocity of b_2-frame



Velocity of {b$_2$}

Angular Vel.

$$ {}^{n}\boldsymbol{\omega}_{b_2/n} = {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{n}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} $$

Linear Vel.

$$ {}^{n}\mathbf{v}_{O_2/E} = {}^{n}\boldsymbol{\omega}_{b_1/n} \times {}^{n}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{v}_{O_1/E} $$

${}^{n}\mathbf{k}_{b_2}$ **Rotation axis of O$_2$ joint decomposed in inertial frame (unit vector of z-axis of b$_2$-frame)**

Coordinate transformation from {n} to body fixed frame {b$_2$} by multiplication of rotation matrix ${}^{b_2}\mathbf{R}_n$

Inverse Dynamics of 2-Link Arm
- Velocity of b_2-frame

Velocity of $\{b_2\}$

$$ {}^{n}\boldsymbol{\omega}_{b_2/n} = {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{n}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} $$

$$ {}^{n}\mathbf{v}_{O_2/E} = {}^{n}\boldsymbol{\omega}_{b_1/n} \times {}^{n}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{v}_{O_1/E} $$

Multiply rotation matrix $\quad {}^{b_2}\mathbf{R}_n$

$$ {}^{b_2}\mathbf{R}_n \cdot {}^{n}\boldsymbol{\omega}_{b_2/n} = \boxed{{}^{b_2}\mathbf{R}_n} \cdot {}^{b}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{R}_n \cdot {}^{n}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} $$

$$ {}^{b_2}\mathbf{R}_n \cdot {}^{n}\mathbf{v}_{O_2/E} = \boxed{{}^{b_2}\mathbf{R}_n} \cdot {}^{n}\boldsymbol{\omega}_{b_1/n} \times \boxed{{}^{b_2}\mathbf{R}_n} \cdot {}^{n}\mathbf{r}_{O_2/O_1} + \boxed{{}^{b_2}\mathbf{R}_n} \cdot {}^{n}\mathbf{v}_{O_1/E} $$

$$ {}^{b_2}\mathbf{R}_n = {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{R}_n $$

$$ {}^{b_2}\mathbf{R}_n \cdot {}^{n}\boldsymbol{\omega}_{b_2/n} = \boxed{{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{R}_n} \cdot {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{R}_n \cdot {}^{n}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} $$

$$ {}^{b_2}\mathbf{R}_n \cdot {}^{n}\mathbf{v}_{O_2/E} = \boxed{{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{R}_n} \cdot {}^{n}\boldsymbol{\omega}_{b_1/n} \times \boxed{{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{R}_n} \cdot {}^{n}\mathbf{r}_{O_2/O_1} + \boxed{{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{R}_n} \cdot {}^{n}\mathbf{v}_{O_1/E} $$

Inverse Dynamics of 2-Link Arm
- Velocity of b_2-frame

Velocity of $\{b_2\}$

$$\boxed{\begin{array}{l} {}^{n}\boldsymbol{\omega}_{b_2/n} = {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{n}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} \\ \hdashline {}^{n}\mathbf{v}_{O_2/E} = {}^{n}\boldsymbol{\omega}_{b_1/n}\times{}^{n}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{v}_{O_1/E} \end{array}}$$

$${}^{b_2}\mathbf{R}_{n}\cdot{}^{n}\boldsymbol{\omega}_{b_2/n} = \boxed{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{R}_{n}}\cdot{}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{R}_{n}\cdot{}^{n}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

$${}^{b_2}\mathbf{R}_{n}\cdot{}^{n}\mathbf{v}_{O_2/E} = \boxed{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{R}_{n}}\cdot{}^{n}\boldsymbol{\omega}_{b_1/n}\times\boxed{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{R}_{n}}\cdot{}^{n}\mathbf{r}_{O_2/O_1} + \boxed{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{R}_{n}}\cdot{}^{n}\mathbf{v}_{O_1/E}$$

$${}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

$${}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}$$

$${}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

$${}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}$$

Inverse Dynamics of 2-Link Arm

- Velocity of b$_2$-frame

Velocity of {b$_2$}

$$\boxed{\begin{array}{l} {}^{n}\boldsymbol{\omega}_{b_2/n} = {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{n}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} \\ \hdashline {}^{n}\mathbf{v}_{O_2/E} = {}^{n}\boldsymbol{\omega}_{b_1/n} \times {}^{n}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{v}_{O_1/E} \end{array}}$$

$$\Downarrow$$

$${}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1}$$

$${}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1} \cdot \left({}^{b_1}\boldsymbol{\omega}_{b_1/n} \times {}^{b_1}\mathbf{r}_{O_2/O_1} \right) + {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{v}_{O_1/E}$$

$$\Downarrow$$

$${}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1}$$

$${}^{b_2}\mathbf{v}_{O_2/E} = -{}^{b_2}\mathbf{R}_{b_1} \cdot \left({}^{b_1}\mathbf{r}_{O_2/O_1} \times {}^{b_1}\boldsymbol{\omega}_{b_1/n} \right) + {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{v}_{O_1/E}$$

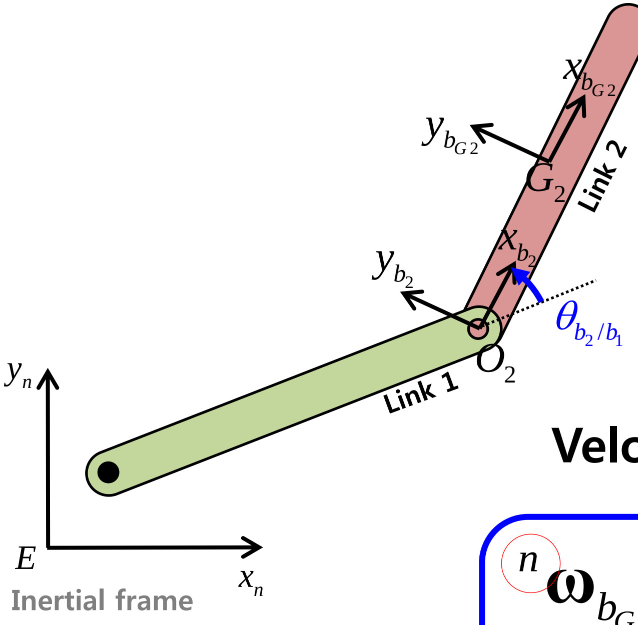
$$\Downarrow$$

$$\begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{b_2/n} \end{bmatrix} = \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_1} & 0 \\ -{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{r}_{O_2/O_1} \times & {}^{b_2}\mathbf{R}_{b_1} \end{bmatrix} \begin{bmatrix} {}^{b_1}\boldsymbol{\omega}_{b_1/n} \\ {}^{b_1}\mathbf{v}_{b_1/n} \end{bmatrix} + \begin{bmatrix} {}^{b_2}\mathbf{k}_{b_2} \\ 0 \end{bmatrix} \dot{\theta}_{b_2/b_1}$$

⇨ $${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

Inverse Dynamics of 2-Link Arm

- Velocity of b_{G2}-frame



cf) Velocity of {b$_2$}

$$ {}^{n}\boldsymbol{\omega}_{b_2/n} = {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{n}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} $$

$$ {}^{n}\mathbf{v}_{O_2/E} = {}^{n}\boldsymbol{\omega}_{b_1/n} \times {}^{n}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{v}_{O_1/E} $$

Velocity of {b$_{G2}$}

$$ {}^{n}\boldsymbol{\omega}_{b_{G2}/n} = {}^{n}\boldsymbol{\omega}_{b_2/n} $$

$$ {}^{n}\mathbf{v}_{G_2/E} = {}^{n}\boldsymbol{\omega}_{b_2/n} \times {}^{n}\mathbf{r}_{G_2/O_2} + {}^{n}\mathbf{v}_{O_2/E} $$

Coordinate transformation from {n} to body fixed frame {G$_2$} by multiplication of rotation matrix ${}^{b_{G2}}\mathbf{R}_n$

Inverse Dynamics of 2-Link Arm

- Velocity of b_{G2}-frame

Velocity of $\{b_{G2}\}$

$$\boxed{\begin{aligned} {}^{\textcircled{n}}\boldsymbol{\omega}_{b_{G2}/n} &= {}^{\textcircled{n}}\boldsymbol{\omega}_{b_2/n} \\ {}^{\textcircled{n}}\mathbf{v}_{G_2/E} &= {}^{\textcircled{n}}\boldsymbol{\omega}_{b_2/n} \times {}^{\textcircled{n}}\mathbf{r}_{G_2/O_2} + {}^{\textcircled{n}}\mathbf{v}_{O_2/E} \end{aligned}}$$

Multiply rotation matrix ${}^{b_{G2}}\mathbf{R}_n$

$${}^{b_{G2}}\mathbf{R}_n \cdot {}^{n}\boldsymbol{\omega}_{b_{G2}/n} = \boxed{{}^{b_{G2}}\mathbf{R}_n} \cdot {}^{n}\boldsymbol{\omega}_{b_2/n}$$

$${}^{b_{G2}}\mathbf{R}_n \cdot {}^{n}\mathbf{v}_{G_2/E} = \boxed{{}^{b_{G2}}\mathbf{R}_n} \cdot {}^{n}\boldsymbol{\omega}_{b_2/n} \times \boxed{{}^{b_{G2}}\mathbf{R}_n} \cdot {}^{n}\mathbf{r}_{G_2/O_2} + \boxed{{}^{b_{G2}}\mathbf{R}_n} \cdot {}^{n}\mathbf{v}_{O_2/E}$$

$${}^{b_{G2}}\mathbf{R}_n = {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{R}_n$$

$${}^{b_{G2}}\mathbf{R}_n \cdot {}^{n}\boldsymbol{\omega}_{b_{G2}/n} = \boxed{{}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{R}_n} \cdot {}^{n}\boldsymbol{\omega}_{b_{O2}/n}$$

$${}^{b_{G2}}\mathbf{R}_n \cdot {}^{n}\mathbf{v}_{G_2/E} = \boxed{{}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{R}_n} \cdot {}^{n}\boldsymbol{\omega}_{b_2/n} \times \boxed{{}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{R}_n} \cdot {}^{n}\mathbf{r}_{G_2/O_2} + \boxed{{}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{R}_n} \cdot {}^{n}\mathbf{v}_{O_2/E}$$

Inverse Dynamics of 2-Link Arm

- Velocity of b_{G2}-frame

Velocity of $\{b_{G2}\}$

$$\boxed{\begin{aligned} {}^{n}\boldsymbol{\omega}_{b_{G2}/n} &= {}^{n}\boldsymbol{\omega}_{b_2/n} \\ {}^{n}\mathbf{v}_{G_2/E} &= {}^{n}\boldsymbol{\omega}_{b_2/n} \times {}^{n}\mathbf{r}_{G_2/O_2} + {}^{n}\mathbf{v}_{O_2/E} \end{aligned}}$$

Multiply rotation matrix ${}^{b_{G2}}\mathbf{R}_n$

$$ {}^{b_{G2}}\mathbf{R}_n \cdot {}^{n}\boldsymbol{\omega}_{b_{G2}/n} = \boxed{{}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{R}_n} \cdot {}^{n}\boldsymbol{\omega}_{b_{O2}/n}$$

$$ {}^{b_{G2}}\mathbf{R}_n \cdot {}^{n}\mathbf{v}_{G_2/E} = \boxed{{}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{R}_n} \cdot {}^{n}\boldsymbol{\omega}_{b_2/n} \times \boxed{{}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{R}_n} \cdot {}^{n}\mathbf{r}_{G_2/O_2} + \boxed{{}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{R}_n} \cdot {}^{n}\mathbf{v}_{O_2/E}$$

$$ {}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n} = {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\boldsymbol{\omega}_{b_2/n}$$

$$ {}^{b_{G2}}\mathbf{v}_{b_{G2}/0} = {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\boldsymbol{\omega}_{b_2/n} \times {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{r}_{G_2/O_2} + {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{v}_{O_2/E}$$

$$ {}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n} = {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\boldsymbol{\omega}_{b_2/n}$$

$$ {}^{b_{G2}}\mathbf{v}_{G_2/E} = -{}^{b_{G2}}\mathbf{R}_{b_2} \cdot \left({}^{b_2}\mathbf{r}_{G_2/O_2} \times {}^{b_2}\boldsymbol{\omega}_{b_2/n} \right) + {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{v}_{O_2/E}$$

Inverse Dynamics of 2-Link Arm

- Velocity of b$_{G2}$-frame

Velocity of {b$_{G2}$}

$$
\begin{aligned}
{}^{n}\boldsymbol{\omega}_{b_{G2}/n} &= {}^{n}\boldsymbol{\omega}_{b_2/n} \\
{}^{n}\mathbf{v}_{G_2/E} &= {}^{n}\boldsymbol{\omega}_{b_2/n} \times {}^{n}\mathbf{r}_{G_2/O_2} + {}^{n}\mathbf{v}_{O_2/E}
\end{aligned}
$$

Multiply rotation matrix $^{b_{G2}}\mathbf{R}_n$

$$
\begin{aligned}
{}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n} &= {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\
{}^{b_{G2}}\mathbf{v}_{G_2/E} &= -{}^{b_{G2}}\mathbf{R}_{b_2} \cdot \left({}^{b_2}\mathbf{r}_{G_2/O_2} \times {}^{b_2}\boldsymbol{\omega}_{b_2/n} \right) + {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{v}_{O_2/E}
\end{aligned}
$$

$$
\begin{bmatrix} {}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n} \\ {}^{b_{G2}}\mathbf{v}_{G_2/E} \end{bmatrix} = \begin{bmatrix} {}^{b_{G2}}\mathbf{R}_{b_2} & 0 \\ -{}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{r}_{G_2/O_2} \times & {}^{b_{G2}}\mathbf{R}_{b_2} \end{bmatrix} \begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{O_2/E} \end{bmatrix} \quad \Rightarrow \quad {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}
$$

Inverse Dynamics of 2-Link Arm
- Acceleration of b_2-frame

Velocity of {b_2}

$$\boxed{{}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}$$

$${}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}$$

Angular Acceleration of {b_2}

the time derivative of $\boxed{{}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}$

$$\frac{d}{dt}\left({}^{b_2}\boldsymbol{\omega}_{b_2/n}\right) = \frac{d}{dt}\left({}^{b_2}\mathbf{R}_{b_1}\right)\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{R}_{b_1}\cdot\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\right) + \frac{d}{dt}\left({}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}\right)$$

$$\frac{d}{dt}{}^{b_2}\mathbf{R}_{b_1} = {}^{b_2}\boldsymbol{\omega}_{b_1/b_2}\times{}^{b_2}\mathbf{R}_{b_1}$$

$$\underline{\frac{d}{dt}\left({}^{b_2}\boldsymbol{\omega}_{b_2/n}\right)} = {}^{b_2}\boldsymbol{\omega}_{b_1/b_2}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{R}_{b_1}\cdot\underline{\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\right)} + \underline{\frac{d}{dt}\left({}^{b_2}\mathbf{k}_{b_2}\right)}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{k}_{b_2}\cdot\underline{\frac{d}{dt}\left(\dot{\theta}_{b_2/b_1}\right)}$$

$\frac{d}{dt}(\mathbf{a}) = \dot{\mathbf{a}}$ **, where $\mathbf{a}$ is an arbitrary vector.**

$$\underline{{}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n}} = {}^{b_2}\boldsymbol{\omega}_{b_1/b_2}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{R}_{b_1}\cdot\underline{{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}} + \underline{{}^{b_2}\dot{\mathbf{k}}_{b_2}}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{k}_{b_2}\cdot\underline{\ddot{\theta}_{b_2/b_1}}$$

Inverse Dynamics of 2-Link Arm

- Acceleration of b_2-frame

Velocity of {b_2}

$$\boxed{{}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}$$

$${}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}$$

Angular Acceleration of {b_2}

the time derivative of $\boxed{{}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}$

$${}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = \underline{{}^{b_2}\boldsymbol{\omega}_{b_1/b_2}}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1}$$

$${}^{b_2}\boldsymbol{\omega}_{b_1/b_2} = -{}^{b_2}\boldsymbol{\omega}_{b_2/b_1}$$

$${}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = \underline{\underline{-{}^{b_2}\boldsymbol{\omega}_{b_2/b_1}}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1}$$

$${}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = \underline{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\boldsymbol{\omega}_{b_2/b_1}} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1}$$

Inverse Dynamics of 2-Link Arm

- Acceleration of b$_2$-frame

Velocity of {b$_2$}

$$\boxed{{}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}$$

$${}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}$$

Angular Acceleration of {b$_2$}

the time derivative of $\boxed{{}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}$

$${}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times\underline{{}^{b_2}\boldsymbol{\omega}_{b_2/b_1}} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1}$$

$${}^{b_2}\boldsymbol{\omega}_{b_2/b_1} = {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

$${}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = \underline{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times\underline{{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}} + \underline{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1}$$

$$\boxed{{}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = \underline{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}} + \underline{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1}}$$

Inverse Dynamics of 2-Link Arm

- Acceleration of b_2-frame

Velocity of $\{b_2\}$

$${}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

$$\boxed{{}^{b_2}\mathbf{v}_{O_2/E} = \underline{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}}$$

$${}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}$$

$$\boxed{{}^{b_2}\mathbf{v}_{O_2/E} = \underline{{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right)} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}}$$

Linear Acceleration of $\{b_2\}$

the time derivative of $\boxed{{}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}}$

$$\frac{d}{dt}\left({}^{b_2}\mathbf{v}_{O_2/E}\right) = \frac{d}{dt}\left({}^{b_2}\mathbf{R}_{b_1}\right)\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + \frac{d}{dt}\left({}^{b_2}\mathbf{R}_{b_1}\right)\cdot{}^{b_1}\mathbf{v}_{O_1/E} + {}^{b_2}\mathbf{R}_{b_1}\cdot\frac{d}{dt}\left({}^{b_1}\mathbf{v}_{O_1/E}\right)$$

Inverse Dynamics of 2-Link Arm

- Acceleration of b_2-frame

Velocity of $\{b_2\}$

$${}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

$$\boxed{{}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}}$$

Linear Acceleration of $\{b_2\}$

the time derivative of $\boxed{{}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}}$

$$\frac{d}{dt}\left({}^{b_2}\mathbf{v}_{O_2/E}\right) = \underline{\frac{d}{dt}\left({}^{b_2}\mathbf{R}_{b_1}\right)}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + \underline{\frac{d}{dt}\left({}^{b_2}\mathbf{R}_{b_1}\right)}\cdot{}^{b_1}\mathbf{v}_{O_1/E} + {}^{b_2}\mathbf{R}_{b_1}\cdot\frac{d}{dt}\left({}^{b_1}\mathbf{v}_{O_1/E}\right)$$

$$\frac{d}{dt}{}^{b_2}\mathbf{R}_{b_1} = {}^{b_2}\boldsymbol{\omega}_{b_1/b_2}\times{}^{b_2}\mathbf{R}_{b_1}$$

$$\frac{d}{dt}\left({}^{b_2}\mathbf{v}_{O_2/E}\right) = \underline{{}^{b_2}\boldsymbol{\omega}_{b_1/b_2}\times{}^{b_2}\mathbf{R}_{b_1}}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\underline{\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right)} + \underline{{}^{b_2}\boldsymbol{\omega}_{b_1/b_2}\times{}^{b_2}\mathbf{R}_{b_1}}\cdot{}^{b_1}\mathbf{v}_{O_1/E} + {}^{b_2}\mathbf{R}_{b_1}\cdot\frac{d}{dt}\left({}^{b_1}\mathbf{v}_{O_1/E}\right)$$

$$\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) = \frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\right)\times{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_1}\boldsymbol{\omega}_{b_1/n}\times\cancel{\frac{d}{dt}\left({}^{b_1}\mathbf{r}_{O_2/O_1}\right)} = \frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\right)\times{}^{b_1}\mathbf{r}_{O_2/O_1}$$

$$\frac{d}{dt}\left({}^{b_2}\mathbf{v}_{O_2/E}\right) = {}^{b_2}\boldsymbol{\omega}_{b_1/b_2}\times{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\underline{\left(\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\right)\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right)} + {}^{b_2}\boldsymbol{\omega}_{b_1/b_2}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E} + {}^{b_2}\mathbf{R}_{b_1}\cdot\frac{d}{dt}\left({}^{b_1}\mathbf{v}_{O_1/E}\right)$$

Inverse Dynamics of 2-Link Arm
- Acceleration of b_2-frame

Velocity of {b_2}

$$ {}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} $$

$$ \boxed{{}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}} $$

Linear Acceleration of {b_2}

the time derivative of $\boxed{{}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}}$

$$ \frac{d}{dt}\left({}^{b_2}\mathbf{v}_{O_2/E}\right) = \underline{{}^{b_2}\boldsymbol{\omega}_{b_1/b_2}}\times{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\left(\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\right)\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + \underline{{}^{b_2}\boldsymbol{\omega}_{b_1/b_2}}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E} + {}^{b_2}\mathbf{R}_{b_1}\cdot\frac{d}{dt}\left({}^{b_1}\mathbf{v}_{O_1/E}\right) $$

$$ {}^{b_2}\boldsymbol{\omega}_{b_1/b_2} = -{}^{b_2}\boldsymbol{\omega}_{b_2/b_1} $$

$$ \frac{d}{dt}\left({}^{b_2}\mathbf{v}_{O_2/E}\right) = \underline{\underline{-{}^{b_2}\boldsymbol{\omega}_{b_1/b_2}}\times{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right)} + {}^{b_2}\mathbf{R}_{b_1}\left(\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\right)\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) \underline{\underline{-{}^{b_2}\boldsymbol{\omega}_{b_2/b_1}}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}} + {}^{b_2}\mathbf{R}_{b_1}\cdot\frac{d}{dt}\left({}^{b_1}\mathbf{v}_{O_1/E}\right) $$

$$ \frac{d}{dt}\left({}^{b_2}\mathbf{v}_{O_2/E}\right) = \underline{{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right)\times{}^{b_2}\boldsymbol{\omega}_{b_1/b_2}} + {}^{b_2}\mathbf{R}_{b_1}\left(\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\right)\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) \underline{+{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}\times{}^{b_2}\boldsymbol{\omega}_{b_2/b_1}} + {}^{b_2}\mathbf{R}_{b_1}\cdot\frac{d}{dt}\left({}^{b_1}\mathbf{v}_{O_1/E}\right) $$

Inverse Dynamics of 2-Link Arm

- Acceleration of b_2-frame

Velocity of $\{b_2\}$

$$ {}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} $$

$$ \boxed{ {}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E} } $$

Linear Acceleration of $\{b_2\}$

the time derivative of $\boxed{ {}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E} }$

$$ \frac{d}{dt}\left({}^{b_2}\mathbf{v}_{O_2/E}\right) = {}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right)\times\underline{{}^{b_2}\boldsymbol{\omega}_{b_1/b_2}} + {}^{b_2}\mathbf{R}_{b_1}\left(\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\right)\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}\times\underline{{}^{b_2}\boldsymbol{\omega}_{b_2/b_1}} + {}^{b_2}\mathbf{R}_{b_1}\cdot\frac{d}{dt}\left({}^{b_1}\mathbf{v}_{O_1/E}\right) $$

$$ {}^{b_2}\boldsymbol{\omega}_{b_2/b_1} = {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} $$

$$ \underline{\frac{d}{dt}\left({}^{b_2}\mathbf{v}_{O_2/E}\right)} = {}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right)\times\underline{{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}} + {}^{b_2}\mathbf{R}_{b_1}\left(\underline{\frac{d}{dt}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\right)}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}\times\underline{{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}} + {}^{b_2}\mathbf{R}_{b_1}\cdot\underline{\frac{d}{dt}\left({}^{b_1}\mathbf{v}_{O_1/E}\right)} $$

$\frac{d}{dt}(\mathbf{a}) = \dot{\mathbf{a}}$, where $\mathbf{a}$ is an arbitrary vector.

$$ \underline{{}^{b_2}\dot{\mathbf{v}}_{O_2/E}} = {}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right)\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{R}_{b_1}\left(\underline{{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot\underline{{}^{b_1}\dot{\mathbf{v}}_{O_1/E}} $$

Inverse Dynamics of 2-Link Arm

- Acceleration of b_2-frame

Velocity of $\{b_2\}$

$${}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

$$\boxed{{}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}+{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}}$$

Linear Acceleration of $\{b_2\}$

the time derivative of $\boxed{{}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}+{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}}$

$${}^{b_2}\dot{\mathbf{v}}_{O_2/E} = \underline{{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right)}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + \underline{{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}\times{}^{b_1}\mathbf{r}_{O_2/O_1}\right)} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\mathbf{v}}_{O_1/E}$$

$${}^{b_2}\dot{\mathbf{v}}_{O_2/E} = \underbrace{\underline{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}_{(1)} + \underbrace{\underline{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}}}_{(2)} + \underbrace{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}_{(3)} + \underbrace{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\mathbf{v}}_{O_1/E}}_{(4)}$$

$$\boxed{{}^{b_2}\dot{\mathbf{v}}_{O_2/E} = \underbrace{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}}_{(2)} + \underbrace{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\mathbf{v}}_{O_1/E}}_{(4)} + \underbrace{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}_{(1)} + \underbrace{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}}_{(3)}}$$

Inverse Dynamics of 2-Link Arm
- Acceleration of b$_2$-frame

Velocity of {b$_2$}

$$ {}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} $$

$$ {}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\boldsymbol{\omega}_{b_1/n} \times {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{v}_{O_1/E} $$

Acceleration of {b$_2$}

the time derivative of **Velocity of {b$_2$}**

$$ {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = \underbrace{{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}}_{(1)} + \underbrace{{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\boldsymbol{\omega}_{b_1/n} \times {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1}}_{(2)} + \underbrace{{}^{b_2}\dot{\mathbf{k}}_{b_2} \cdot \dot{\theta}_{b_2/b_1}}_{(3)} + \underbrace{{}^{b_2}\mathbf{k}_{b_2} \cdot \ddot{\theta}_{b_2/b_1}}_{(4)} $$

$$ {}^{b_2}\dot{\mathbf{v}}_{O_2/E} = \underbrace{{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} \times {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{r}_{O_2/O_1}}_{(1)} + {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\dot{\mathbf{v}}_{O_1/E} + \underbrace{{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\boldsymbol{\omega}_{b_1/n} \times {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{r}_{O_2/O_1} \times {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} + {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{v}_{O_1/E} \times {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1}}_{(2)} $$

$$ {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} \times {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} = 0 $$

$$ {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = \underbrace{{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}}_{(1)} \quad \underbrace{+ {}^{b_2}\mathbf{k}_{b_2} \cdot \ddot{\theta}_{b_2/b_1}}_{(4)} + \underbrace{{}^{b_2}\dot{\mathbf{k}}_{b_2} \cdot \dot{\theta}_{b_2/b_1}}_{(3)} + \underbrace{{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\boldsymbol{\omega}_{b_1/n} \times {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1}}_{(2)} + {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} \times {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} $$

$$ {}^{b_2}\dot{\mathbf{v}}_{O_2/E} = \underbrace{-{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\mathbf{r}_{O_2/O_1} \times {}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}\right)}_{(1)} + {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\dot{\mathbf{v}}_{O_1/E} + \underbrace{\left({}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\boldsymbol{\omega}_{b_1/n} \times {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{v}_{O_1/E}\right) \times {}^{b_2}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1}}_{(2)} $$

Inverse Dynamics of 2-Link Arm

- Acceleration of b$_2$-frame

Velocity of {b$_2$}

$$ {}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} $$

$$ {}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E} $$

Acceleration of {b$_2$}

the time derivative of Velocity of {b$_2$}

$$ {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + \underline{{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{2\;b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}} $$

$$ {}^{b_2}\dot{\mathbf{v}}_{O_2/E} = -{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\mathbf{r}_{O_2/O_1}\times{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\mathbf{v}}_{O_1/E} + \left({}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}\right)\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} $$

$$ {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + \underline{\overbrace{\left({}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}\right)}^{{}^{b_2}\boldsymbol{\omega}_{b_2/n}}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}} $$

$$ {}^{b_2}\dot{\mathbf{v}}_{O_2/E} = -{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\mathbf{r}_{O_2/O_1}\times{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\mathbf{v}}_{O_1/E} + \underbrace{\left({}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}\right)}_{{}^{2}\mathbf{v}_{2/0}}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} $$

$$ {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\boldsymbol{\omega}_{b_2/n}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} $$

$$ {}^{b_2}\dot{\mathbf{v}}_{O_2/E} = -{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\mathbf{r}_{O_2/O_1}\times{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\mathbf{v}}_{O_1/E} + {}^{b_2}\mathbf{v}_{O_2/E}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} $$

Inverse Dynamics of 2-Link Arm

- Acceleration of b_2-frame

Acceleration of $\{b_2\}$

$$
\begin{aligned}
{}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} &= {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\boldsymbol{\omega}_{b_2/n}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} \\
{}^{b_2}\dot{\mathbf{v}}_{O_2/E} &= -{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\mathbf{r}_{O_2/O_1}\times{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\mathbf{v}}_{O_1/E} + {}^{b_2}\mathbf{v}_{O_2/E}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}
\end{aligned}
$$

$$
\begin{bmatrix} {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} \\ {}^{b_2}\dot{\mathbf{v}}_{O_2/E} \end{bmatrix} = \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_1} & 0 \\ {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}\times & {}^{b_2}\mathbf{R}_{b_1} \end{bmatrix} \begin{bmatrix} {}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} \\ {}^{b_1}\dot{\mathbf{v}}_{O_1/E} \end{bmatrix} + \begin{bmatrix} {}^{b_2}\mathbf{k}_{b_2} \\ 0 \end{bmatrix}\ddot{\theta}_{b_2/b_1} + \begin{bmatrix} {}^{b_2}\dot{\mathbf{k}}_{b_2} \\ 0 \end{bmatrix}\dot{\theta}_{b_2/b_1} + \begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n}\times & 0 \\ {}^{b_2}\mathbf{v}_{O_2/E}\times & {}^{b_2}\boldsymbol{\omega}_{b_2/n}\times \end{bmatrix}\begin{bmatrix} {}^{b_2}\mathbf{k}_{b_2} \\ 0 \end{bmatrix}\dot{\theta}_{b_2/b_1}
$$

$$
{}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2}\cdot\ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times\mathbf{S}_{b_2}\cdot\dot{q}_2
$$

where

$$
{}^{b_2}\hat{\mathbf{a}}_{b_2} = \begin{bmatrix} {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} \\ {}^{b_2}\dot{\mathbf{v}}_{O_2/E} \end{bmatrix},\ {}^{b_2}\mathbf{X}_{b_1} = \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_1} & 0 \\ {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}\times & {}^{b_2}\mathbf{R}_{b_1} \end{bmatrix},\ \mathbf{S}_{b_2} = \begin{bmatrix} {}^{b_2}\mathbf{k}_{b_2} \\ 0 \end{bmatrix},\ q_2 = \theta_{b_2/b_1},\ {}^{b_2}\hat{\mathbf{v}}_{b_2}\times = \begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{O_2/E} \end{bmatrix}\times = \begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n}\times & 0 \\ {}^{b_2}\mathbf{v}_{O_2/E}\times & {}^{b_2}\boldsymbol{\omega}_{b_2/n}\times \end{bmatrix}
$$

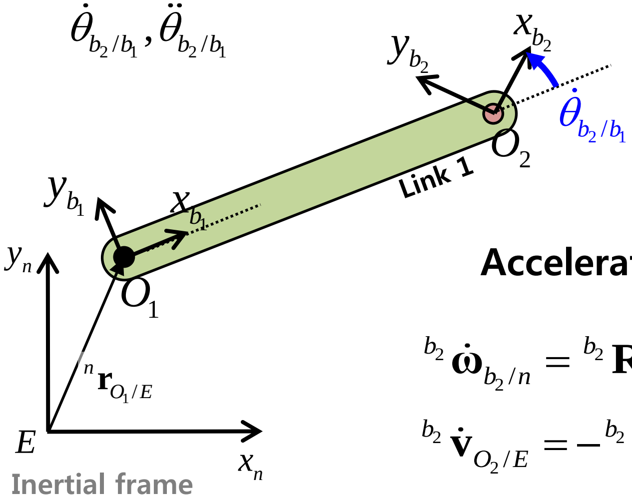
$\overset{\circ}{\mathrm{S}}_{b_2}$ can be regarded as the apparent rates of change of S_{b_2} as viewed by an observer having a velocity of ${}^{b_2}\hat{\mathbf{v}}_{b_2}$

Inverse Dynamics of 2-Link Arm

- Acceleration of b_{G2}-frame

Given: Acceleration of $\{b_1\}$

$\dot{\theta}_{b_2/b_1}, \ddot{\theta}_{b_2/b_1}$



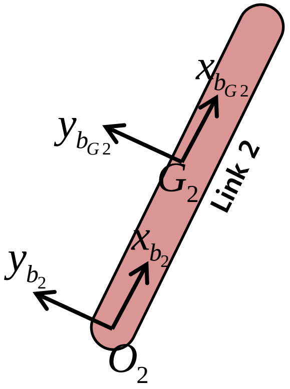
Acceleration of $\{b_2\}$

$$^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\ddot{\theta}_{b_2/b_1} + {}^{b_2}\dot{\mathbf{k}}_{b_2}\cdot\dot{\theta}_{b_2/b_1} + {}^{b_2}\boldsymbol{\omega}_{b_2/n}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

$$^{b_2}\dot{\mathbf{v}}_{O_2/E} = -{}^{b_2}\mathbf{R}_{b_1}\left({}^{b_1}\mathbf{r}_{O_2/O_1}\times{}^{b_1}\dot{\boldsymbol{\omega}}_{b_1/n}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\dot{\mathbf{v}}_{O_1/E} + {}^{b_2}\mathbf{v}_{O_2/E}\times{}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

Given: Acceleration of $\{b_2\}$

$\dot{\theta}_{b_{G2}/b_2}, \ddot{\theta}_{b_{G2}/b_2} = 0$

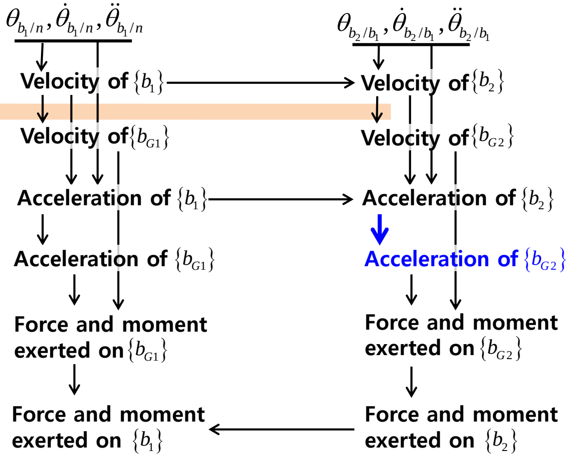


Acceleration of $\{b_{G2}\}$ can be calculated by using abovementioned equations.

$$^{b_{G2}}\dot{\boldsymbol{\omega}}_{b_{G2}/n} = {}^{b_{G2}}\mathbf{R}_{b_2}\cdot{}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} + \cancel{{}^{b_{G2}}\mathbf{k}_{b_{G2}}\cdot\ddot{\theta}_{b_{G2}/b_2}} + \cancel{{}^{b_{G2}}\dot{\mathbf{k}}_{b_{G2}}\cdot\dot{\theta}_{b_{G2}/b_2}} + {}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}\times\cancel{{}^{b_{G2}}\mathbf{k}_{b_{G2}}\cdot\dot{\theta}_{b_{G2}/b_2}}$$

$$^{b_{G2}}\dot{\mathbf{v}}_{G_2/E} = -{}^{b_{G2}}\mathbf{R}_{b_2}\left({}^{b_2}\mathbf{r}_{G_2/O_2}\times{}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n}\right) + {}^{b_{G2}}\mathbf{R}_{b_2}\cdot{}^{b_2}\dot{\mathbf{v}}_{O_2/E} + {}^{b_{G2}}\mathbf{v}_{G_2/E}\times{}^{b_{G2}}\cancel{\mathbf{k}_{b_{G2}}\cdot\dot{\theta}_{b_{G2}/b_2}}$$

Inverse Dynamics of 2-Link Arm
- Acceleration of b_{G2}-frame



$\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \ddot{\theta}_{b_1/n}$

Velocity of $\{b_1\}$

Velocity of $\{b_{G1}\}$

Acceleration of $\{b_1\}$

Acceleration of $\{b_{G1}\}$

Force and moment exerted on $\{b_{G1}\}$

Force and moment exerted on $\{b_1\}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \ddot{\theta}_{b_2/b_1}$

Velocity of $\{b_2\}$

Velocity of $\{b_{G2}\}$

Acceleration of $\{b_2\}$

Acceleration of $\{b_{G2}\}$

Force and moment exerted on $\{b_{G2}\}$

Force and moment exerted on $\{b_2\}$

Acceleration of $\{b_{G2}\}$

$${}^{b_{G2}}\dot{\boldsymbol{\omega}}_{b_{G2}/n} = {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} + {}^{b_{G2}}\cancel{\mathbf{k}_{b_{G2}} \cdot \ddot{\theta}_{b_{G2}/b_2}} + {}^{b_{G2}}\cancel{\dot{\mathbf{k}}_{b_{G2}} \cdot \dot{\theta}_{b_{G2}/b_2}} + {}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n} \times {}^{b_{G2}}\cancel{\mathbf{k}_{b_{G2}} \cdot \dot{\theta}_{b_{G2}/b_2}}$$

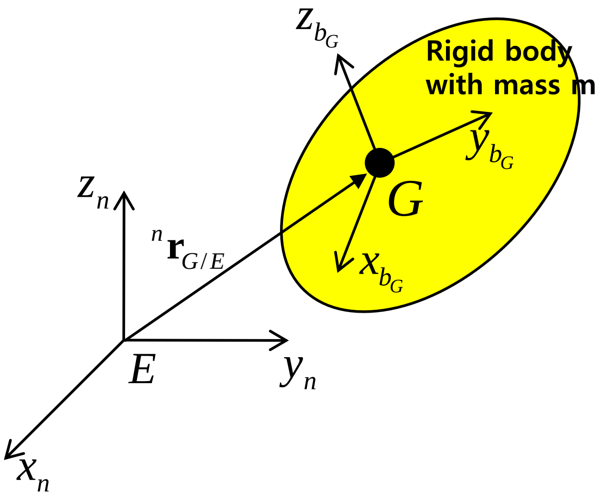
$${}^{b_{G2}}\dot{\mathbf{v}}_{G_2/E} = -{}^{b_{G2}}\mathbf{R}_{b_2}\left({}^{b_2}\mathbf{r}_{G_2/O_2} \times {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n}\right) + {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\dot{\mathbf{v}}_{O_2/E} + {}^{b_{G2}}\mathbf{v}_{G_2/E} \times {}^{b_{G2}}\cancel{\mathbf{k}_{b_{G2}} \cdot \dot{\theta}_{b_{G2}/b_2}}$$

$$\boxed{\begin{aligned} {}^{b_{G2}}\dot{\boldsymbol{\omega}}_{b_{G2}/n} &= {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} \\ {}^{b_{G2}}\dot{\mathbf{v}}_{G_2/E} &= -{}^{b_{G2}}\mathbf{R}_{b_2}\left({}^{b_2}\mathbf{r}_{G_2/O_2} \times {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n}\right) + {}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\dot{\mathbf{v}}_{O_2/E} \end{aligned}}$$

$$\begin{bmatrix} {}^{b_{G2}}\dot{\boldsymbol{\omega}}_{b_{G2}/n} \\ {}^{b_{G2}}\dot{\mathbf{v}}_{G_2/E} \end{bmatrix} = \begin{bmatrix} {}^{b_{G2}}\mathbf{R}_{b_2} & 0 \\ -{}^{b_{G2}}\mathbf{R}_{b_2} \cdot {}^{b_2}\mathbf{r}_{G_2/O_2} \times & {}^{b_{G2}}\mathbf{R}_{b_2} \end{bmatrix} \begin{bmatrix} {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} \\ {}^{b_2}\dot{\mathbf{v}}_{O_2/E} \end{bmatrix} \Rightarrow {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2}$$

Inverse Dynamics of 2-Link Arm

- Force exerted on the b_{G2}-frame



$n-frame$: Inertial Reference Frame

$b_G-frame$: Body fixed frame. Origin of b-frame G is on center of mass.

In accordance with Newton's 2nd law $\quad {}^{n}\mathbf{F}_G = m \cdot {}^{n}\mathbf{a}_{G/E}$

We want to calculate ${}^{b_G}\mathbf{F}_G$ with given variables $\boxed{{}^{b_G}\boldsymbol{\omega}_{b/n}, {}^{b_G}\mathbf{v}_{G/E}}, \boxed{\frac{d}{dt}\left({}^{b_G}\boldsymbol{\omega}_{b/n}\right), \frac{d}{dt}\left({}^{b_G}\mathbf{v}_{G/E}\right)}.$

How we can use the Newton's 2nd law to calculate ${}^{b_G}\mathbf{F}_G$?

Inverse Dynamics of 2-Link Arm

- Force exerted on the b_{G2}-frame

Given: $\boxed{{}^{b_G}\boldsymbol{\omega}_{b_G/n}}, \boxed{{}^{b_G}\mathbf{v}_{G/E}}, \frac{d}{dt}\left({}^{b_G}\boldsymbol{\omega}_{b_G/n}\right), \boxed{\frac{d}{dt}\left({}^{b_G}\mathbf{v}_{G/E}\right)}$

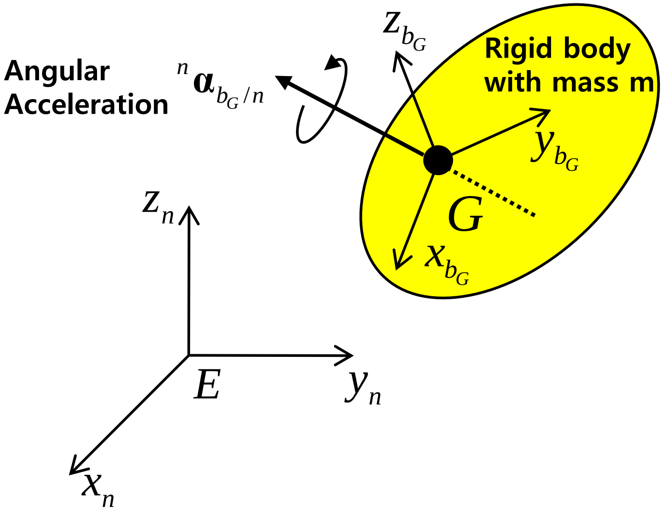
$$\frac{d}{dt}\left({}^{b_G}\boldsymbol{\omega}_{b/n}\right) = {}^{b_G}\dot{\boldsymbol{\omega}}_{b/n}, \frac{d}{dt}\left({}^{b_G}\mathbf{v}_{G/E}\right) = {}^{b_G}\dot{\mathbf{v}}_{G/E}$$

Find: $\boxed{{}^{b_G}\mathbf{F}_G}$

Position Vector	${}^{n}\mathbf{r}_{G/E}$	${}^{n}\mathbf{r}_{G/E}$
Absolute Linear Velocity Vector	$\frac{d}{dt}\,{}^{n}\mathbf{r}_{G/E} = {}^{n}\mathbf{v}_{G/E}$	$\frac{d}{dt}\,{}^{n}\mathbf{r}_{G/E} = {}^{n}\mathbf{v}_{G/E} = {}^{n}\mathbf{R}_{b_G}\cdot{}^{b_G}\mathbf{v}_{G/E}$
Absolute Linear Acceleration Vector	$\frac{d}{dt}\,{}^{n}\mathbf{v}_{G/E} = {}^{n}\mathbf{a}_{G/E}$	$\frac{d}{dt}\left({}^{n}\mathbf{R}_{b_G}\cdot{}^{b_G}\mathbf{v}_{G/E}\right) = \frac{d}{dt}\left({}^{n}\mathbf{R}_{b_G}\right)\cdot{}^{b_G}\mathbf{v}_{G/E} + {}^{n}\mathbf{R}_{b_G}\cdot\frac{d}{dt}\left({}^{b_G}\mathbf{v}_{G/E}\right)$ $= {}^{E}\boldsymbol{\omega}_{b_G/n}\times{}^{n}\mathbf{R}_{b_G}\cdot{}^{G}\mathbf{v}_{G/E} + {}^{n}\mathbf{R}_{b_G}\cdot{}^{b_G}\dot{\mathbf{v}}_{G/E}$
Force Vector (decomposed in n-frame)	${}^{n}\mathbf{F}_G = m\cdot{}^{n}\mathbf{a}_{G/E}$	${}^{n}\mathbf{F}_G = m\left({}^{n}\boldsymbol{\omega}_{b_G/n}\times{}^{n}\mathbf{R}_{b_G}\cdot{}^{b_G}\mathbf{v}_{G/E} + {}^{n}\mathbf{R}_{b_G}\cdot{}^{b_G}\dot{\mathbf{v}}_{G/E}\right)$
Force Vector (decomposed in b_G-frame)	${}^{b_G}\mathbf{R}_n\cdot{}^{n}\mathbf{F}_G = m\cdot{}^{b_G}\mathbf{R}_n\cdot{}^{n}\mathbf{a}_{G/E}$ ${}^{b_G}\mathbf{F}_G = m\cdot{}^{b_G}\mathbf{a}_{G/E}$	${}^{b_G}\mathbf{R}_n\cdot{}^{n}\mathbf{F}_G = m\cdot{}^{b_G}\mathbf{R}_n\cdot\left({}^{n}\boldsymbol{\omega}_{b_G/n}\times{}^{n}\mathbf{R}_{b_G}\cdot{}^{b_G}\mathbf{v}_{G/E} + {}^{n}\mathbf{R}_{b_G}\cdot{}^{b_G}\dot{\mathbf{v}}_{G/E}\right)$ $\boxed{\boxed{{}^{b_G}\mathbf{F}_G} = m\cdot\left(\boxed{{}^{b_G}\boldsymbol{\omega}_{b_G/n}}\times\boxed{{}^{b_G}\mathbf{v}_{G/E}}+\boxed{{}^{b_G}\dot{\mathbf{v}}_{G/E}}\right)}$

Inverse Dynamics of 2-Link Arm

- Moment about certain axis through point G_2



$n-frame$: Inertial Reference Frame

$b_G - frame$: Body fixed frame. Origin of b-frame G is on center of mass.

