Chapter 3 Turbulent Diffusion

Contents

Objectives

• present equations and concepts for turbulent diffusion processes

• present similar approaches for the turbulent diffusion and shear flow dispersion

3.1 Introduction

- Mass introduced at a point will spread much faster in turbulent flow than in laminar flow.

- Velocities and pressures measured at a point in the fluid are unsteady and possess a random component.

Use **Navier-Stokes Eq**. to explain turbulent flow

◆ Turbulent flow: Irregularity, randomness ↔ coherent structure

Diffusivity

High Reynolds number

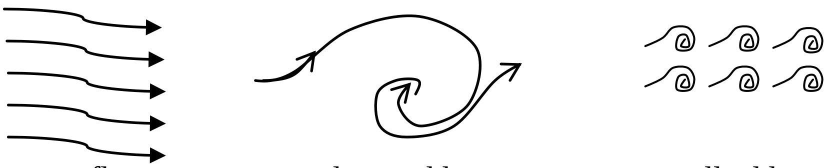
3-D fluctuations ↔ tendency to be isotropic

Dissipation of kinetic energy

Continuum phenomenon

Feature of flow ↔ property of fluid (ρ, μ,)

◆ Scale of turbulence



mean flow → large eddy → small eddy → heat

generation of turbulence — energy cascade — dissipation by viscosity

In equilibrium, transfer rate = dissipation rate

◆ Reynolds experiment

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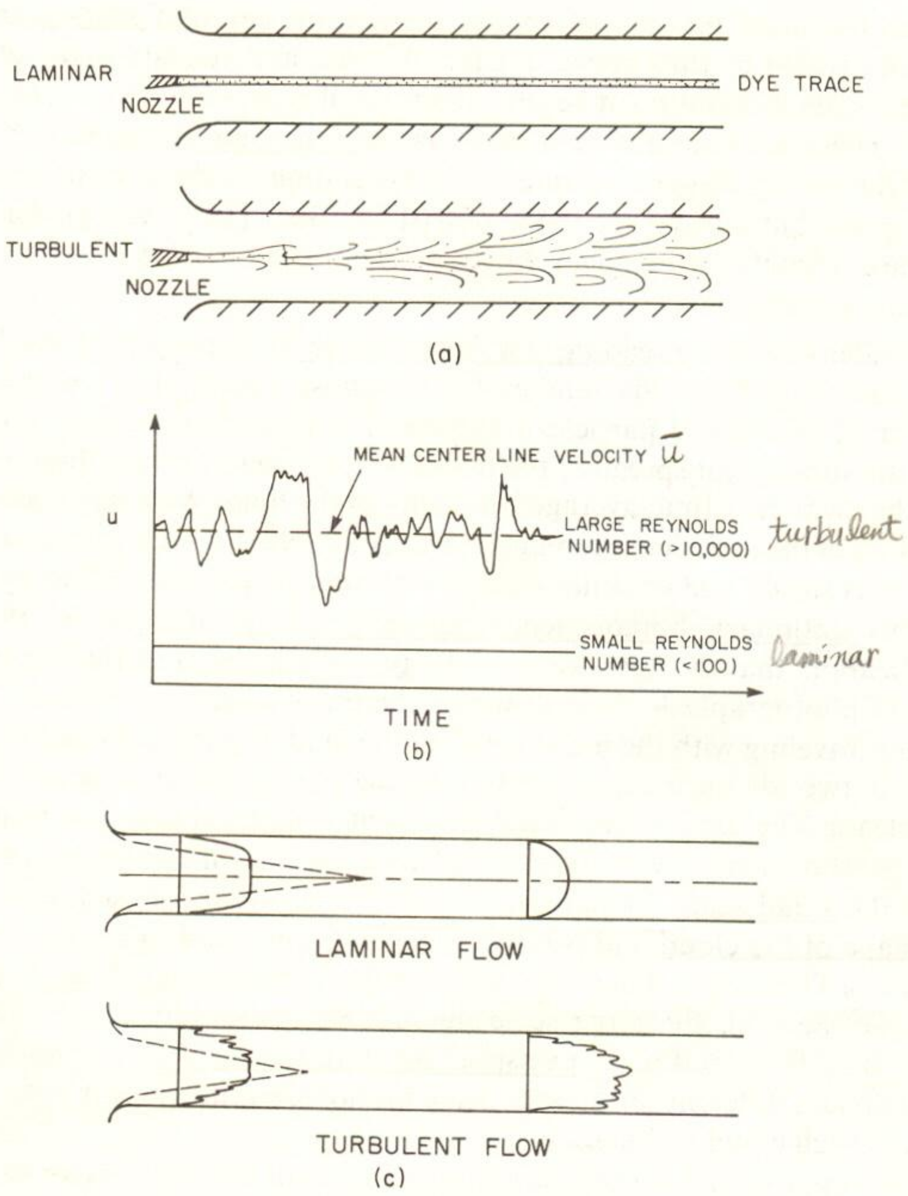