In accordance with Newton-Euler equation $${}^{n}\mathbf{M}_G = {}^{n}\mathbf{I}_G \cdot {}^{n}\boldsymbol{\alpha}_{b_G/n} + {}^{n}\boldsymbol{\omega}_{b_G/n} \times {}^{n}\mathbf{I}_G \cdot {}^{n}\boldsymbol{\omega}_{b_G/n}$$

We want to calculate ${}^{b_G}\mathbf{M}_G$ with given variables $\boxed{{}^{b_G}\boldsymbol{\omega}_{b_G/n}}, \boxed{\frac{d}{dt}\left({}^{b_G}\boldsymbol{\omega}_{b_G/n}\right)}$.

How we can use the Newton-Euler equation to calculate ${}^{b_G}\mathbf{M}_G$?

Inverse Dynamics of 2-Link Arm

- Moment about certain axis through point G_2

Given: $\boxed{{}^{b_G}\boldsymbol{\omega}_{b_G/n}}, \boxed{\frac{d}{dt}\left({}^{b_G}\boldsymbol{\omega}_{b_G/n}\right)}$

Find: $\boxed{{}^{b_G}\mathbf{M}_G}$

$$\frac{d}{dt}\left({}^{b_G}\boldsymbol{\omega}_{b/n}\right) = {}^{b_G}\dot{\boldsymbol{\omega}}_{b/n}$$

Absolute Angular Velocity Vector

$${}^{n}\boldsymbol{\omega}_{b_G/n}$$

$${}^{n}\boldsymbol{\omega}_{b_G/n} = {}^{n}\mathbf{R}_{b_G} \cdot {}^{b_G}\boldsymbol{\omega}_{b_G/n}$$

Absolute Angular Acceleration Vector

$$\frac{d}{dt}\left({}^{n}\boldsymbol{\omega}_{b_G/n}\right) = {}^{n}\boldsymbol{\alpha}_{b_G/n}$$

$$\begin{aligned}\frac{d}{dt}\left({}^{n}\mathbf{R}_{b_G} \cdot {}^{b_G}\boldsymbol{\omega}_{b_G/n}\right) &= \frac{d}{dt}\left({}^{n}\mathbf{R}_{b_G}\right) \cdot {}^{b_G}\boldsymbol{\omega}_{b_G/n} + {}^{n}\mathbf{R}_{b_G} \cdot \frac{d}{dt}\left({}^{b_G}\boldsymbol{\omega}_{b_G/n}\right)\\ &= {}^{b_G}\boldsymbol{\omega}_{b_G/n} \times {}^{n}\mathbf{R}_{b_G} \cdot {}^{b_G}\boldsymbol{\omega}_{b_G/n} + {}^{n}\mathbf{R}_{b_G} \cdot {}^{b_G}\dot{\boldsymbol{\omega}}_{b_G/n}\\ &= {}^{b_G}\boldsymbol{\omega}_{b_G/n} \times {}^{n}\boldsymbol{\omega}_{b_G/n} + {}^{n}\mathbf{R}_{b_G} \cdot {}^{b_G}\dot{\boldsymbol{\omega}}_{b_G/n}\\ &= {}^{n}\mathbf{R}_{b_G} \cdot {}^{b_G}\dot{\boldsymbol{\omega}}_{b_G/n}\end{aligned}$$

Moment Vector
(decomposed in n-frame)

$$\begin{aligned}{}^{\boxed{n}}\mathbf{M}_G &= {}^{n}\mathbf{I}_G \cdot {}^{n}\boldsymbol{\alpha}_{b_G/n}\\ &+ {}^{n}\boldsymbol{\omega}_{b_G/n} \times {}^{n}\mathbf{I}_G \cdot {}^{n}\boldsymbol{\omega}_{b_G/n}\end{aligned}$$

$$\begin{aligned}{}^{\boxed{n}}\mathbf{M}_G &= {}^{n}\mathbf{I}_G \cdot {}^{n}\mathbf{R}_{b_G} \cdot {}^{b_G}\dot{\boldsymbol{\omega}}_{b_G/n}\\ &+ {}^{n}\mathbf{R}_{b_G} \cdot {}^{b_G}\boldsymbol{\omega}_{b_G/n} \times {}^{n}\mathbf{I}_G \cdot {}^{n}\mathbf{R}_{b_G} \cdot {}^{b_G}\boldsymbol{\omega}_{b_G/n}\end{aligned}$$

Moment Vector
(decomposed in b_G-frame)

$$\begin{aligned}{}^{\boxed{b_G}}\mathbf{M}_G &= {}^{b_G}\mathbf{I}_G \cdot {}^{b_G}\boldsymbol{\alpha}_{b_G/n}\\ &+ {}^{b_G}\boldsymbol{\omega}_{b_G/n} \times {}^{b_G}\mathbf{I}_G \cdot {}^{b_G}\boldsymbol{\omega}_{b_G/n}\end{aligned}$$

$$\boxed{{}^{\boxed{b_G}}\mathbf{M}_G} = {}^{b_G}\mathbf{I}_G \cdot \boxed{{}^{b_G}\dot{\boldsymbol{\omega}}_{b_G/n}} + \boxed{{}^{b_G}\boldsymbol{\omega}_{b_G/n}} \times {}^{b_G}\mathbf{I}_G \cdot \boxed{{}^{b_G}\boldsymbol{\omega}_{b_G/n}}$$

Inverse Dynamics of 2-Link Arm

- Force and moment exerted on G_2

Given: $\boxed{{}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}}$, $\boxed{{}^{b_{G2}}\mathbf{v}_{G_2/E}}$, $\boxed{\dfrac{d}{dt}\left({}^{b_{G2}}\boldsymbol{\omega}_{b_{G_2}/n}\right)}$, $\boxed{\dfrac{d}{dt}\left({}^{b_{G2}}\mathbf{v}_{G_2/E}\right)}$

Find: $\boxed{{}^{b_{G2}}\mathbf{F}_{G_2}}$ $\boxed{{}^{b_{G2}}\mathbf{M}_{G_2}}$

$$\boxed{{}^{b_{G2}}\mathbf{M}_{G_2}} = {}^{b_{G2}}\mathbf{I}_{G_2}\cdot\boxed{{}^{b_{G2}}\dot{\boldsymbol{\omega}}_{b_{G2}/n}} + \boxed{{}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}} \times {}^{b_{G2}}\mathbf{I}_{G_2}\cdot\boxed{{}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}}$$

$$\boxed{{}^{b_{G2}}\mathbf{F}_{G_2}} = m_2\cdot\left(\boxed{{}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}}\times\boxed{{}^{b_{G2}}\mathbf{v}_{G_2/E}} + \boxed{{}^{b_{G2}}\dot{\mathbf{v}}_{G_2/E}}\right)$$

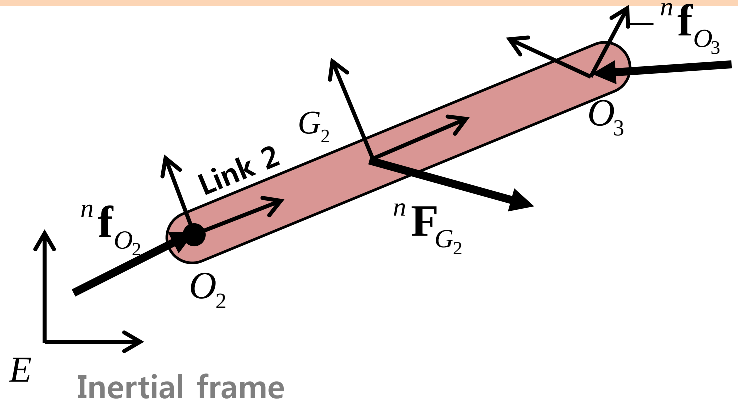
$$\boxed{\begin{aligned}{}^{b_{G2}}\mathbf{M}_{G_2} &= {}^{b_{G2}}\mathbf{I}_{G_2}\cdot{}^{b_{G2}}\dot{\boldsymbol{\omega}}_{b_{G2}/n} + {}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}\times{}^{b_{G2}}\mathbf{I}_{G_2}\cdot{}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}\\ {}^{b_{G2}}\mathbf{F}_{G_2} &= m_2\cdot{}^{b_{G2}}\dot{\mathbf{v}}_{G_2/E} + m_2\cdot{}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}\times{}^{b_{G2}}\mathbf{v}_{G_2/E}\end{aligned}}$$

$$\begin{bmatrix}{}^{b_{G2}}\mathbf{M}_{G_2}\\ {}^{b_{G2}}\mathbf{F}_{G_2}\end{bmatrix} = \begin{bmatrix}{}^{b_{G2}}\mathbf{I}_{G_2} & 0\\ 0 & m_2\cdot\mathbf{1}\end{bmatrix}\begin{bmatrix}{}^{b_{G2}}\dot{\boldsymbol{\omega}}_{b_{G2}/n}\\ {}^{b_{G2}}\dot{\mathbf{v}}_{G_2/E}\end{bmatrix} + \begin{bmatrix}{}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}\times & {}^{b_{G2}}\mathbf{v}_{G_2/E}\times\\ 0 & {}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}\times\end{bmatrix}\begin{bmatrix}{}^{b_{G2}}\mathbf{I}_{G_2} & 0\\ 0 & m_2\cdot\mathbf{1}\end{bmatrix}\begin{bmatrix}{}^{b_{G2}}\boldsymbol{\omega}_{b_{G2}/n}\\ {}^{b_{G2}}\mathbf{v}_{G_2/E}\end{bmatrix}\quad ,where\ \mathbf{1}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$${}^{b_{G2}}\hat{\mathbf{f}}^{B}_{G_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}\times^{*}{}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}$$

Inverse Dynamics of 2-Link Arm

- Force and moment exerted on O_2



${}^{n}\mathbf{F}_{G_2}$: Resultant Force exerted on G_2

${}^{n}\mathbf{f}_{O_2}$: Force exerted on O_2 by link 1

$-{}^{n}\mathbf{f}_{O_3}$: Reaction Force exerted on O_3 by link 3

$${}^{n}\mathbf{F}_{G_2} = {}^{n}\mathbf{f}_{O_2} - {}^{n}\mathbf{f}_{O_3}$$

$${}^{n}\mathbf{f}_{O_2} = {}^{n}\mathbf{F}_{G_2} + {}^{n}\mathbf{f}_{O_3}$$

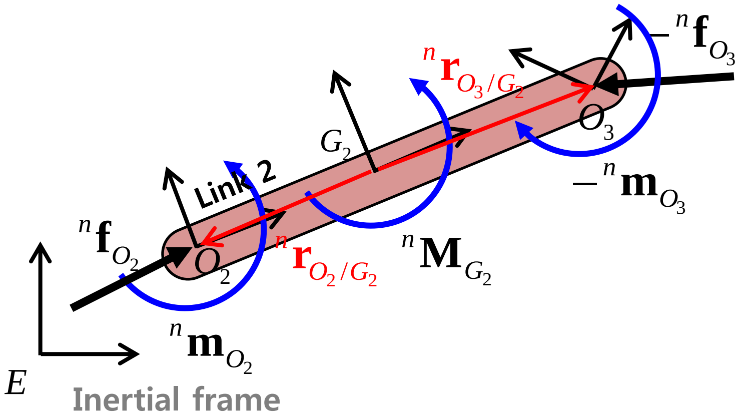
$${}^{b_2}\mathbf{R}_{n} \cdot {}^{n}\mathbf{f}_{O_2} = {}^{b_2}\mathbf{R}_{n} \cdot {}^{n}\mathbf{F}_{G_2} + {}^{b_2}\mathbf{R}_{n} \cdot {}^{n}\mathbf{f}_{O_3}$$

$${}^{b_2}\mathbf{f}_{O_2} = {}^{b_2}\mathbf{F}_{G_2} + {}^{b_2}\mathbf{f}_{O_3}$$

$$\boxed{{}^{b_2}\mathbf{f}_{O_2} = {}^{b_2}\mathbf{R}_{b_{G2}} \cdot {}^{b_{G2}}\mathbf{F}_{G_2} + {}^{b_2}\mathbf{R}_{b_3} \cdot {}^{b_3}\mathbf{f}_{O_3}}$$

Inverse Dynamics of 2-Link Arm

- Force and moment exerted on O_2



${}^{n}\mathbf{M}_{G_2}$: Resultant moment about certain axis through point G_2

${}^{n}\mathbf{f}_{O_2}$: Force exerted on O_2 by link 1

${}^{n}\mathbf{m}_{O_2}$: Moment about certain axis through point O_2 exerted on link 2 by link 1

$-{}^{n}\mathbf{f}_{O_3}$: Reaction Force exerted on O_3 from link 3

$-{}^{n}\mathbf{m}_{O_3}$: Reaction Moment about certain axis through point O_3 exerted on link 2 by link 3

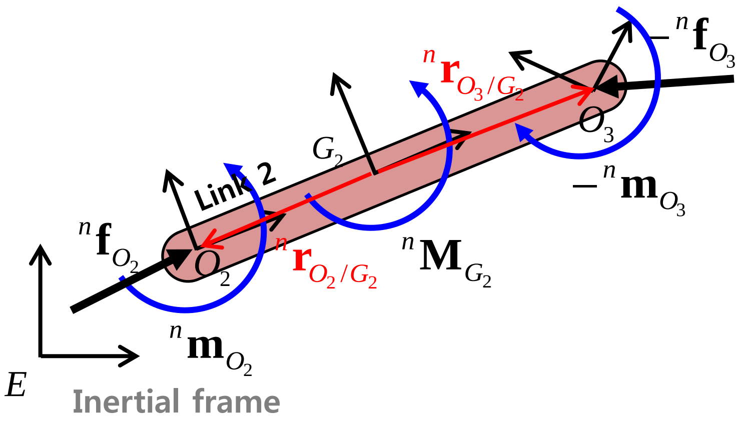
$${}^{n}\mathbf{M}_{G_2} = {}^{n}\mathbf{m}_{O_2} \underline{+{}^{n}\mathbf{r}_{O_2/G_2} \times {}^{n}\mathbf{f}_{O_2}} - {}^{n}\mathbf{m}_{O_3} \underline{-{}^{n}\mathbf{r}_{O_3/G_2} \times {}^{n}\mathbf{f}_{O_3}}$$

Moment about certain axis
through point G_2
caused by ${}^{n}\mathbf{f}_{O_2}$

Moment about certain axis
through point G_2
caused by $-{}^{n}\mathbf{f}_{O_3}$

Inverse Dynamics of 2-Link Arm

- Force and moment exerted on O_2



$${}^n\mathbf{M}_{G_2} = {}^n\mathbf{m}_{O_2} + {}^n\mathbf{r}_{O_2/G_2} \times {}^n\mathbf{f}_{O_2} - {}^n\mathbf{m}_{O_3} - {}^n\mathbf{r}_{O_3/G_2} \times {}^n\mathbf{f}_{O_3}$$

$${}^n\mathbf{m}_{O_2} = {}^n\mathbf{M}_{G_2} - {}^n\mathbf{r}_{O_2/G_2} \times {}^n\mathbf{f}_{O_2} + {}^n\mathbf{m}_{O_3} + {}^n\mathbf{r}_{O_3/G_2} \times {}^n\mathbf{f}_{O_3}$$

$${}^0\mathbf{f}_2 = {}^0\mathbf{F}_{G_2} + {}^0\mathbf{f}_3$$

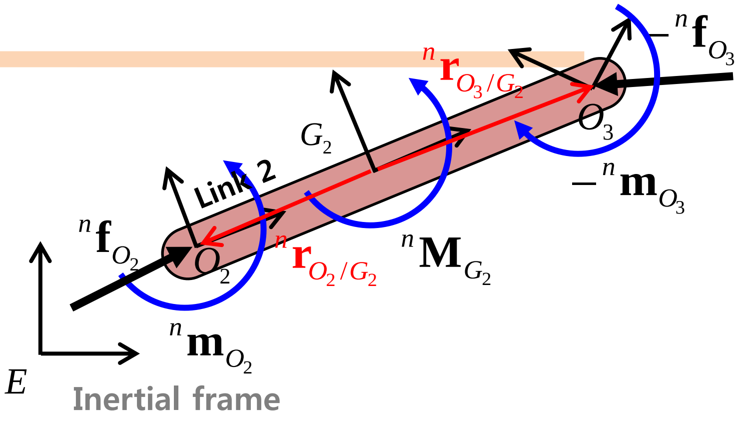
$${}^n\mathbf{m}_{O_2} = {}^n\mathbf{M}_{G_2} \underline{- {}^n\mathbf{r}_{O_2/G_2} \times \left({}^n\mathbf{F}_{G_2} + {}^n\mathbf{f}_{O_3} \right)} + {}^n\mathbf{m}_{O_3} + {}^n\mathbf{r}_{O_3/G_2} \times {}^n\mathbf{f}_{O_3}$$

$${}^n\mathbf{m}_{O_2} = {}^n\mathbf{M}_{G_2} \underline{- {}^n\mathbf{r}_{O_2/G_2} \times {}^n\mathbf{F}_{G_2} \underbrace{- {}^n\mathbf{r}_{O_2/G_2} \times {}^n\mathbf{f}_{O_3}}_{(1)}} + \underbrace{{}^n\mathbf{m}_{O_3}}_{(2)} + {}^n\mathbf{r}_{O_3/G_2} \times {}^n\mathbf{f}_{O_3}$$

$${}^n\mathbf{m}_{O_2} = {}^n\mathbf{M}_{G_2} - {}^n\mathbf{r}_{O_2/G_2} \times {}^n\mathbf{F}_{G_2} + \underbrace{{}^n\mathbf{m}_{O_3}}_{(2)} \underbrace{- {}^n\mathbf{r}_{O_2/G_2} \times {}^n\mathbf{f}_{O_3}}_{(1)} + {}^n\mathbf{r}_{O_3/G_2} \times {}^n\mathbf{f}_{O_3}$$

Inverse Dynamics of 2-Link Arm

- Force and moment exerted on O_2



$${}^n\mathbf{m}_{O_2} = {}^n\mathbf{M}_{G_2} - {}^n\mathbf{r}_{O_2/G_2}\times{}^n\mathbf{F}_{G_2} + {}^n\mathbf{m}_{O_3} \underline{- {}^n\mathbf{r}_{O_2/G_2}\times{}^n\mathbf{f}_{O_3} + {}^n\mathbf{r}_{O_3/G_2}\times{}^n\mathbf{f}_{O_3}}$$

$$\Downarrow$$

$${}^n\mathbf{m}_{O_2} = {}^n\mathbf{M}_{G_2} - {}^n\mathbf{r}_{O_2/G_2}\times{}^n\mathbf{F}_{G_2} + {}^n\mathbf{m}_{O_3} \underline{-\left({}^n\mathbf{r}_{O_2/G_2} \underline{- {}^n\mathbf{r}_{O_3/G_2}}\right)\times{}^n\mathbf{f}_{O_3}}$$

$$\Downarrow$$

$${}^n\mathbf{m}_{O_2} = {}^n\mathbf{M}_{G_2} - {}^n\mathbf{r}_{O_2/G_2}\times{}^n\mathbf{F}_{G_2} + {}^n\mathbf{m}_{O_3} \underline{-\left({}^n\mathbf{r}_{O_2/G_2} \underline{+ {}^n\mathbf{r}_{G_2/O_3}}\right)}\times{}^n\mathbf{f}_{O_3}$$

$$\Downarrow$$

$${}^n\mathbf{m}_{O_2} = {}^n\mathbf{M}_{G_2} - {}^n\mathbf{r}_{O_2/G_2}\times{}^n\mathbf{F}_{G_2} + {}^n\mathbf{m}_{O_3} \underline{- {}^n\mathbf{r}_{O_2/O_3}}\times{}^n\mathbf{f}_{O_3}$$

$$\Downarrow$$

$${}^{b_2}\mathbf{m}_{O_2} = \boxed{{}^{b_2}}\mathbf{M}_{G_2} - \boxed{{}^{b_2}}\mathbf{r}_{O_2/G_2}\times\boxed{{}^{b_2}}\mathbf{F}_{G_2} + \boxed{{}^{b_2}}\mathbf{m}_{O_3} - \boxed{{}^{b_2}}\mathbf{r}_{O_2/O_3}\times\boxed{{}^{b_2}}\mathbf{f}_{O_3}$$

$$\Downarrow$$

$${}^{b_2}\mathbf{m}_{O_2} = \boxed{{}^{b_2}\mathbf{R}_{b_{G2}}\cdot{}^{b_{G2}}}\mathbf{M}_{G_2} \underline{- \boxed{{}^{b_2}\mathbf{R}_{b_{G2}}\cdot{}^{b_{G2}}}\mathbf{r}_{O_2/G_2}\times\boxed{{}^{b_2}\mathbf{R}_{b_{G2}}\cdot{}^{b_{G2}}}\mathbf{F}_{G_2}} + \boxed{{}^{b_2}\mathbf{R}_{b_3}\cdot{}^{b_3}}\mathbf{m}_{O_3} - \boxed{{}^{b_2}\mathbf{R}_{b_3}\cdot{}^{b_3}}\mathbf{r}_{O_2/O_3}\times\boxed{{}^{b_2}\mathbf{R}_{b_3}\cdot{}^{b_3}}\mathbf{f}_{O_3}$$

$$\Downarrow$$

$$\boxed{{}^{b_2}\mathbf{m}_{O_2} = {}^{b_2}\mathbf{R}_{b_{G2}}\cdot{}^{b_{G2}}\mathbf{M}_{G_2} \underline{- {}^{b_2}\mathbf{R}_{b_{G2}}\cdot\left({}^{b_{G2}}\mathbf{r}_{O_2/G_2}\times{}^{b_{G2}}\mathbf{F}_{G_2}\right)} + {}^{b_2}\mathbf{R}_{b_3}\cdot{}^{b_3}\mathbf{m}_{O_3} - {}^{b_2}\mathbf{R}_{b_3}\cdot{}^{b_3}\mathbf{r}_{O_2/O_3}\times{}^{b_2}\mathbf{R}_{b_3}\cdot{}^{b_3}\mathbf{f}_{O_3}}$$

Inverse Dynamics of 2-Link Arm

- Force and moment exerted on O_2

$$\boxed{{}^{b_2}\mathbf{m}_{O_2} = {}^{b_2}\mathbf{R}_{b_{G2}} \cdot {}^{b_{G2}}\mathbf{M}_{G_2} - {}^{b_2}\mathbf{R}_{b_{G2}} \cdot \left({}^{b_{G2}}\mathbf{r}_{O_2/G_2} \times {}^{b_{G2}}\mathbf{F}_{G_2} \right) + {}^{b_2}\mathbf{R}_{b_3} \cdot {}^{b_3}\mathbf{m}_{O_3} - {}^{b_2}\mathbf{R}_{b_3} \cdot {}^{b_3}\mathbf{r}_{O_2/O_3} \times {}^{b_2}\mathbf{R}_{b_3} \cdot {}^{b_3}\mathbf{f}_{O_3}}$$

$$\boxed{{}^{b_2}\mathbf{f}_{O_2} = {}^{b_2}\mathbf{R}_{b_{G2}} \cdot {}^{b_{G2}}\mathbf{F}_{G_2} + {}^{b_2}\mathbf{R}_{b_3} \cdot {}^{b_3}\mathbf{f}_{O_3}}$$

$$\begin{bmatrix} {}^{b_2}\mathbf{m}_{O_2} \\ {}^{b_2}\mathbf{f}_{O_2} \end{bmatrix} = \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_{G2}} & -{}^{b_2}\mathbf{R}_{b_{G2}} \cdot {}^{b_{G2}}\mathbf{r}_{O_2/G_2} \times \\ 0 & {}^{b_2}\mathbf{R}_{b_{G2}} \end{bmatrix} \begin{bmatrix} {}^{b_{G2}}\mathbf{M}_{G_2} \\ {}^{b_{G2}}\mathbf{F}_{G_2} \end{bmatrix} + \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_3} & -{}^{b_2}\mathbf{R}_{b_3} \cdot {}^{b_3}\mathbf{r}_{O_2/O_3} \times \\ 0 & {}^{b_2}\mathbf{R}_{b_3} \end{bmatrix} \begin{bmatrix} {}^{b_3}\mathbf{m}_{O_3} \\ {}^{b_3}\mathbf{f}_{O_3} \end{bmatrix}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{f}}^{B}_{G_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

Inverse Dynamics of 2-Link Arm

- Input torque of joint 2 for link 2

$$\begin{bmatrix} {}^{b_2}\mathbf{m}_{O_2} \\ {}^{b_2}\mathbf{f}_{O_2} \end{bmatrix} = \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_{G2}} & -{}^{b_2}\mathbf{R}_{b_{G2}} \cdot {}^{b_{G2}}\mathbf{r}_{O_2/G_2} \times \\ 0 & {}^{b_2}\mathbf{R}_{b_{G2}} \end{bmatrix} \begin{bmatrix} {}^{b_{G2}}\mathbf{M}_{G_2} \\ {}^{b_{G2}}\mathbf{F}_{G_2} \end{bmatrix} + \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_3} & -{}^{b_2}\mathbf{R}_{b_3} \cdot {}^{b_3}\mathbf{r}_{O_2/O_3} \times \\ 0 & {}^{b_2}\mathbf{R}_{b_3} \end{bmatrix} \begin{bmatrix} {}^{b_3}\mathbf{m}_{O_3} \\ {}^{b_3}\mathbf{f}_{O_3} \end{bmatrix}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{f}}^{B}_{G_2} + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

Input torque of joint 2 for link 2

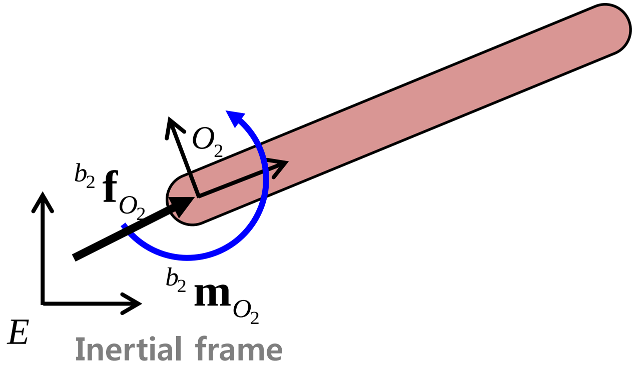
The moment and force exerted on point O$_2$ are calculated. However, what we are interested in is only to moment which can be generated by joint 2.

The moment, which is generated by joint 2, can be evaluated from the scalar product of ${}^{b_2}\mathbf{m}_{O_2}$ and ${}^{b_2}\mathbf{k}_{b_2}$,the revolute axis of joint 2.

$$\tau_2 = {}^{b_2}\mathbf{k}_{b_2}{}^{T} \cdot {}^{b_2}\mathbf{m}_{O_2} + \mathbf{0}^T \cdot {}^{b_2}\mathbf{f}_{O_2}$$

$$\tau_2 = \begin{bmatrix} {}^{b_2}\mathbf{k}_{b_2}{}^{T} & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} {}^{b_2}\mathbf{m}_{O_2} \\ {}^{b_2}\mathbf{f}_{O_2} \end{bmatrix}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$



Inverse Dynamics of 2-Link Arm
- Summary

Given: **Kinematic Model**

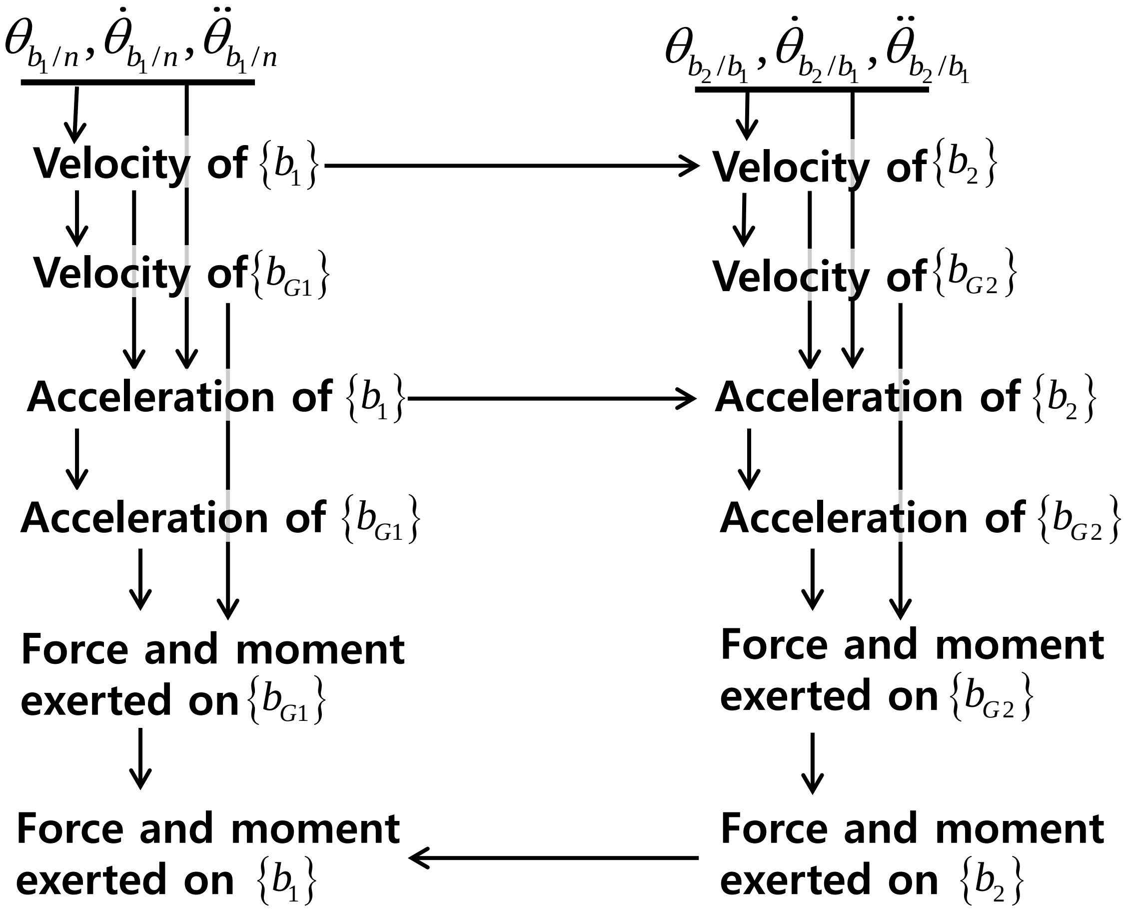
$\theta_{1/0}\ ,\dot{\theta}_{1/0},\ddot{\theta}_{1/0}$

$\theta_{2/1}\ ,\dot{\theta}_{2/1},\ddot{\theta}_{2/1}$

Generalized Coordinates

Find: **input torque for link 1** - τ_1

input torque for link 2 - τ_2



$\theta_{b_1/n},\dot{\theta}_{b_1/n},\ddot{\theta}_{b_1/n}$

Velocity of $\{b_1\}$

Velocity of $\{b_{G1}\}$

Acceleration of $\{b_1\}$

Acceleration of $\{b_{G1}\}$

Force and moment exerted on $\{b_{G1}\}$

Force and moment exerted on $\{b_1\}$

$\theta_{b_2/b_1},\dot{\theta}_{b_2/b_1},\ddot{\theta}_{b_2/b_1}$

Velocity of $\{b_2\}$

$$^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2}\cdot\dot{q}_2$$

Velocity of $\{b_{G2}\}$

$$^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}$$

Acceleration of $\{b_2\}$

$$^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2}\cdot\ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times\mathbf{S}_{b_2}\cdot\dot{q}_2$$

Acceleration of $\{b_{G2}\}$

$$^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2}$$

Force and moment exerted on $\{b_{G2}\}$

$$^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}\times^{*}{}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}$$

Force and moment exerted on $\{b_2\}$

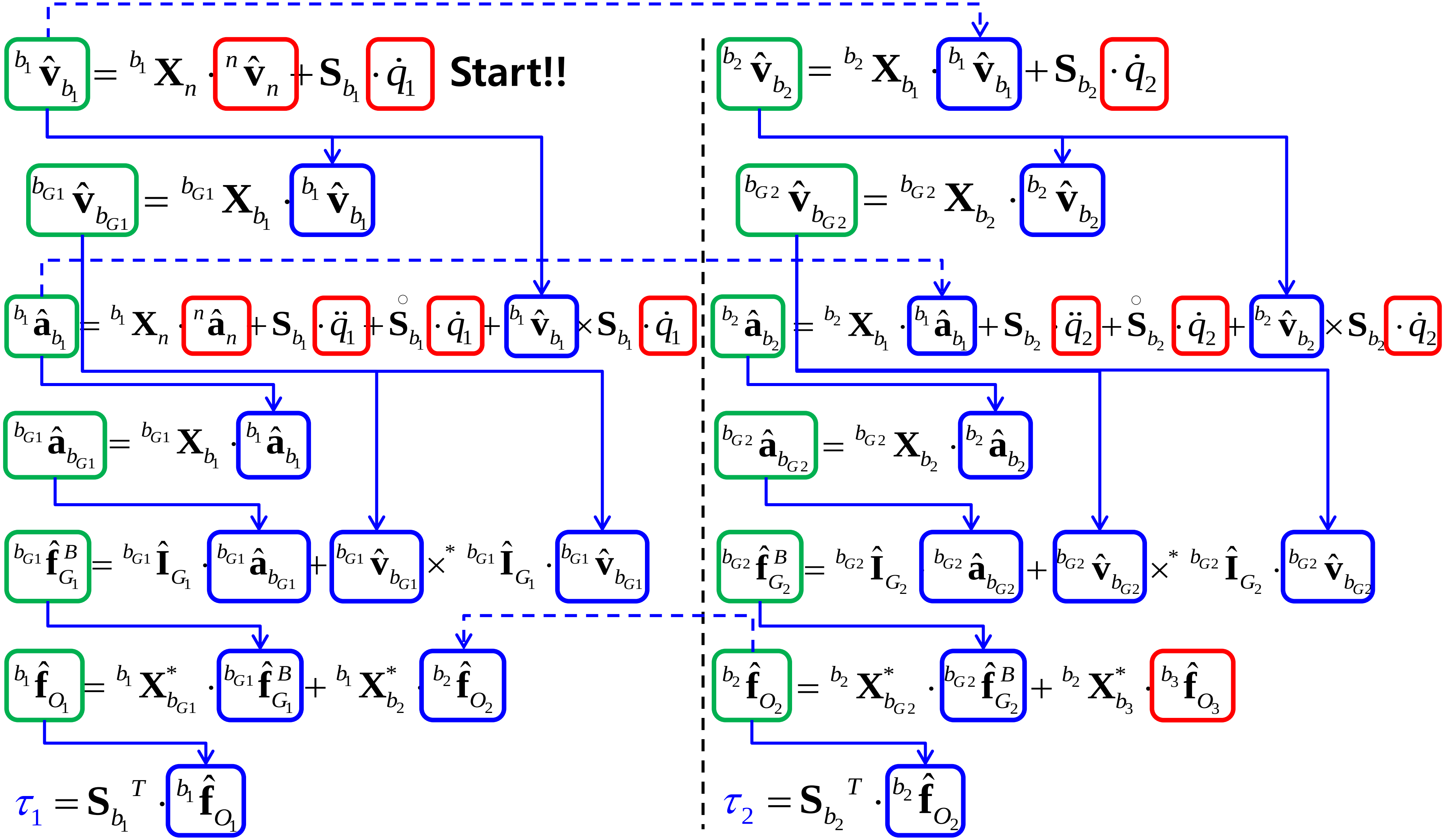
$$^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}}\cdot{}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T}\cdot{}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Inverse Dynamics of 2-Link Arm

- Summary

Given: ${}^{i}\mathbf{X}_{i},\ \mathbf{S}_{i},\ {}^{i}\hat{\mathbf{I}}_{i},\ and\ \boxed{}$ **Find:** τ_2, τ_2



$$\boxed{{}^{b_1}\hat{\mathbf{v}}_{b_1}} = {}^{b_1}\mathbf{X}_{n}\cdot\boxed{{}^{n}\hat{\mathbf{v}}_{n}} + \mathbf{S}_{b_1}\boxed{\cdot\dot{q}_1}$$

Start!!

$$\boxed{{}^{b_{G1}}\hat{\mathbf{v}}_{b_{G1}}} = {}^{b_{G1}}\mathbf{X}_{b_1}\cdot\boxed{{}^{b_1}\hat{\mathbf{v}}_{b_1}}$$

$$\boxed{{}^{b_1}\hat{\mathbf{a}}_{b_1}} = {}^{b_1}\mathbf{X}_{n}\boxed{{}^{n}\hat{\mathbf{a}}_{n}} + \mathbf{S}_{b_1}\boxed{\cdot\ddot{q}_1} + \overset{\circ}{\mathbf{S}}_{b_1}\boxed{\cdot\dot{q}_1} + \boxed{{}^{b_1}\hat{\mathbf{v}}_{b_1}}\times\mathbf{S}_{b_1}\boxed{\cdot\dot{q}_1}$$

$$\boxed{{}^{b_{G1}}\hat{\mathbf{a}}_{b_{G1}}} = {}^{b_{G1}}\mathbf{X}_{b_1}\cdot\boxed{{}^{b_1}\hat{\mathbf{a}}_{b_1}}$$

$$\boxed{{}^{b_{G1}}\hat{\mathbf{f}}^{B}_{G_1}} = {}^{b_{G1}}\hat{\mathbf{I}}_{G_1}\cdot\boxed{{}^{b_{G1}}\hat{\mathbf{a}}_{b_{G1}}} + \boxed{{}^{b_{G1}}\hat{\mathbf{v}}_{b_{G1}}}\times^{*}\ {}^{b_{G1}}\hat{\mathbf{I}}_{G_1}\cdot\boxed{{}^{b_{G1}}\hat{\mathbf{v}}_{b_{G1}}}$$

$$\boxed{{}^{b_1}\hat{\mathbf{f}}_{O_1}} = {}^{b_1}\mathbf{X}^{*}_{b_{G1}}\cdot\boxed{{}^{b_{G1}}\hat{\mathbf{f}}^{B}_{G_1}} + {}^{b_1}\mathbf{X}^{*}_{b_2}\cdot\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2}}$$

$$\tau_1 = \mathbf{S}_{b_1}{}^{T}\cdot\boxed{{}^{b_1}\hat{\mathbf{f}}_{O_1}}$$

$$\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}} = {}^{b_2}\mathbf{X}_{b_1}\cdot\boxed{{}^{b_1}\hat{\mathbf{v}}_{b_1}} + \mathbf{S}_{b_2}\boxed{\cdot\dot{q}_2}$$

$$\boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}}$$

$$\boxed{{}^{b_2}\hat{\mathbf{a}}_{b_2}} = {}^{b_2}\mathbf{X}_{b_1}\cdot\boxed{{}^{b_1}\hat{\mathbf{a}}_{b_1}} + \mathbf{S}_{b_2}\boxed{\ddot{q}_2} + \overset{\circ}{\mathbf{S}}_{b_2}\boxed{\cdot\dot{q}_2} + \boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}}\times\mathbf{S}_{b_2}\boxed{\cdot\dot{q}_2}$$

$$\boxed{{}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot\boxed{{}^{b_2}\hat{\mathbf{a}}_{b_2}}$$

$$\boxed{{}^{b_{G2}}\hat{\mathbf{f}}^{B}_{G_2}} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\boxed{{}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}}} + \boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}}\times^{*}\ {}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot\boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2}} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}}\cdot\boxed{{}^{b_{G2}}\hat{\mathbf{f}}^{B}_{G_2}} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot\boxed{{}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T}\cdot\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2}}$$

5.7 Recursive Newton-Euler Formulation using Spatial Vector
(Forward Dynamics – Propagation Methods)

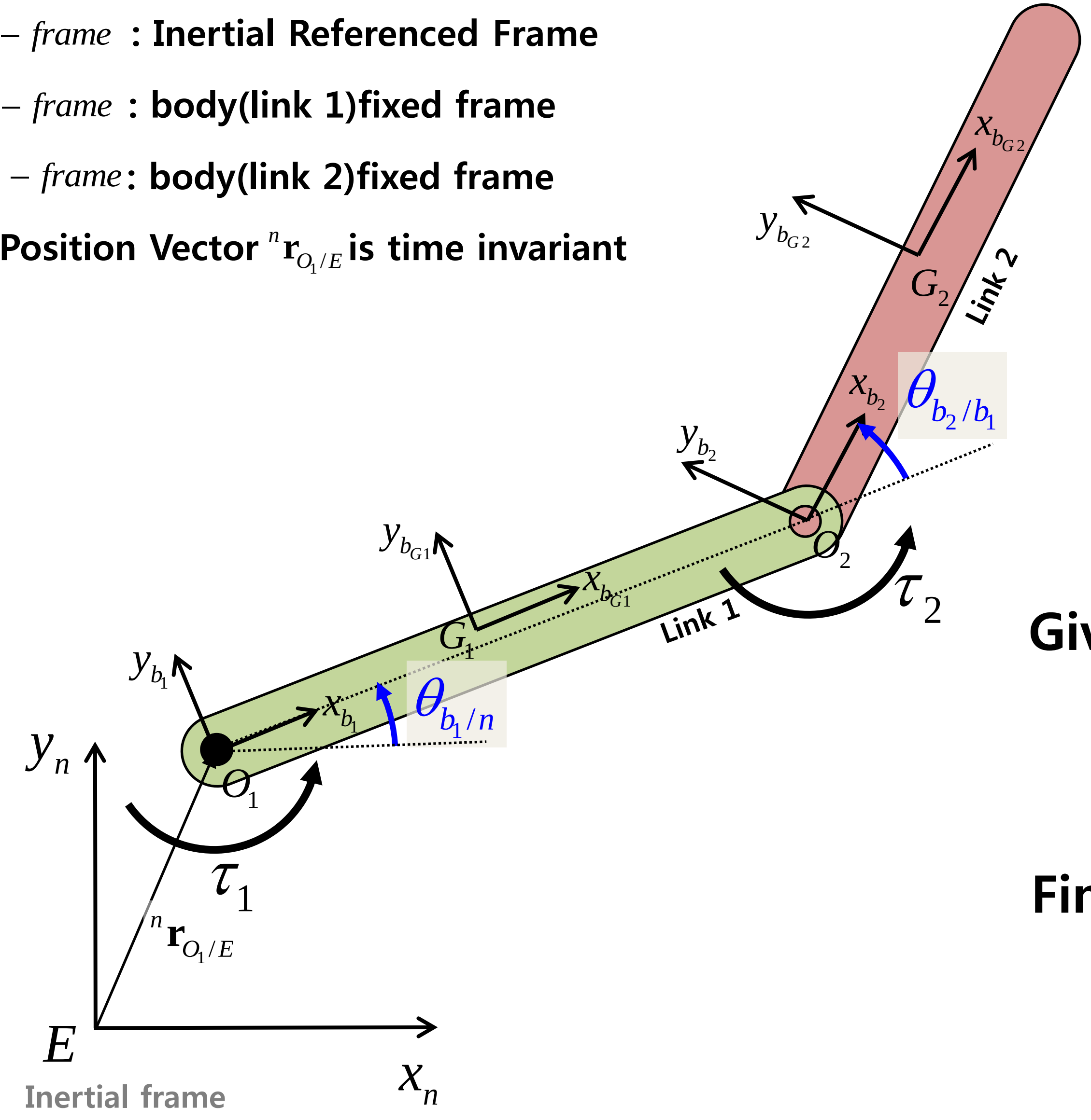
Forward Dynamics - Propagation Methods
- Example of 2 Link Arm

$n-frame$: **Inertial Referenced Frame**

$b_1-frame$: **body(link 1)fixed frame**

$b_2-frame$: **body(link 2)fixed frame**

Position Vector ${}^n\mathbf{r}_{O_1/E}$ **is time invariant**



$$\theta_{b_1/n} = q_1$$
$$\theta_{b_2/b_1} = q_2$$

Generalized Coordinates

Given: **Kinematic Model**

$$\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$$
$$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$$

Find: $\ddot{\theta}_{b_1/n}$

$$\ddot{\theta}_{b_2/b_1}$$

Forward Dynamics - Propagation Methods
- Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2}\cdot\dot{q}_2 \longleftarrow \text{Velocity of}\{b_2\}$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2} \longleftarrow \text{Velocity of}\{b_{G2}\}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2}\cdot\ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times\mathbf{S}_{b_2}\cdot\dot{q}_2 \longleftarrow \text{Acceleration of }\{b_2\}$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2} \longleftarrow \text{Acceleration of }\{b_{G2}\}$$

$${}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}\times^{*}{}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \longleftarrow$$

Force and moment
exerted on $\{b_{G2}\}$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}}\cdot{}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3} \longleftarrow$$

Force and moment
exerted on $\{b_2\}$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T}\cdot{}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2}\cdot\ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times\mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$${}^{b_{G2}}\hat{\mathbf{f}}^{B_2}_{G_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}\times^{*}{}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}}\cdot{}^{b_{G2}}\hat{\mathbf{f}}^{B_2}_{G_2} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T}\cdot{}^{b_2}\hat{\mathbf{f}}_{O_2}$$

To derive the simplified version of the equations, the equations will be manipulated.