◆ Kolmogorov's universal equilibrium theory of turbulence

- Behavior of the intermediate scale is governed only by the transfer of energy which, in turn, is exactly balanced by dissipation at the very small scales.

ε = time rate of energy dissipation per unit mass

$$[\varepsilon] = \left[\frac{energy}{time}\frac{1}{Mass}\right] = \left[\frac{F\,L}{T}\frac{1}{M}\right] = \left[\frac{ML^2T^{-2}}{T}\frac{1}{M}\right] = \left[L^2T^{-3}\right]$$

ν = kinematic viscosity $= \dfrac{\mu}{\rho} = \left[\dfrac{ML^{-1}T^{-1}}{ML^{-3}}\right] = \left[L^2T^{-1}\right]$

Kolmogorov scales: Use ε and ν to represent different scales

→ length scale $\propto \varepsilon,\ \nu$

i) length $= \left(\dfrac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} = \left[\dfrac{L^6T^{-3}}{L^2T^{-3}}\right]^{\frac{1}{4}} = [L]$

ii) time $= \left(\dfrac{\nu}{\varepsilon}\right)^{\frac{1}{2}} = \left[\dfrac{L^2T^{-1}}{L^2T^{-3}}\right]^{\frac{1}{2}} = [T]$

iii) velocity $= \dfrac{\left(\dfrac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}}{\left(\dfrac{\nu}{\varepsilon}\right)^{\frac{1}{2}}} = (\nu\varepsilon)^{\frac{1}{4}} = \left[(L^2T^{-1})(L^2T^{-3})\right]^{\frac{1}{4}} = \left[LT^{-1}\right]$

For open ocean,

$$\varepsilon = 0.01 cm^2/\sec^3 \;\; ; \quad \nu = 0.01 cm^2/s \quad (20°C)$$

→ dissipation length scale ≈ 0.1cm

time scale ≈ 1sec

velocity scale ≈ 0.1cm/sec

◆ Spreading of a slug of tracer in a high Reynolds number flow

(1) Small scale fluctuations, which are different for each cloud, distort the shape of the cloud and produce steep concentration differences over short distance.

→ These small scale irregularities are smoothed out by molecular diffusion.

(2) Large scale fluctuations transport the entire cloud.