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$\boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$${}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} \times^{*} {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}_{b_{G2}}^{*} \cdot {}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} + {}^{b_2}\mathbf{X}_{b_3}^{*} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$\boxed{^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$$^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$$^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \boxed{^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \boxed{^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} \Rightarrow {}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}\right) \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}\right)$$

$$^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^T \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$ {}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2 $$

$$ \boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} $$

$$ {}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2 $$

$$ {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} $$

$$ {}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} \times^{*} {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} \Rightarrow \boxed{{}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2}} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) \times^{*} {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) $$

$$ {}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}_{b_{G2}}^{*} \cdot \boxed{{}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2}} + {}^{b_2}\mathbf{X}_{b_3}^{*} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3} $$

$$ \tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2} $$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$\boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$${}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \boxed{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}} \Rightarrow \boxed{{}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2}} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}\right) \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}\right)$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}_{b_{G2}}^* \cdot \boxed{{}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2}} + {}^{b_2}\mathbf{X}_{b_3}^* \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3} \Rightarrow \boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}_{b_{G2}}^* \cdot \left({}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}\right) \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}\right)\right) + {}^{b_2}\mathbf{X}_{b_3}^* \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^T \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot \left({}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) \right) + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\Downarrow$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot \left({}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_2} \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{v}}_{b_2} \right) + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$$\Downarrow$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot \left({}^{b_{G2}}\hat{\mathbf{v}}_{b_2} \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{v}}_{b_2} \right) + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$$\Downarrow \qquad {}^{b_2}\mathbf{X}^*_{b_{G2}} \left({}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \right) = \left({}^{b_2}\hat{\mathbf{v}}_{b_2} \times^* {}^{b_2}\hat{\mathbf{I}}_{O_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right)$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_2}\hat{\mathbf{v}}_{b_2} \times^* {}^{b_2}\hat{\mathbf{I}}_{O_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot \left({}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) \right) + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \boxed{\cdot} {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_2}\hat{\mathbf{v}}_{b_2} \times^* {}^{b_2}\hat{\mathbf{I}}_{O_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$${}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\mathbf{X}_{b_{G2}} = {}^{b_2}\mathbf{X}_{b_2}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\mathbf{X}_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_2}\hat{\mathbf{v}}_{b_2} \times^* {}^{b_2}\hat{\mathbf{I}}_{O_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$${}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\mathbf{X}_{b_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^* {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$$^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$$^{b_{G2}}\hat{\mathbf{f}}^{B_2}_{G_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \times^{*} {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}$$

$$^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{f}}^{B_2}_{G_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3} \Rightarrow \boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}} \cdot \left({}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) \times^{*} {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) \right) + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$ {}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2 $$

$$ {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} $$

$$ {}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2 $$

$$ {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} $$

$$ {}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \times^{*} {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} $$

$$ {}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3} \Rightarrow \boxed{ {}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}} \cdot \left({}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) \times^{*} {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot \left({}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \right) \right) + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3} } $$

$$ \Downarrow $$

$$ \boxed{ {}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3} } $$

$$ \tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2} $$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$${}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \times^{*} {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3} \Rightarrow \boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2}\cdot\ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times\mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$${}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}\times^{*}{}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times^{*}{}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}}\cdot{}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3} \;⇨\; \boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = \underbrace{{}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times^{*}{}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}}_{{}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2}} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T}\cdot{}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$$^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$$^{b_{G2}}\hat{\mathbf{f}}^{B_2}_{G_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}$$

$$^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^* {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$$^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{f}}^{B_2}_{G_2} + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3} \Rightarrow$$

$$\boxed{^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^* {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\Downarrow$$

$$\boxed{^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^T \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2}\cdot\ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times\mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$${}^{b_{G2}}\hat{\mathbf{f}}^{B_2}_{G_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}\times^{*}{}^{b_{G2}}\hat{\mathbf{I}}_{G_2}\cdot{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \quad \Leftarrow \quad {}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times^{*}{}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}}\cdot{}^{b_{G2}}\hat{\mathbf{f}}^{B_2}_{G_2} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3} \quad \Leftarrow \quad \boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T}\cdot{}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} + {}^{b_2}\mathbf{X}_{b_3}^{*} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2}$$

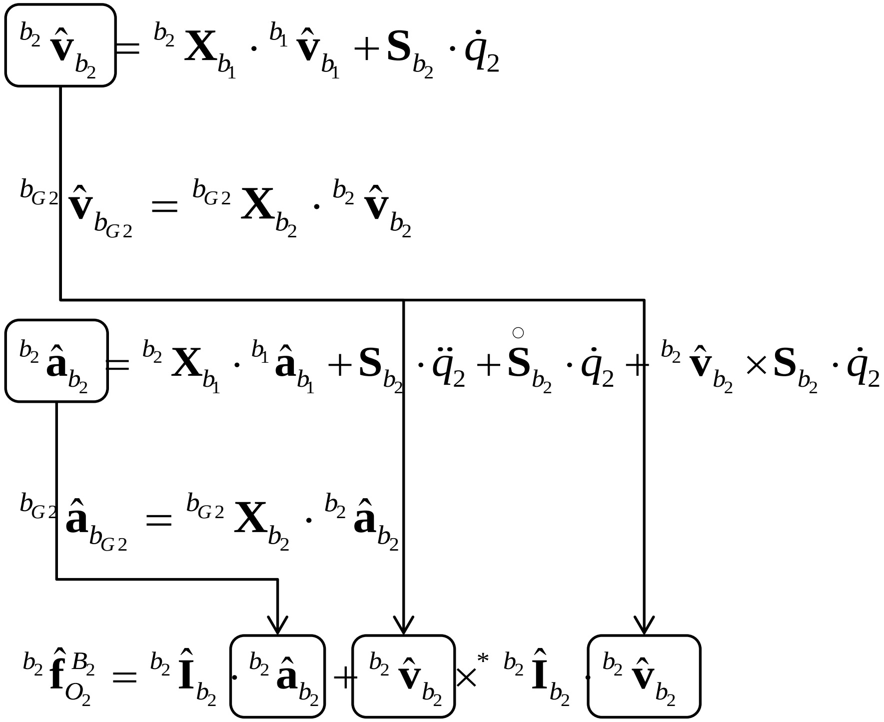
$${}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + \boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot \boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics



$$\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$$\boxed{{}^{b_2}\hat{\mathbf{a}}_{b_2}} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2}\cdot\ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times\mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2}$$

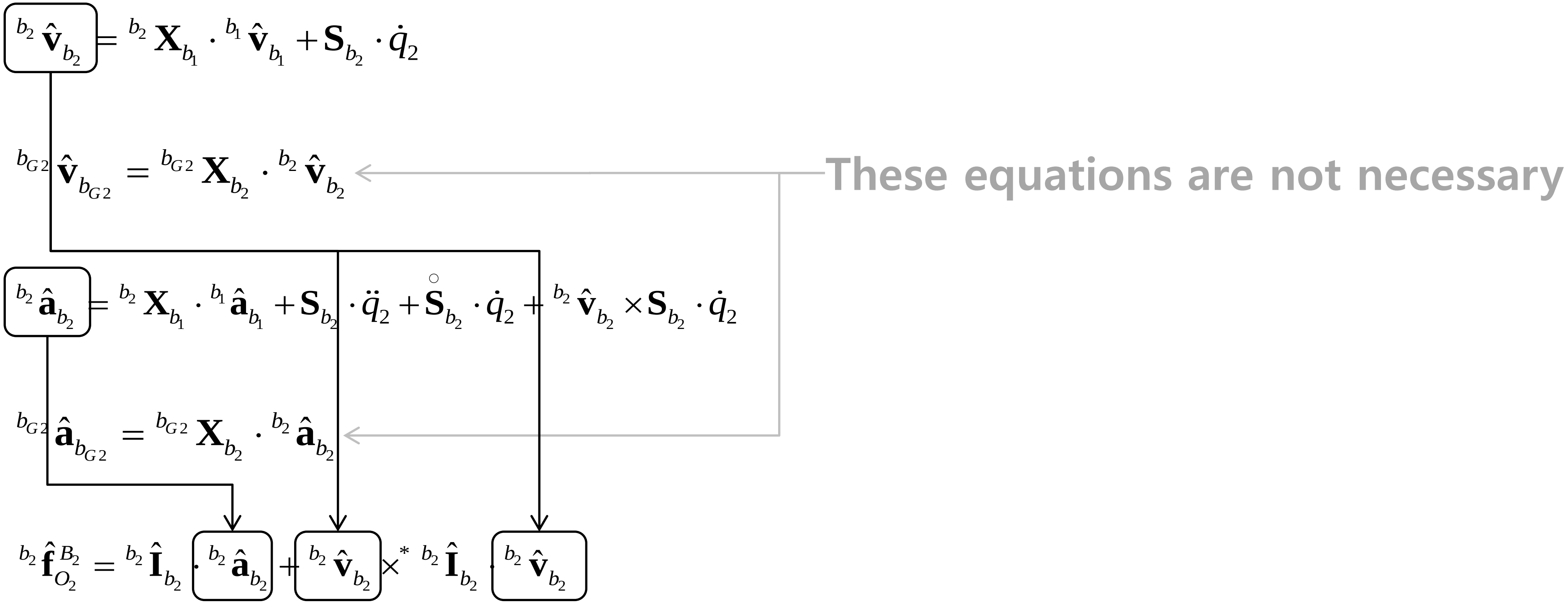
$${}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot\boxed{{}^{b_2}\hat{\mathbf{a}}_{b_2}} + \boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}}\times^{*}{}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T}\cdot{}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics



$$\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}$$

These equations are not necessary

$$\boxed{{}^{b_2}\hat{\mathbf{a}}_{b_2}} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2}\cdot\ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times\mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot\boxed{{}^{b_2}\hat{\mathbf{a}}_{b_2}} + \boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}}\times^{*}{}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T}\cdot{}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2}\cdot\dot{q}_2$$

$$\cancel{{}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}}$$

These equations are not necessary

$$\boxed{{}^{b_2}\hat{\mathbf{a}}_{b_2}} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2}\cdot\ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times\mathbf{S}_{b_2}\cdot\dot{q}_2$$

$$\cancel{{}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2}}$$

$${}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot\boxed{{}^{b_2}\hat{\mathbf{a}}_{b_2}} + \boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}}\times^{*}{}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T}\cdot{}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} + {}^{b_2}\mathbf{X}_{b_3}^{*} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$ {}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2 $$

$$ {}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2 $$

$$ {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} $$

$$ \boxed{ {}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} + {}^{b_2}\mathbf{X}_{b_3}^{*} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3} } $$

$$ \tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2} $$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

$$\theta_{b_1/n} = q_1$$

$$\theta_{b_2/b_1} = q_2$$

Given: **Kinematic Model**

$$\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$$

$$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$$

Find: $\ddot{\theta}_{b_1/n}$

$$\ddot{\theta}_{b_2/b_1}$$

Equations from Inverse Dynamics

$$^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$$\boxed{^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} + {}^{b_2}\mathbf{X}_{b_3}^{*} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$\theta_{b_1/n} = q_1$$

$$\theta_{b_2/b_1} = q_2$$

Given: **Kinematic Model**

$$\boxed{\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}}, \tau_1, \tau_2$$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

$$^{b_2}\hat{\mathbf{v}}_{b_2} = \boxed{^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2}$$

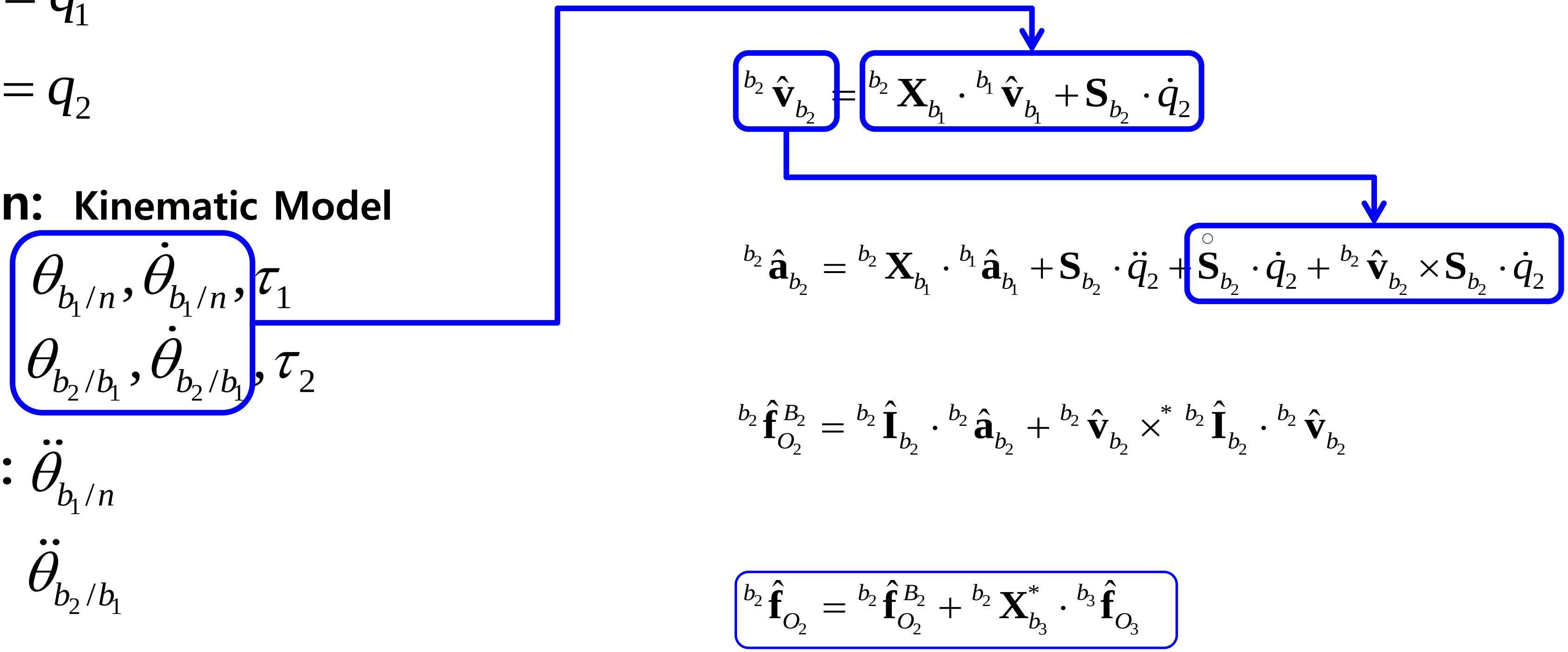
$$^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$$\boxed{^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics



Equations from Inverse Dynamics

$$\theta_{b_1/n} = q_1$$

$$\theta_{b_2/b_1} = q_2$$

Given: **Kinematic Model**

$$\boxed{\begin{matrix}\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \\ \theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}\end{matrix}} \begin{matrix}\tau_1 \\ \tau_2\end{matrix}$$

Find: $\ddot{\theta}_{b_1/n}$

$$\ddot{\theta}_{b_2/b_1}$$

$$\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}} = \boxed{{}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \boxed{\overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

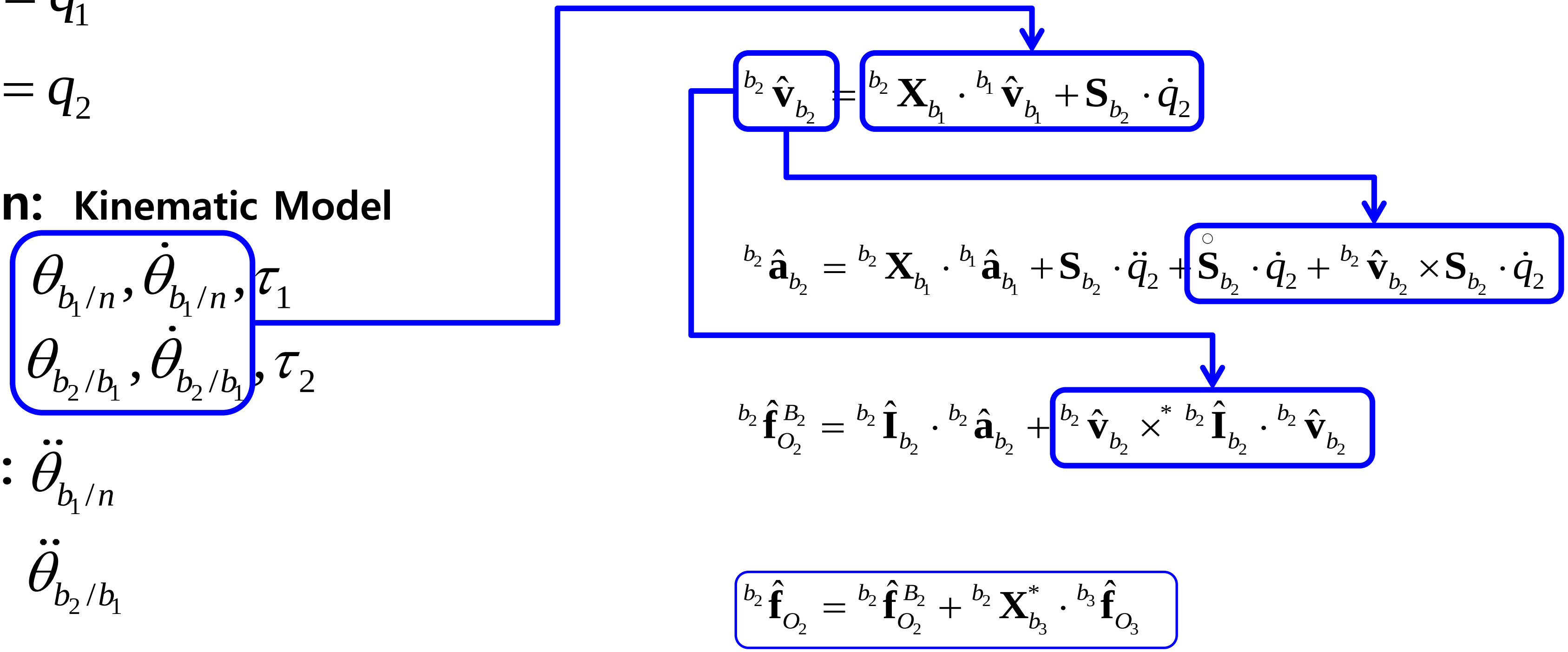
$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} + {}^{b_2}\mathbf{X}_{b_3}^{*} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

$$\theta_{b_1/n} = q_1$$

$$\theta_{b_2/b_1} = q_2$$



Given: Kinematic Model

$$\boxed{\begin{matrix}\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \\ \theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}\end{matrix}} \begin{matrix}\tau_1 \\ , \tau_2\end{matrix}$$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Equations from Inverse Dynamics

$$\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}} = \boxed{{}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2}$$

$$ {}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \boxed{\overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2}$$

$$ {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + \boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} + {}^{b_2}\mathbf{X}_{b_3}^{*} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

$$\theta_{b_1/n} = q_1$$

$$\theta_{b_2/b_1} = q_2$$

Given: **Kinematic Model**

$$\boxed{\begin{array}{l}\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \\ \theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}\end{array}} \begin{array}{l}, \tau_1 \\ , \tau_2\end{array}$$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Equations from Inverse Dynamics

$$\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}} = \boxed{{}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2}$$

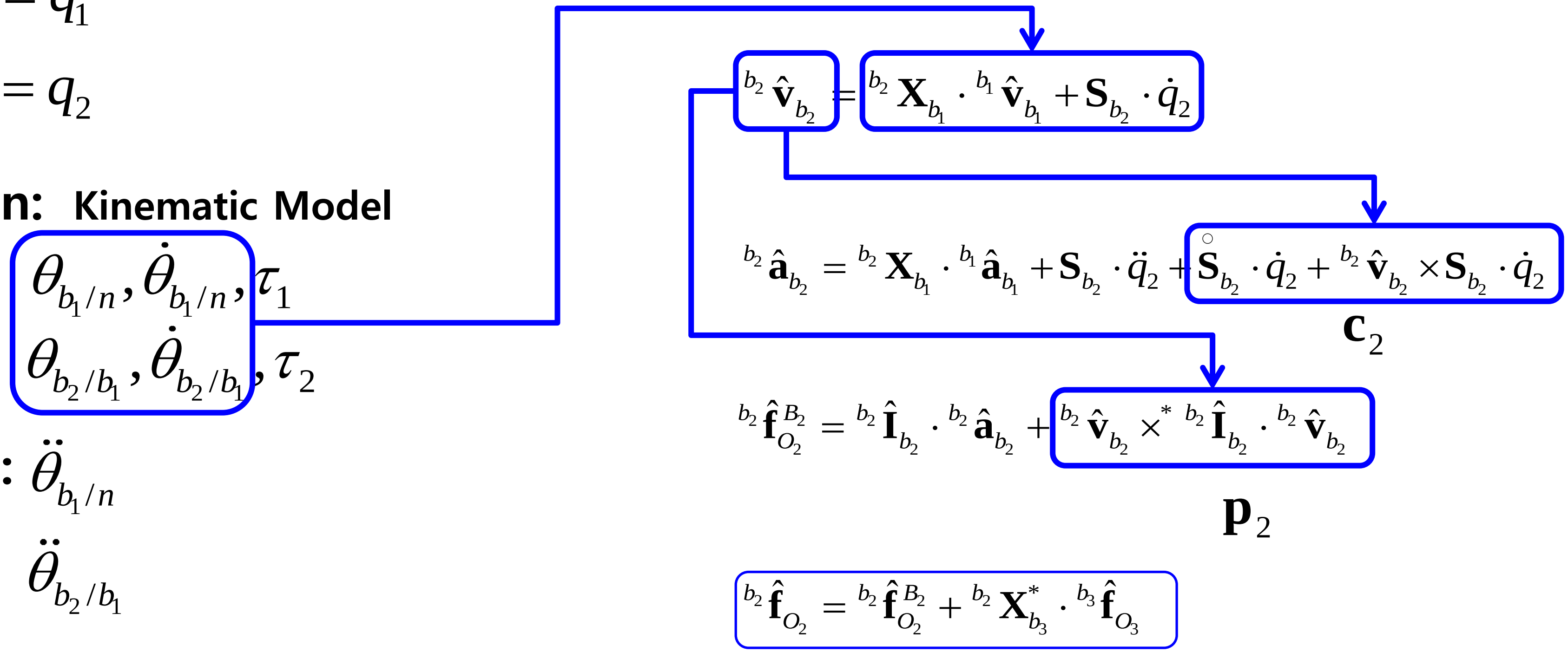
$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \underbrace{\boxed{\overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2}}_{\mathbf{c}_2}$$

$${}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + \underbrace{\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}}}_{\mathbf{p}_2}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics



$$\theta_{b_1/n} = q_1$$

$$\theta_{b_2/b_1} = q_2$$

Given: Kinematic Model

$$\boxed{\begin{array}{l}\theta_{b_1/n}, \dot{\theta}_{b_1/n} \\ \theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}\end{array}} \begin{array}{l}, \tau_1 \\ , \tau_2\end{array}$$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Equations from Inverse Dynamics

$$\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2}} = \boxed{{}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \underset{\mathbf{c}_2}{\boxed{\overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2}}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + \underset{\mathbf{p}_2}{\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}}}$$

$$\boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} + {}^{b_2}\mathbf{X}_{b_3}^{*} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

We can calculate $\mathbf{c}_2, \mathbf{p}_2$ in advance. So, $\mathbf{c}_2, \mathbf{p}_2$ can be considered as the known variables.

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$$ {}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \underbrace{\boxed{\overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2}}_{\mathbf{c}_2} $$

$$ {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + \underbrace{\boxed{{}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}}}_{\mathbf{p}_2} $$

$$ \boxed{{}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} + {}^{b_2}\mathbf{X}_{b_3}^{*} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}} $$

$$ \tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2} $$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^{*} {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}^{B_2}_{O_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_i}\hat{\mathbf{a}}_{b_i} = \mathbf{a}_1 \qquad {}^{b_i}\mathbf{X}_{b_j} = {}^{i}\mathbf{X}_j \qquad \mathbf{S}_{b_i} = \mathbf{S}_i \qquad {}^{b_i}\hat{\mathbf{I}}_{b_i} = \mathbf{I}_i \qquad {}^{b_i}\hat{\mathbf{f}}_{O_i} = \mathbf{f}_i$$

$${}^{b_i}\hat{\mathbf{f}}_{O_i}^{B_i} = \mathbf{f}_i^{B} \qquad {}^{b_i}\mathbf{X}^{*}_{b_j} = {}^{i}\mathbf{X}^{*}_j \qquad \overset{\circ}{\mathbf{S}}_{b_i}\cdot\dot{q}_i + {}^{b_i}\hat{\mathbf{v}}_{b_i}\times\mathbf{S}_{b_i}\cdot\dot{q}_i = \mathbf{c}_i \qquad {}^{b_i}\hat{\mathbf{v}}_{b_i}\times^{*}{}^{b_i}\hat{\mathbf{I}}_{b_i}\cdot{}^{b_i}\hat{\mathbf{v}}_{b_i} = \mathbf{p}_i$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2}\cdot\ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times\mathbf{S}_{b_2}\cdot\dot{q}_2$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot{}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2}\times^{*}{}^{b_2}\hat{\mathbf{I}}_{b_2}\cdot{}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} + {}^{b_2}\mathbf{X}^{*}_{b_3}\cdot{}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T}\cdot{}^{b_2}\hat{\mathbf{f}}_{O_2}$$

Forward Dynamics - Propagation Methods
- Simplification of Equations from Inverse Dynamics

Equations from Inverse Dynamics

$${}^{b_i}\hat{\mathbf{a}}_{b_i} = \mathbf{a}_1 \qquad {}^{b_i}\mathbf{X}_{b_j} = {}^{i}\mathbf{X}_j \qquad \mathbf{S}_{b_i} = \mathbf{S}_i \qquad {}^{b_i}\hat{\mathbf{I}}_{b_i} = \mathbf{I}_i \qquad {}^{b_i}\hat{\mathbf{f}}_{O_i} = \mathbf{f}_i$$

$${}^{b_i}\hat{\mathbf{f}}_{O_i}^{B_i} = \mathbf{f}_i^B \qquad {}^{b_i}\mathbf{X}^*_{b_j} = {}^{i}\mathbf{X}^*_j \qquad \overset{\circ}{\mathbf{S}}_{b_i} \cdot \dot{q}_i + {}^{b_i}\hat{\mathbf{v}}_{b_i} \times \mathbf{S}_{b_i} \cdot \dot{q}_i = \mathbf{c}_i \qquad {}^{b_i}\hat{\mathbf{v}}_{b_i} \times^* {}^{b_i}\hat{\mathbf{I}}_{b_i} \cdot {}^{b_i}\hat{\mathbf{v}}_{b_i} = \mathbf{p}_i$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$$\mathbf{a}_2 = {}^{2}\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} = {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times^* {}^{b_2}\hat{\mathbf{I}}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$$\mathbf{f}_2^B = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\hat{\mathbf{f}}_{O_2}^{B_2} + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3}$$

$$\mathbf{f}_2 = \mathbf{f}_2^B + {}^{2}\mathbf{X}^*_3 \cdot \mathbf{f}_3$$

$$\tau_2 = \mathbf{S}_{b_2}{}^T \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

Forward Dynamics - Propagation Methods
- Simplified version of Equations from Inverse Dynamics

Equations from Inverse Dynamics

Equations for link 1

$$\mathbf{a}_1 = {}^{1}\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + {}^{1}\mathbf{X}_2^* \cdot \mathbf{f}_2$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^{2}\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2^B = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\mathbf{f}_2 = \mathbf{f}_2^B + {}^{2}\mathbf{X}_3^* \cdot \mathbf{f}_3$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

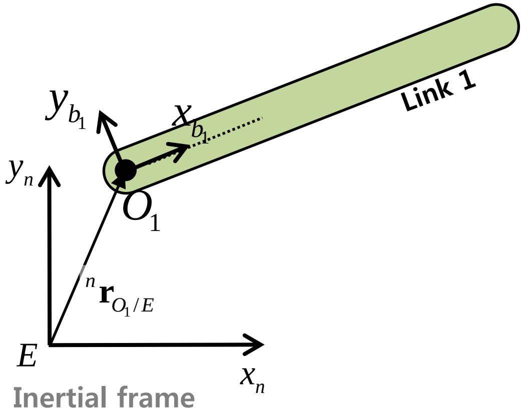
Forward Dynamics - Propagation Methods
- Examples of 1 link arm

Forward Dynamics of 1 link arm

$\theta_{b_1/n} = q_1$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

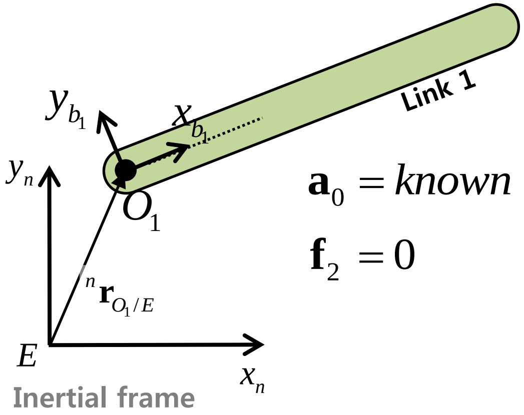
- Examples of 1 link arm

Forward Dynamics of 1 link arm

$\theta_{b_1/n} = q_1$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



$\mathbf{a}_0 = known$

$\mathbf{f}_2 = 0$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + \cancel{{}^1\mathbf{X}_2^* \cdot \mathbf{f}_2}$$

Because the link 1 is leave of the arm, the force exerted on the next link by the link 1 is equal to zero

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

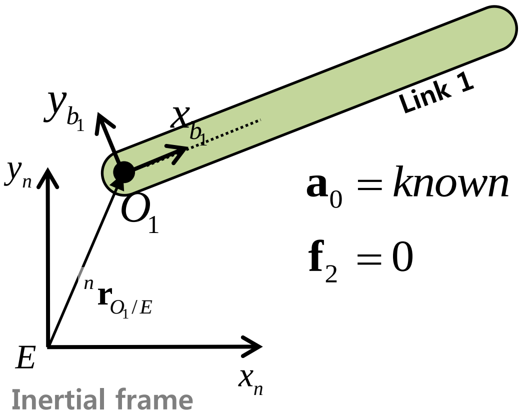
- Examples of 1 link arm

Forward Dynamics of 1 link arm

$\theta_{b_1/n} = q_1$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



$\mathbf{a}_0 = known$

$\mathbf{f}_2 = 0$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B$$

$$\tau_1 = \mathbf{S}_1^{\ T} \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

- Examples of 1 link arm

Forward Dynamics of 1 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^{\,B} = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

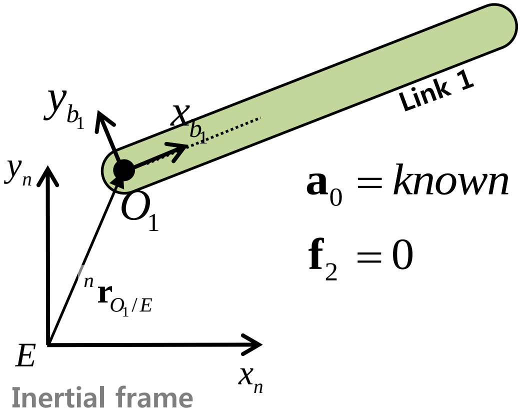
$$\mathbf{f}_1 = \mathbf{f}_1^{\,B}$$

$$\tau_1 = \mathbf{S}_1^{\,T} \cdot \mathbf{f}_1$$

$$\theta_{b_1/n} = q_1$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



$\mathbf{a}_0 = known$

$\mathbf{f}_2 = 0$

Forward Dynamics - Propagation Methods

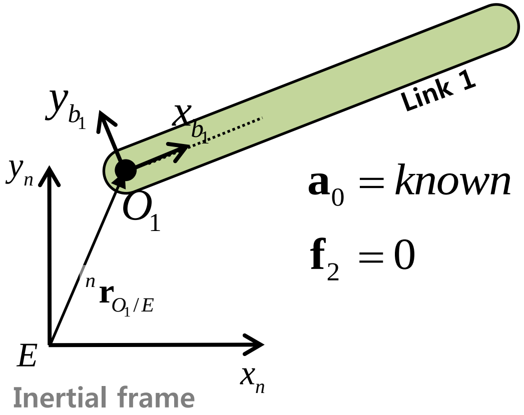
- Examples of 1 link arm

Forward Dynamics of 1 link arm

$$\theta_{b_1/n} = q_1$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



$\mathbf{a}_0 = known$

$\mathbf{f}_2 = 0$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B$$

$$\Rightarrow \boxed{\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1}$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

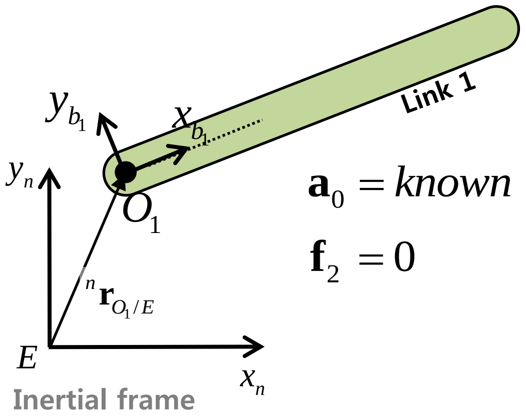
- Examples of 1 link arm

Forward Dynamics of 1 link arm

$$\theta_{b_1/n} = q_1$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



$\mathbf{a}_0 = known$

$\mathbf{f}_2 = 0$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B$$

$$\Rightarrow \boxed{\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1}$$

Because the link 1 is leave of the arm, this manipulation is possible.

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

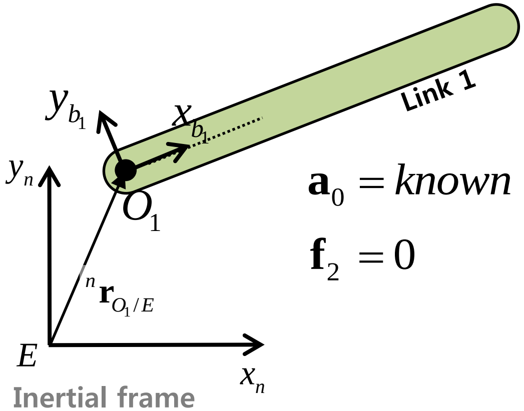
- Examples of 1 link arm

Forward Dynamics of 1 link arm

$\theta_{b_1/n} = q_1$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



$\mathbf{a}_0 = known$

$\mathbf{f}_2 = 0$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

- Examples of 1 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

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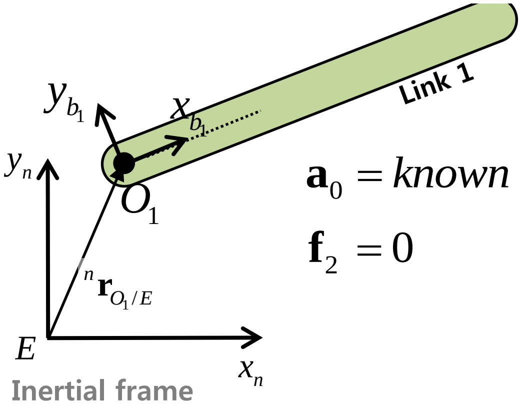

$$\mathbf{a}_0 = known$$

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Forward Dynamics of 1 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 \quad \Rightarrow \quad \mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

- Examples of 1 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$$\theta_{b_1/n} = q_1$$

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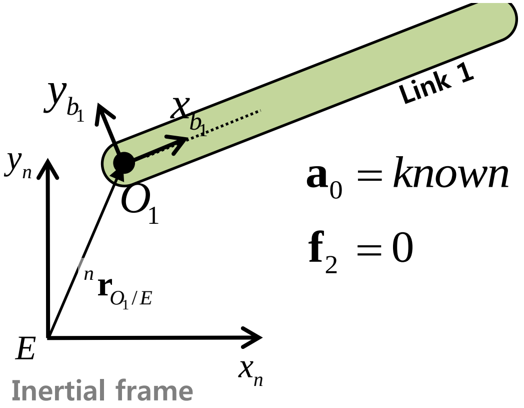

$\mathbf{a}_0 = known$

$\mathbf{f}_2 = 0$

Forward Dynamics of 1 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 \Rightarrow \mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1 \Rightarrow \boxed{\tau_1 = \mathbf{S}_1{}^T \left(\mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)}$$

Forward Dynamics - Propagation Methods

- Examples of 1 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^{1}\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

Forward Dynamics of 1 link arm

$$\theta_{b_1/n} = q_1$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

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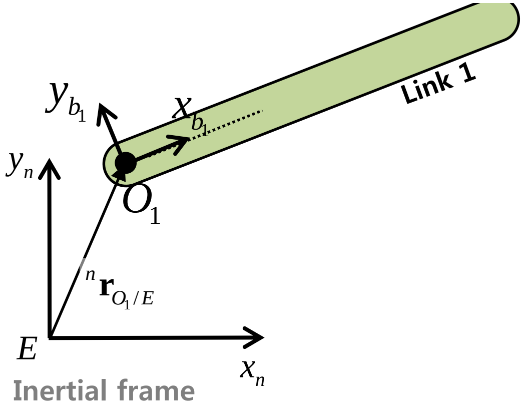

Equations for link 1

$$\mathbf{a}_1 = {}^{1}\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\boxed{\tau_1 = \mathbf{S}_1^{\ T} \left(\mathbf{I}_1 \cdot \left({}^{1}\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)}$$

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Forward Dynamics - Propagation Methods

- Examples of 1 link arm

Forward Dynamics of 1 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

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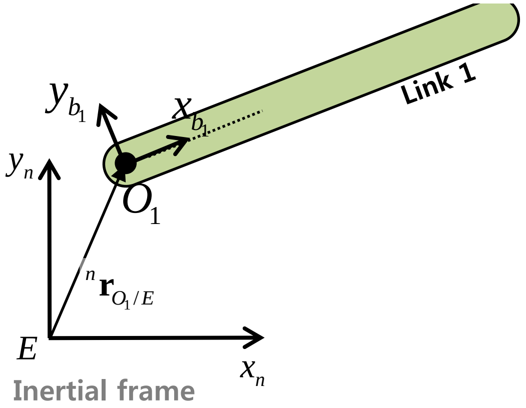
Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

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$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

$$\boxed{\tau_1 = \mathbf{S}_1{}^T \left(\mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)}$$

$$\tau_1 = \mathbf{S}_1{}^T \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{I}_1 \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{p}_1 \right)$$

Forward Dynamics - Propagation Methods

- Examples of 1 link arm

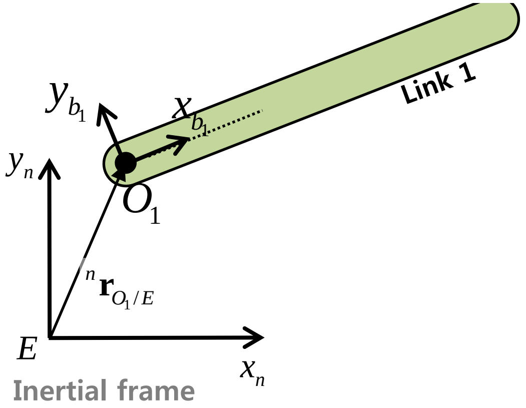
Forward Dynamics of 1 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$$\theta_{b_1/n} = q_1$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

$$\boxed{\tau_1 = \mathbf{S}_1{}^T \left(\mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)}$$

$$\Downarrow$$

$$\tau_1 = \mathbf{S}_1{}^T \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{I}_1 \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{p}_1 \right)$$

$$\Downarrow$$

$$\tau_1 = \mathbf{S}_1^T \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right) + \mathbf{S}_1^T \mathbf{I}_1 \mathbf{S}_1 \cdot \ddot{q}_1$$

Forward Dynamics - Propagation Methods

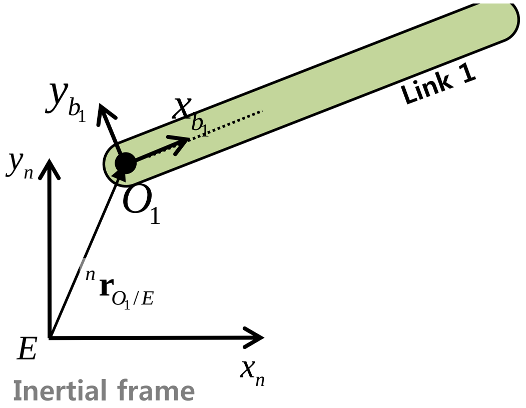
- Examples of 1 link arm

Forward Dynamics of 1 link arm

$$\theta_{b_1/n} = q_1$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\tau_1 = \mathbf{S}_1^{\,T} \cdot \mathbf{f}_1$$

$$\boxed{\tau_1 = \mathbf{S}_1^{\,T} \left(\mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)}$$

$$\tau_1 = \mathbf{S}_1^{\,T} \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{I}_1 \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{p}_1 \right)$$

$$\tau_1 = \mathbf{S}_1^T \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right) + \mathbf{S}_1^T \mathbf{I}_1 \mathbf{S}_1 \cdot \ddot{q}_1$$

$$\mathbf{S}_1^T \mathbf{I}_1 \mathbf{S}_1 \cdot \ddot{q}_1 = \tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)$$

Forward Dynamics - Propagation Methods

- Examples of 1 link arm

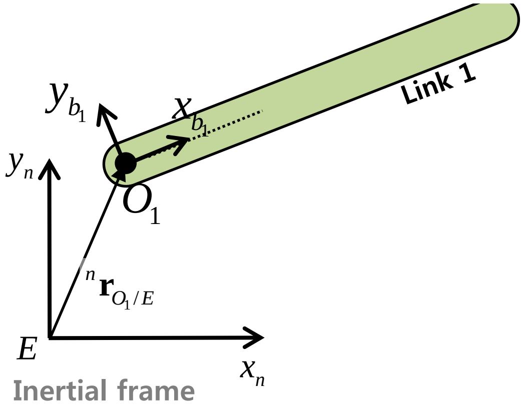
Forward Dynamics of 1 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$$\theta_{b_1/n} = q_1$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



Equations for **link 1**

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

$$\boxed{\tau_1 = \mathbf{S}_1{}^T \left(\mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)}$$

$$\mathbf{S}_1^T \mathbf{I}_1 \mathbf{S}_1 \cdot \ddot{q}_1 = \tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)$$

Forward Dynamics - Propagation Methods

- Examples of 1 link arm

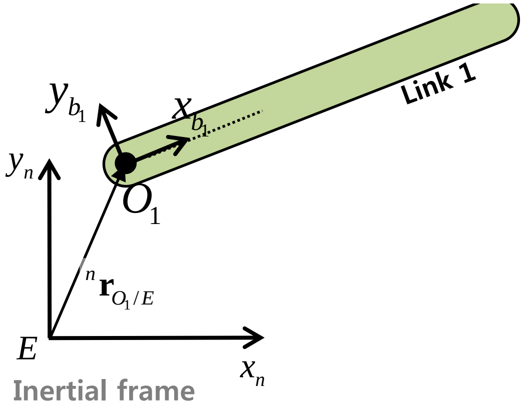
Forward Dynamics of 1 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$$\theta_{b_1/n} = q_1$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



Equations for **link 1**

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\boxed{\tau_1 = \mathbf{S}_1{}^T \left(\mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)}$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{S}_1^T \mathbf{I}_1 \mathbf{S}_1 \cdot \ddot{q}_1 = \tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

$$\ddot{q}_1 = \left(\mathbf{S}_1^T \mathbf{I}_1 \mathbf{S}_1 \right)^{-1} \left(\tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right) \right) \text{---(1)}$$

Because the link 1 is leave of the arm, it is possible to derive the eq. (1)

Forward Dynamics - Propagation Methods

- Examples of 1 link arm

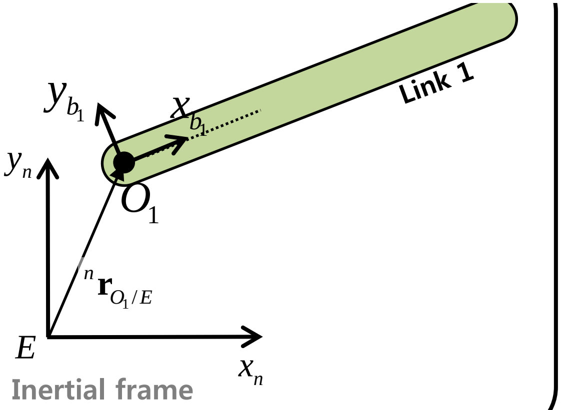
Forward Dynamics of 1 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$$\theta_{b_1/n} = q_1$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

Find: $\ddot{\theta}_{b_1/n}$



Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\boxed{\tau_1 = \mathbf{S}_1{}^T \left(\mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)}$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{S}_1^T \mathbf{I}_1 \mathbf{S}_1 \cdot \ddot{q}_1 = \tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right)$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

$$\boxed{\ddot{q}_1} = \left(\mathbf{S}_1^T \mathbf{I}_1 \mathbf{S}_1 \right)^{-1} \left(\tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1 \left({}^1\mathbf{X}_0 \mathbf{a}_0 + \mathbf{c}_1 \right) + \mathbf{p}_1 \right) \right) \text{---(1)}$$

Because the link 1 is leave of the arm, it is possible to derive the eq. (1)

We can solve the Eq. (1)!

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2^B = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\mathbf{f}_2 = \mathbf{f}_2^B + {}^2\mathbf{X}_3^* \cdot \mathbf{f}_3$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1\right) + \mathbf{p}_1$$

$\theta_{b_1/n} = q_1$

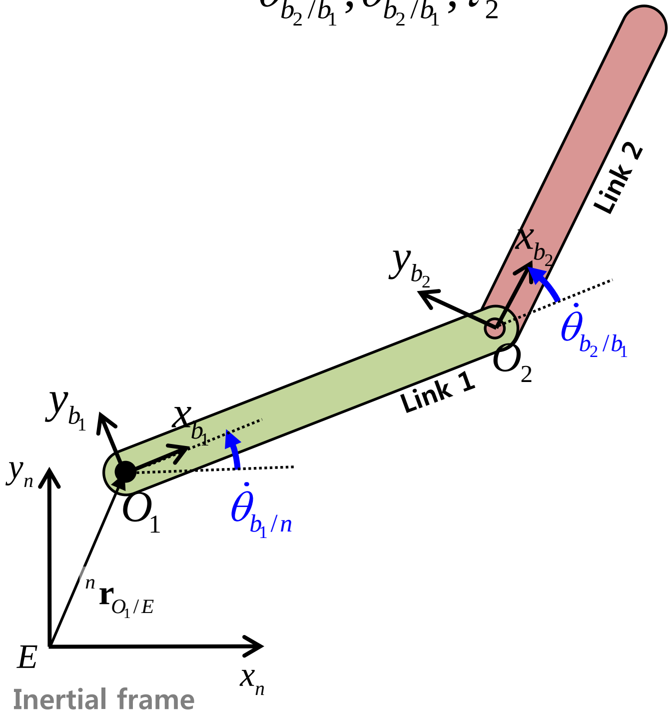
$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$



Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^{1}\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^{2}\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2^B = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\mathbf{f}_2 = \mathbf{f}_2^B + \cancel{{}^{2}\mathbf{X}_3^* \cdot \mathbf{f}_3}$$

Because the link 2 is leave of the arm, the force exerted on the next link by the link 2 is equal to zero

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2^B = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\mathbf{f}_2 = \mathbf{f}_2^B$$

$$\tau_2 = \mathbf{S}_2^{\ T} \cdot \mathbf{f}_2$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2^B = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\mathbf{f}_2 = \mathbf{f}_2^B$$

$$\tau_2 = \mathbf{S}_2^T \cdot \mathbf{f}_2$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^{1}\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^{2}\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\boxed{\begin{aligned} \mathbf{f}_2^B &= \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2 \\ \mathbf{f}_2 &= \mathbf{f}_2^B \end{aligned}} \Rightarrow \boxed{\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2}$$

Because the link 1 is leave of the arm, this manipulation is possible.

$$\tau_2 = \mathbf{S}_2^{T} \cdot \mathbf{f}_2$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2^{\ T} \cdot \mathbf{f}_2$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$$\theta_{b_1/n} = q_1 \qquad \theta_{b_2/b_1} = q_2$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2 \quad \Rightarrow \quad \mathbf{f}_2 = \mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2 \quad \Rightarrow \quad \mathbf{f}_2 = \mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2 \quad \Rightarrow \quad \boxed{\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

Forward Dynamics - Propagation Methods
- Examples of 2 link arm

$\theta_{b_1/n} = q_1$
$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$
$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$
$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\boxed{\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\boxed{\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{I}_2 \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{p}_2 \right)$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

Forward Dynamics - Propagation Methods
- Examples of 2 link arm

Forward Dynamics of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\boxed{\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\Downarrow$$

$$\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{I}_2 \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{p}_2 \right)$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\Downarrow$$

$$\tau_2 = \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) + \mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \cdot \ddot{q}_2$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$$\theta_{b_1/n} = q_1$$

$$\theta_{b_2/b_1} = q_2$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\boxed{\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

$$\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{I}_2 \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{p}_2 \right)$$

$$\tau_2 = \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) + \mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \cdot \ddot{q}_2$$

$$\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \cdot \ddot{q}_2 = \tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\boxed{\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

$$\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \cdot \ddot{q}_2 = \tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\boxed{\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

⇓

$$\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \cdot \ddot{q}_2 = \tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

⇓

$$\boxed{\ddot{q}_2} = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) \right) \text{---(1)}$$

Because the link 2 is leave of the arm, it is possible to derive the eq. (1)

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\boxed{\tau_2 = \mathbf{S}_2{}^T \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

$$\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \cdot \ddot{q}_2 = \tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)$$

$$\boxed{\ddot{q}_2} = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \boxed{\mathbf{a}_1} + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) \right) \text{---(1)}$$

Because the link 2 is leave of the arm, it is possible to derive the eq. (1)

However, Since a_1 is unknown, we can not solve the eq. (1)

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \left({}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1 \right) + \mathbf{p}_1$$

$\theta_{b_1/n} = q_1$

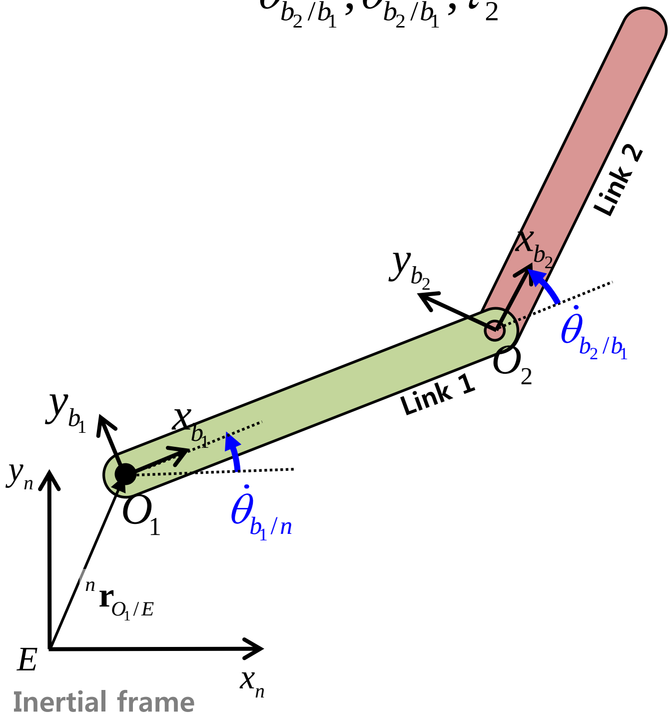
$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$



Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$$\theta_{b_1/n} = q_1 \qquad \theta_{b_2/b_1} = q_2$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ $\quad$ $\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$ $\quad$ $\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

Because the link 1 is not leave of the arm, the force $\mathbf{f}_2$,exerted on the next link by the link 1, is not equal to zero

$$\mathbf{f}_1 = \mathbf{f}_1^B + \boxed{{}^1\mathbf{X}_2^* \cdot \mathbf{f}_2}$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

Because the link 1 is not leave of the arm, the force $\mathbf{f}_2$,exerted on the next link by the link 1, is not equal to zero

$$\mathbf{f}_1 = \mathbf{f}_1^B + \boxed{{}^1\mathbf{X}_2^* \cdot \mathbf{f}_2}$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods
- Examples of 2 link arm

$$\theta_{b_1/n} = q_1$$
$$\theta_{b_2/b_1} = q_2$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + \boxed{{}^1\mathbf{X}_2^* \cdot \mathbf{f}_2}$$

Because the link 1 is not leave of the arm, the force $\mathbf{f}_2$,exerted on the next link by the link 1, is not equal to zero

$$\boxed{\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2}$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^{1}\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^{B} = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^{B} + \boxed{{}^{1}\mathbf{X}_2^{*} \cdot \mathbf{f}_2}$$

Because the link 1 is not leave of the arm, the force $\mathbf{f}_2$,exerted on the next link by the link 1, is not equal to zero

From Equations for link 2 $\mathbf{f}_2 = \mathbf{I}_2 \cdot \left({}^{2}\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 + {}^{1}\mathbf{X}_2^{*} \cdot \mathbf{f}_2$$

$$\tau_1 = \mathbf{S}_1^{T} \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

Because the link 1 is not leave of the arm, the force $\mathbf{f}_2$,exerted on the next link by the link 1, is not equal to zero

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + \boxed{{}^1\mathbf{X}_2^* \cdot \mathbf{f}_2}$$

From Equations for link 2 $\mathbf{f}_2 = \mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$

Substituting

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + \boxed{{}^1\mathbf{X}_2^* \cdot \mathbf{f}_2}$$

Because the link 1 is not leave of the arm, the force $\mathbf{f}_2$,exerted on the next link by the link 1, is not equal to zero

From Equations for link 2 $\mathbf{f}_2 = \mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods
- Examples of 2 link arm

$$\theta_{b_1/n} = q_1$$
$$\theta_{b_2/b_1} = q_2$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$

$$\Rightarrow \quad \mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$

$$\Rightarrow \quad \mathbf{f}_1 = \mathbf{I}_1 \cdot \boxed{\mathbf{a}_1} + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \boxed{\mathbf{a}_1} + \mathbf{S}_2 \cdot \boxed{\ddot{q}_2} + \mathbf{c}_2\right) + \mathbf{p}_2\right)$$

unknowns

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

There are two unknown variables in the equation.
We will eliminate one unknown variable $\ddot{q}_2$.

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \boxed{\mathbf{a}_1} + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \boxed{\mathbf{a}_1} + \mathbf{S}_2 \cdot \boxed{\ddot{q}_2} + \mathbf{c}_2\right) + \mathbf{p}_2\right)$$

unknowns

There are two unknown variables in the equation.
We will eliminate one unknown variable $\ddot{q}_2$.

Forward Dynamics - Propagation Methods
- Examples of 2 link arm

$$\boxed{\mathbf{f}_1 = \mathbf{I}_1 \boxed{\cdot \mathbf{a}_1} + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \boxed{\cdot \mathbf{a}_1} + \mathbf{S}_2 \boxed{\cdot \ddot{q}_2} + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

unknowns

There are two unknown variables in the equation.
We will eliminate one unknown variable $\ddot{q}_2$.

Substituting following equation into $\ddot{q}_2$.

$$\ddot{q}_2 = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) \right)$$

← This equation is from the equations for link 2

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$$\boxed{\mathbf{f}_1 = \mathbf{I}_1 \boxed{\cdot \mathbf{a}_1} + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \boxed{\cdot \mathbf{a}_1} + \mathbf{S}_2 \boxed{\cdot \ddot{q}_2} + \mathbf{c}_2\right) + \mathbf{p}_2\right)}$$

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← This equation is from the equations for link 2

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2\right) + \mathbf{p}_2\right)\right) + \mathbf{c}_2\right) + \mathbf{p}_2\right)$$

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$$\mathbf{f}_1 = \mathbf{I}_1 \boxed{\cdot \mathbf{a}_1} + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \boxed{\cdot \mathbf{a}_1} + \mathbf{S}_2 \boxed{\cdot \ddot{q}_2} + \mathbf{c}_2\right) + \mathbf{p}_2\right)$$

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$$\ddot{q}_2 = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2\right) + \mathbf{p}_2\right)\right)$$

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$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2\right) + \mathbf{p}_2\right)\right) + \mathbf{c}_2\right) + \mathbf{p}_2\right)$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2\right) + \mathbf{p}_2\right)\right) + \mathbf{c}_2\right) + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2$$

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$$\boxed{\mathbf{f}_1 = \mathbf{I}_1 \boxed{\cdot \mathbf{a}_1} + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \boxed{\cdot \mathbf{a}_1} + \mathbf{S}_2 \boxed{\cdot \ddot{q}_2} + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

Substituting $\ddot{q}_2 = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) \right)$

$$\Downarrow$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) \right) + \mathbf{c}_2 \right) + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2$$

$$\Downarrow$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) \right) + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2$$

$$\Downarrow$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) \right) + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2$$

$$\Downarrow$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \, {}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2 \right) \right) + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2$$

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$$\boxed{\mathbf{f}_1 = \mathbf{I}_1 \boxed{\cdot \mathbf{a}_1} + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \boxed{\cdot \mathbf{a}_1} + \mathbf{S}_2 \boxed{\cdot \ddot{q}_2} + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

Substituting $\ddot{q}_2 = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) \right)$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1 \mathbf{a}_1 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2 \right) \right) + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1 \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2 \right) \right) + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \mathbf{a}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2 \right) \right)$$

$$\mathbf{f}_1 = \left(\mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1 \right) \mathbf{a}_1 + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2 \right) \right)$$

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- Examples of 2 link arm

$$\boxed{\mathbf{f}_1 = \mathbf{I}_1 \cdot \boxed{\mathbf{a}_1} + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \boxed{\mathbf{a}_1} + \mathbf{S}_2 \cdot \boxed{\ddot{q}_2} + \mathbf{c}_2 \right) + \mathbf{p}_2 \right)}$$

Substituting $\ddot{q}_2 = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2 \right) + \mathbf{p}_2 \right) \right)$

$$\Downarrow$$

$$\mathbf{f}_1 = \underbrace{\left(\mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1 \right)}_{\mathbf{I}_1^A} \mathbf{a}_1 + \underbrace{\mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2 \right) \right)}_{\mathbf{p}_1^A}$$

$$\boxed{\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A}$$

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2 \right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2 \right) \right)$$

Forward Dynamics - Propagation Methods
- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$

$$\mathbf{f}_1 = \mathbf{I}_1 \cdot \boxed{\mathbf{a}_1} + \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \left(\mathbf{I}_2 \cdot \left({}^2\mathbf{X}_1 \cdot \boxed{\mathbf{a}_1} + \mathbf{S}_2 \cdot \boxed{\ddot{q}_2} + \mathbf{c}_2\right) + \mathbf{p}_2\right)$$

unknowns

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$!!!!

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2\right)\right)$$

Forward Dynamics - Propagation Methods

- Examples of 2 link arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Forward Dynamics of 2 link arm

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\boxed{\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A}$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

The equation of the link 1 is manipulated in same pattern with the equation of the link 2, which is leave of the arm.

So, we can derive the following equation.

$$\boxed{\ddot{q}_1} = \left(\mathbf{S}_1^T \mathbf{I}_1^A \mathbf{S}_1\right)^{-1} \left(\tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1^A \left({}^1\mathbf{X}_0 \mathbf{a}_0 + \mathbf{c}_1\right) + \mathbf{p}_1^A\right)\right)$$

We can solve the equation!!

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2\right)\right)$$

5.8 Forward Dynamics Summary

Forward Dynamics of 2-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Equations for link 1

$$\mathbf{a}_1 = {}^{1}\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^{\,B} = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^{\,B} + {}^{1}\mathbf{X}_2^{*} \cdot \mathbf{f}_2$$

$$\tau_1 = \mathbf{S}_1^{\,T} \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^{2}\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\boxed{\begin{aligned} \mathbf{f}_2^{\,B} &= \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2 \\ \mathbf{f}_2 &= \mathbf{f}_2^{\,B} + \cancel{{}^{2}\mathbf{X}_3^{*} \cdot \mathbf{f}_3} \end{aligned}} \Rightarrow \boxed{\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2}$$

Because the link 2 is the leave of the arm, it is possible

$$\tau_2 = \mathbf{S}_2^{\,T} \cdot \mathbf{f}_2$$

Forward Dynamics of 2-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2^T \cdot \mathbf{f}_2$$

Forward Dynamics of 2-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$

!!!!

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$

The equations are simplified as if the link 1 is the leave of the arm!

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2^T \cdot \mathbf{f}_2$$

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2\right)\right)$$

Forward Dynamics of 2-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2\right)\right)$$

Forward Dynamics of 2-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2^T \cdot \mathbf{f}_2$$

$$\ddot{q}_2 = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \mathbf{a}_1 + \mathbf{c}_2\right) + \mathbf{p}_2\right)\right) \text{-(1)}$$

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2\right)\right)$$

Forward Dynamics of 2-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\ddot{q}_2 = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \boxed{\mathbf{a}_1} + \mathbf{c}_2\right) + \mathbf{p}_2\right)\right) \text{-(1)}$$

Since a1 is unknown, the equation (1) can not be solved

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2\right)\right)$$

Forward Dynamics of 2-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\boxed{\ddot{q}_1} = \left(\mathbf{S}_1^T \mathbf{I}_1^A \mathbf{S}_1\right)^{-1} \left(\tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1^A \left({}^1\mathbf{X}_0 \mathbf{a}_0 + \mathbf{c}_1\right) + \mathbf{p}_1^A\right)\right) \text{-(2)}$$

$$\ddot{q}_2 = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \boxed{\mathbf{a}_1} + \mathbf{c}_2\right) + \mathbf{p}_2\right)\right) \text{-(1)}$$

We can solve the equation (2)

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2\right)\right)$$

Forward Dynamics of 2-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Equations for link 1

$$\boxed{\mathbf{a}_1} = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \boxed{\ddot{q}_1} + \mathbf{c}_1$$

a$_1$ can be calculated

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\boxed{\ddot{q}_1} = \left(\mathbf{S}_1^T \mathbf{I}_1^A \mathbf{S}_1\right)^{-1} \left(\tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1^A \left({}^1\mathbf{X}_0 \mathbf{a}_0 + \mathbf{c}_1\right) + \mathbf{p}_1^A\right)\right) \text{-(2)}$$

$$\ddot{q}_2 = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \boxed{\mathbf{a}_1} + \mathbf{c}_2\right) + \mathbf{p}_2\right)\right) \text{-(1)}$$

We can solve the equation (2)

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2\right)\right)$$

Forward Dynamics of 2-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$ **Find:** $\ddot{\theta}_{b_1/n}$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$ $\ddot{\theta}_{b_2/b_1}$

Equations for link 1

$$\boxed{\mathbf{a}_1} = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

a$_1$ can be calculated

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\boxed{\ddot{q}_1} = \left(\mathbf{S}_1^T \mathbf{I}_1^A \mathbf{S}_1\right)^{-1} \left(\tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1^A \left({}^1\mathbf{X}_0 \mathbf{a}_0 + \mathbf{c}_1\right) + \mathbf{p}_1^A\right)\right) \text{-(2)}$$

We can solve the equation (2)

$$\boxed{\ddot{q}_2} = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \boxed{\mathbf{a}_1} + \mathbf{c}_2\right) + \mathbf{p}_2\right)\right) \text{-(1)}$$

We can solve the equation (1)

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2\right)\right)$$

Forward Dynamics of 2-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

Equations for link 1

$$\boxed{\mathbf{a}_1} = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$\mathbf{a}_1$ can be calculated

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2 = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\boxed{\ddot{q}_1} = \left(\mathbf{S}_1^T \mathbf{I}_1^A \mathbf{S}_1\right)^{-1} \left(\tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1^A \left({}^1\mathbf{X}_0 \mathbf{a}_0 + \mathbf{c}_1\right) + \mathbf{p}_1^A\right)\right) \text{-(2)}$$

We can solve the equation (2)

$$\boxed{\ddot{q}_2} = \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \left({}^2\mathbf{X}_1 \boxed{\mathbf{a}_1} + \mathbf{c}_2\right) + \mathbf{p}_2\right)\right) \text{-(1)}$$

We can solve the equation (1)

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2 \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2 \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2 \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2 \mathbf{c}_2 + \mathbf{p}_2\right)\right)$$

Forward Dynamics of 3-Link Arm

$$\theta_{b_1/n} = q_1 \qquad \theta_{b_2/b_1} = q_2 \qquad \theta_{b_3/b_2} = q_3$$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$; $\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$; $\theta_{b_3/b_2}, \dot{\theta}_{b_3/b_2}, \tau_3$

Find: $\ddot{\theta}_{b_1/n}$; $\ddot{\theta}_{b_2/b_1}$; $\ddot{\theta}_{b_3/b_2}$

Equations for link 1

$$\boxed{\mathbf{a}_1} = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \boxed{\ddot{q}_1} + \mathbf{c}_1$$

(3)

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

$$\boxed{\ddot{q}_1} = \left(\mathbf{S}_1^T \mathbf{I}_1^A \mathbf{S}_1\right)^{-1} \left(\tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1^A \left({}^1\mathbf{X}_0 \mathbf{a}_0 + \mathbf{c}_1\right) + \mathbf{p}_1^A\right)\right)$$

(2)

Equations for link 2

$$\boxed{\mathbf{a}_2} = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \boxed{\ddot{q}_2} + \mathbf{c}_2$$

(5)

$$\mathbf{f}_2 = \mathbf{I}_2^A \cdot \mathbf{a}_2 + \mathbf{p}_2^A$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\boxed{\ddot{q}_2} = \left(\mathbf{S}_2^T \mathbf{I}_2^A \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2^A \left({}^2\mathbf{X}_1 \boxed{\mathbf{a}_1} + \mathbf{c}_2\right) + \mathbf{p}_2^A\right)\right)$$

(4)

Equations for link 3

$$\mathbf{a}_3 = {}^3\mathbf{X}_2 \cdot \mathbf{a}_2 + \mathbf{S}_3 \cdot \ddot{q}_3 + \mathbf{c}_3$$

$$\mathbf{f}_3 = \mathbf{I}_3 \cdot \mathbf{a}_3 + \mathbf{p}_3$$

$$\tau_3 = \mathbf{S}_3{}^T \cdot \mathbf{f}_3$$

$$\boxed{\ddot{q}_3} = \left(\mathbf{S}_3^T \mathbf{I}_3 \mathbf{S}_3\right)^{-1} \left(\tau_3 - \mathbf{S}_3^T \left(\mathbf{I}_3 \left({}^2\mathbf{X}_2 \boxed{\mathbf{a}_2} + \mathbf{c}_3\right) + \mathbf{p}_3\right)\right)$$

(6)

(1)

$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2^A \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2^A \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2^A \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2^A \, {}^2\mathbf{X}_1$$

$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2^A + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2^A \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2^A \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2^A \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2^A \mathbf{c}_2 + \mathbf{p}_2^A\right)\right)$$

$$\mathbf{I}_2^A = \mathbf{I}_2 + {}^2\mathbf{X}_3^* \cdot \mathbf{I}_3 \cdot {}^3\mathbf{X}_2 - {}^2\mathbf{X}_3^* \cdot \mathbf{I}_3 \cdot \mathbf{S}_3 \cdot \left(\mathbf{S}_3^T \mathbf{I}_3 \mathbf{S}_3\right)^{-1} \mathbf{S}_3^T \mathbf{I}_3 \, {}^3\mathbf{X}_2$$

$$\mathbf{p}_2^A = \mathbf{p}_2 + {}^2\mathbf{X}_3^* \cdot \mathbf{p}_3 + {}^2\mathbf{X}_3^* \cdot \mathbf{I}_3 \cdot \mathbf{c}_3 + {}^2\mathbf{X}_3^* \cdot \mathbf{I}_3 \cdot \mathbf{S}_3 \cdot \left(\mathbf{S}_3^T \mathbf{I}_3 \mathbf{S}_3\right)^{-1} \left(\tau_3 - \mathbf{S}_3^T \left(\mathbf{I}_3 \mathbf{c}_3 + \mathbf{p}_3\right)\right)$$

Naval Architecture & Ocean Engineering

Topics in ship design automation

6. Offshore Floating Wind Turbine

Prof. Kyu-Yeul Lee

September, 2010

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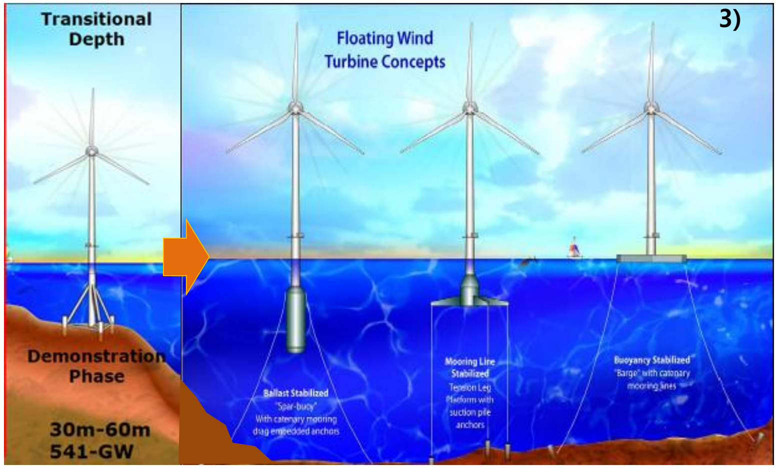
6.1 Introduction to Offshore Floating Wind Turbine

1.1 Motivation :
Pilot Project for Offshore Wind Turbine Farm

wind speed 6.7~7.5m
depth : about 20m



✓**Offshore Wind Turbine Farm**[1)]

✓Marine Operation Cost[2)]

- about 55% of the initial cost

✓Floating Offshore Wind Turbine (in the near future)

- **At some water depth, floating platforms will be more economical than fixed bottom substructure**

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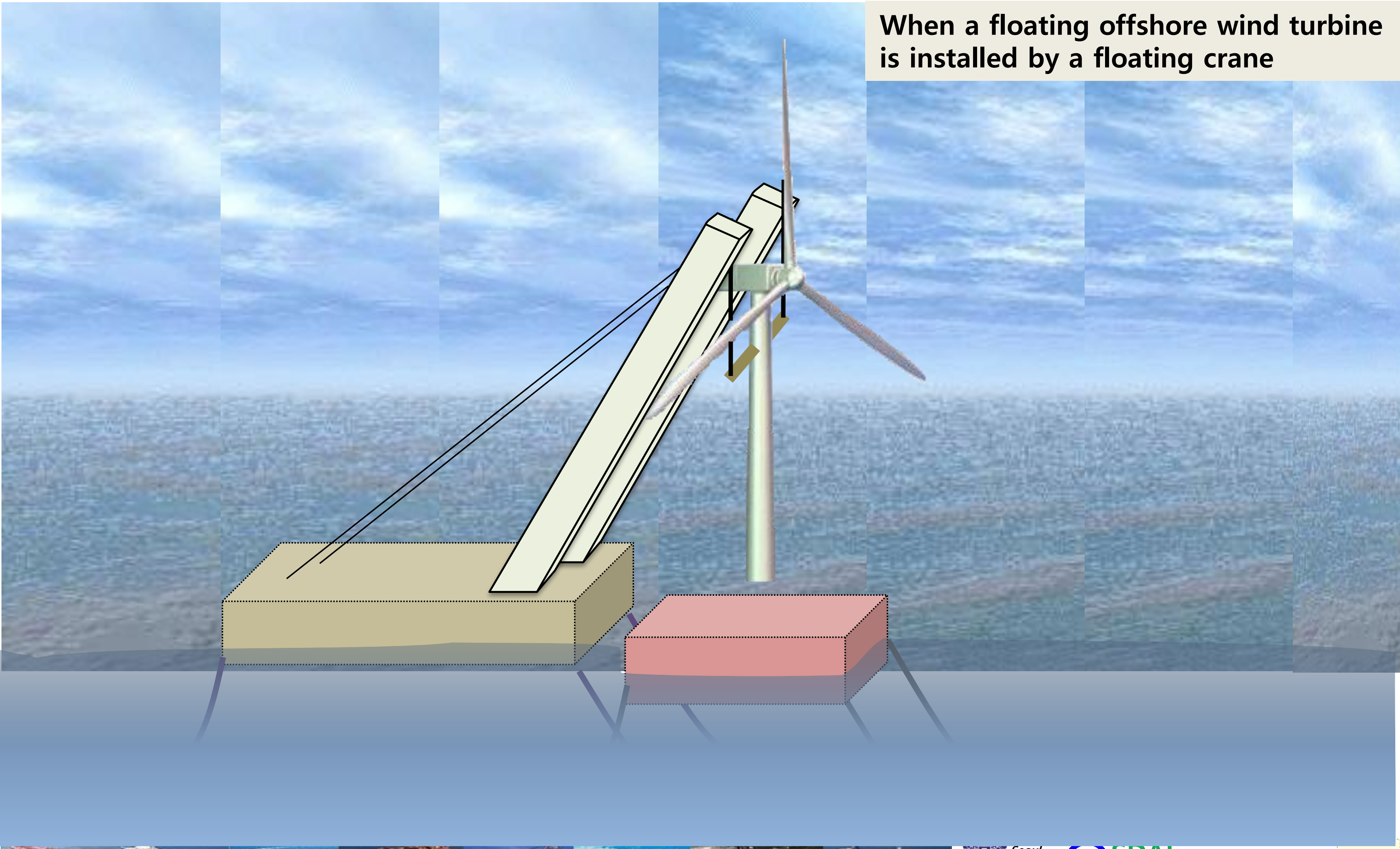
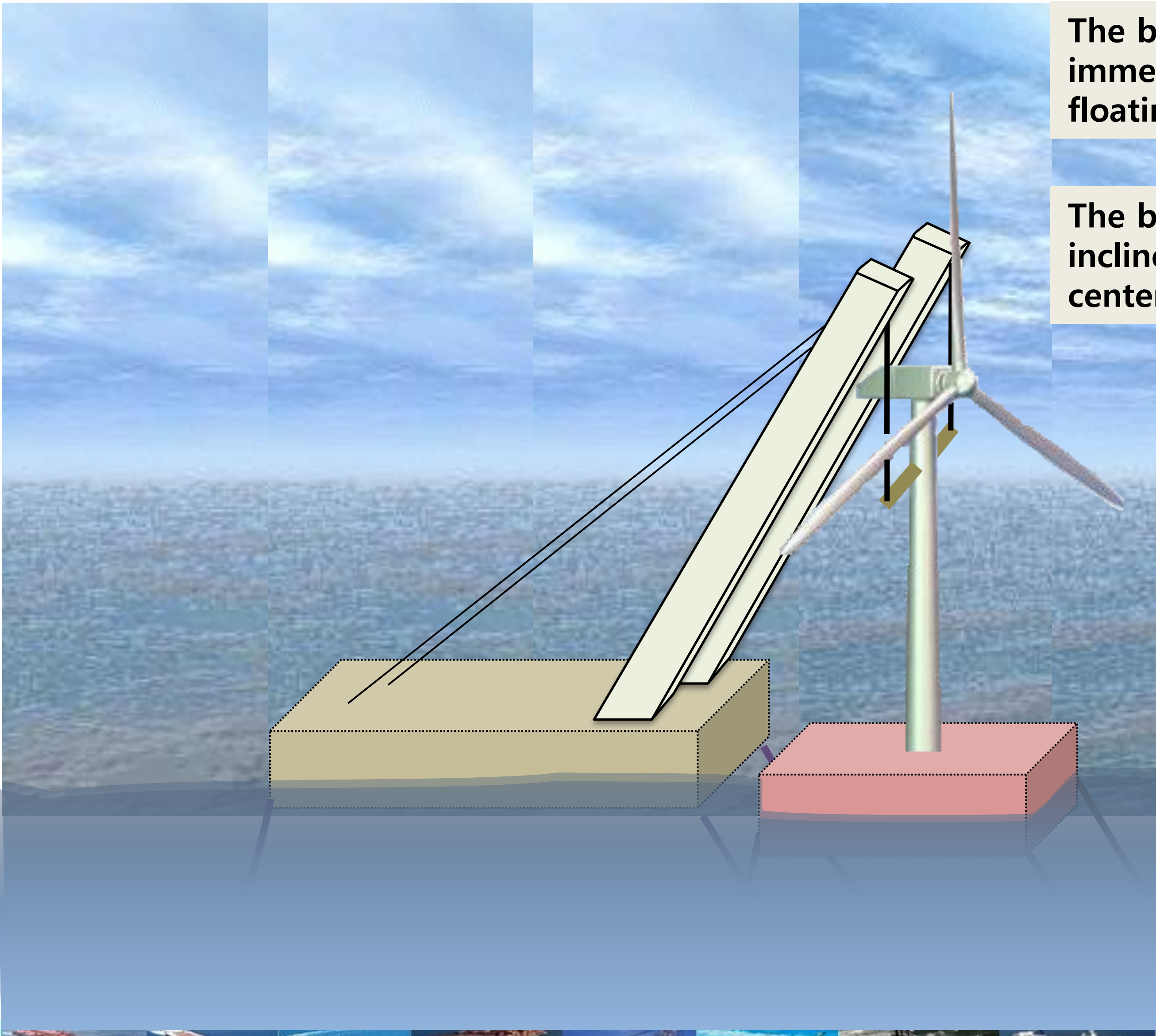
1) 아시아 경제 , 2010.11.2일자 기사
2) Fingersh, L., Hand, M. and Laxson, A., 2006, Wind Turbine Design Cost and Scaling Model, Technical Report, NREL 500-40566, 2006
3) Figure from Jonkman, J., Dynamic Modeling and Loads Analysis of an Offshore Floating Turbine, NREL-TP-500-41958, 2007

1.2 Floating Offshore Wind Turbine in Marine Operations



When a floating offshore wind turbine is installed by a floating crane

1.2 Floating Offshore Wind Turbine in Marine Operations

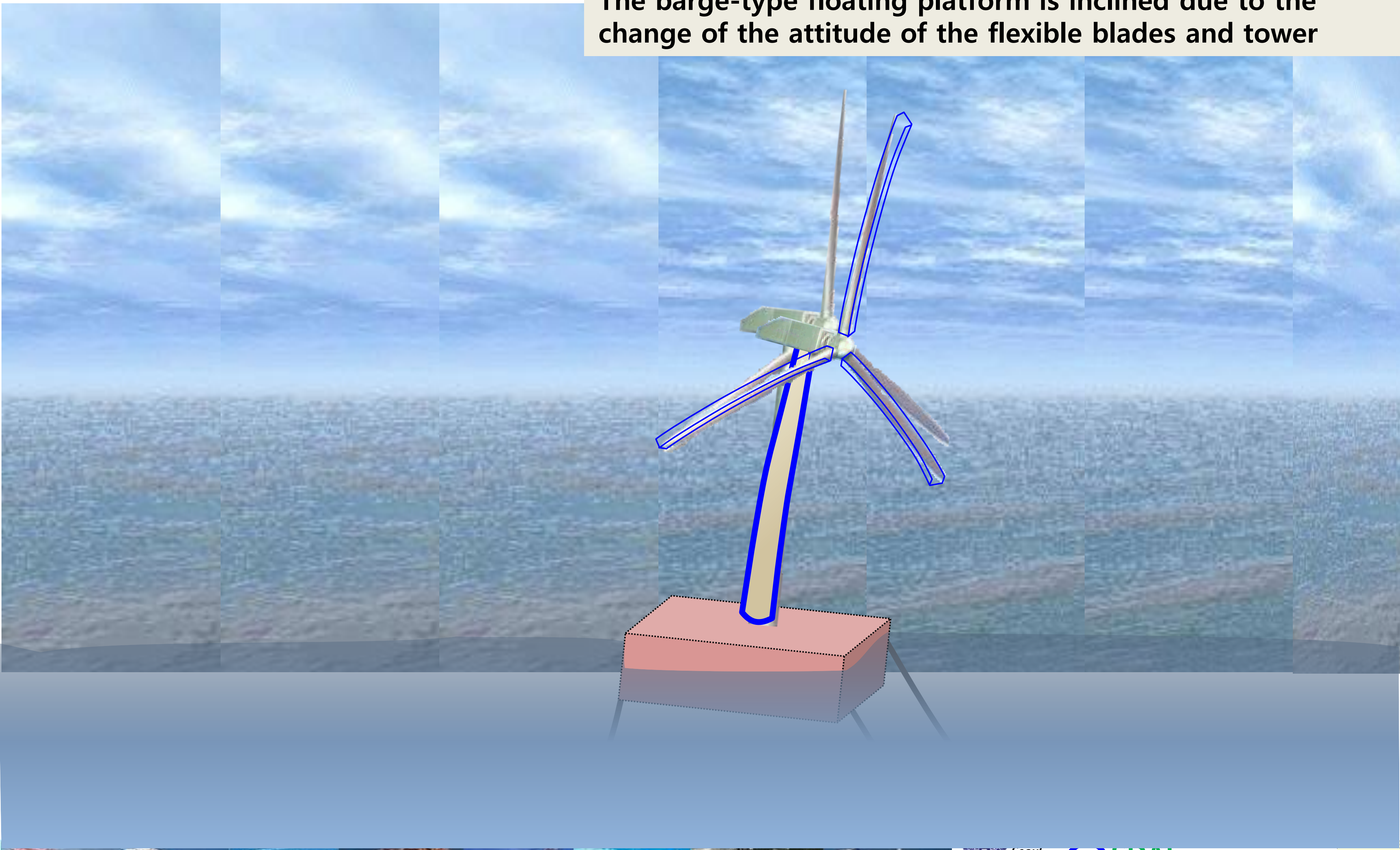


The barge-type floating platform is immersed by the weight of the floating offshore wind turbine

The barge-type floating platform is inclined due to the difference of the center of mass and buoyancy.