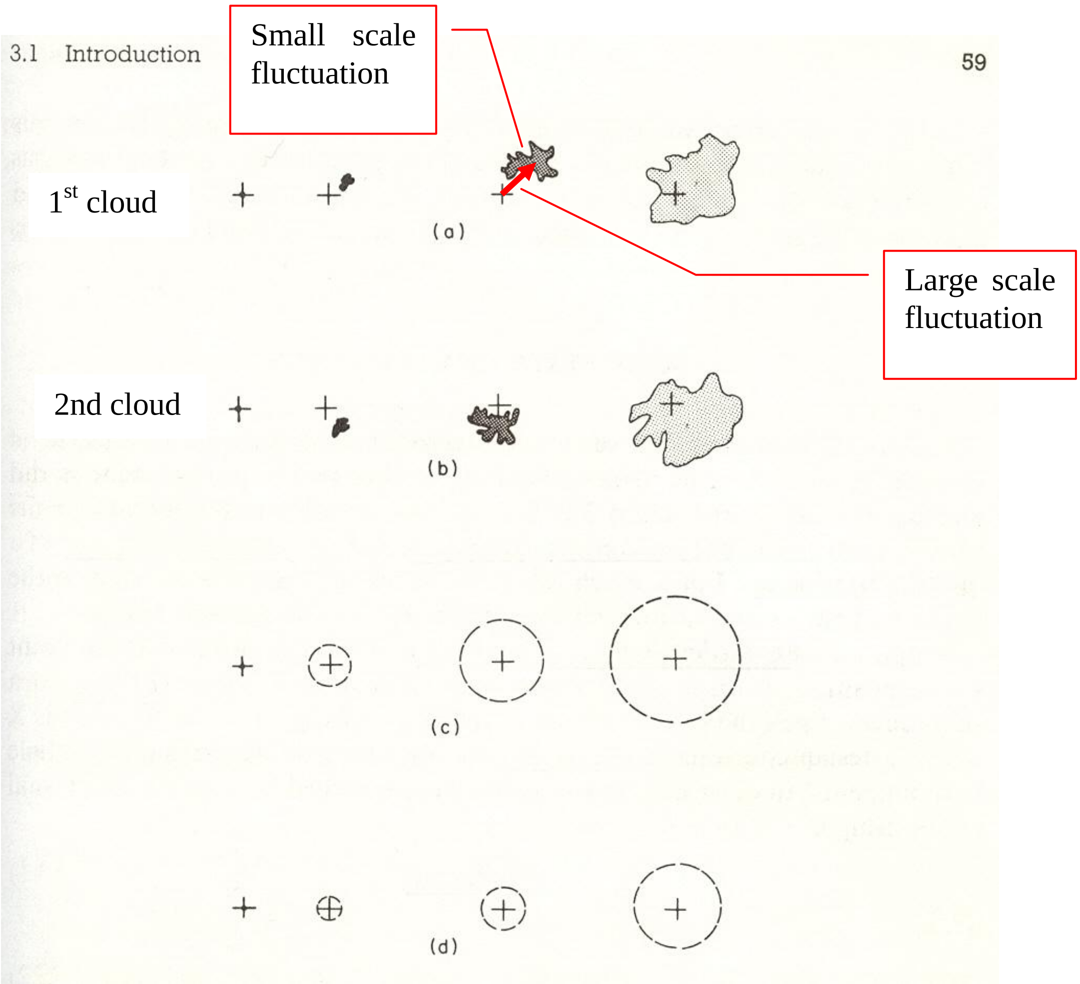
→ The largest scale of motion is slightly larger than the largest cloud.

(3) Ensemble average = mean over many trials

- average the random motions over a long period time
- average out the effects of the largest eddies

→ The center of mass tends to return to the origin through the process of averaging.

→ Final result is the **larger spread** in turbulent flow than in laminar flow.



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Small scale fluctuation
Large scale fluctuation
1st cloud
2nd cloud
(a)
(b)
(c)
(d)

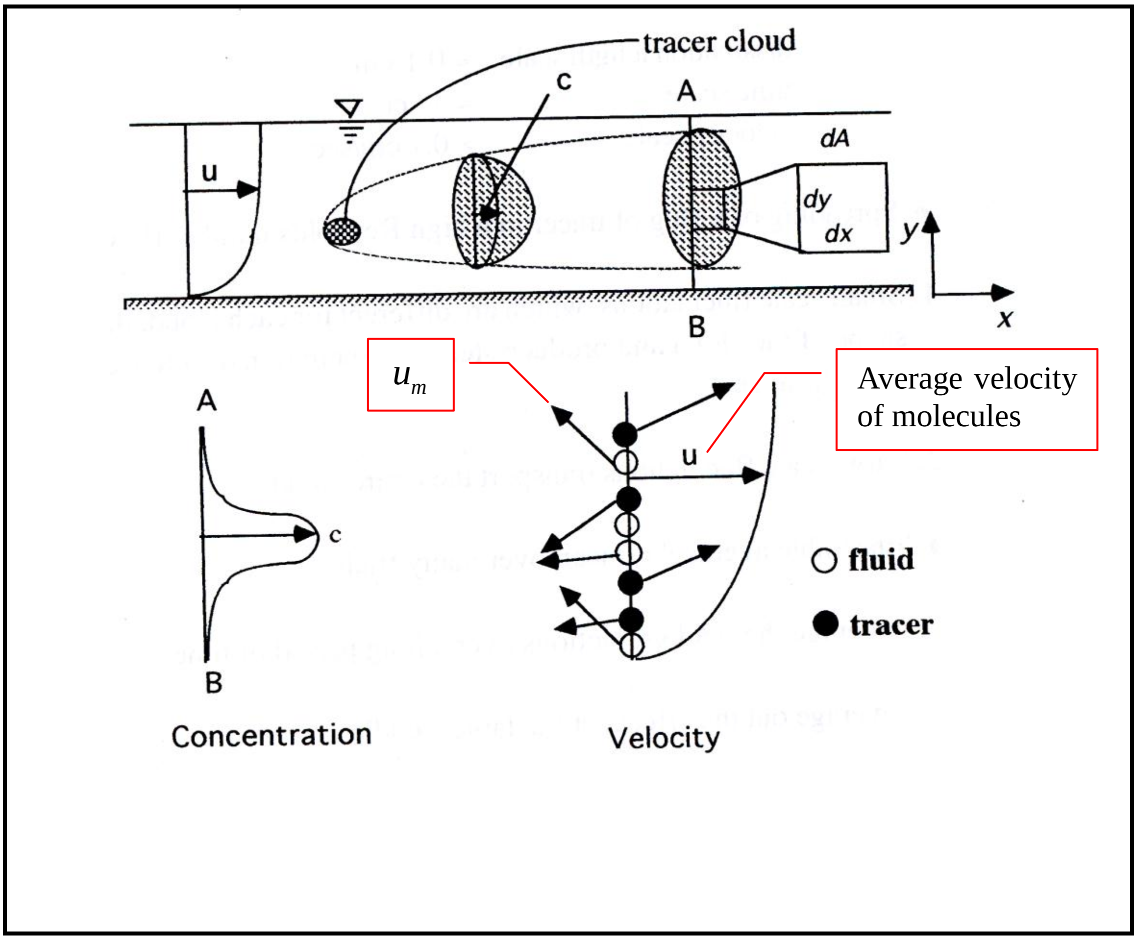
3.2 Unified View of Diffusion and Dispersion