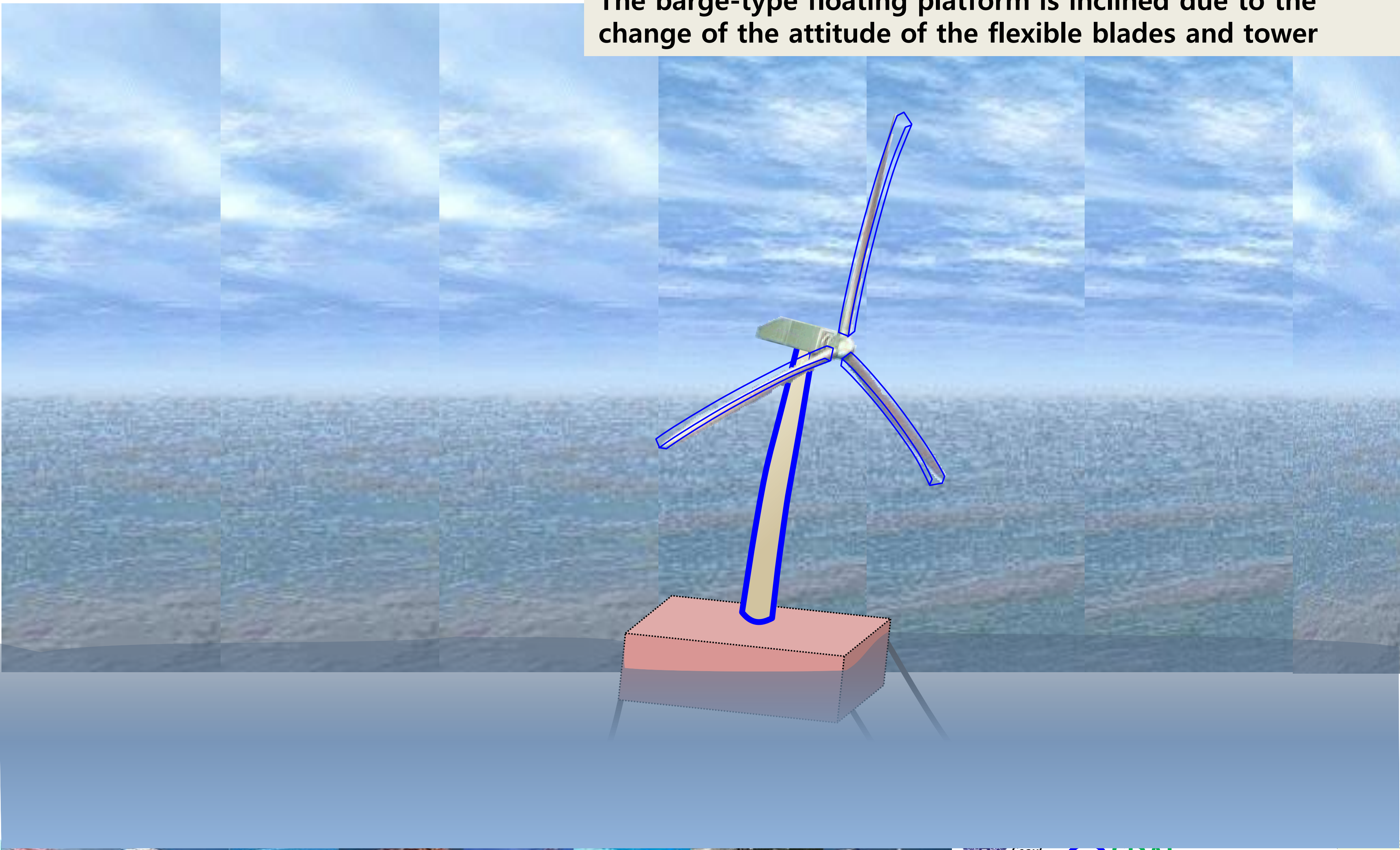
1.2 Floating Offshore Wind Turbine in Marine Operations

The barge-type floating platform is inclined due to the change of the attitude of the flexible blades and tower



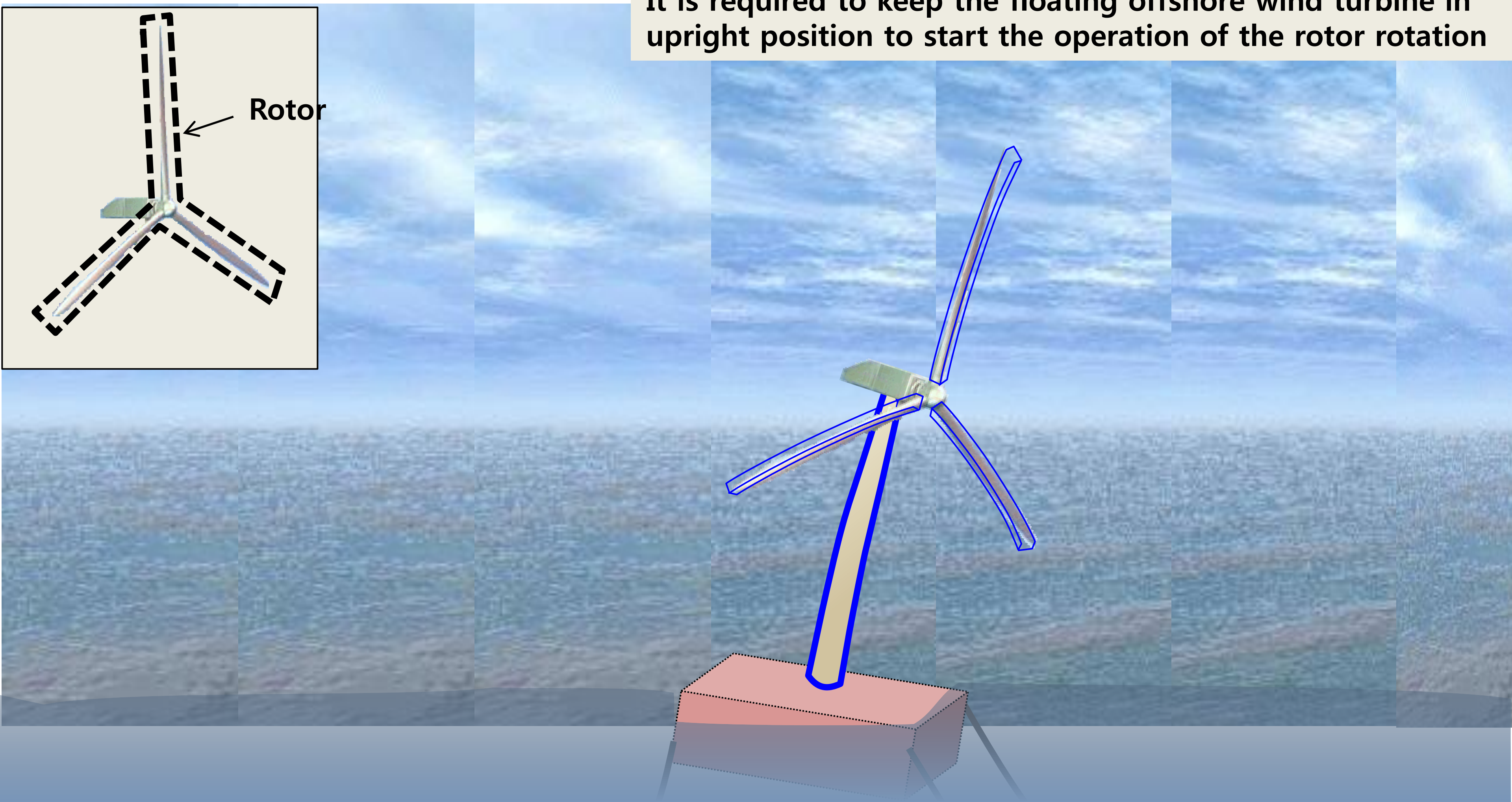
1.2 Floating Offshore Wind Turbine in Marine Operations

The barge-type floating platform is inclined due to the change of the attitude of the flexible blades and tower



1.2 Floating Offshore Wind Turbine in Marine Operations

It is required to keep the floating offshore wind turbine in upright position to start the operation of the rotor rotation



1.2 Floating Offshore Wind Turbine in Marine Operations

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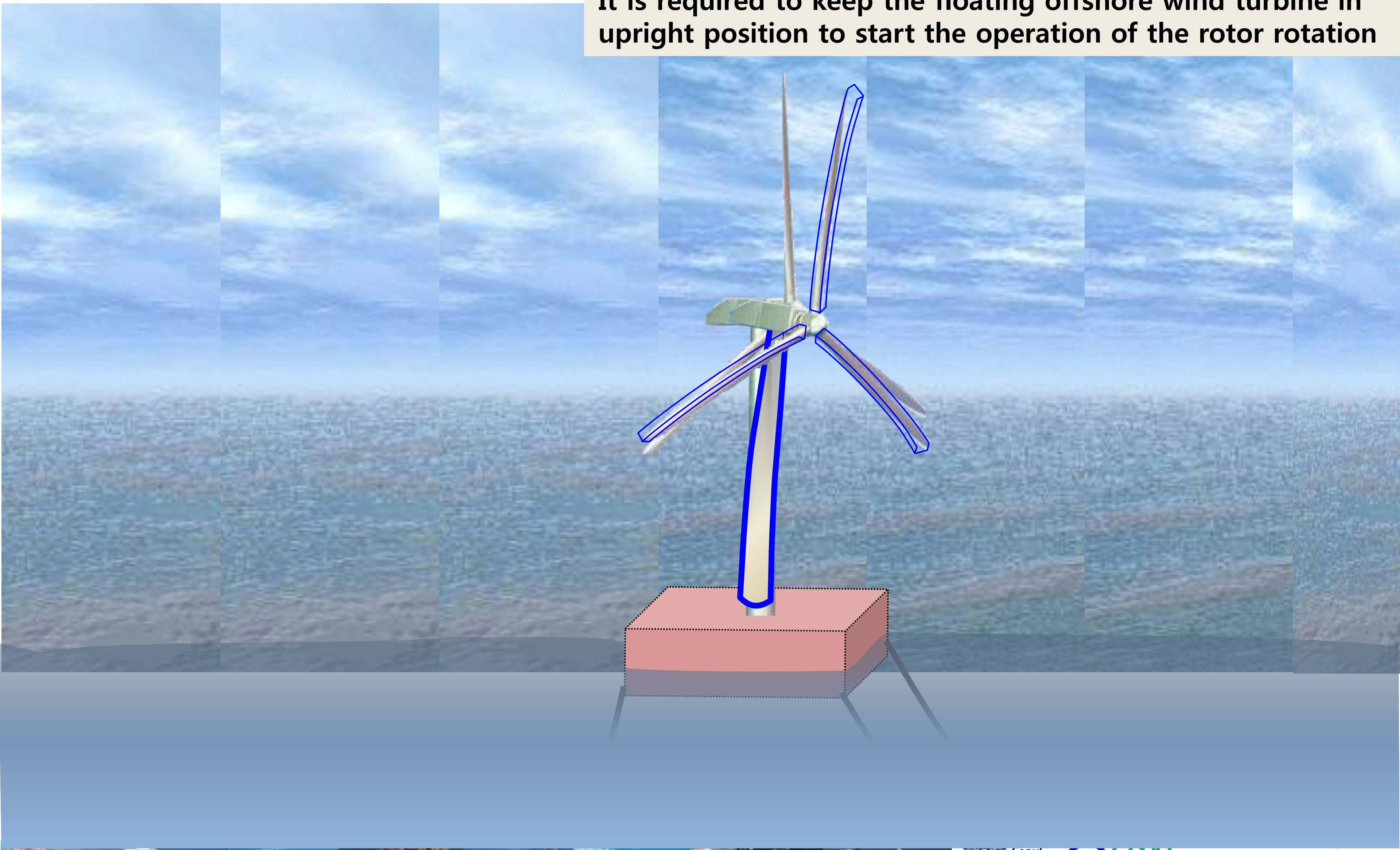
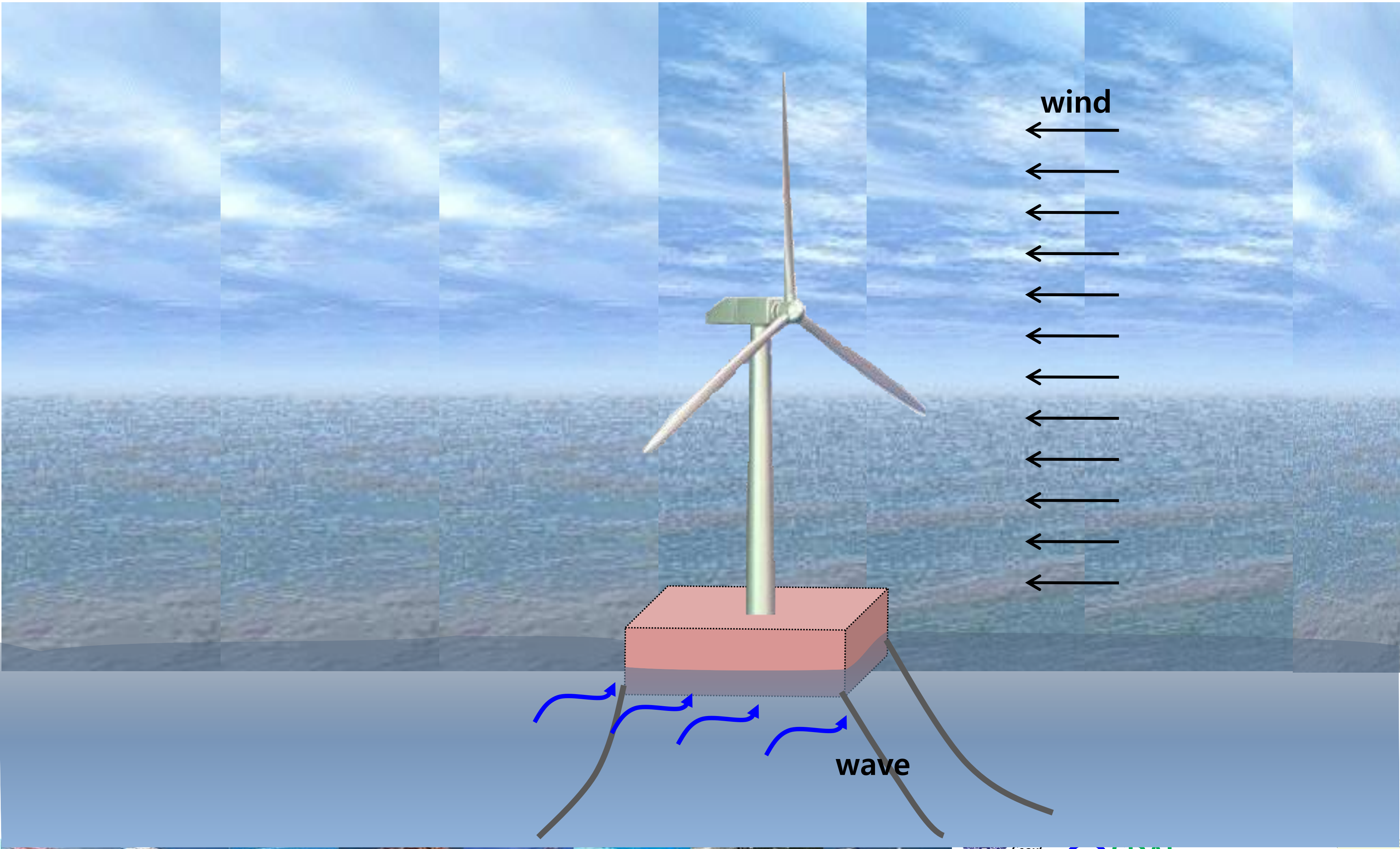
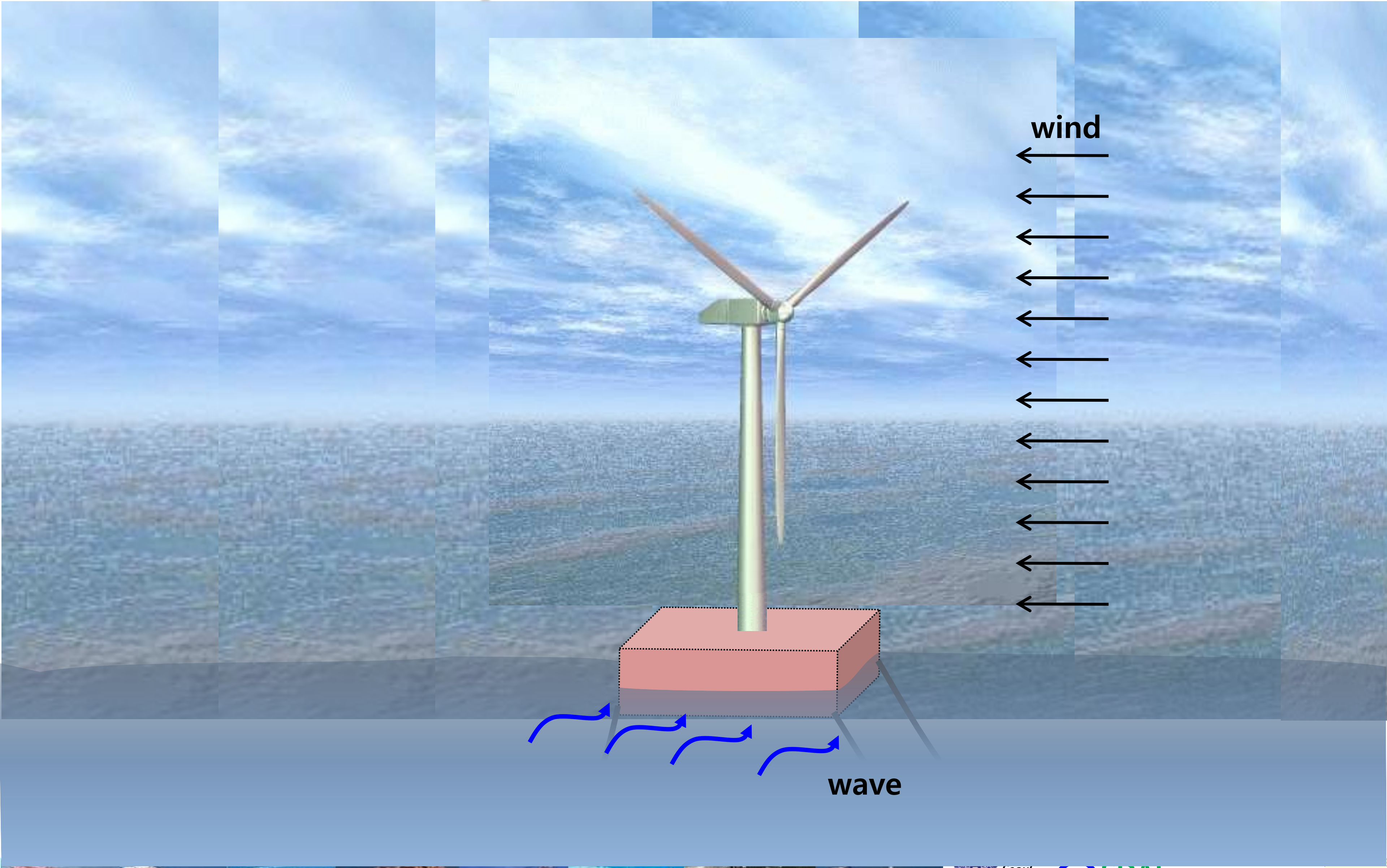


1.2 Floating Offshore Wind Turbine in Marine Operations



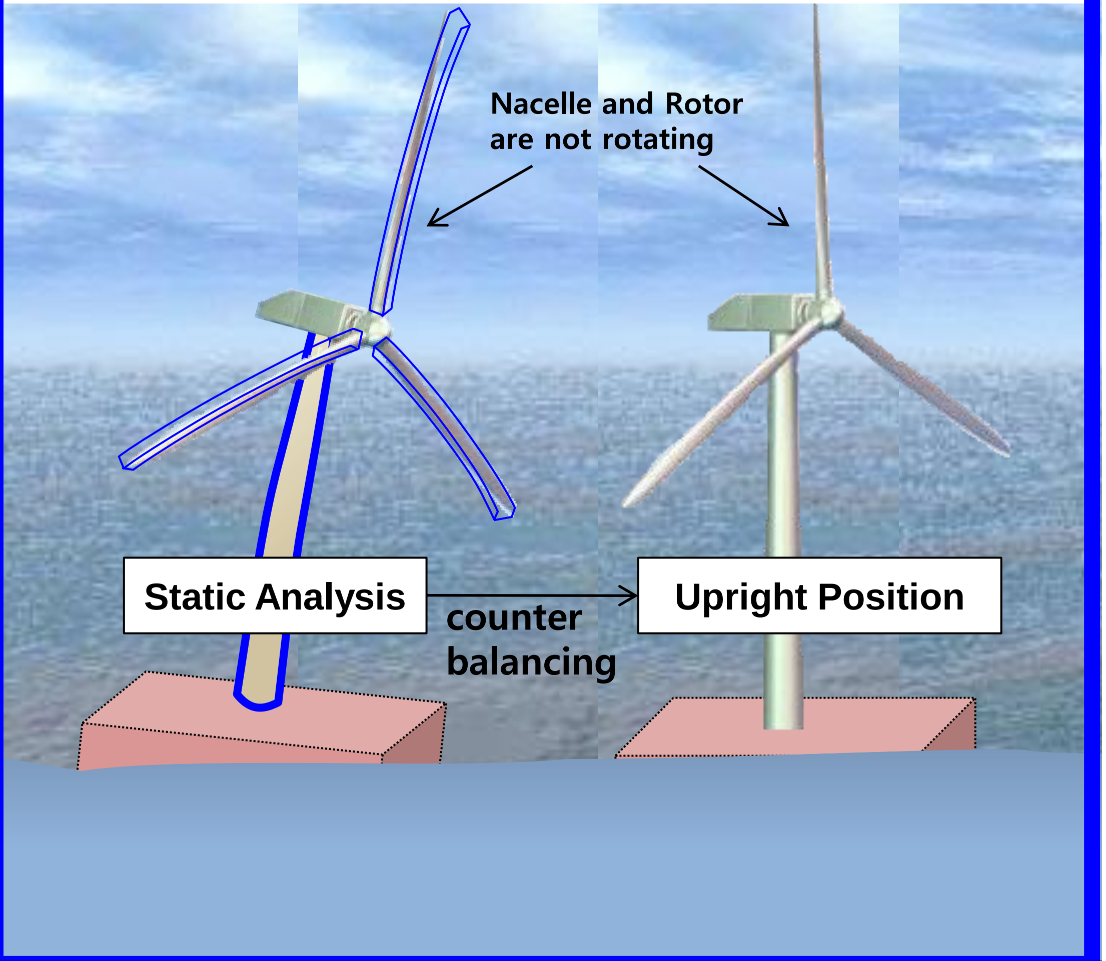
1.2 Floating Offshore Wind Turbine in Marine Operations



1.3 Objective and Scope

Static Analysis

To keep the floating offshore wind turbine in upright position, we have to calculate the initial position and attitude by hydrostatic equilibrium and structural equilibrium



Example of Counter Balancing

Why is it necessary to calculate the position and attitude of a floating body in a large inclination for the static analysis?

<before lifting a heavy cargo>

<after lifting a heavy cargo>

Floating crane
L : 110m, B : 45m, T : 7.5m
Capacity 3,600 Mg
Light Weight : 9,500 Mg

Heavy Cargo : 2,608 Mg

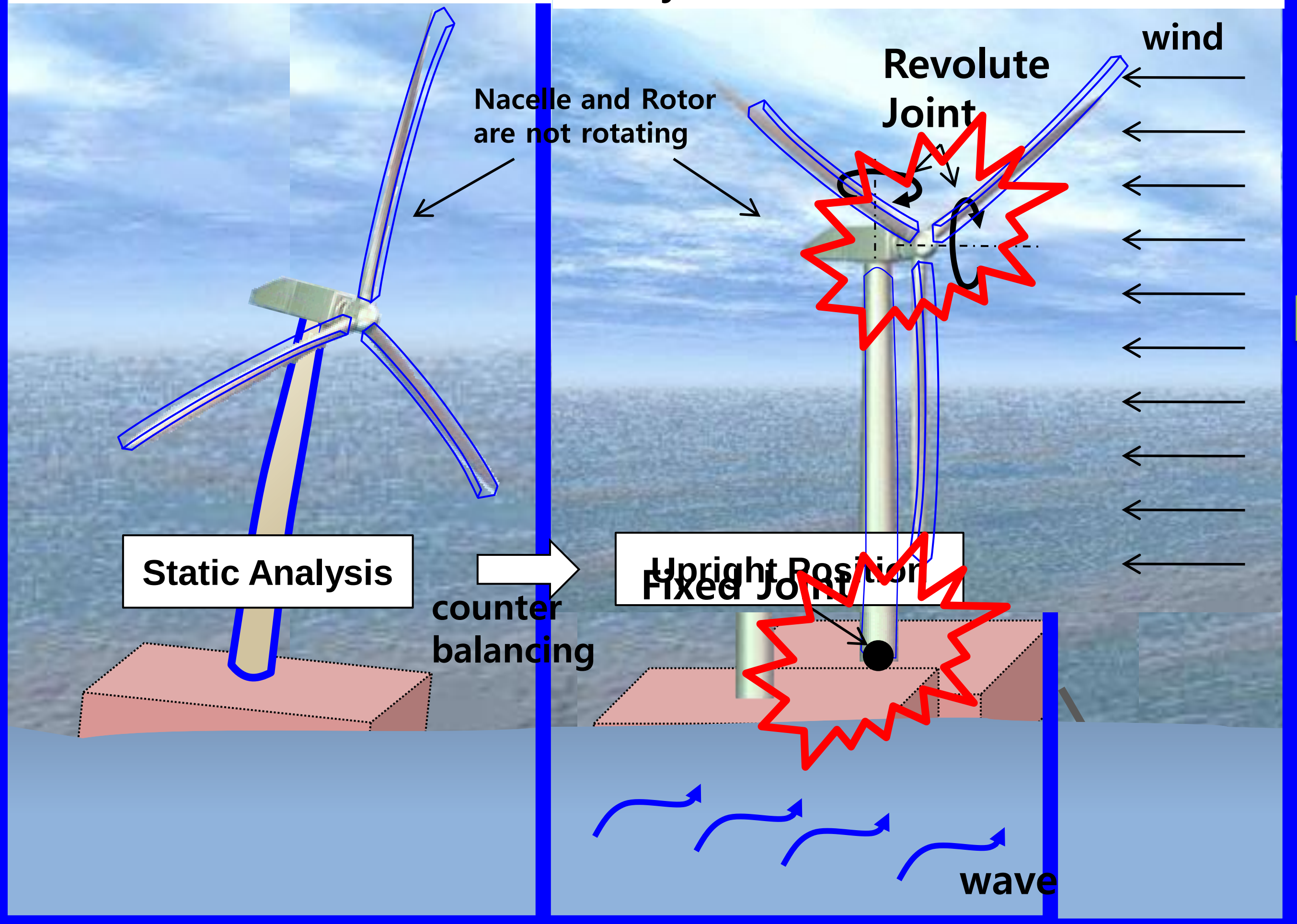
1.3 Objective and Scope

Static Analysis

To keep the floating offshore w position, we have to calculate t attitude by hydrostatic equilibr equilibrium

Dynamic Response Analysis

Dynamic motion of the floating offshore wind turbine which consists of the rigid and flexible bodies with the revolute and fixed joints.

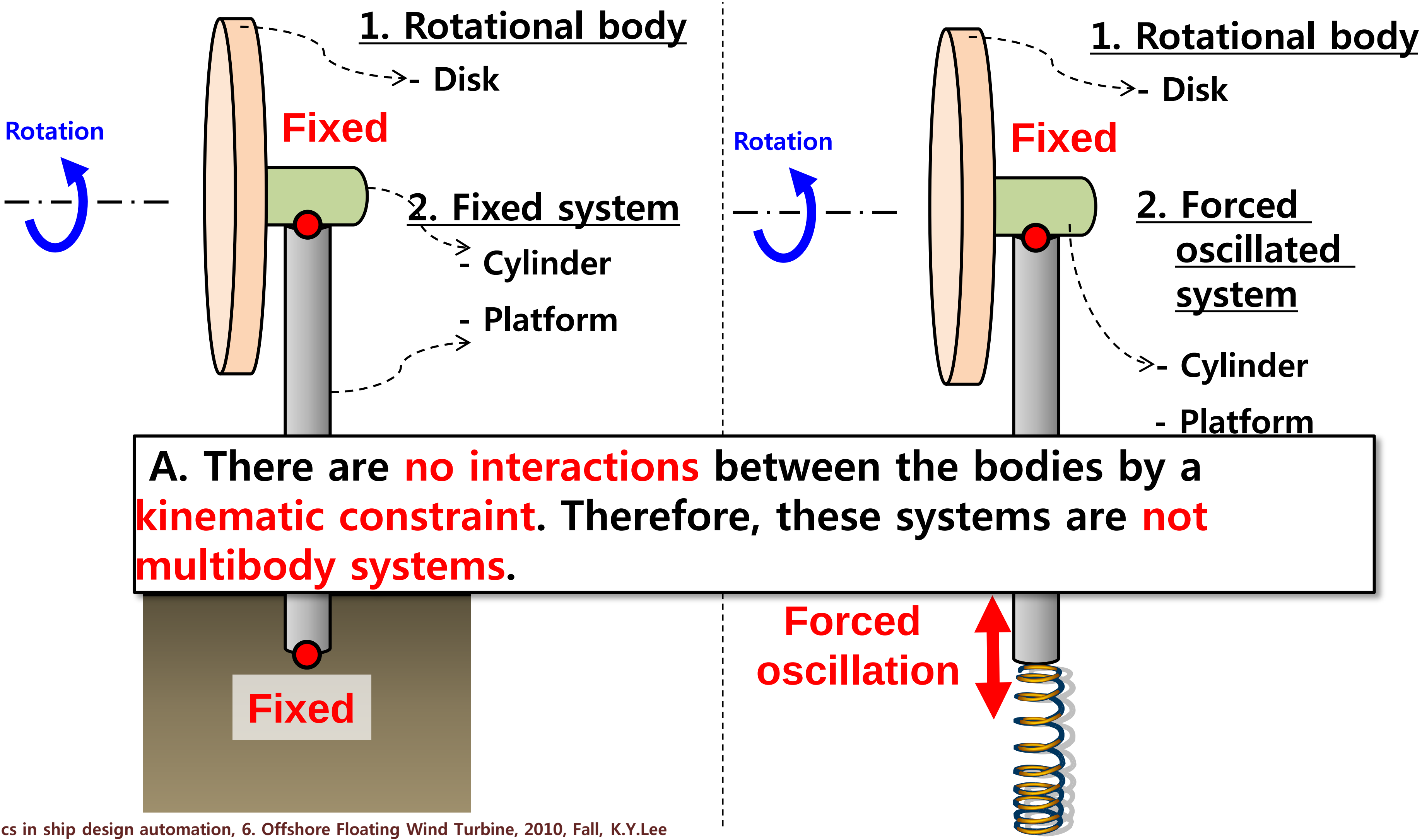


Flexible Multibody

2. Equations of Motion of Flexible Multibody System :

Comparison of multibody and non-multibody systems

non-multibody systems



Rotation

Fixed

1. Rotational body
   - Disk
2. Fixed system
   - Cylinder
   - Platform

Fixed

Rotation

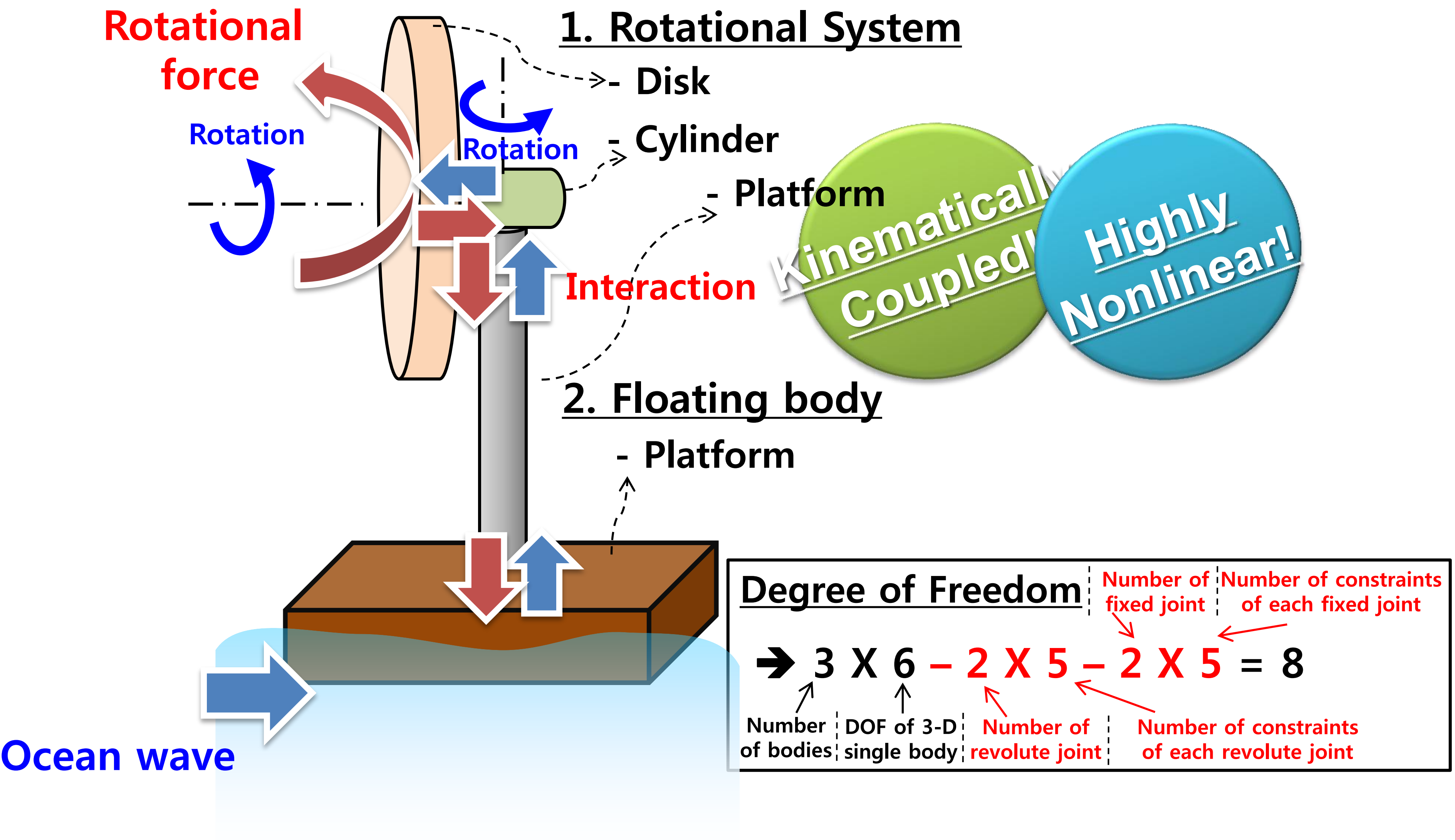
Fixed

1. Rotational body
   - Disk
2. Forced oscillated system
   - Cylinder
   - Platform

Forced oscillation

A. There are no interactions between the bodies by a kinematic constraint. Therefore, these systems are not multibody systems.

2. Equations of Motion of Flexible Multibody System : Comparison of multibody and non-multibody systems



1. Rotational System

- Disk
- Cylinder
- Platform

2. Floating body

- Platform

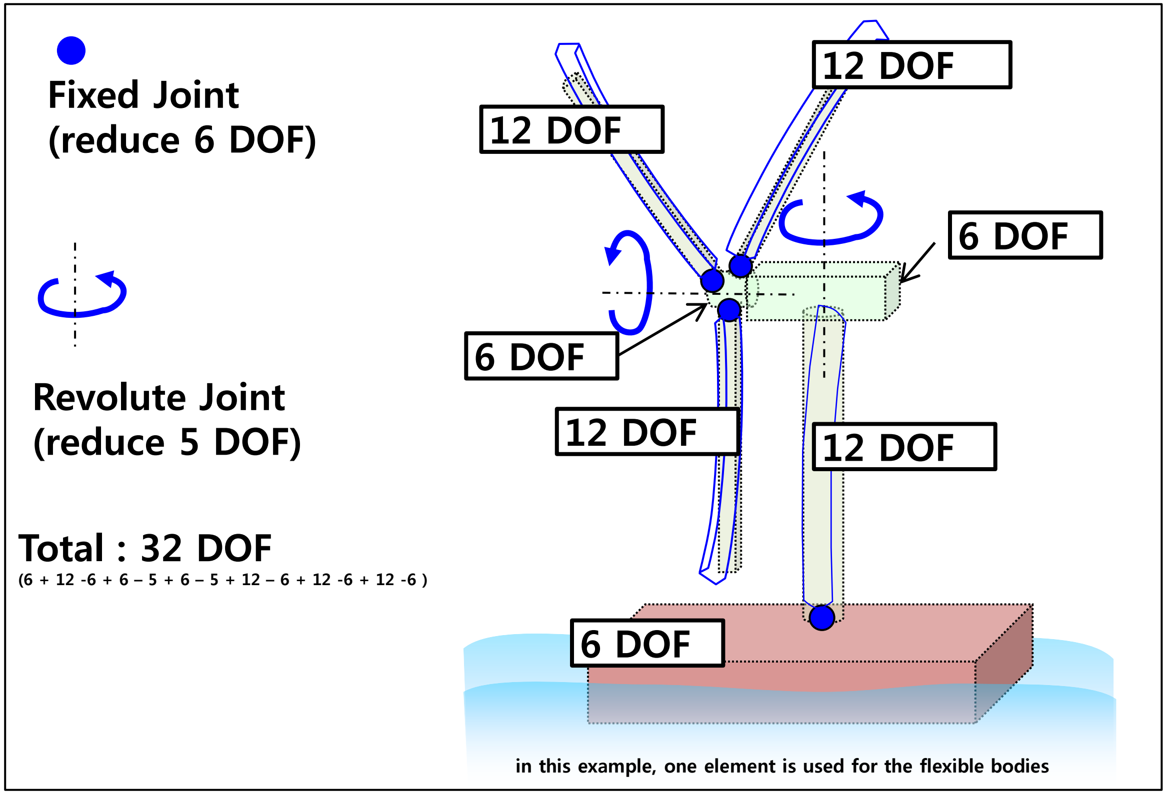
Degree of Freedom

$$\rightarrow \underbrace{3}_{\text{Number of bodies}} \times \underbrace{6}_{\text{DOF of 3-D single body}} - \underbrace{2}_{\text{Number of revolute joint}} \times \underbrace{5}_{\text{Number of constraints of each revolute joint}} - \overbrace{2}^{\text{Number of fixed joint}} \times \overbrace{5}^{\text{Number of constraints of each fixed joint}} = 8$$

2. Equations of Motion of Flexible Multibody System: "Multibody" in Mechanical Engineering

DOF : Degree of Freedom

The motion of a multibody system is kinematically constrained because of different types of joints[1)]



Total : 32 DOF

(6 + 12 -6 + 6 – 5 + 6 – 5 + 12 – 6 + 12 -6 + 12 -6)

in this example, one element is used for the flexible bodies

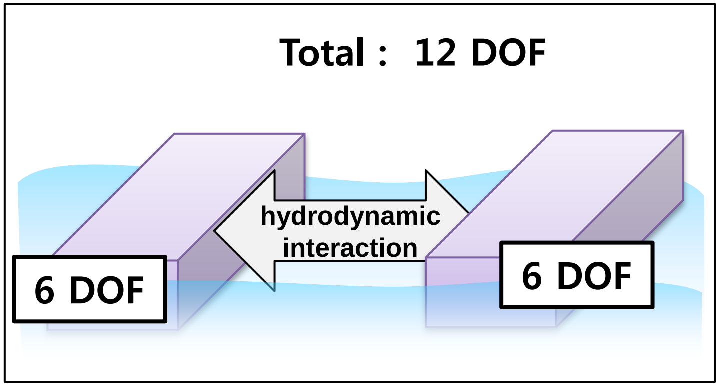
Reference:

1) Shabana,A.A. "Dynamics of Multibody Systems,", Third Edition, Cambridge Univ. Press. 2005, p.1.

2) Shabana,A.A., "Computational Dynamics", John Wiley & Sons, 1994, p.3

"The bodies in a mechanical system are not free to have arbitrary displacements because they are connected by joints or force elements. While a force element such as springs and dampers may significantly affect the motion of the bodies in one or more directions. As a consequence, a force element does not reduce the number of independent coordinates required to describe the configuration of the system. ...The joints reduce the number of independent coordinates of the system since they prevent motion in some directions.

C.f) "Multiple floating body" in Hydrodynamics



Reference :

1) Kim, K.H., Kim, Y., Kim, M.S. (2009) "Numerical Analysis on Motion Responses of Adjacent Multiple Floating Bodies by Using Rankine Panel Method, IJOPE, Vol.19 , No.2.

"In multiple-body problem, the DOF is determined by the number of multiple bodies, For example, if there are 2 freely floating ships and each body is rigid, it has 12 DOF."

2) Fang, MC and Chen, GR (2001). "The Relative Motion and Wave Elevation Between two Floating Structures in Waves," Proc 11th Int Offshore and Polar Eng, ISOPE, Vol 1, pp 361-368.

"Including the three-dimensional hydrodynamic interaction between two moving ships, a set of equations of twelve coupled motions were obtained.."

3) Hong, SY, Kim, JH, Cho, YR and Kim, YS (2005). "Numerical and Experimental Study on Hydrodynamic Interaction of Side-by-side Moored Multiple Vessels," Ocean Eng, Vol 32, pp 783-801.

"multi-body hydrodynamic interaction problem", " 6 x number of bodies DOF assuming each body behaves as a rigid body"

6.2 Equations of Motion for Offshore Floating Wind Turbine Using Embedding Formulation

2.1 Equations of Motion
Reference Frame defined with respect to Center of Mass G

The floating offshore wind turbine consists of the barge type floating platform, tower, nacelle, hub, and three blades with the fixed and revolute joints.

At this moment, all the bodies are regarded as rigid bodies

The body fixed reference frame are defined at the center of mass of each body

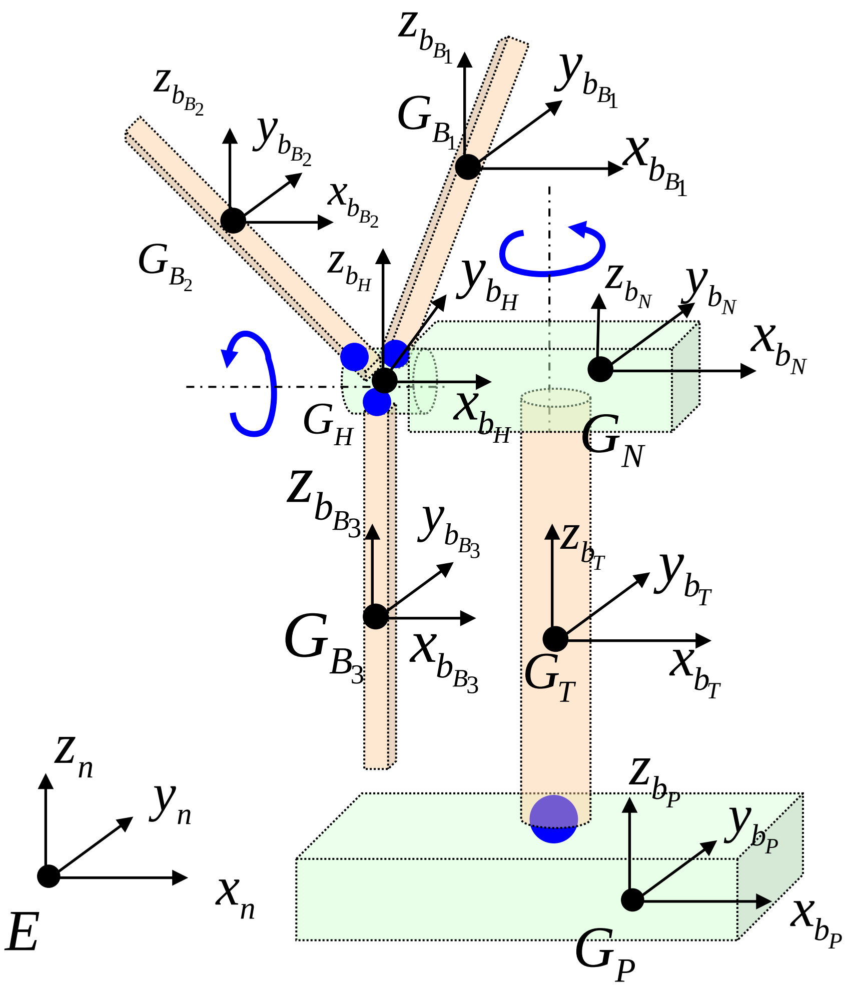
● **Fixed Joint**

⟳ **Revolute Joint**

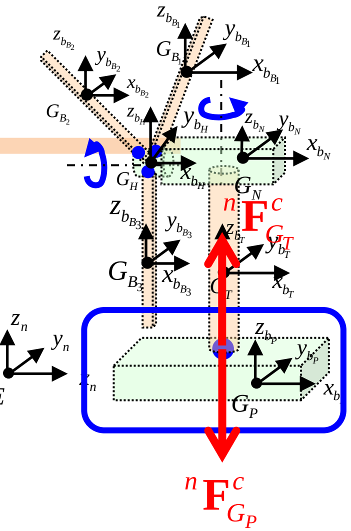
G **: center of mass of each body**

<subindex>

P : Platform(barge type)
T : Tower
N : Nacelle
H : Hub
B1, B2, B3 : Blade1, 2, 3



2.1 Equations of Motion
Equations of Motion of Multibody with respect to Center of Mass



Example) Newton-Euler equation for the platform

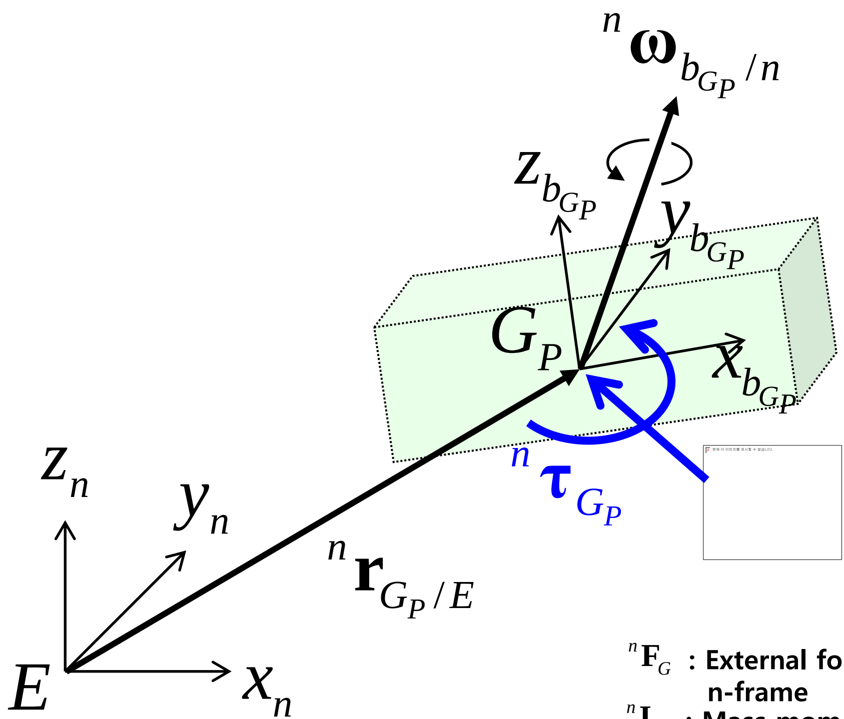
$$m\ {}^n\ddot{\mathbf{r}}_{G_P/E} = {}^n\mathbf{F}_{G_P} + {}^n\mathbf{F}^c_{G_P}$$

$${}^n\mathbf{I}_G\ {}^n\dot{\boldsymbol{\omega}}_{b_{G_P}/n} + {}^n\boldsymbol{\omega}_{b_{G_P}/n} \times {}^n\mathbf{I}_G\ {}^n\boldsymbol{\omega}_{b_{G_P}/n} = {}^n\boldsymbol{\tau}_{G_P} + {}^n\boldsymbol{\tau}^c_{G_P}$$

$$\mathbf{r}_{G_P/E} \equiv \left[x_{G_P/E}\,; y_{G_P/E}\,; z_{G_P/E}\right]^T_E$$

$$\dot{\mathbf{r}}_{G_P/E} = \left[\dot{x}_{G_P/E}\ ; \dot{y}_{G_P/E}\ ; \dot{z}_{G_P/E}\right]^T$$

$$\ddot{\mathbf{r}}_{G_P/E} = \left[\ddot{x}_{G_P/E}\ ; \ddot{y}_{G_P/E}\ ; \ddot{z}_{G_P/E}\right]^T$$



$$\boldsymbol{\chi}_{b_{G_P}/n} \equiv \left[\int \omega_{b_{G_P}/n,x}\,dt\ ;\ \int \omega_{b_{G_P}/n,y}\,dt\ ;\ \int \omega_{b_{G_P}/n,z}\,dt\right]^T$$

$$\boldsymbol{\omega}_{b_{G_P}/n} = \left[\omega_{b_{G_P}/n,x}\ ;\omega_{b_{G_P}/n,y}\ \ ;\omega_{b_{G_P}/n,z}\right]^T$$

$$\dot{\boldsymbol{\omega}}_{b_{G_P}/n} = \left[\dot{\omega}_{b_{G_P}/n,x}\ ;\dot{\omega}_{b_{G_P}/n,y}\ \ ;\dot{\omega}_{b_{G_P}/n,z}\right]^T$$