- Similarities among the various types of diffusion and dispersion are shown.

- Diffusion and dispersion are actually advective transport mechanisms.

3.2.1 Molecular Diffusion

◆ 2-D open-channel flow



To write the mass balance equation, we need to know how many fluid molecules and how many tracer molecules pass through and the direction and spread of each molecule.

→ molecular approach → statistical manner

◆ Continuum approach

- Assume fluid carries tracer through at a rate depending on the concentration, *c*, and the fluid velocity, *u*.
- However, the fluid *u*, cannot completely represent the tracer movement because the velocity, *u*, does not account for the movement of the molecules which have directions and speeds different from *u*.
- Molecular diffusion accounts for the difference between the true molecular motion and the manner chosen to represent the motion. (i.e., by *u*)

$$\Delta u = u_m - u$$

Thus, mass flux by this **velocity difference** is

$$j = \Delta u\, c$$

Now, apply Fick' law

- transport called molecular diffusion is proportional to the concentration gradient.

$$j_m = \Delta u\, c \propto \frac{\partial c}{\partial x}$$

$$j_m = -D_m \frac{\partial c}{\partial x} \qquad \text{(a)}$$

D_m = constant of proportionality = molecular diffusivity

Now, consider advection by **mean motion**

$$j_x = cu - D_m \frac{\partial c}{\partial x} \qquad \text{(a)}$$

Then, substituting (a) into mass conservation equation yields 2-D advection-diffusion equation as

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D_m \frac{\partial^2 c}{\partial x^2} + D_m \frac{\partial^2 c}{\partial y^2} \tag{3.1}$$

By mean motion

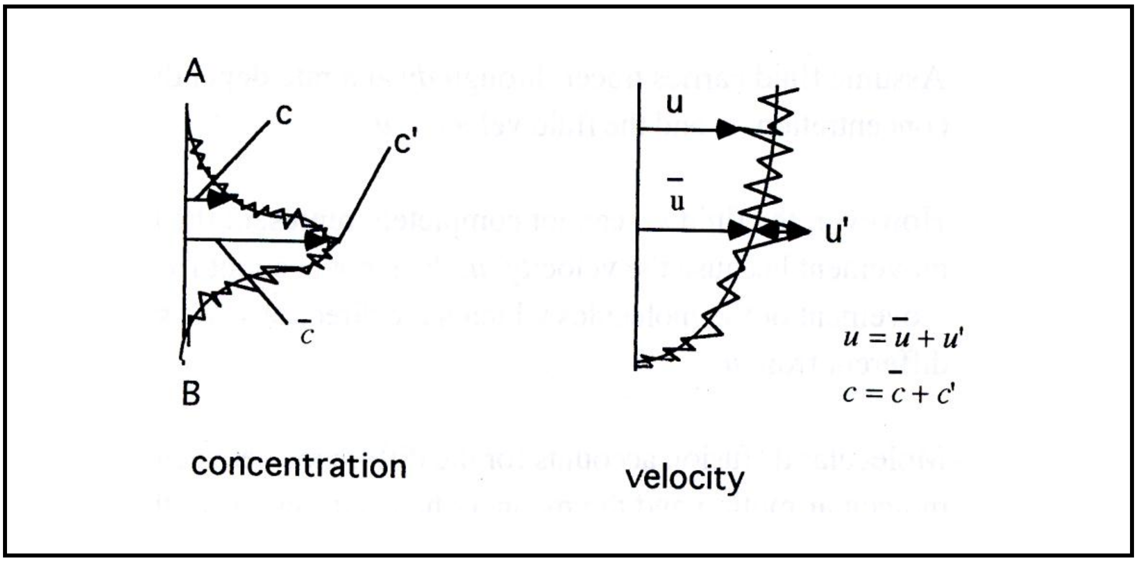
① $\frac{\partial c}{\partial t}$ = time rate of change of concentration at a point

② $u\frac{\partial c}{\partial x}$ = advection of tracer with the fluid

By velocity fluctuation

③ $D_m \frac{\partial^2 c}{\partial x^2}$, $D_m \frac{\partial^2 c}{\partial y^2}$ = molecular diffusion

3.2.2 Turbulent Diffusion



Decompose velocity and concentration into mean and fluctuation

$$u = \bar{u} + u'$$

$$c = \bar{c} + c' \qquad \text{(b)}$$

$$v = v' \qquad \text{(assume only fluctuation in } y\text{-direction)}$$

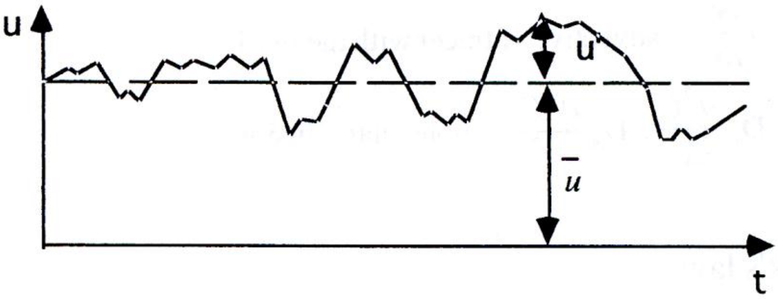
$\bar{u}$, $\bar{c}$ = time-averaged values of u and c

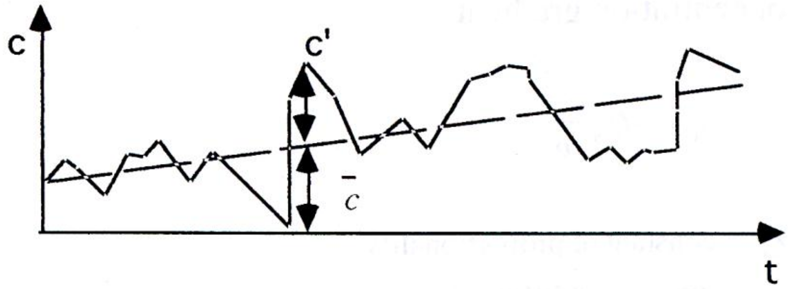
$$\bar{u} \equiv \frac{1}{T}\int_0^T u dt$$

$$\bar{u}' = \bar{v}' = \bar{c}' = 0$$

where T = averaging time interval

$$\begin{cases} 10^0 \sim 10^2 \sec & \text{for open channel flow} \\ 10^{-1} \sim 10^0 \sec & \text{for pipe flow} \end{cases}$$





Conservative form

For 2-D flow, the advection-diffusion equation is

$$\frac{\partial c}{\partial t}+\frac{\partial uc}{\partial x}+\frac{\partial vc}{\partial y}=D_m\frac{\partial^2 c}{\partial x^2}+D_m\frac{\partial^2 c}{\partial y^2} \tag{3.2}$$