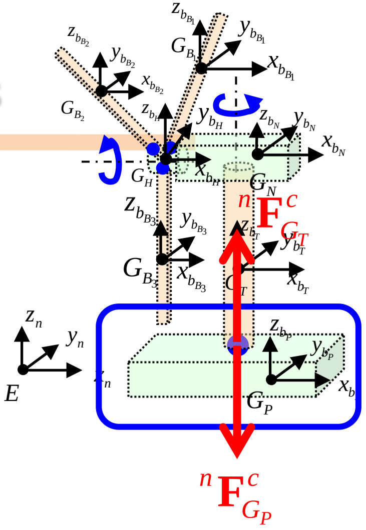
$$\boldsymbol{\omega}_{b_{G_P}/n} = \mathbf{G}\dot{\boldsymbol{\gamma}}$$

n-frame: Earth-fixed inertial frame
b-frame: body-fixed frame
m: mass of a body
${}^n\mathbf{r}_{G/E}$: Position vector of point G with respect to point E decomposed in n-frame

${}^n\mathbf{F}_G$: External force acting on point G decomposed in n-frame
${}^n\mathbf{I}_G$: Mass moment of inertia about certain axis through point G decomposed in n-frame
${}^n\boldsymbol{\omega}_{b/n}$: Angular velocity of b-frame with respect to n-frame decomposed in n-frame
${}^n\boldsymbol{\tau}_G$: External moment about certain axes through point G decomposed in n-frame

$\boldsymbol{\gamma}$: ZYX-Euler angle
${}^n\mathbf{F}^c_G$: Constraint force acting on point G decomposed in n-frame
${}^n\boldsymbol{\tau}^c_G$: Constraint moment about certain axes through point G decomposed in n-frame

2.1 Equations of Motion
Equations of Motion of Multibody with respect to Center of Mass



Newton-Euler equation for the platform

$$m\,{}^{n}\ddot{\mathbf{r}}_{G_P/E} = {}^{n}\mathbf{F}_{G_P} + {}^{n}\mathbf{F}_{G_P}^{c}$$

$${}^{n}\mathbf{I}_G\,{}^{n}\dot{\boldsymbol{\omega}}_{b_{G_P}/n} + {}^{n}\boldsymbol{\omega}_{b_{G_P}/n} \times {}^{n}\mathbf{I}_G\,{}^{n}\boldsymbol{\omega}_{b_{G_P}/n} = {}^{n}\boldsymbol{\tau}_{G_P} + {}^{n}\boldsymbol{\tau}_{G_P}^{c}$$

If we rewrite the force and moment equation in matrix form,

$$\underbrace{\begin{bmatrix} \mathbf{m}_P & 0 \\ 0 & \mathbf{I}_{G_P} \end{bmatrix}}_{\mathbf{M}_P} \underbrace{\begin{bmatrix} \ddot{\mathbf{r}}_{G_P/E} \\ \dot{\boldsymbol{\omega}}_{b_{G_P}/n} \end{bmatrix}}_{\ddot{\mathbf{s}}_P} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \tilde{\boldsymbol{\omega}}_{b_{G_P}/n}\mathbf{I}_{G_P} \end{bmatrix}}_{\mathbf{Q}_P} \underbrace{\begin{bmatrix} \dot{\mathbf{r}}_{G_P/E} \\ \boldsymbol{\omega}_{b_{G_P}/n} \end{bmatrix}}_{\dot{\mathbf{s}}_P} = \underbrace{\begin{bmatrix} \mathbf{F}_{G_P} \\ \boldsymbol{\tau}_{G_P} \end{bmatrix}}_{\boxed{\mathbf{F}_P}} + \underbrace{\begin{bmatrix} \mathbf{F}_{G_P}^{c} \\ \boldsymbol{\tau}_{G_P}^{c} \end{bmatrix}}_{\boxed{\mathbf{F}_P^{C}}}$$

$,\ \tilde{\boldsymbol{\omega}}_{b_{G_P}/n}$ **is the skew matrix of** $\boldsymbol{\omega}_{b_{G_P}/n}$

external force & moment

constraint force & moment

$$\boldsymbol{\omega}\times\mathbf{r} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underbrace{\boxed{\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}}_{\tilde{\boldsymbol{\omega}}} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$\mathbf{M}_P\ddot{\mathbf{s}}_P + \mathbf{Q}_P\dot{\mathbf{s}}_P = \mathbf{F}_P + \mathbf{F}_P^{C}$$

$$\dot{\mathbf{s}}_P = \left[\dot{x}_{G_P/E}\ ;\dot{y}_{G_P/E}\ ;\dot{z}_{G_P/E}\ ;\omega_{b_{G_P}/n,x}\ ;\omega_{b_{G_P}/n,y}\ ;\omega_{b_{G_P}/n,z}\right]^T$$

$$\ddot{\mathbf{s}}_P = \left[\ddot{x}_{G_P/E}\ ;\ddot{y}_{G_P/E}\ ;\ddot{z}_{G_P/E}\ ;\dot{\omega}_{b_{G_P}/n,x}\ ;\dot{\omega}_{b_{G_P}/n,y}\ ;\dot{\omega}_{b_{G_P}/n,z}\right]^T$$

2.1 Equations of Motion

Force and Moment Equations for Floating Offshore Wind Turbine with respect to Center of Mass

Newton-Euler equation for the platform

$$\mathbf{M}_P\ddot{\mathbf{s}}_P+\mathbf{Q}_P\dot{\mathbf{s}}_P=\mathbf{F}_P+\mathbf{F}_P^c$$

Newton-Euler equation for the tower

$$\mathbf{M}_T\ddot{\mathbf{s}}_T+\mathbf{Q}_T\dot{\mathbf{s}}_T=\mathbf{F}_T+\mathbf{F}_T^c$$

Newton-Euler equation for the nacelle

$$\mathbf{M}_N\ddot{\mathbf{s}}_N+\mathbf{Q}_N\dot{\mathbf{s}}_N=\mathbf{F}_N+\mathbf{F}_N^c$$

Newton-Euler equation for the hub

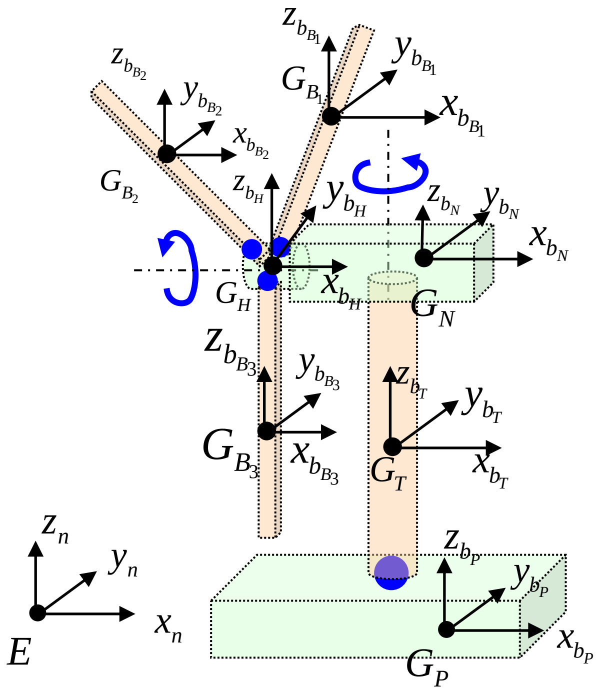
$$\mathbf{M}_H\ddot{\mathbf{s}}_H+\mathbf{Q}_H\dot{\mathbf{s}}_H=\mathbf{F}_H+\mathbf{F}_H^c$$

Newton-Euler equation for the blade 1, 2, 3

$$\mathbf{M}_{B1}\ddot{\mathbf{s}}_{B1}+\mathbf{Q}_{B1}\dot{\mathbf{s}}_{B1}=\mathbf{F}_{B1}+\mathbf{F}_{B1}^c$$

$$\mathbf{M}_{B2}\ddot{\mathbf{s}}_{B2}+\mathbf{Q}_{B2}\dot{\mathbf{s}}_{B2}=\mathbf{F}_{B2}+\mathbf{F}_{B2}^c$$

$$\mathbf{M}_{B3}\ddot{\mathbf{s}}_{B3}+\mathbf{Q}_{B3}\dot{\mathbf{s}}_{B3}=\mathbf{F}_{B3}+\mathbf{F}_{B3}^c$$

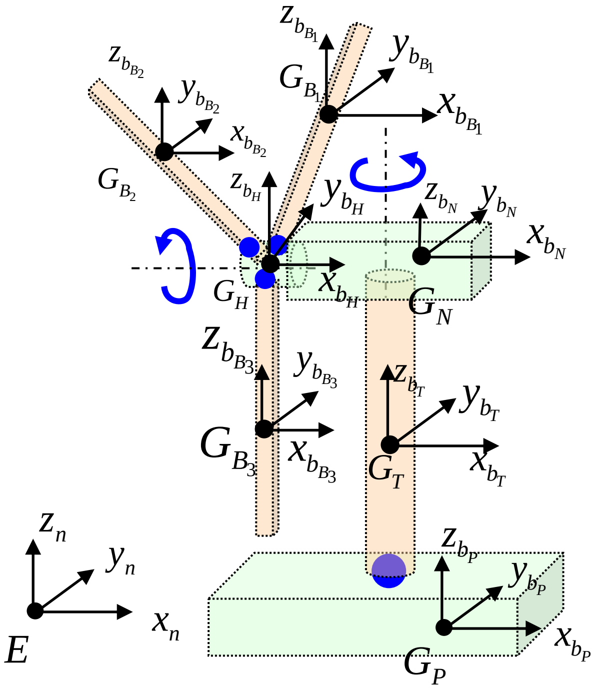


2.1 Equations of Motion
Force and Moment Equations for Floating Offshore Wind Turbine with respect to Center of Mass

Newton-Euler equation for the Floating Offshore Wind Turbine

$$\begin{bmatrix} \mathbf{M}_P &&&&&& \\ & \mathbf{M}_T &&&&& \\ && \mathbf{M}_N &&&& \\ &&& \mathbf{M}_H &&& \\ &&&& \mathbf{M}_{B1} && \\ &&&&& \mathbf{M}_{B2} & \\ &&&&&& \mathbf{M}_{B3} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{s}}_P \\ \dot{\mathbf{s}}_T \\ \dot{\mathbf{s}}_N \\ \dot{\mathbf{s}}_H \\ \dot{\mathbf{s}}_{B1} \\ \dot{\mathbf{s}}_{B2} \\ \dot{\mathbf{s}}_{B3} \end{bmatrix} +$$

$$\begin{bmatrix} \mathbf{Q}_P &&&&&& \\ & \mathbf{Q}_T &&&&& \\ && \mathbf{Q}_N &&&& \\ &&& \mathbf{Q}_H &&& \\ &&&& \mathbf{Q}_{B1} && \\ &&&&& \mathbf{Q}_{B2} & \\ &&&&&& \mathbf{Q}_{B3} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{s}}_P \\ \ddot{\mathbf{s}}_T \\ \ddot{\mathbf{s}}_N \\ \ddot{\mathbf{s}}_H \\ \ddot{\mathbf{s}}_{B1} \\ \ddot{\mathbf{s}}_{B2} \\ \ddot{\mathbf{s}}_{B3} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_P \\ \mathbf{F}_T \\ \mathbf{F}_N \\ \mathbf{F}_H \\ \mathbf{F}_{B1} \\ \mathbf{F}_{B2} \\ \mathbf{F}_{B3} \end{bmatrix} + \begin{bmatrix} \mathbf{F}_P^c \\ \mathbf{F}_T^c \\ \mathbf{F}_N^c \\ \mathbf{F}_H^c \\ \mathbf{F}_{B1}^c \\ \mathbf{F}_{B2}^c \\ \mathbf{F}_{B3}^c \end{bmatrix}$$



2.1 Equations of Motion
Force and Moment Equations for Floating Offshore Wind Turbine with respect to Center of Mass

Newton-Euler equation for the Floating Offshore Wind Turbine

$$\mathbf{M}\ddot{\mathbf{s}} + \mathbf{Q}\dot{\mathbf{s}} = \mathbf{F} + \mathbf{F}^C$$

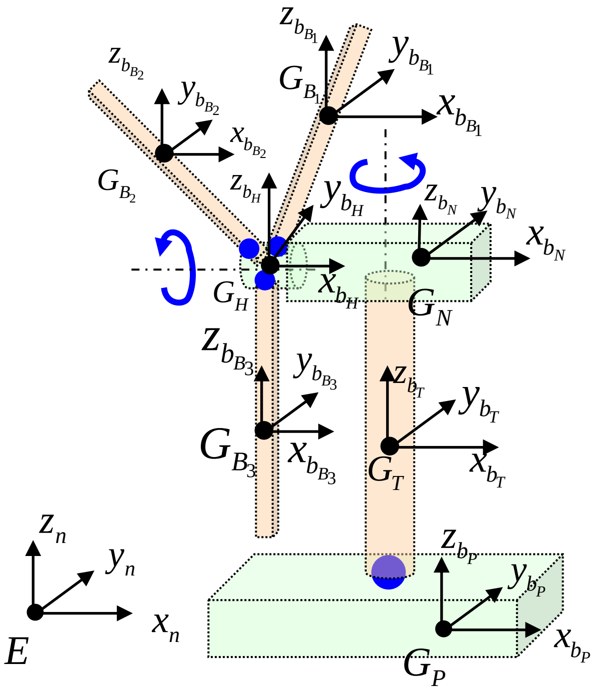
where,

$$\mathbf{s} = \left[\mathbf{s}_P \,;\, \mathbf{s}_T \,;\, \mathbf{s}_N \,;\, \mathbf{s}_H \,;\, \mathbf{s}_{B1} \,;\, \mathbf{s}_{B2} \,;\, \mathbf{s}_{B3}\right]^T$$

$$\mathbf{F} = \left[\mathbf{F}_P \,;\, \mathbf{F}_T \,;\, \mathbf{F}_N \,;\, \mathbf{F}_H \,;\, \mathbf{F}_{B1} \,;\, \mathbf{F}_{B2} \,;\, \mathbf{F}_{B3}\right]^T$$

$$\mathbf{F}^c = \left[\mathbf{F}_P^c \,;\, \mathbf{F}_T^c \,;\, \mathbf{F}_N^c \,;\, \mathbf{F}_H^c \,;\, \mathbf{F}_{B1}^c \,;\, \mathbf{F}_{B2}^c \,;\, \mathbf{F}_{B3}^c\right]^T$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_P & & & & & & \\ & \mathbf{M}_T & & & & & \\ & & \mathbf{M}_N & & & & \\ & & & \mathbf{M}_H & & & \\ & & & & \mathbf{M}_{B1} & & \\ & & & & & \mathbf{M}_{B2} & \\ & & & & & & \mathbf{M}_{B3} \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_P & & & & & & \\ & \mathbf{Q}_T & & & & & \\ & & \mathbf{Q}_N & & & & \\ & & & \mathbf{Q}_H & & & \\ & & & & \mathbf{Q}_{B1} & & \\ & & & & & \mathbf{Q}_{B2} & \\ & & & & & & \mathbf{Q}_{B3} \end{bmatrix}$$



cf) Newton-Euler equation for the platform

$$\underbrace{\begin{bmatrix} \mathbf{m}_P & 0 \\ 0 & \mathbf{I}_{G_P} \end{bmatrix}}_{\mathbf{M}_P} \underbrace{\begin{bmatrix} \ddot{\mathbf{r}}_{G_P/E} \\ \dot{\boldsymbol{\omega}}_{b_{G_P}/n} \end{bmatrix}}_{\ddot{\mathbf{s}}_P} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \tilde{\boldsymbol{\omega}}_{b_{G_P}/n}\mathbf{I}_{G_P} \end{bmatrix}}_{\mathbf{Q}_P} \underbrace{\begin{bmatrix} \dot{\mathbf{r}}_{G_P/E} \\ \boldsymbol{\omega}_{b_{G_P}/n} \end{bmatrix}}_{\dot{\mathbf{s}}_P} = \underbrace{\begin{bmatrix} \mathbf{F}_{G_P} \\ \boldsymbol{\tau}_{G_P} \end{bmatrix}}_{\mathbf{F}_P} + \underbrace{\begin{bmatrix} \mathbf{F}_{G_P}^c \\ \boldsymbol{\tau}_{G_P}^c \end{bmatrix}}_{\mathbf{F}_P^C}$$

Derivation of Equations of Motion with respect to the Center of Mass G for Rigid Body

$$\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}=\mathbf{F}+\mathbf{F}^c$$

D'Alembert principle (Dynamic Equilibrium)

$$(\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}-\mathbf{F}-\mathbf{F}^c)=0$$

Virtual displacement, Virtual work

"Virtual displacement" is defined as an arbitrary displacement which does not affect the force system , each of the forces remains constant in magnitude and direction.

the work done by the forces acting on the particle during a virtual displacement is called the "virtual work". The virtual work done is zero if the system is in equilibrium, since the resultant force vanishes.

$$\delta\mathbf{s}^T(\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}-\mathbf{F}-\mathbf{F}^c)=0 \Rightarrow$$

it means the inner product

$$\mathbf{r}_{G/E}\cdot(\mathbf{m}\,\ddot{\mathbf{r}}_{G/E}-\mathbf{F}_G-\mathbf{F}_G^c)$$

$$\boldsymbol{\chi}_{b_G/n}\cdot(\mathbf{I}_G\,\dot{\boldsymbol{\omega}}_{b_G/n}+\boldsymbol{\omega}_{b_G/n}\times\mathbf{I}_G\,\boldsymbol{\omega}_{b_G/n}-\boldsymbol{\tau}_G-\boldsymbol{\tau}_G^c)$$

$$\delta\mathbf{s}\cdot\mathbf{F}^c=0$$

suppress the constraint force based on that "virtual displacement is perpendicular to the constraint force"*

$$\boxed{\delta\mathbf{s}^T(\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}-\mathbf{F})=0 \quad ,\boxed{\tilde{\mathbf{C}}(\mathbf{s})=0}} \longrightarrow$$

The coordinates in $\mathbf{s}$ are not independent.

equations of motion derived with respect to the center of mass and the kinematic constraint

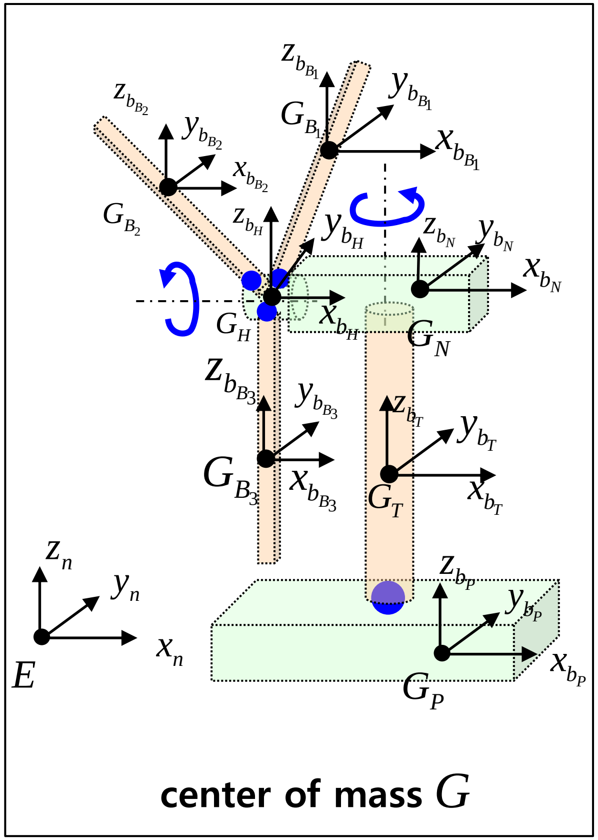
---

*Haug,E.J., Intermediate Dynamics, Prentice-Hall, 1992, p15
: the constraint reaction force must be perpendicular to the curve or surface along which the particle constrained to move. This suggests that the constraint reaction force may be suppressed by taking the scalar product of both sides of the equations of motion with vectors that are tangent to the curve or surface

2.1 Equations of Motion

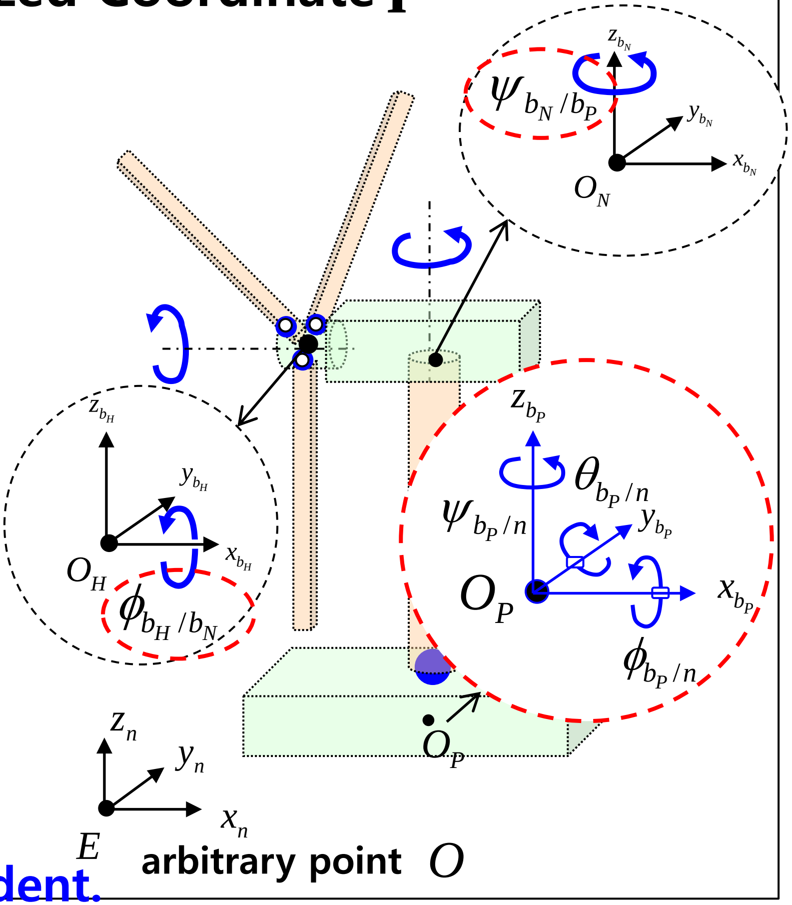
Derivation of Equations of Motion with respect to the Arbitrary Point O for Rigid Body

Any coordinates can be used to specify the configuration of the floating offshore wind turbine. We take the a set of coordinates, called generalized coordinates, for each body at the center or end of the geometry for the advantage that we could recognize the position of the origin.[1) 2)], For the rotational angles, ZYX-Euler angles are used.



Generalized Coordinate $\mathbf{q}$

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} = \begin{bmatrix} x_{O_P/E} \\ y_{O_P/E} \\ z_{O_P/E} \\ \phi_{b_P/n} \\ \theta_{b_P/n} \\ \psi_{b_P/n} \\ \psi_{b_N/b_P} \\ \phi_{b_H/b_N} \end{bmatrix}$$



Define the generalized coordinate $\mathbf{q}$ for all the coordinates in $\mathbf{q}$ to be independent.

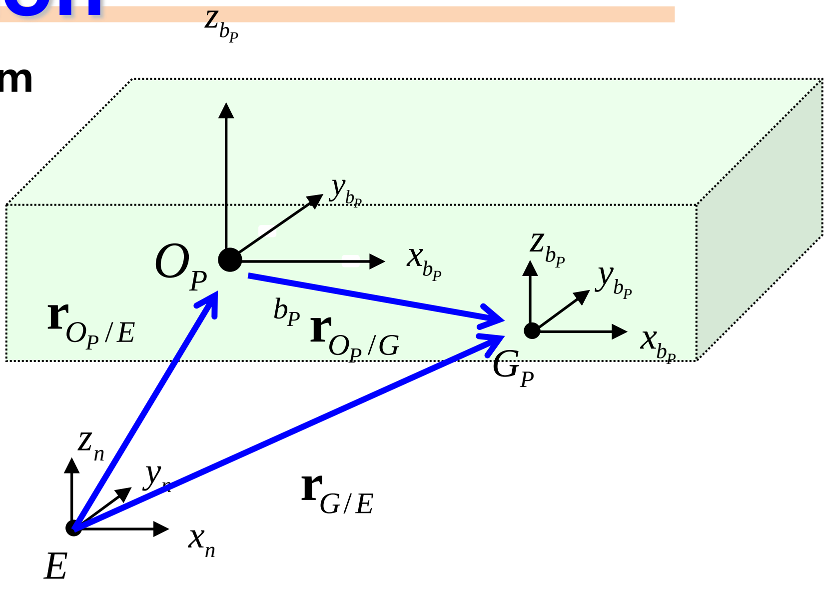
---

1) Haug, E.J., intermediate Dynamics, Prentice-Hall, 1992, p.37
Cartesian coordinates represent the most fundamental and generally applicable set of coordinates to locate point in space. In numerous applications, however, such as orbital motion of a satellite around the earth or motion that is expected to follow some curve or surface in space, a set of coordinates that define position, called generalized coordinates, may be used to advantage.

2) Greenwood, D.T., Principles of Dynamics, Second Edition, 1988, p.241
Any set of numbers which serve to specify the configuration of a system are examples of generalized coordinates. …Note that the term generalized coordinates can refer to any of commonly used coordinate systems, but it can also refer to any set of parameters which serve to specify the configuration of a system.

2.1 Equations of Motion
Coordinates Transformation

Example) Coordinate Transformation for the platform



$$\mathbf{r}_{G_P/E} = \mathbf{r}_{O_P/E} + {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{O_P/G_P}$$

↓

differentiate w.r.t time

$$\dot{\mathbf{r}}_{G_P/E} = \dot{\mathbf{r}}_{O_P/E} + \boldsymbol{\omega}_{b_P/n} \times {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{O_P/G_P}$$

$$= \dot{\mathbf{r}}_{O_P/E} + \left(\mathbf{G}\dot{\boldsymbol{\gamma}}\right) \times {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{O_P/G_P}$$

$$= \dot{\mathbf{r}}_{O_P/E} - {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{O_P/G_P} \times \mathbf{G}\dot{\boldsymbol{\gamma}}$$

↓

$$\boxed{\dot{\mathbf{r}}_{G_P/E} = \dot{\mathbf{r}}_{O_P/E} - {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{O_P/G_P} \times \mathbf{G}\dot{\boldsymbol{\gamma}}}$$

$$\boxed{\boldsymbol{\omega}_{b_{G_P}/n} = \mathbf{G}\dot{\boldsymbol{\gamma}}}$$

in matrix form

$$\underbrace{\begin{bmatrix}\dot{\mathbf{r}}_{G_P/E}\\ \boldsymbol{\omega}_{b_{G_P}/n}\end{bmatrix}}_{\dot{\mathbf{s}}} = \underbrace{\begin{bmatrix}\mathbf{I} & -{}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{O_P/G_P} \times \mathbf{G}\\ \mathbf{O} & \mathbf{G}\end{bmatrix}}_{\mathbf{J}}\underbrace{\begin{bmatrix}\dot{\mathbf{r}}_{O_P/E}\\ \dot{\boldsymbol{\gamma}}\end{bmatrix}}_{\dot{\mathbf{q}}}$$

Coordinates Transformation relation ⇨ $\dot{\mathbf{s}} = \mathbf{J}\dot{\mathbf{q}}$

2.1 Equations of Motion
Coordinates Transformation

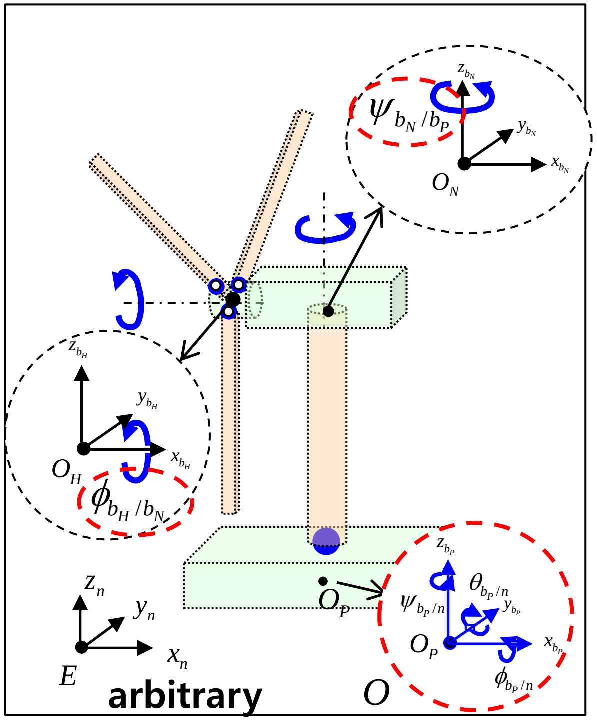
equation of motion derived with respect to the center of mass G and kinematic constraint

$$\delta\mathbf{s}\cdot(\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}-\mathbf{F})=0, \qquad \tilde{\mathbf{C}}(\mathbf{s})=0$$

Coordinates Transformation

$$\dot{\mathbf{s}}=\mathbf{J}\dot{\mathbf{q}}\ ,\delta\mathbf{s}=\mathbf{J}\delta\mathbf{q},\ \ddot{\mathbf{s}}=\mathbf{J}\ddot{\mathbf{q}}+\dot{\mathbf{J}}\dot{\mathbf{q}},$$

where $\mathbf{q}=\left[x_{O_P/E}\ \ y_{O_P/E}\ \ z_{O_P/E}\ \ \phi_{b_P/n}\ \ \theta_{b_P/n}\ \ \psi_{b_P/n}\ \ \psi_{b_N/b_P}\ \ \phi_{b_H/b_N}\right]^T$



arbitrary point

$$(\mathbf{J}\delta\mathbf{q})\cdot\left(\mathbf{M}\left[\mathbf{J}\ddot{\mathbf{q}}+\dot{\mathbf{J}}\dot{\mathbf{q}}\right]+\mathbf{Q}\left[\mathbf{J}\dot{\mathbf{q}}\right]-\mathbf{F}\right)=0$$

$$\delta\mathbf{q}^T\mathbf{J}^T\left(\mathbf{M}\left[\mathbf{J}\ddot{\mathbf{q}}+\dot{\mathbf{J}}\dot{\mathbf{q}}\right]+\mathbf{Q}(\mathbf{J}\dot{\mathbf{q}})-\mathbf{F}\right)=0$$

$$\delta\mathbf{q}^T(\mathbf{J}^T\mathbf{M}\mathbf{J}\ddot{\mathbf{q}}+\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}+\mathbf{J}^T\mathbf{Q}\mathbf{J}\dot{\mathbf{q}}-\mathbf{J}^T\mathbf{F})=0$$

$$\delta\mathbf{q}^T(\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}})=0$$

$$,\bar{\mathbf{M}}=\mathbf{J}^T\mathbf{M}\mathbf{J},\ \bar{\mathbf{k}}=\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}+\mathbf{J}^T\mathbf{Q}\mathbf{J}\dot{\mathbf{q}},\ \bar{\mathbf{F}}=\mathbf{J}^T\mathbf{F}$$

equations of motion derived with respect to the arbitrary point *O*

$\bar{\mathbf{k}}=\bar{\mathbf{k}}(\mathbf{q},\dot{\mathbf{q}})$: Gyroscopic and Coriolis Force Vector

Derivation of Equations of Motion with respect to the Arbitrary Point O for Rigid Body

$$\delta \mathbf{q}^T(\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}})=0$$

$$,\bar{\mathbf{M}}=\mathbf{J}^T\mathbf{M}\mathbf{J},\ \bar{\mathbf{k}}=\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}+\mathbf{J}^T\mathbf{Q}\mathbf{J}\dot{\mathbf{q}},\ \bar{\mathbf{F}}=\mathbf{J}^T\mathbf{F}$$

↓ Since all the coordinates in $\mathbf{q}$ are independent.

$$\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}}=0$$

which is called “embedding formulation” derived with respect to the arbitrary point *O*

6.3 Equations of Motion for Offshore Floating Wind Turbine Using Augmented Formulation

2.1 Equations of Motion
Reference Frame defined with respect to Center of Mass G

The floating offshore wind turbine consists of the barge type floating platform, tower, nacelle, hub, and three blades with the fixed and revolute joints.

At this moment, all the bodies are regarded as rigid bodies

The body fixed reference frame are defined at the center of mass of each body

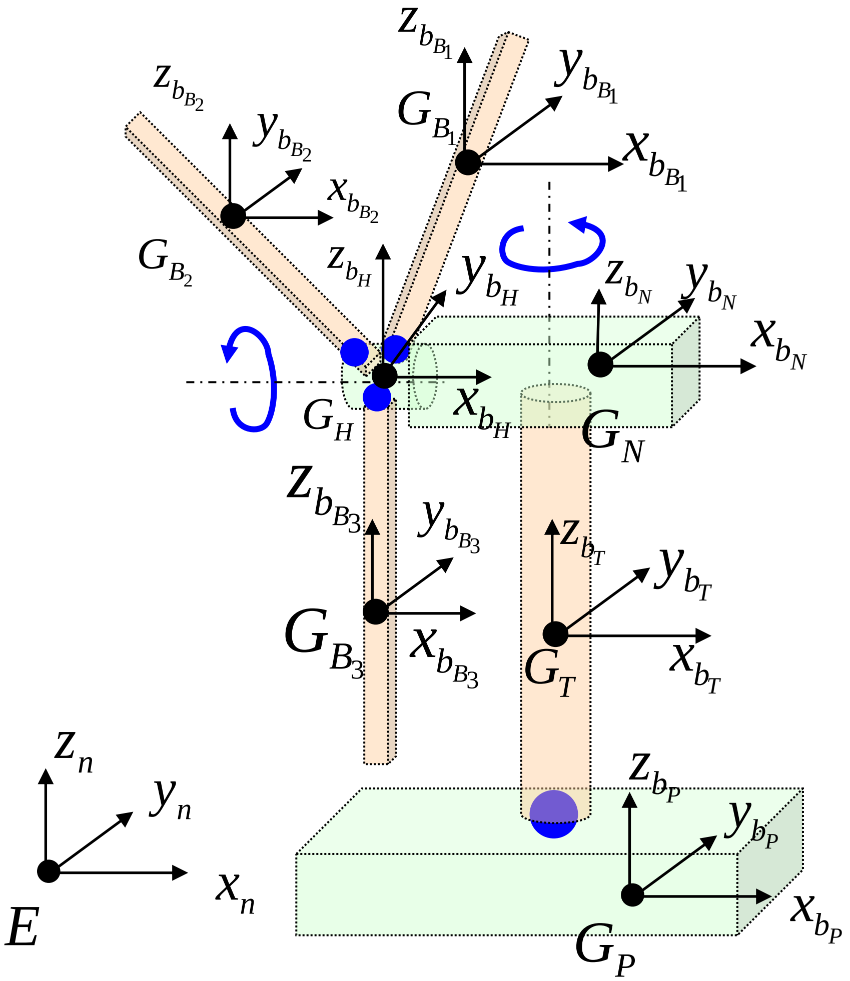
● **Fixed Joint**

↻ **Revolute Joint**

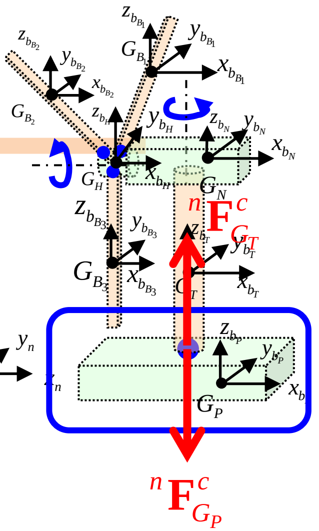
G **: center of mass of each body**

<subindex>

P : Platform(barge type)
T : Tower
N : Nacelle
H : Hub
B1, B2, B3 : Blade1, 2, 3



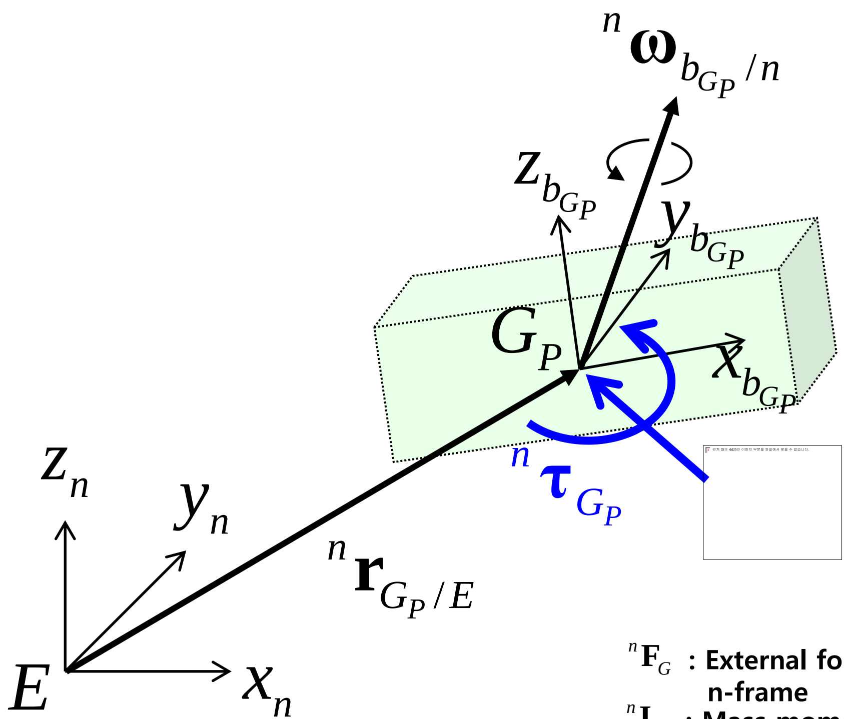
2.1 Equations of Motion
Equations of Motion of Multibody with respect to Center of Mass



Example) Newton-Euler equation for the platform

$$m\;{}^n\ddot{\mathbf{r}}_{G_P/E} = {}^n\mathbf{F}_{G_P} + {}^n\mathbf{F}^c_{G_P}$$

$${}^n\mathbf{I}_G\;{}^n\dot{\boldsymbol{\omega}}_{b_{G_P}/n} + {}^n\boldsymbol{\omega}_{b_{G_P}/n} \times {}^n\mathbf{I}_G\;{}^n\boldsymbol{\omega}_{b_{G_P}/n} = {}^n\boldsymbol{\tau}_{G_P} + {}^n\boldsymbol{\tau}^c_{G_P}$$



$$\mathbf{r}_{G_P/E} \equiv \left[x_{G_P/E}\,;y_{G_P/E}\,;z_{G_P/E}\right]^T_E$$

$$\dot{\mathbf{r}}_{G_P/E} = \left[\dot{x}_{G_P/E}\;;\dot{y}_{G_P/E}\;;\dot{z}_{G_P/E}\right]^T$$

$$\ddot{\mathbf{r}}_{G_P/E} = \left[\ddot{x}_{G_P/E}\;;\ddot{y}_{G_P/E}\;;\ddot{z}_{G_P/E}\right]^T$$

$$\boldsymbol{\chi}_{b_{G_P}/n} \equiv \left[\int\omega_{b_{G_P}/n,x}\,dt\;;\;\int\omega_{b_{G_P}/n,y}\,dt\;;\;\int\omega_{b_{G_P}/n,z}\,dt\right]^T$$

$$\boldsymbol{\omega}_{b_{G_P}/n} = \left[\omega_{b_{G_P}/n,x}\;\;;\omega_{b_{G_P}/n,y}\;\;;\omega_{b_{G_P}/n,z}\right]^T$$

$$\dot{\boldsymbol{\omega}}_{b_{G_P}/n} = \left[\dot{\omega}_{b_{G_P}/n,x}\;\;;\dot{\omega}_{b_{G_P}/n,y}\;\;;\dot{\omega}_{b_{G_P}/n,z}\right]^T$$

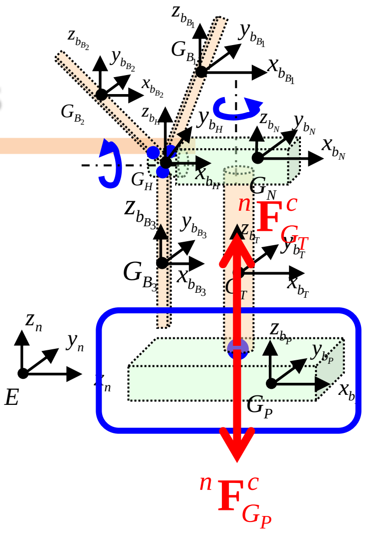
$$\boldsymbol{\omega}_{b_{G_P}/n} = \mathbf{G}\dot{\boldsymbol{\gamma}}$$

n-frame: Earth-fixed inertial frame
b-frame: body-fixed frame
m: mass of a body
${}^n\mathbf{r}_{G/E}$: Position vector of point G with respect to point E decomposed in n-frame

${}^n\mathbf{F}_G$: External force acting on point G decomposed in n-frame
${}^n\mathbf{I}_G$: Mass moment of inertia about certain axis through point G decomposed in n-frame
${}^n\boldsymbol{\omega}_{b/n}$: Angular velocity of b-frame with respect to n-frame decomposed in n-frame
${}^n\boldsymbol{\tau}_G$: External moment about certain axes through point G decomposed in n-frame

$\boldsymbol{\gamma}$: ZYX-Euler angle
${}^n\mathbf{F}^c_G$: Constraint force acting on point G decomposed in n-frame
${}^n\boldsymbol{\tau}^c_G$: Constraint moment about certain axes through point G decomposed in n-frame

2.1 Equations of Motion
Equations of Motion of Multibody with respect to Center of Mass



Newton-Euler equation for the platform

$$m\,{}^n\ddot{\mathbf{r}}_{G_P/E} = {}^n\mathbf{F}_{G_P} + {}^n\mathbf{F}^c_{G_P}$$

$${}^n\mathbf{I}_G\,{}^n\dot{\boldsymbol{\omega}}_{b_{G_P}/n} + {}^n\boldsymbol{\omega}_{b_{G_P}/n} \times {}^n\mathbf{I}_G\,{}^n\boldsymbol{\omega}_{b_{G_P}/n} = {}^n\boldsymbol{\tau}_{G_P} + {}^n\boldsymbol{\tau}^c_{G_P}$$

If we rewrite the force and moment equation in matrix form,

$$\underbrace{\begin{bmatrix} \mathbf{m}_P & 0 \\ 0 & \mathbf{I}_{G_P} \end{bmatrix}}_{\mathbf{M}_P} \underbrace{\begin{bmatrix} \ddot{\mathbf{r}}_{G_P/E} \\ \dot{\boldsymbol{\omega}}_{b_{G_P}/n} \end{bmatrix}}_{\ddot{\mathbf{s}}_P} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \tilde{\boldsymbol{\omega}}_{b_{G_P}/n}\mathbf{I}_{G_P} \end{bmatrix}}_{\mathbf{Q}_P} \underbrace{\begin{bmatrix} \dot{\mathbf{r}}_{G_P/E} \\ \boldsymbol{\omega}_{b_{G_P}/n} \end{bmatrix}}_{\dot{\mathbf{s}}_P} = \underbrace{\begin{bmatrix} \mathbf{F}_{G_P} \\ \boldsymbol{\tau}_{G_P} \end{bmatrix}}_{\boxed{\mathbf{F}_P}} + \underbrace{\begin{bmatrix} \mathbf{F}^c_{G_P} \\ \boldsymbol{\tau}^c_{G_P} \end{bmatrix}}_{\boxed{\mathbf{F}^C_P}}$$

, $\tilde{\boldsymbol{\omega}}_{b_{G_P}/n}$ **is the skew matrix of** $\boldsymbol{\omega}_{b_{G_P}/n}$

external force & moment

constraint force & moment

$$\boldsymbol{\omega}\times\mathbf{r} = \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} \times \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underbrace{\boxed{\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}}_{\tilde{\boldsymbol{\omega}}} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$\mathbf{M}_P\ddot{\mathbf{s}}_P + \mathbf{Q}_P\dot{\mathbf{s}}_P = \mathbf{F}_P + \mathbf{F}^C_P$$

$$\dot{\mathbf{s}}_P = \left[\dot{x}_{G_P/E}\ ;\dot{y}_{G_P/E}\ ;\dot{z}_{G_P/E}\ \ ;\omega_{b_{G_P}/n,x}\ \ ;\omega_{b_{G_P}/n,y}\ \ ;\omega_{b_{G_P}/n,z}\right]^T$$

$$\ddot{\mathbf{s}}_P = \left[\ddot{x}_{G_P/E}\ ;\ddot{y}_{G_P/E}\ ;\ddot{z}_{G_P/E}\ \ ;\dot{\omega}_{b_{G_P}/n,x}\ \ ;\dot{\omega}_{b_{G_P}/n,y}\ \ ;\dot{\omega}_{b_{G_P}/n,z}\right]^T$$

2.1 Equations of Motion
Force and Moment Equations for Floating Offshore Wind Turbine with respect to Center of Mass

Newton-Euler equation for the platform

$$\mathbf{M}_P \ddot{\mathbf{s}}_P + \mathbf{Q}_P \dot{\mathbf{s}}_P = \mathbf{F}_P + \mathbf{F}_P^c$$

Newton-Euler equation for the tower

$$\mathbf{M}_T \ddot{\mathbf{s}}_T + \mathbf{Q}_T \dot{\mathbf{s}}_T = \mathbf{F}_T + \mathbf{F}_T^c$$

Newton-Euler equation for the nacelle

$$\mathbf{M}_N \ddot{\mathbf{s}}_N + \mathbf{Q}_N \dot{\mathbf{s}}_N = \mathbf{F}_N + \mathbf{F}_N^c$$

Newton-Euler equation for the hub

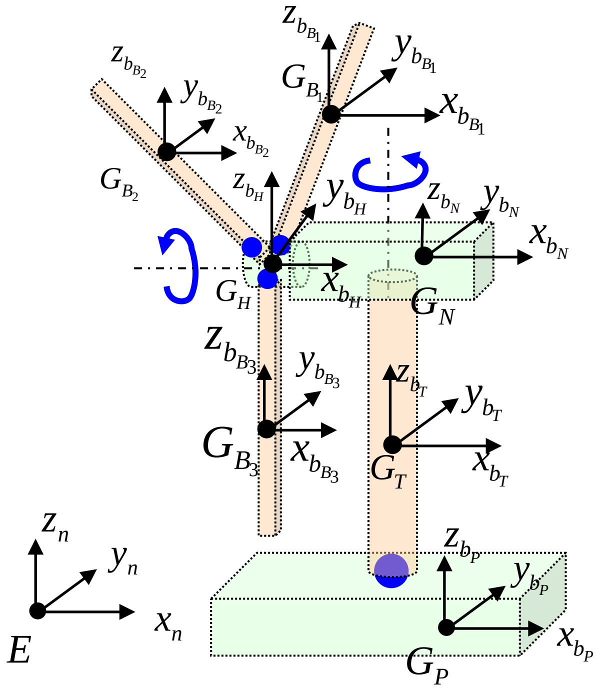
$$\mathbf{M}_H \ddot{\mathbf{s}}_H + \mathbf{Q}_H \dot{\mathbf{s}}_H = \mathbf{F}_H + \mathbf{F}_H^c$$

Newton-Euler equation for the blade 1, 2, 3

$$\mathbf{M}_{B1} \ddot{\mathbf{s}}_{B1} + \mathbf{Q}_{B1} \dot{\mathbf{s}}_{B1} = \mathbf{F}_{B1} + \mathbf{F}_{B1}^c$$

$$\mathbf{M}_{B2} \ddot{\mathbf{s}}_{B2} + \mathbf{Q}_{B2} \dot{\mathbf{s}}_{B2} = \mathbf{F}_{B2} + \mathbf{F}_{B2}^c$$

$$\mathbf{M}_{B3} \ddot{\mathbf{s}}_{B3} + \mathbf{Q}_{B3} \dot{\mathbf{s}}_{B3} = \mathbf{F}_{B3} + \mathbf{F}_{B3}^c$$

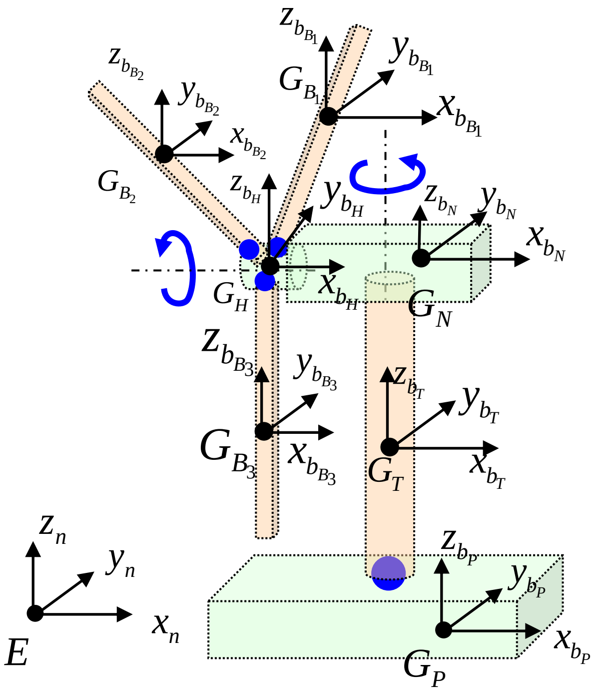


2.1 Equations of Motion
Force and Moment Equations for Floating Offshore Wind Turbine with respect to Center of Mass

Newton-Euler equation for the Floating Offshore Wind Turbine

$$\begin{bmatrix} \mathbf{M}_P & & & & & & \\ & \mathbf{M}_T & & & & & \\ & & \mathbf{M}_N & & & & \\ & & & \mathbf{M}_H & & & \\ & & & & \mathbf{M}_{B1} & & \\ & & & & & \mathbf{M}_{B2} & \\ & & & & & & \mathbf{M}_{B3} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{s}}_P \\ \dot{\mathbf{s}}_T \\ \dot{\mathbf{s}}_N \\ \dot{\mathbf{s}}_H \\ \dot{\mathbf{s}}_{B1} \\ \dot{\mathbf{s}}_{B2} \\ \dot{\mathbf{s}}_{B3} \end{bmatrix} +$$

$$\begin{bmatrix} \mathbf{Q}_P & & & & & & \\ & \mathbf{Q}_T & & & & & \\ & & \mathbf{Q}_N & & & & \\ & & & \mathbf{Q}_H & & & \\ & & & & \mathbf{Q}_{B1} & & \\ & & & & & \mathbf{Q}_{B2} & \\ & & & & & & \mathbf{Q}_{B3} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{s}}_P \\ \ddot{\mathbf{s}}_T \\ \ddot{\mathbf{s}}_N \\ \ddot{\mathbf{s}}_H \\ \ddot{\mathbf{s}}_{B1} \\ \ddot{\mathbf{s}}_{B2} \\ \ddot{\mathbf{s}}_{B3} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_P \\ \mathbf{F}_T \\ \mathbf{F}_N \\ \mathbf{F}_H \\ \mathbf{F}_{B1} \\ \mathbf{F}_{B2} \\ \mathbf{F}_{B3} \end{bmatrix} + \begin{bmatrix} \mathbf{F}_P^c \\ \mathbf{F}_T^c \\ \mathbf{F}_N^c \\ \mathbf{F}_H^c \\ \mathbf{F}_{B1}^c \\ \mathbf{F}_{B2}^c \\ \mathbf{F}_{B3}^c \end{bmatrix}$$



2.1 Equations of Motion
Force and Moment Equations for Floating Offshore Wind Turbine with respect to Center of Mass

Newton-Euler equation for the Floating Offshore Wind Turbine

$$\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}=\mathbf{F}+\mathbf{F}^{C}$$

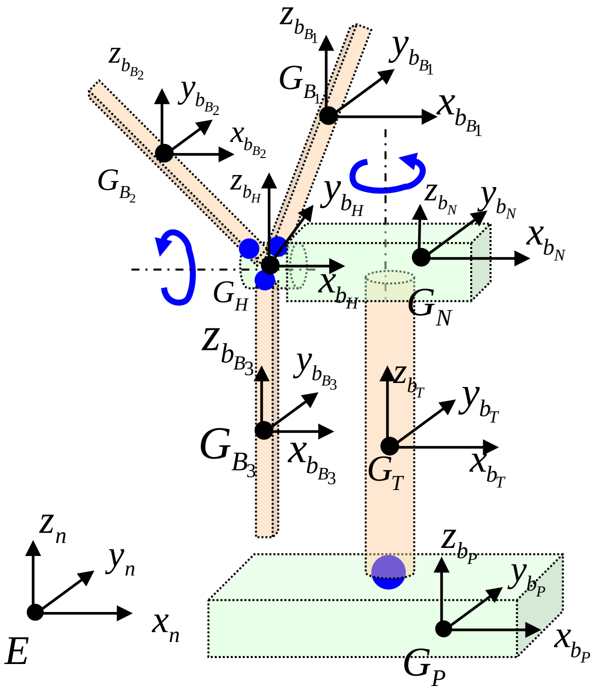
where,

$$\mathbf{s}=\left[\mathbf{s}_P\,;\,\mathbf{s}_T\,;\,\mathbf{s}_N\,;\,\mathbf{s}_H\,;\,\mathbf{s}_{B1}\,;\,\mathbf{s}_{B2}\,;\,\mathbf{s}_{B3}\right]^T$$

$$\mathbf{F}=\left[\mathbf{F}_P\,;\,\mathbf{F}_T\,;\,\mathbf{F}_N\,;\,\mathbf{F}_H\,;\,\mathbf{F}_{B1}\,;\,\mathbf{F}_{B2}\,;\,\mathbf{F}_{B3}\right]^T$$

$$\mathbf{F}^c=\left[\mathbf{F}_P^c\,;\,\mathbf{F}_T^c\,;\,\mathbf{F}_N^c\,;\,\mathbf{F}_H^c\,;\,\mathbf{F}_{B1}^c\,;\,\mathbf{F}_{B2}^c\,;\,\mathbf{F}_{B3}^c\right]^T$$

$$\mathbf{M}=\begin{bmatrix}\mathbf{M}_P &&&&&& \\ & \mathbf{M}_T &&&&& \\ && \mathbf{M}_N &&&& \\ &&& \mathbf{M}_H &&& \\ &&&& \mathbf{M}_{B1} && \\ &&&&& \mathbf{M}_{B2} & \\ &&&&&& \mathbf{M}_{B3}\end{bmatrix}\quad \mathbf{Q}=\begin{bmatrix}\mathbf{Q}_P &&&&&& \\ & \mathbf{Q}_T &&&&& \\ && \mathbf{Q}_N &&&& \\ &&& \mathbf{Q}_H &&& \\ &&&& \mathbf{Q}_{B1} && \\ &&&&& \mathbf{Q}_{B2} & \\ &&&&&& \mathbf{Q}_{B3}\end{bmatrix}$$



cf) Newton-Euler equation for the platform

$$\underbrace{\begin{bmatrix}\mathbf{m}_P & 0\\ 0 & \mathbf{I}_{G_P}\end{bmatrix}}_{\mathbf{M}_P}\underbrace{\begin{bmatrix}\ddot{\mathbf{r}}_{G_P/E}\\ \dot{\boldsymbol{\omega}}_{b_{G_P}/n}\end{bmatrix}}_{\ddot{\mathbf{s}}_P}+\underbrace{\begin{bmatrix}0 & 0\\ 0 & \tilde{\boldsymbol{\omega}}_{b_{G_P}/n}\mathbf{I}_{G_P}\end{bmatrix}}_{\mathbf{Q}_P}\underbrace{\begin{bmatrix}\dot{\mathbf{r}}_{G_P/E}\\ \boldsymbol{\omega}_{b_{G_P}/n}\end{bmatrix}}_{\dot{\mathbf{s}}_P}=\underbrace{\begin{bmatrix}\mathbf{F}_{G_P}\\ \boldsymbol{\tau}_{G_P}\end{bmatrix}}_{\mathbf{F}_P}+\underbrace{\begin{bmatrix}\mathbf{F}_{G_P}^c\\ \boldsymbol{\tau}_{G_P}^c\end{bmatrix}}_{\mathbf{F}_P^C}$$

Derivation of Equations of Motion with respect to the Center of Mass G for Rigid Body

$$\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}=\mathbf{F}+\mathbf{F}^{c}$$

$\downarrow$ **D'Alembert principle (Dynamic Equilibrium)**

$$(\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}-\mathbf{F}-\mathbf{F}^{c})=0$$

$\downarrow$ **Virtual displacement, Virtual work**

"Virtual displacement" is defined as an arbitrary displacement which does not affect the force system , each of the forces remains constant in magnitude and direction.

the work done by the forces acting on the particle during a virtual displacement is called the "virtual work". The virtual work done is zero if the system is in equilibrium, since the resultant force vanishes.

$$\delta\mathbf{s}^{T}(\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}-\mathbf{F}-\mathbf{F}^{c})=0 \Rightarrow$$

it means the inner product

$$\mathbf{r}_{G/E}\cdot(\mathbf{m}\,\ddot{\mathbf{r}}_{G/E}-\mathbf{F}_{G}-\mathbf{F}_{G}^{c})$$

$$\boldsymbol{\chi}_{b_G/n}\cdot(\mathbf{I}_{G}\,\dot{\boldsymbol{\omega}}_{b_G/n}+\boldsymbol{\omega}_{b_G/n}\times\mathbf{I}_{G}\,\boldsymbol{\omega}_{b_G/n}-\boldsymbol{\tau}_{G}-\boldsymbol{\tau}_{G}^{c})$$

$\downarrow$ $$\delta\mathbf{s}\cdot\mathbf{F}^{c}=0$$ **suppress the constraint force based on that "virtual displacement is perpendicular to the constraint force"***

$$\boxed{\delta\mathbf{s}^{T}(\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}-\mathbf{F})=0 \quad ,\boxed{\tilde{\mathbf{C}}(\mathbf{s})=0}} \longrightarrow$$

The coordinates in s are not independent.

equations of motion derived with respect to the center of mass and the kinematic constraint

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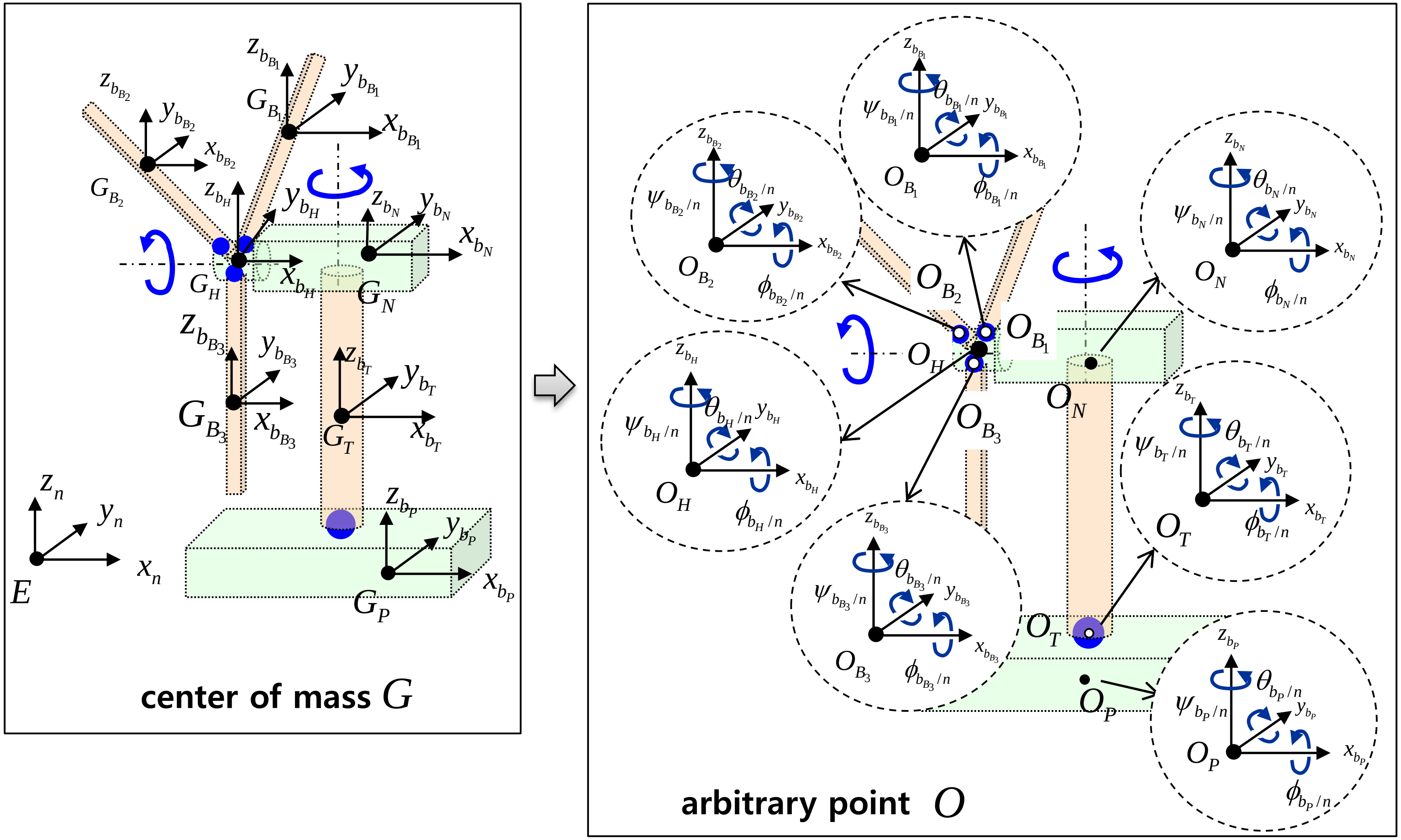
*Haug,E.J., Intermediate Dynamics, Prentice-Hall, 1992, p15
: the constraint reaction force must be perpendicular to the curve or surface along which the particle constrained to move. This suggests that the constraint reaction force may be suppressed by taking the scalar product of both sides of the equations of motion with vectors that are tangent to the curve or surface

Derivation of Equations of Motion with respect to the Arbitrary Point O for Rigid Body

참고자료

Any coordinates can be used to specify the configuration of the floating offshore wind turbine. We take the a set of coordinates, called generalized coordinates, for each body at the center or end of the geometry for the advantage that we could recognize the position of the origin.1) 2), For the rotational angles, ZYX-Euler angles are used.



center of mass G

arbitrary point O

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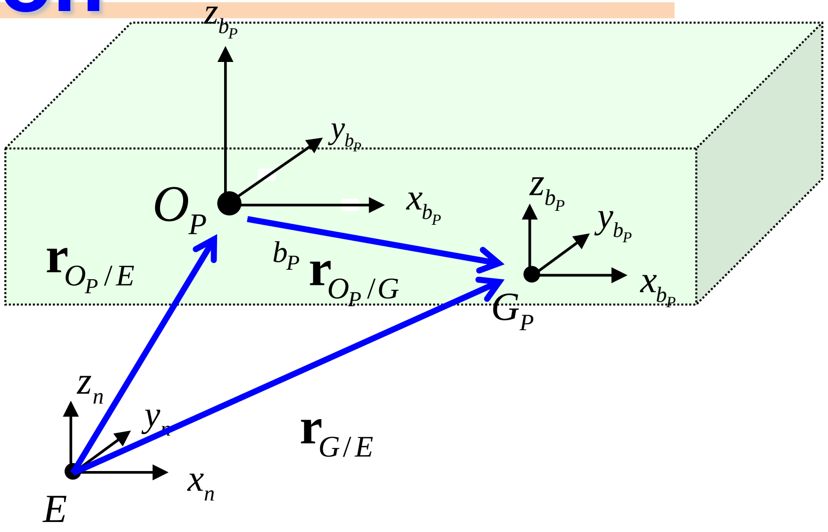
1) Haug, E.J., intermediate Dynamics, Prentice-Hall, 1992, p.37
Cartesian coordinates represent the most fundamental and generally applicable set of coordinates to locate point in space. In numerous applications, however, such as orbital motion of a satellite around the earth or motion that is expected to follow some curve or surface in space, a set of coordinates that define position, called generalized coordinates, may be used to advantage.

2) Greenwood, D.T., Principles of Dynamics, Second Edition, 1988, p.241
Any set of numbers which serve to specify the configuration of a system are examples of generalized coordinates. …Note that the term generalized coordinates can refer to any of commonly used coordinate systems, but it can also refer to any set of parameters which serve to specify the configuration of a system.

2.1 Equations of Motion Coordinates Transformation



$$\mathbf{r}_{O_P/E} = \mathbf{r}_{G/E} + {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{G/O_P}$$

$$\downarrow$$

differentiate w.r.t time

$$\dot{\mathbf{r}}_{O_P/E} = \dot{\mathbf{r}}_{G/E} + \boldsymbol{\omega}_{b_P/n} \times {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{G/O_P}$$

$$= \dot{\mathbf{r}}_{G/E} + \left(\mathbf{G}\dot{\boldsymbol{\gamma}}\right) \times {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{G/O_P}$$

$$= \dot{\mathbf{r}}_{G/E} - {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{G/O_P} \times \mathbf{G}\dot{\boldsymbol{\gamma}}$$

$$\downarrow$$

$$\left.\begin{array}{l}\boxed{\dot{\mathbf{r}}_{O_P/E} = \dot{\mathbf{r}}_{G/E} - {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{G/O_P} \times \mathbf{G}\dot{\boldsymbol{\gamma}}} \\ \boxed{\boldsymbol{\omega}_{b_G/n} = \mathbf{G}\dot{\boldsymbol{\gamma}}}\end{array}\right\} \underset{\dot{\mathbf{s}}}{\begin{bmatrix}\dot{\mathbf{r}}_{O_P/E} \\ \boldsymbol{\omega}_{b_G/n}\end{bmatrix}} = \underset{\mathbf{J}}{\begin{bmatrix}\mathbf{I} & -{}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{G/O_P} \times \mathbf{G} \\ \mathbf{O} & \mathbf{G}\end{bmatrix}} \underset{\dot{\mathbf{q}}}{\begin{bmatrix}\dot{\mathbf{r}}_{G/E} \\ \dot{\boldsymbol{\gamma}}\end{bmatrix}}$$

in matrix form

Coordinates Transformation relation ⇨ $\dot{\mathbf{s}} = \mathbf{J}\dot{\mathbf{q}}$

2.1 Equations of Motion
Coordinates Transformation

equation of motion derived with respect to the center of mass G and kinematic constraint

$$\delta\mathbf{s}\cdot(\mathbf{M}\ddot{\mathbf{s}}+\mathbf{Q}\dot{\mathbf{s}}-\mathbf{F})=0, \qquad \tilde{\mathbf{C}}(\mathbf{s})=0$$

Coordinates Transformation

$$\dot{\mathbf{s}}=\mathbf{J}\dot{\mathbf{q}}\ ,\delta\mathbf{s}=\mathbf{J}\delta\mathbf{q},\ \ddot{\mathbf{s}}=\mathbf{J}\ddot{\mathbf{q}}+\dot{\mathbf{J}}\dot{\mathbf{q}},$$

where $\mathbf{q}=\begin{bmatrix}\mathbf{q}_P & \mathbf{q}_T & \mathbf{q}_N & \mathbf{q}_H & \mathbf{q}_{B_1} & \mathbf{q}_{B_2} & \mathbf{q}_{B_3}\end{bmatrix}^T$

$$\downarrow$$

$$(\mathbf{J}\delta\mathbf{q})\cdot\left(\mathbf{M}\left[\mathbf{J}\ddot{\mathbf{q}}+\dot{\mathbf{J}}\dot{\mathbf{q}}\right]+\mathbf{Q}\left[\mathbf{J}\dot{\mathbf{q}}\right]-\mathbf{F}\right)=0$$

$$\downarrow$$

$$\delta\mathbf{q}^T\mathbf{J}^T\left(\mathbf{M}\left[\mathbf{J}\ddot{\mathbf{q}}+\dot{\mathbf{J}}\dot{\mathbf{q}}\right]+\mathbf{Q}(\mathbf{J}\dot{\mathbf{q}})-\mathbf{F}\right)=0$$

$$\downarrow$$

$$\delta\mathbf{q}^T(\mathbf{J}^T\mathbf{M}\mathbf{J}\ddot{\mathbf{q}}+\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}+\mathbf{J}^T\mathbf{Q}\mathbf{J}\dot{\mathbf{q}}-\mathbf{J}^T\mathbf{F})=0$$

$$\downarrow$$

$$\delta\mathbf{q}^T(\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}})=0 \qquad ,\mathbf{C}(\mathbf{q})=0$$

$$,\bar{\mathbf{M}}=\mathbf{J}^T\mathbf{M}\mathbf{J},\ \bar{\mathbf{k}}=\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}+\mathbf{J}^T\mathbf{Q}\mathbf{J}\dot{\mathbf{q}},\ \bar{\mathbf{F}}=\mathbf{J}^T\mathbf{F}$$

equations of motion derived with respect to the arbitrary point *O* and the kinematic constraint.

$\bar{\mathbf{k}}=\bar{\mathbf{k}}(\mathbf{q},\dot{\mathbf{q}})$: Gyroscopic and Coriolis Force Vector

Derivation of Equations of Motion with respect to the Arbitrary Point O for Rigid Body

$$\delta\mathbf{q}^T(\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}})=0 \qquad ,\mathbf{C}(\mathbf{q})=0$$

$$,\bar{\mathbf{M}}=\mathbf{J}^T\mathbf{M}\mathbf{J},\ \bar{\mathbf{k}}=\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}+\mathbf{J}^T\mathbf{Q}\mathbf{J}\dot{\mathbf{q}},\ \bar{\mathbf{F}}=\mathbf{J}^T\mathbf{F}$$

$\downarrow$ **if all the coordinates in $\mathbf{q}$ are independent.**

$$\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}}=0$$

which is called "embedding formulation" derived with respect to the arbitrary point *O*

2.1 Equations of Motion

Derivation of Equations of Motion with respect to the Arbitrary Point O for Rigid Body

$$\boxed{\begin{gathered}\delta\mathbf{q}^T(\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}})=0 \qquad ,\mathbf{C}(\mathbf{q})=0 \\ ,\bar{\mathbf{M}}=\mathbf{J}^T\mathbf{M}\mathbf{J},\ \bar{\mathbf{k}}=\mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}}+\mathbf{J}^T\mathbf{Q}\mathbf{J}\dot{\mathbf{q}},\ \bar{\mathbf{F}}=\mathbf{J}^T\mathbf{F}\end{gathered}}$$

↓

if the generalized coordinates are not independent.
we introduce the Lagrange multiplier $\boldsymbol{\lambda}$ with the variation of the kinematic constraint equation $\mathbf{C}(\mathbf{q})=0$ to remove the dependent coordinates 1)

$$\delta\mathbf{q}^T(\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}})+\boldsymbol{\lambda}\delta\mathbf{C}=0 \qquad ,\delta\mathbf{C}=\frac{\partial\mathbf{C}}{\partial\mathbf{q}}\delta\mathbf{q}$$

↓

$$\delta\mathbf{q}^T(\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}}+\mathbf{C}_{\mathbf{q}}^T\boldsymbol{\lambda})=0$$

↓ **Set λ to make remove the dependent coordinates ,then the only independent coordinates are left**

$$\boxed{\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}}+\mathbf{C}_{\mathbf{q}}^T\boldsymbol{\lambda}=0} \quad \boxed{,\mathbf{C}(\mathbf{q})=0}$$

---

1) Lanczos, C., The Variational Principle of Mechanics, Fourth Edition, 1970, Dover, p.44, “Lagrange multiplier method”, “method of the undetermined multiplier”

Derivation of Equations of Motion with respect to the Arbitrary Point O for Rigid Body

$$\boxed{\bar{\mathbf{M}}\ddot{\mathbf{q}}+\bar{\mathbf{k}}-\bar{\mathbf{F}}+\mathbf{C}_{\mathbf{q}}^{T}\boldsymbol{\lambda}=0}\quad \boxed{,\mathbf{C}(\mathbf{q})=0}$$

Because the Lagrange multipliers are added, the equations are less than the variables. Therefore, more equations are required.

If we derivative twice the kinematic constraint equation w.r.t time, we obtain,

$$if\ \ \mathbf{C}(\mathbf{q},t)=0$$

$$\frac{d\mathbf{C}(\mathbf{q},t)}{dt}=\frac{\partial\mathbf{C}(\mathbf{q},t)}{\partial t}+\frac{\partial\mathbf{C}(\mathbf{q},t)}{\partial\mathbf{q}}\frac{\partial\mathbf{q}}{\partial t}$$

$$=\mathbf{C}_{t}+\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}$$

$$\frac{d\left(\mathbf{C}_{t}+\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)}{dt}=\frac{\partial\left(\mathbf{C}_{t}+\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)}{\partial t}+\frac{\partial\left(\mathbf{C}_{t}+\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)}{\partial\mathbf{q}}\frac{\partial\mathbf{q}}{\partial t}$$

$$=\mathbf{C}_{tt}+\left(\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{t}+\left(\mathbf{C}_{\mathbf{q}t}+\left(\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\right)\dot{\mathbf{q}}$$

$$=\mathbf{C}_{tt}+\mathbf{C}_{\mathbf{q}t}\dot{\mathbf{q}}+\mathbf{C}_{\mathbf{q}}\ddot{\mathbf{q}}+\mathbf{C}_{\mathbf{q}t}\dot{\mathbf{q}}+\left(\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}}$$

$$=\mathbf{C}_{tt}+2\mathbf{C}_{\mathbf{q}t}\dot{\mathbf{q}}+\mathbf{C}_{\mathbf{q}}\ddot{\mathbf{q}}+\left(\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}}$$

$$if\ \ \mathbf{C}(\mathbf{q})=0$$

$$\mathbf{C}_{\mathbf{q}}\ddot{\mathbf{q}}+\left(\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}}=0$$

$$\boxed{\mathbf{C}_{\mathbf{q}}\ddot{\mathbf{q}}=-\left(\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}}}$$

in matrix form

$$\begin{bmatrix}\bar{\mathbf{M}}(\mathbf{q}) & \mathbf{C}_{\mathbf{q}}^{T}(\mathbf{q})\\ \mathbf{C}_{\mathbf{q}}(\mathbf{q}) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}}\\ \boldsymbol{\lambda}\end{bmatrix}=\begin{bmatrix}\bar{\mathbf{F}}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},t)-\bar{\mathbf{k}}(\mathbf{q},\dot{\mathbf{q}})\\ -(\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}}\end{bmatrix}$$

which is called “augmented formulation” derived with respect to the arbitrary point O

Derivation of Equations of Motion with respect to the Arbitrary Point O for Rigid Body

Equations of Motion

$$\bar{\mathbf{M}}\ddot{\mathbf{q}} + \bar{\mathbf{k}} - \bar{\mathbf{F}} + \mathbf{C}_{\mathbf{q}}^T\boldsymbol{\lambda} = 0$$

Kinematic Constraint: $\mathbf{C}(\mathbf{q}) = 0$

1) Fixed Joint

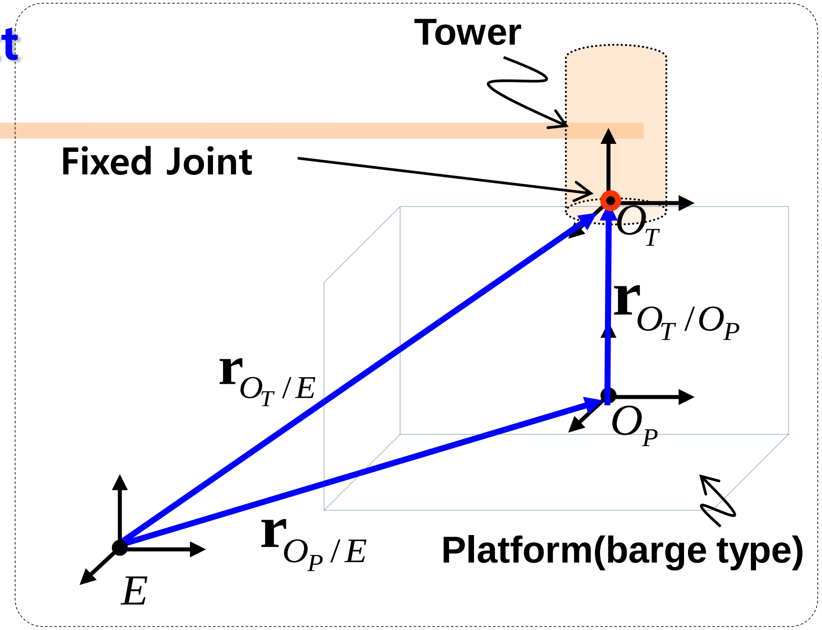
$$\mathbf{r}_{O_P/E} + {}^{n}\mathbf{R}_{b_P}\,{}^{b_P}\mathbf{r}_{O_T/O_P} = \mathbf{r}_{O_T/E}$$

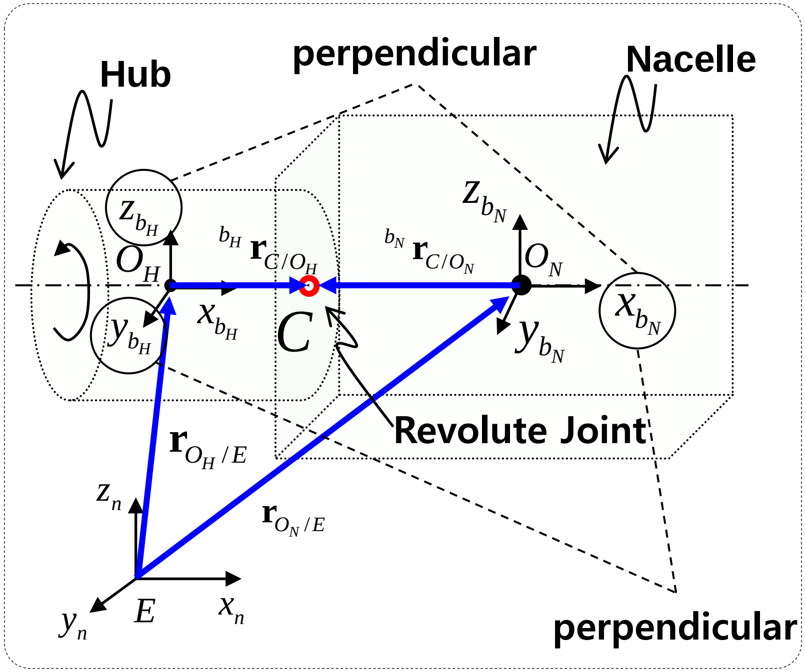
2) Revolute Joint

$$\mathbf{r}_{O_N/E} + {}^{n}\mathbf{R}_{b_N}\,{}^{b_N}\mathbf{r}_{C/O_N} = \mathbf{r}_{O_H/E} + {}^{n}\mathbf{R}_{b_H}\,{}^{b_H}\mathbf{r}_{C/O_H}$$

$$\left({}^{n}\mathbf{R}_{b_N}\,{}^{b_N}y_{b_N}\right)\bullet\left({}^{n}\mathbf{R}_{b_H}\,{}^{b_H}x_{b_H}\right) = 0$$

$$\left({}^{n}\mathbf{R}_{b_N}\,{}^{b_N}z_{b_N}\right)\bullet\left({}^{n}\mathbf{R}_{b_H}\,{}^{b_H}x_{b_H}\right) = 0$$



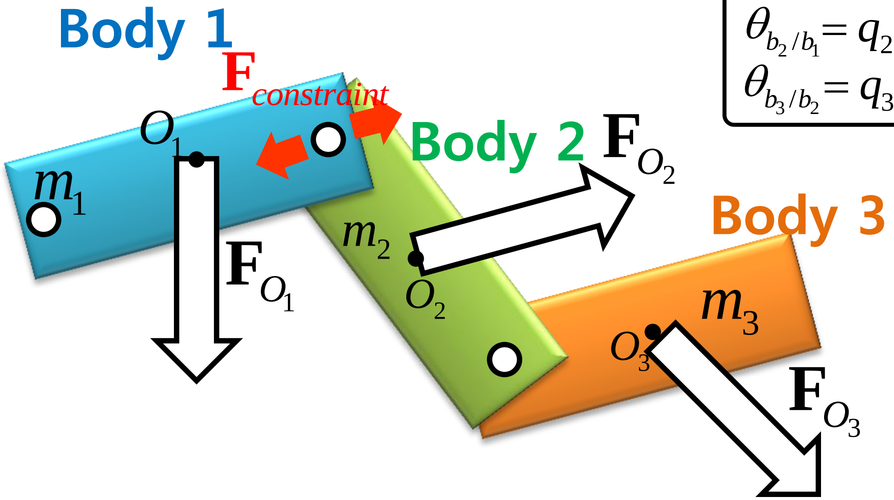


note:

${}^{n}\mathbf{R}_{b_P}$ rotational transformation matrix from the b_P- frame to inertial n-frame

6.4 Equations of Motion for Offshore Floating Wind Turbine Using Recursive Formulation

Inverse Dynamics of 3-Link Arm



$$\theta_{b_1/n} = q_1 \qquad \theta_{b_2/b_1} = q_2 \qquad \theta_{b_3/b_2} = q_3$$

Given:

$$\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \ddot{\theta}_{b_1/n}$$

$$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \ddot{\theta}_{b_2/b_1}$$

$$\theta_{b_3/b_2}, \dot{\theta}_{b_3/b_2}, \ddot{\theta}_{b_3/b_2}$$

Find:

$$\tau_1 \qquad \tau_2 \qquad \tau_3$$

$$\mathbf{p} = \mathbf{p}\left(\theta, \dot{\theta}\right)$$

$$\mathbf{c} = \mathbf{c}\left(\theta, \dot{\theta}\right)$$

$$\mathbf{S} : known$$

$$\mathbf{X} = \left(\theta\right)$$

Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$

$$\mathbf{f}_1^{\,B} = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^{\,B} + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$

$$\tau_1 = \mathbf{S}_1^{\,T} \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$

$$\mathbf{f}_2^{\,B} = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\mathbf{f}_2 = \mathbf{f}_2^{\,B} + {}^2\mathbf{X}_3^* \cdot \mathbf{f}_3$$

$$\tau_2 = \mathbf{S}_2^{\,T} \cdot \mathbf{f}_2$$

Equations for link 3

$$\mathbf{a}_3 = {}^3\mathbf{X}_2 \cdot \mathbf{a}_2 + \mathbf{S}_3 \cdot \ddot{q}_3 + \mathbf{c}_3$$

$$\mathbf{f}_3^{\,B} = \mathbf{I}_3 \cdot \mathbf{a}_3 + \mathbf{p}_3$$

$$\mathbf{f}_3 = \mathbf{f}_3^{\,B} + {}^3\mathbf{X}_4^* \cdot \mathbf{f}_4$$

$$\tau_3 = \mathbf{S}_3^{\,T} \cdot \mathbf{f}_3$$

Forward Dynamics of 3-Link Arm

$\theta_{b_1/n} = q_1$

$\theta_{b_2/b_1} = q_2$

$\theta_{b_3/b_2} = q_3$

Given: $\theta_{b_1/n}, \dot{\theta}_{b_1/n}, \tau_1$

$\theta_{b_2/b_1}, \dot{\theta}_{b_2/b_1}, \tau_2$

$\theta_{b_3/b_2}, \dot{\theta}_{b_3/b_2}, \tau_3$

Find: $\ddot{\theta}_{b_1/n}$

$\ddot{\theta}_{b_2/b_1}$

$\ddot{\theta}_{b_3/b_2}$

$\mathbf{p} = \mathbf{p}(\theta, \dot{\theta})$ $\mathbf{S} : known$

$\mathbf{c} = \mathbf{c}(\theta, \dot{\theta})$ $\mathbf{X} = (\theta)$

Equations for link 1

$$\boxed{\mathbf{a}_1} = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \boxed{\ddot{q}_1} + \mathbf{c}_1$$

(3)

$$\mathbf{f}_1 = \mathbf{I}_1^A \cdot \mathbf{a}_1 + \mathbf{p}_1^A$$

$$\tau_1 = \mathbf{S}_1{}^T \cdot \mathbf{f}_1$$

$$\boxed{\ddot{q}_1} = \left(\mathbf{S}_1^T \mathbf{I}_1^A \mathbf{S}_1\right)^{-1} \left(\tau_1 - \mathbf{S}_1^T \left(\mathbf{I}_1^A \left({}^1\mathbf{X}_0 \mathbf{a}_0 + \mathbf{c}_1\right) + \mathbf{p}_1^A\right)\right)$$

(2)

Equations for link 2

$$\boxed{\mathbf{a}_2} = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \boxed{\ddot{q}_2} + \mathbf{c}_2$$

(5)

$$\mathbf{f}_2 = \mathbf{I}_2^A \cdot \mathbf{a}_2 + \mathbf{p}_2^A$$

$$\tau_2 = \mathbf{S}_2{}^T \cdot \mathbf{f}_2$$

$$\boxed{\ddot{q}_2} = \left(\mathbf{S}_2^T \mathbf{I}_2^A \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2^A \left({}^2\mathbf{X}_1 \boxed{\mathbf{a}_1} + \mathbf{c}_2\right) + \mathbf{p}_2^A\right)\right)$$

(4)

Equations for link 3

$$\mathbf{a}_3 = {}^3\mathbf{X}_2 \cdot \mathbf{a}_2 + \mathbf{S}_3 \cdot \ddot{q}_3 + \mathbf{c}_3$$

$$\mathbf{f}_3 = \mathbf{I}_3 \cdot \mathbf{a}_3 + \mathbf{p}_3$$

$$\tau_3 = \mathbf{S}_3{}^T \cdot \mathbf{f}_3$$

$$\boxed{\ddot{q}_3} = \left(\mathbf{S}_3^T \mathbf{I}_3 \mathbf{S}_3\right)^{-1} \left(\tau_3 - \mathbf{S}_3^T \left(\mathbf{I}_3 \left({}^2\mathbf{X}_2 \boxed{\mathbf{a}_2} + \mathbf{c}_3\right) + \mathbf{p}_3\right)\right)$$

(6)

(1)

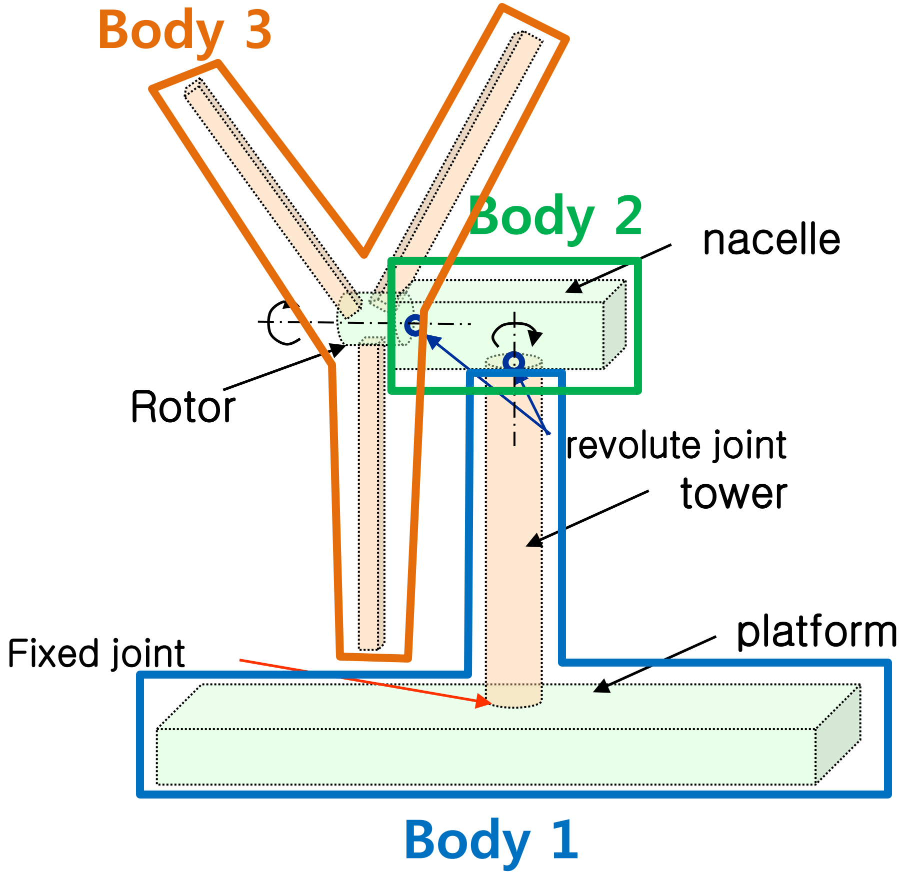
$$\mathbf{I}_1^A = \mathbf{I}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2^A \cdot {}^2\mathbf{X}_1 - {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2^A \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2^A \mathbf{S}_2\right)^{-1} \mathbf{S}_2^T \mathbf{I}_2^A \, {}^2\mathbf{X}_1$$

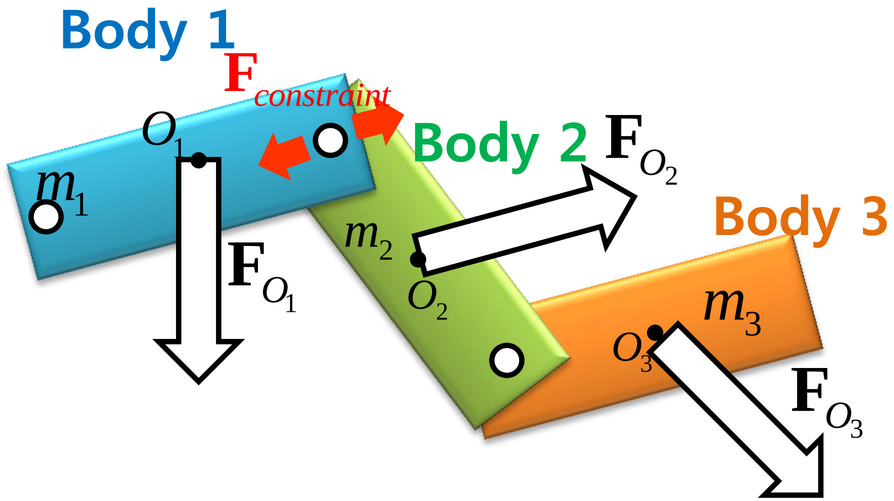
$$\mathbf{p}_1^A = \mathbf{p}_1 + {}^1\mathbf{X}_2^* \cdot \mathbf{p}_2^A + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2^A \cdot \mathbf{c}_2 + {}^1\mathbf{X}_2^* \cdot \mathbf{I}_2^A \cdot \mathbf{S}_2 \cdot \left(\mathbf{S}_2^T \mathbf{I}_2^A \mathbf{S}_2\right)^{-1} \left(\tau_2 - \mathbf{S}_2^T \left(\mathbf{I}_2^A \mathbf{c}_2 + \mathbf{p}_2^A\right)\right)$$

$$\mathbf{I}_2^A = \mathbf{I}_2 + {}^2\mathbf{X}_3^* \cdot \mathbf{I}_3 \cdot {}^3\mathbf{X}_2 - {}^2\mathbf{X}_3^* \cdot \mathbf{I}_3 \cdot \mathbf{S}_3 \cdot \left(\mathbf{S}_3^T \mathbf{I}_3 \mathbf{S}_3\right)^{-1} \mathbf{S}_3^T \mathbf{I}_3 \, {}^3\mathbf{X}_2$$

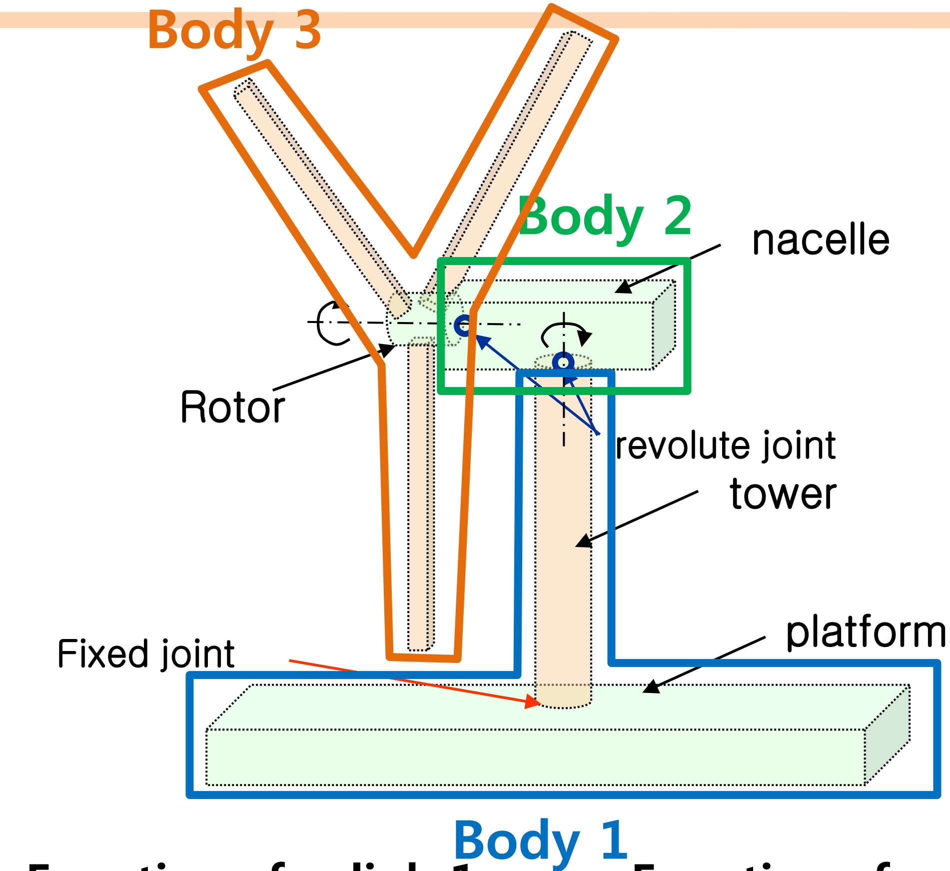
$$\mathbf{p}_2^A = \mathbf{p}_2 + {}^2\mathbf{X}_3^* \cdot \mathbf{p}_3 + {}^2\mathbf{X}_3^* \cdot \mathbf{I}_3 \cdot \mathbf{c}_3 + {}^2\mathbf{X}_3^* \cdot \mathbf{I}_3 \cdot \mathbf{S}_3 \cdot \left(\mathbf{S}_3^T \mathbf{I}_3 \mathbf{S}_3\right)^{-1} \left(\tau_3 - \mathbf{S}_3^T \left(\mathbf{I}_3 \mathbf{c}_3 + \mathbf{p}_3\right)\right)$$

Construction of the Dynamic Equation

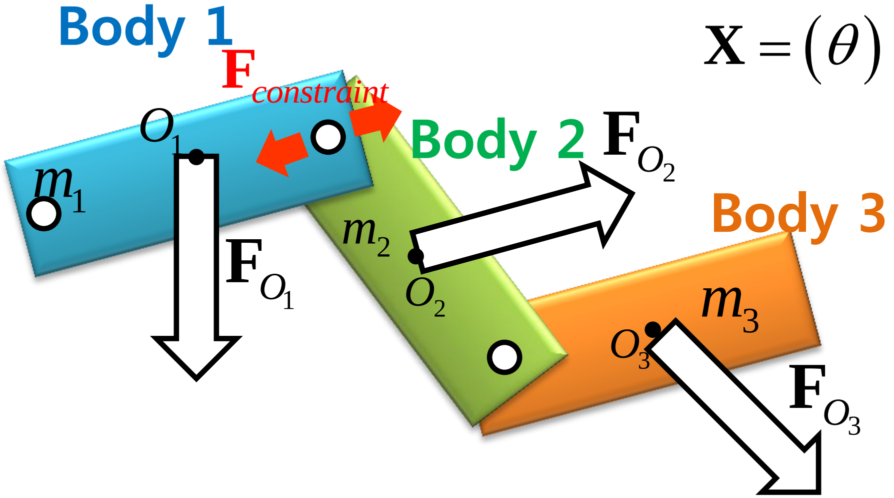




Inverse Dynamics of floating wind turbine



$$\mathbf{p} = \mathbf{p}\left(\theta, \dot{\theta}\right)$$
$$\mathbf{c} = \mathbf{c}\left(\theta, \dot{\theta}\right)$$
$$\mathbf{S} : known$$
$$\mathbf{X} = \left(\theta\right)$$



Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \mathbf{S}_1 \cdot \ddot{q}_1 + \mathbf{c}_1$$
$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$
$$\mathbf{f}_1 = \mathbf{f}_1^B + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$
$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

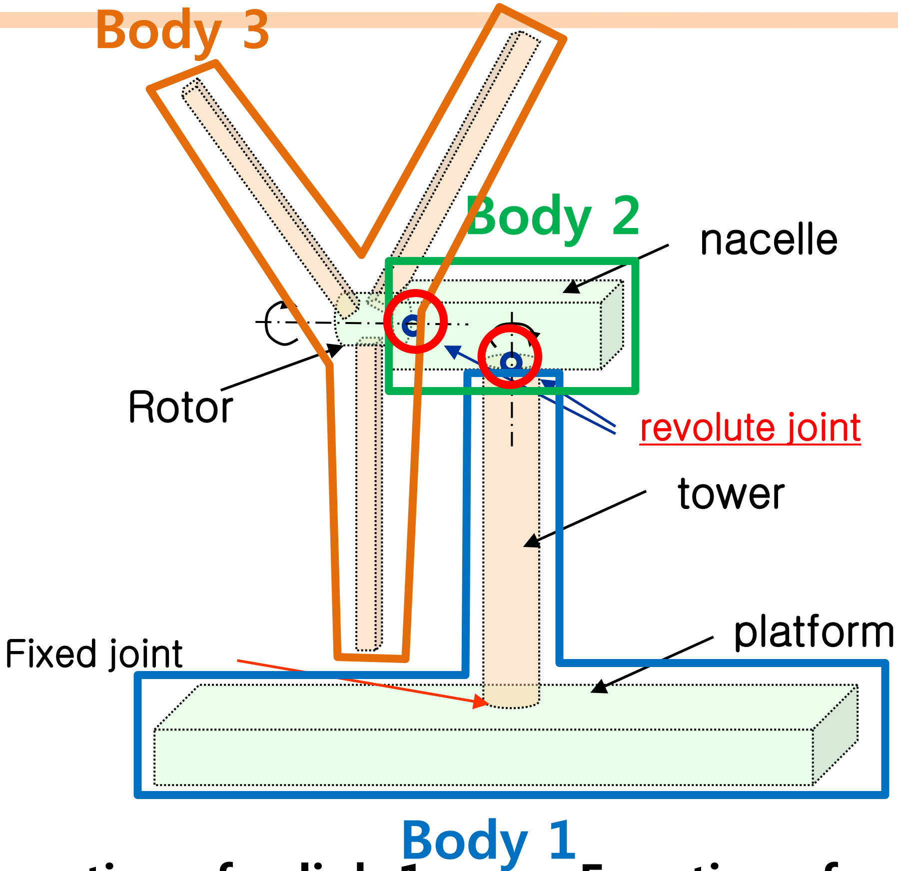
Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \mathbf{S}_2 \cdot \ddot{q}_2 + \mathbf{c}_2$$
$$\mathbf{f}_2^B = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$
$$\mathbf{f}_2 = \mathbf{f}_2^B + {}^2\mathbf{X}_3^* \cdot \mathbf{f}_3$$
$$\tau_2 = \mathbf{S}_2^T \cdot \mathbf{f}_2$$

Equations for link 3

$$\mathbf{a}_3 = {}^3\mathbf{X}_2 \cdot \mathbf{a}_2 + \mathbf{S}_3 \cdot \ddot{q}_3 + \mathbf{c}_3$$
$$\mathbf{f}_3^B = \mathbf{I}_3 \cdot \mathbf{a}_3 + \mathbf{p}_3$$
$$\mathbf{f}_3 = \mathbf{f}_3^B + {}^3\mathbf{X}_4^* \cdot \mathbf{f}_4$$
$$\tau_3 = \mathbf{S}_3^T \cdot \mathbf{f}_3$$

Inverse Dynamics of floating wind turbine

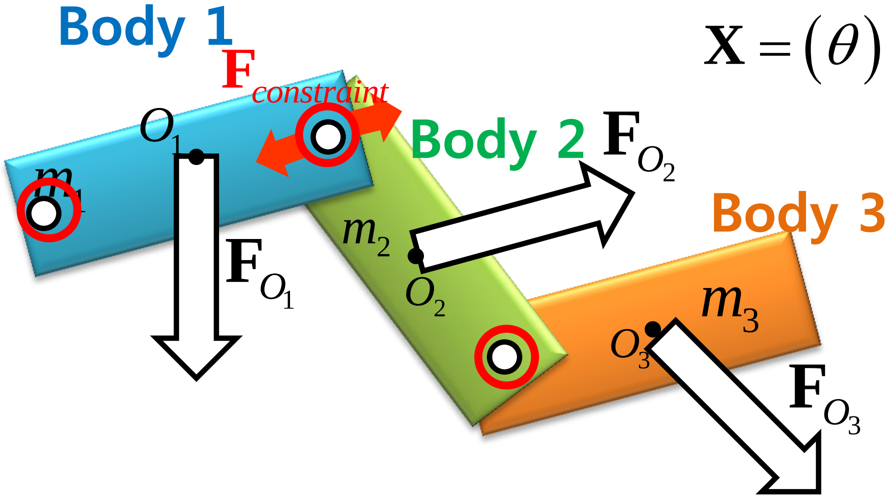


$$\mathbf{p} = \mathbf{p}\left(\theta, \dot{\theta}\right)$$

$$\mathbf{c} = \mathbf{c}\left(\theta, \dot{\theta}\right)$$

$$\mathbf{S} : known$$

$$\mathbf{X} = \left(\theta\right)$$



Equations for link 1

$$\mathbf{a}_1 = {}^1\mathbf{X}_0 \cdot \mathbf{a}_0 + \boxed{\mathbf{S}_1 \cdot \ddot{q}_1} + \mathbf{c}_1$$

$$\mathbf{f}_1^B = \mathbf{I}_1 \cdot \mathbf{a}_1 + \mathbf{p}_1$$

$$\mathbf{f}_1 = \mathbf{f}_1^B + {}^1\mathbf{X}_2^* \cdot \mathbf{f}_2$$

$$\tau_1 = \mathbf{S}_1^T \cdot \mathbf{f}_1$$

Equations for link 2

$$\mathbf{a}_2 = {}^2\mathbf{X}_1 \cdot \mathbf{a}_1 + \boxed{\mathbf{S}_2 \cdot \ddot{q}_2} + \mathbf{c}_2$$

$$\mathbf{f}_2^B = \mathbf{I}_2 \cdot \mathbf{a}_2 + \mathbf{p}_2$$

$$\mathbf{f}_2 = \mathbf{f}_2^B + {}^2\mathbf{X}_3^* \cdot \mathbf{f}_3$$

$$\tau_2 = \mathbf{S}_2^T \cdot \mathbf{f}_2$$

Equations for link 3

$$\mathbf{a}_3 = {}^3\mathbf{X}_2 \cdot \mathbf{a}_2 + \boxed{\mathbf{S}_3 \cdot \ddot{q}_3} + \mathbf{c}_3$$

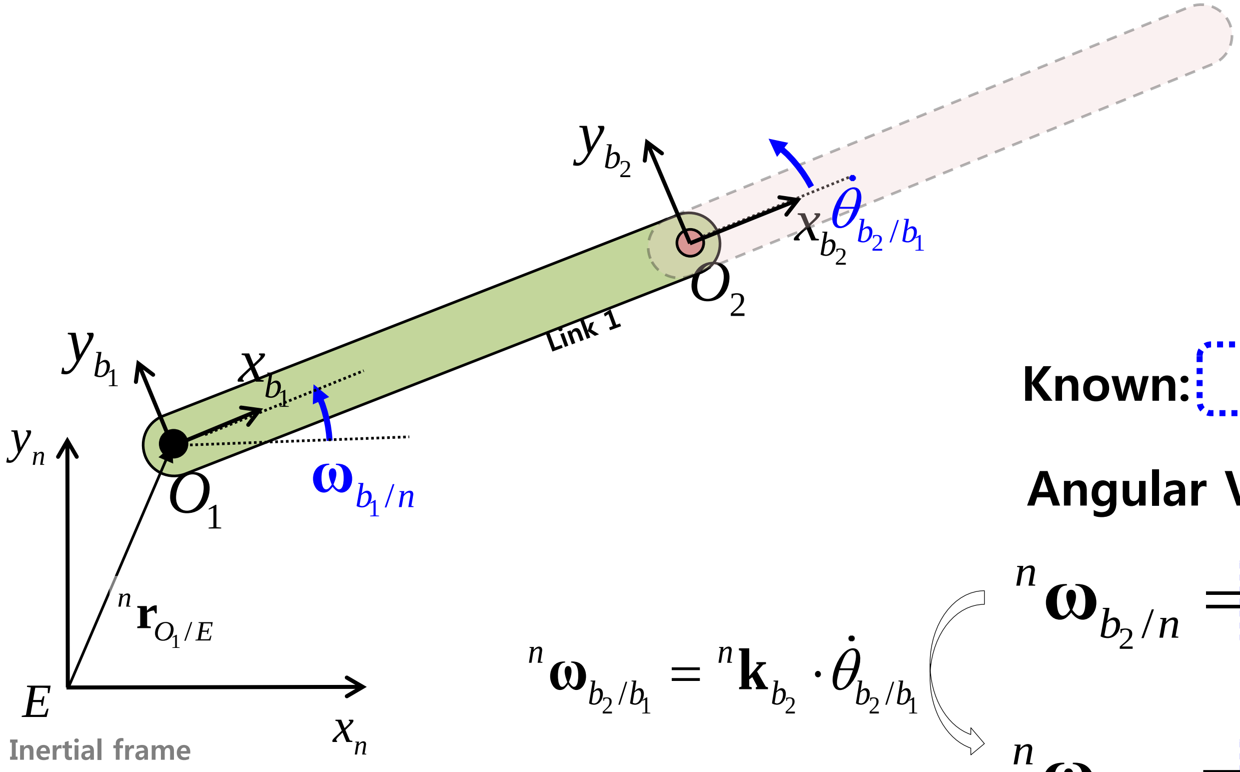
$$\mathbf{f}_3^B = \mathbf{I}_3 \cdot \mathbf{a}_3 + \mathbf{p}_3$$

$$\mathbf{f}_3 = \mathbf{f}_3^B + {}^3\mathbf{X}_4^* \cdot \mathbf{f}_4$$

$$\tau_3 = \mathbf{S}_3^T \cdot \mathbf{f}_3$$

Inverse Dynamics of 2-Link Arm

- Angluar velocity of b_2-frame



Known: ⬚ **Given:** ☐

Angular Velocity of {b$_2$}

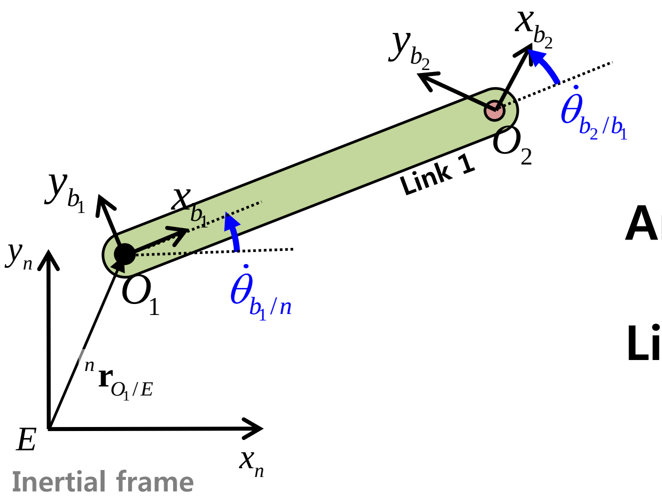
$$ {}^n\boldsymbol{\omega}_{b_2/n} = \boxed{{}^n\boldsymbol{\omega}_{b_1/n}} + {}^n\boldsymbol{\omega}_{b_2/b_1} $$

$$ {}^n\boldsymbol{\omega}_{b_2/b_1} = {}^n\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} $$

$$ {}^n\boldsymbol{\omega}_{b_2/n} = \boxed{{}^n\boldsymbol{\omega}_{b_1/n}} + {}^n\mathbf{k}_{b_2} \cdot \boxed{\dot{\theta}_{b_2/b_1}} $$

Inverse Dynamics of 2-Link Arm

- Velocity of b_2-frame



Velocity of {b_2}

Angular Vel.

$$ {}^{n}\boldsymbol{\omega}_{b_2/n} = {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{n}\mathbf{k}_{b_2} \cdot \dot{\theta}_{b_2/b_1} $$

Linear Vel.

$$ {}^{n}\mathbf{v}_{O_2/E} = {}^{n}\boldsymbol{\omega}_{b_1/n} \times {}^{n}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{v}_{O_1/E} $$

${}^{n}\mathbf{k}_{b_2}$ Rotation axis of O_2 joint decomposed in inertial frame (unit vector of z-axis of b_2-frame)

Coordinate transformation from {n} to body fixed frame {b_2} by multiplication of rotation matrix ${}^{b_2}\mathbf{R}_n$

Inverse Dynamics of 2-Link Arm
- Velocity of b_2-frame

Velocity of $\{b_2\}$

$$\boxed{\begin{array}{l} {}^{n}\boldsymbol{\omega}_{b_2/n} = {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{n}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1} \\ \hdashline {}^{n}\mathbf{v}_{O_2/E} = {}^{n}\boldsymbol{\omega}_{b_1/n} \times {}^{n}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{v}_{O_1/E} \end{array}}$$

$$\Downarrow$$

$${}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

$${}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\mathbf{R}_{b_1}\cdot\left({}^{b_1}\boldsymbol{\omega}_{b_1/n} \times {}^{b_1}\mathbf{r}_{O_2/O_1}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}$$

$$\Downarrow$$

$${}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{\theta}_{b_2/b_1}$$

$${}^{b_2}\mathbf{v}_{O_2/E} = -{}^{b_2}\mathbf{R}_{b_1}\cdot\left({}^{b_1}\mathbf{r}_{O_2/O_1} \times {}^{b_1}\boldsymbol{\omega}_{b_1/n}\right) + {}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{v}_{O_1/E}$$

$$\Downarrow$$

$$\begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{b_2/n} \end{bmatrix} = \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_1} & 0 \\ -{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}\times & {}^{b_2}\mathbf{R}_{b_1} \end{bmatrix} \begin{bmatrix} {}^{b_1}\boldsymbol{\omega}_{b_1/n} \\ {}^{b_1}\mathbf{v}_{b_1/n} \end{bmatrix} + \boxed{\begin{bmatrix} {}^{b_2}\mathbf{k}_{b_2} \\ 0 \end{bmatrix}}\dot{\theta}_{b_2/b_1}$$

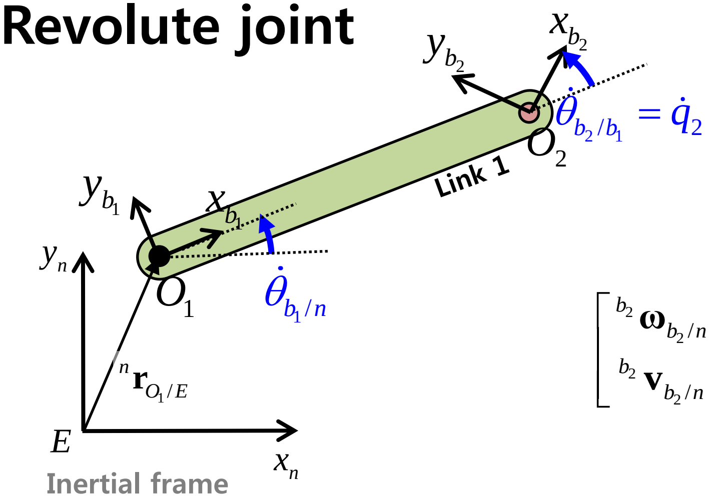
$$\Rightarrow \quad {}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \boxed{\mathbf{S}_{b_2}}\cdot\dot{q}_2$$

In case of revolute joint

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\cdot\dot{q}_2$$

Inverse Dynamics of 2-Link Arm
- Define S matrix according to the joint

Revolute joint

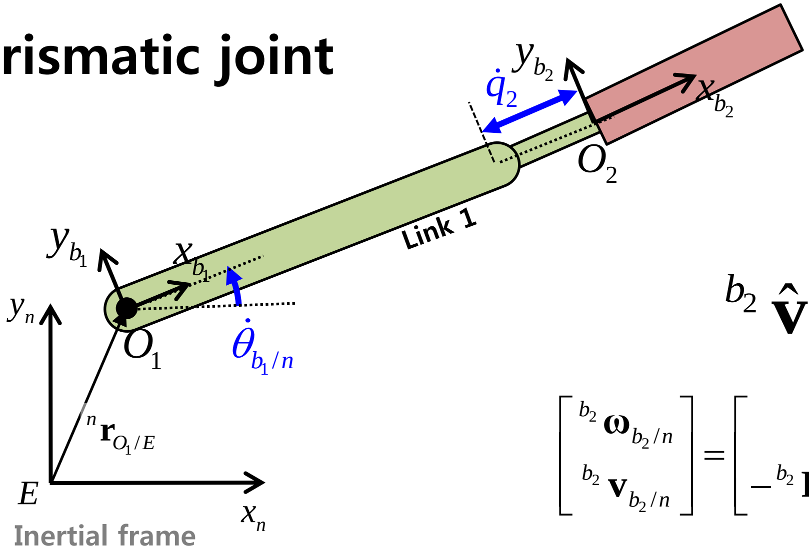


Velocity of {b$_2$}

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \boxed{\mathbf{S}_{b_2}\cdot\dot{q}_2} \rightarrow \boxed{\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}\cdot\dot{q}_2}$$

$$\begin{bmatrix}{}^{b_2}\boldsymbol{\omega}_{b_2/n}\\ {}^{b_2}\mathbf{v}_{b_2/n}\end{bmatrix} = \begin{bmatrix}{}^{b_2}\mathbf{R}_{b_1} & 0\\ -{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}\times & {}^{b_2}\mathbf{R}_{b_1}\end{bmatrix}\begin{bmatrix}{}^{b_1}\boldsymbol{\omega}_{b_1/n}\\ {}^{b_1}\mathbf{v}_{b_1/n}\end{bmatrix} + \begin{bmatrix}{}^{b_2}\mathbf{k}_{b_2}\\ 0\end{bmatrix}\dot{\theta}_{b_2/b_1}$$

Prismatic joint



Velocity of {b$_2$}

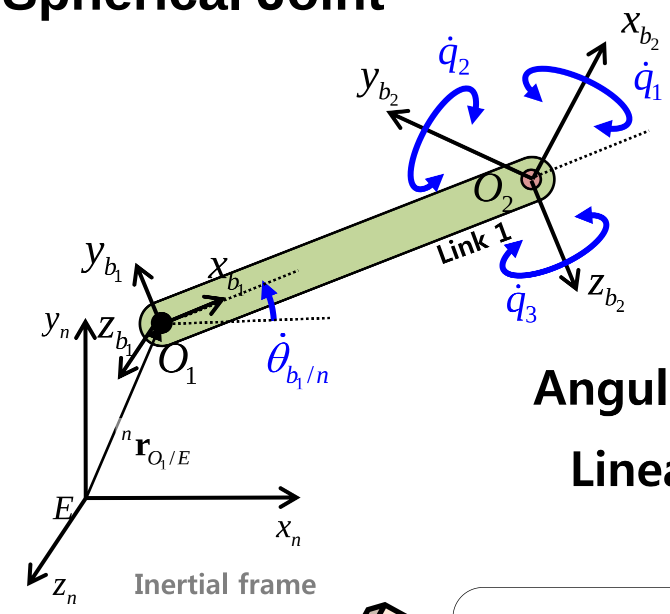
$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \boxed{\mathbf{S}_{b_2}\cdot\dot{q}_2} \rightarrow \boxed{\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}\cdot\dot{q}_2}$$

$$\begin{bmatrix}{}^{b_2}\boldsymbol{\omega}_{b_2/n}\\ {}^{b_2}\mathbf{v}_{b_2/n}\end{bmatrix} = \begin{bmatrix}{}^{b_2}\mathbf{R}_{b_1} & 0\\ -{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}\times & {}^{b_2}\mathbf{R}_{b_1}\end{bmatrix}\begin{bmatrix}{}^{b_1}\boldsymbol{\omega}_{b_1/n}\\ {}^{b_1}\mathbf{v}_{b_1/n}\end{bmatrix} + \begin{bmatrix}0\\ {}^{b_2}\mathbf{i}_{b_2}\end{bmatrix}\dot{\theta}_{b_2/b_1}$$

Inverse Dynamics of 2-Link Arm

- Define S matrix according to the joint

Spherical Joint



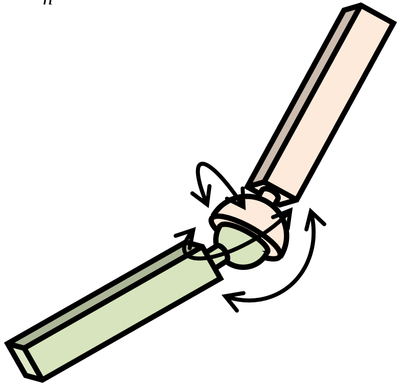
Angular Vel.

$$ {}^{n}\boldsymbol{\omega}_{b_2/n} = {}^{n}\boldsymbol{\omega}_{b_1/n} + {}^{n}\mathbf{i}_{b_2}\cdot\dot{q}_1 + {}^{n}\mathbf{j}_{b_2}\cdot\dot{q}_2 + {}^{n}\mathbf{k}_{b_2}\cdot\dot{q}_3 $$

Linear Vel.

$$ {}^{n}\mathbf{v}_{O_2/E} = {}^{n}\boldsymbol{\omega}_{b_1/n} \times {}^{n}\mathbf{r}_{O_2/O_1} + {}^{n}\mathbf{v}_{O_1/E} $$

Coordinate transformation from {n} to body fixed frame {b$_2$} by multiplication of rotation matrix ${}^{b_2}\mathbf{R}_{n}$



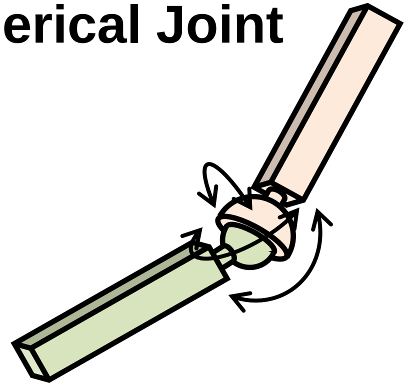
Spherical Joint - 3 degree of freedom

$$ {}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{i}_{b_2}\cdot\dot{q}_1 + {}^{b_2}\mathbf{j}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{q}_3 $$

$$ {}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\boldsymbol{\omega}_{b_1/n} \times {}^{b_2}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{v}_{O_1/E} $$

Inverse Dynamics of 2-Link Arm
- Define S matrix according to the joint

Spherical Joint



$$\boxed{\begin{array}{l} {}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{i}_{b_2}\cdot\dot{q}_1 + {}^{b_2}\mathbf{j}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{q}_3 \\ \hdashline {}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{v}_{O_1/E} \end{array}}$$

$$\begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{b_2/n} \end{bmatrix} = \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_1} & 0 \\ -{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}\times & {}^{b_2}\mathbf{R}_{b_1} \end{bmatrix} \begin{bmatrix} {}^{b_1}\boldsymbol{\omega}_{b_1/n} \\ {}^{b_1}\mathbf{v}_{b_1/n} \end{bmatrix} + \begin{bmatrix} {}^{b_2}\mathbf{i}_{b_2} \\ 0 \end{bmatrix}\dot{q}_1 + \begin{bmatrix} {}^{b_2}\mathbf{j}_{b_2} \\ 0 \end{bmatrix}\dot{q}_2 + \begin{bmatrix} {}^{b_2}\mathbf{k}_{b_2} \\ 0 \end{bmatrix}\dot{q}_3$$

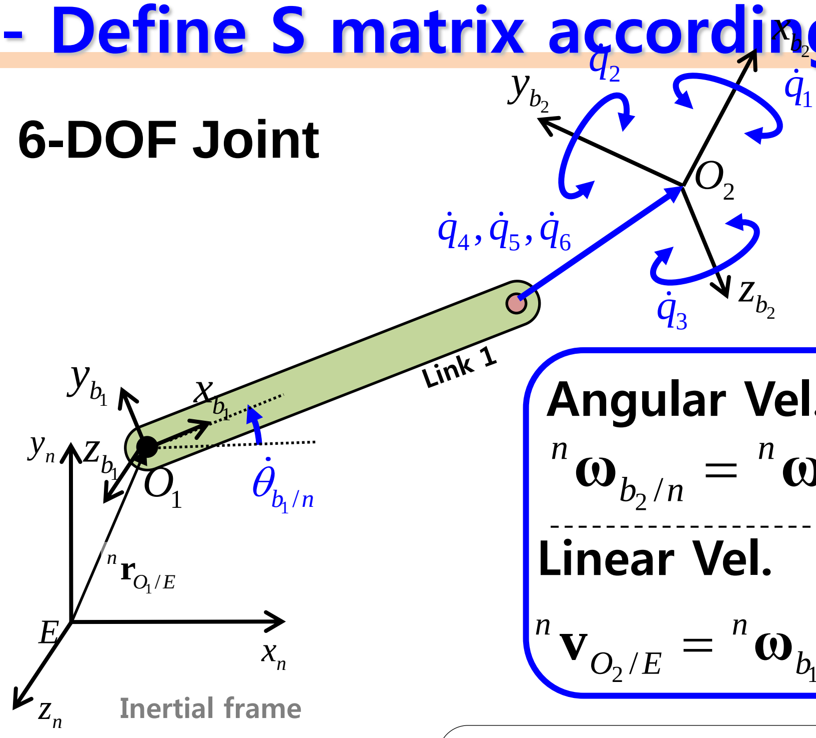
$$\begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{b_2/n} \end{bmatrix} = \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_1} & 0 \\ -{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}\times & {}^{b_2}\mathbf{R}_{b_1} \end{bmatrix} \begin{bmatrix} {}^{b_1}\boldsymbol{\omega}_{b_1/n} \\ {}^{b_1}\mathbf{v}_{b_1/n} \end{bmatrix} + \boxed{\begin{bmatrix} {}^{b_2}\mathbf{i}_{b_2} & {}^{b_2}\mathbf{j}_{b_2} & {}^{b_2}\mathbf{k}_{b_2} \\ 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$\Rightarrow \quad {}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \boxed{\boxed{\mathbf{S}_{b_2}}\cdot\dot{\mathbf{q}}} \longrightarrow \boxed{\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}}$$

Inverse Dynamics of 2-Link Arm

- Define S matrix according to the joint

6-DOF Joint



Angular Vel.

$${}^n\boldsymbol{\omega}_{b_2/n} = {}^n\boldsymbol{\omega}_{b_1/n} + {}^n\mathbf{i}_{b_2}\cdot\dot{q}_1 + {}^n\mathbf{j}_{b_2}\cdot\dot{q}_2 + {}^n\mathbf{k}_{b_2}\cdot\dot{q}_3$$

Linear Vel.

$${}^n\mathbf{v}_{O_2/E} = {}^n\boldsymbol{\omega}_{b_1/n} \times {}^n\mathbf{r}_{O_2/O_1} + {}^n\mathbf{v}_{O_1/E} + {}^n\mathbf{i}_{b_2}\cdot\dot{q}_4 + {}^n\mathbf{j}_{b_2}\cdot\dot{q}_5 + {}^n\mathbf{k}_{b_2}\cdot\dot{q}_6$$

Coordinate transformation from {n} to body fixed frame {b$_2$} by multiplication of rotation matrix ${}^{b_2}\mathbf{R}_n$

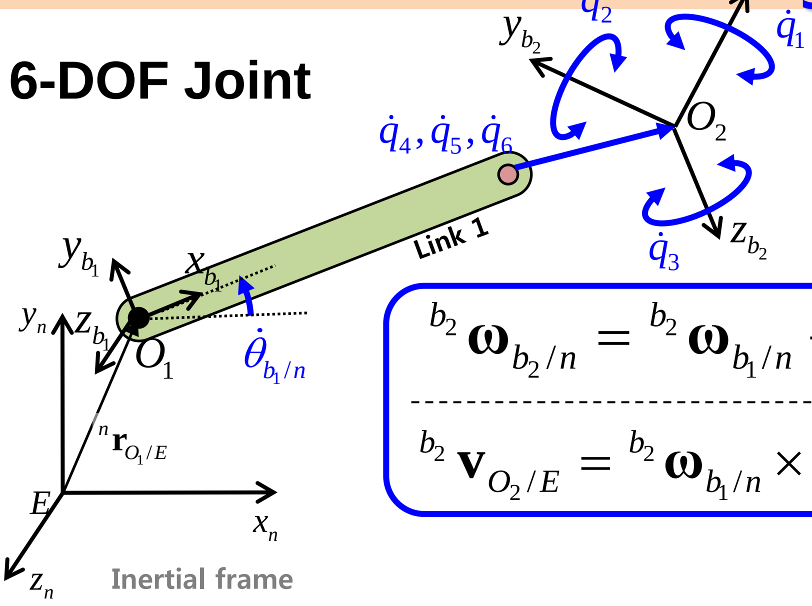
$${}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{i}_{b_2}\cdot\dot{q}_1 + {}^{b_2}\mathbf{j}_{b_2}\cdot\dot{q}_2 + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{q}_3$$

$${}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\boldsymbol{\omega}_{b_1/n} \times {}^{b_2}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{v}_{O_1/E} + {}^{b_2}\mathbf{i}_{b_2}\cdot\dot{q}_4 + {}^{b_2}\mathbf{j}_{b_2}\cdot\dot{q}_5 + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot{q}_6$$

Inverse Dynamics of 2-Link Arm

- Define S matrix according to the joint

6-DOF Joint



$$
\boxed{
\begin{array}{l}
{}^{b_2}\boldsymbol{\omega}_{b_2/n} = {}^{b_2}\boldsymbol{\omega}_{b_1/n} + {}^{b_2}\mathbf{i}_{b_2}\cdot\dot q_1 + {}^{b_2}\mathbf{j}_{b_2}\cdot\dot q_2 + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot q_3 \\
\hdashline
{}^{b_2}\mathbf{v}_{O_2/E} = {}^{b_2}\boldsymbol{\omega}_{b_1/n}\times{}^{b_2}\mathbf{r}_{O_2/O_1} + {}^{b_2}\mathbf{v}_{O_1/E} + {}^{b_2}\mathbf{i}_{b_2}\cdot\dot q_4 + {}^{b_2}\mathbf{j}_{b_2}\cdot\dot q_5 + {}^{b_2}\mathbf{k}_{b_2}\cdot\dot q_6
\end{array}
}
$$

$$
\begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{b_2/n} \end{bmatrix}
=
\begin{bmatrix} {}^{b_2}\mathbf{R}_{b_1} & 0 \\ -{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}\times & {}^{b_2}\mathbf{R}_{b_1} \end{bmatrix}
\begin{bmatrix} {}^{b_1}\boldsymbol{\omega}_{b_1/n} \\ {}^{b_1}\mathbf{v}_{b_1/n} \end{bmatrix}
+ \begin{bmatrix} {}^{b_2}\mathbf{i}_{b_2} \\ 0 \end{bmatrix}\dot q_1
+ \begin{bmatrix} {}^{b_2}\mathbf{j}_{b_2} \\ 0 \end{bmatrix}\dot q_2
+ \begin{bmatrix} {}^{b_2}\mathbf{k}_{b_2} \\ 0 \end{bmatrix}\dot q_3
+ \begin{bmatrix} 0 \\ {}^{b_2}\mathbf{i}_{b_2} \end{bmatrix}\dot q_4
+ \begin{bmatrix} 0 \\ {}^{b_2}\mathbf{j}_{b_2} \end{bmatrix}\dot q_5
+ \begin{bmatrix} 0 \\ {}^{b_2}\mathbf{k}_{b_2} \end{bmatrix}\dot q_6
$$

$$\Downarrow$$

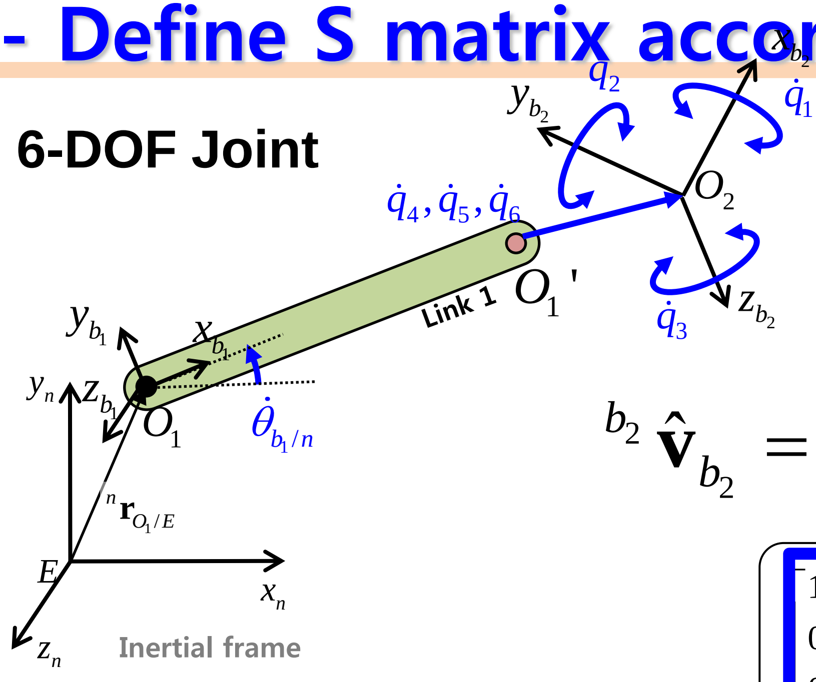
$$
\begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{b_2/n} \end{bmatrix}
=
\begin{bmatrix} {}^{b_2}\mathbf{R}_{b_1} & 0 \\ -{}^{b_2}\mathbf{R}_{b_1}\cdot{}^{b_1}\mathbf{r}_{O_2/O_1}\times & {}^{b_2}\mathbf{R}_{b_1} \end{bmatrix}
\begin{bmatrix} {}^{b_1}\boldsymbol{\omega}_{b_1/n} \\ {}^{b_1}\mathbf{v}_{b_1/n} \end{bmatrix}
+
\boxed{\begin{bmatrix} {}^{b_2}\mathbf{i}_{b_2} & {}^{b_2}\mathbf{j}_{b_2} & {}^{b_2}\mathbf{k}_{b_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & {}^{b_2}\mathbf{i}_{b_2} & {}^{b_2}\mathbf{j}_{b_2} & {}^{b_2}\mathbf{k}_{b_2} \end{bmatrix}}
\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \\ \dot q_4 \\ \dot q_5 \\ \dot q_6 \end{bmatrix}
$$

$$\Downarrow$$

$$
{}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1}\cdot{}^{b_1}\hat{\mathbf{v}}_{b_1} + \boxed{\boxed{\mathbf{S}_{b_2}}\cdot\dot{\mathbf{q}}}
\longrightarrow
\boxed{\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}
\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \\ \dot q_4 \\ \dot q_5 \\ \dot q_6 \end{bmatrix}}
$$

Inverse Dynamics of 2-Link Arm - Define S matrix according to the joint

6-DOF Joint



$$ {}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \boxed{\boxed{\mathbf{S}_{b_2}} \cdot \dot{\mathbf{q}}} $$

$$ \boxed{\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}} $$

Caution!

$$ \boxed{ \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = {}^{b_2}\boldsymbol{\omega}_{b_2/b_1} \qquad \begin{bmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} = {}^{b_2}\mathbf{r}_{O_2/O_1{}'} } $$

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