Substitute (b) into (3.2), then Eq. (3.2) becomes

$$\frac{\partial(\bar{c}+c')}{\partial t}+\frac{\partial\left(\bar{u}+u'\right)(\bar{c}+c')}{\partial x}+\frac{\partial v'(\bar{c}+c')}{\partial y}=D_m\frac{\partial^2(\bar{c}+c')}{\partial x^2}+D_m\frac{\partial^2(\bar{c}+c')}{\partial y^2}$$

$$\frac{\partial \bar{c}}{\partial t}+\frac{\partial}{\partial x}\bar{u}\,\bar{c}=D_m\frac{\partial^2\bar{c}}{\partial x^2}+D_m\frac{\partial^2\bar{c}}{\partial y^2}$$

$$-\frac{\partial c'}{\partial t}-\frac{\partial}{\partial x}(\bar{u}\,c')-\frac{\partial}{\partial x}(u'\bar{c})-\frac{\partial}{\partial x}(u'c')\;-\frac{\partial}{\partial y}(v'\bar{c})-\frac{\partial}{\partial y}(v'c')$$

$$+D_m\frac{\partial^2 c'}{\partial x^2}+D_m\frac{\partial^2 c'}{\partial y^2}$$

Integrate (average) w.r.t. time, and apply Reynolds rule

$$\overline{\frac{\partial \bar{c}}{\partial t}} + \overline{\frac{\partial(\bar{u}\,\bar{c})}{\partial x}} = \overline{D_m \frac{\partial^2 \bar{c}}{\partial x^2}} + \overline{D_m \frac{\partial^2 \bar{c}}{\partial y^2}}$$

$$-\cancel{\overline{\frac{\partial c'}{\partial t}}} + \cancel{\overline{\frac{\partial(\bar{u}\,c')}{\partial x}}} - \cancel{\overline{\frac{\partial(u'\bar{c})}{\partial x}}} - \overline{\frac{\partial u'c'}{\partial x}} - \cancel{\overline{\frac{\partial v'\bar{c}}{\partial y}}} - \overline{\frac{\partial v'c'}{\partial y}}$$

$$+\cancel{D_m \frac{\partial^2 c'}{\partial x^2}} + \cancel{D_m \frac{\partial^2 c'}{\partial y^2}}$$

[Re] Reynolds rules of averages (Schlichting; p460, 371)

$$\bar{\bar{f}} = \bar{f}$$

$$\overline{f+g} = \bar{f} + \bar{g}$$

$$\overline{f \cdot \bar{g}} = \bar{f} \cdot \bar{g}$$

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \bar{f}}{\partial s}$$

$$\int \overline{f ds} = \int \bar{f} ds$$

Drop all zero terms using Reynolds rules of averages

$$\frac{\partial \bar{c}}{\partial t} + \bar{u}\frac{\partial \bar{c}}{\partial x} = D_m \frac{\partial^2 \bar{c}}{\partial x^2} + D_m \frac{\partial^2 \bar{c}}{\partial y^2} + \underbrace{\frac{\partial(\overline{-u'c'})}{\partial x} + \frac{\partial(\overline{-v'c'})}{\partial y}}_{\substack{\text{advective transport} \\ \text{due to } u', v', \text{and } c'}}$$

It is assumed and confirmed experimentally that transport associated with the turbulent fluctuations is proportional to the gradient of average concentration.

$$\overline{u'c'} \approx \frac{\partial \overline{c}}{\partial x} \rightarrow \boxed{\overline{u'c'} = -\varepsilon_x \frac{\partial \overline{c}}{\partial x}}$$

$$\overline{v'c'} = -\varepsilon_y \frac{\partial \overline{c}}{\partial y}$$

ε_x, ε_y = turbulent diffusion coefficient

$$\frac{\partial}{\partial x}\left(-\overline{u'c'}\right) = \frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial \overline{c}}{\partial x}\right)$$

$$\frac{\partial}{\partial y}\left(-\overline{v'c'}\right) = \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial \overline{c}}{\partial y}\right)$$

Assuming that ε_x and ε_y are constant, the mass balance equation for turbulent flow is given as

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\frac{\partial \overline{c}}{\partial x} = (D_m + \varepsilon_x)\frac{\partial^2 \overline{c}}{\partial x^2} + (D_m + \varepsilon_y)\frac{\partial^2 \overline{c}}{\partial y^2} \qquad (3.3)$$

$\boxed{D_m \ll \varepsilon_x, \varepsilon_y}$

Drop overbars, and neglect molecular diffusion terms

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = \varepsilon_x \frac{\partial^2 c}{\partial x^2} + \varepsilon_y \frac{\partial^2 c}{\partial y^2} \qquad (3.4)$$

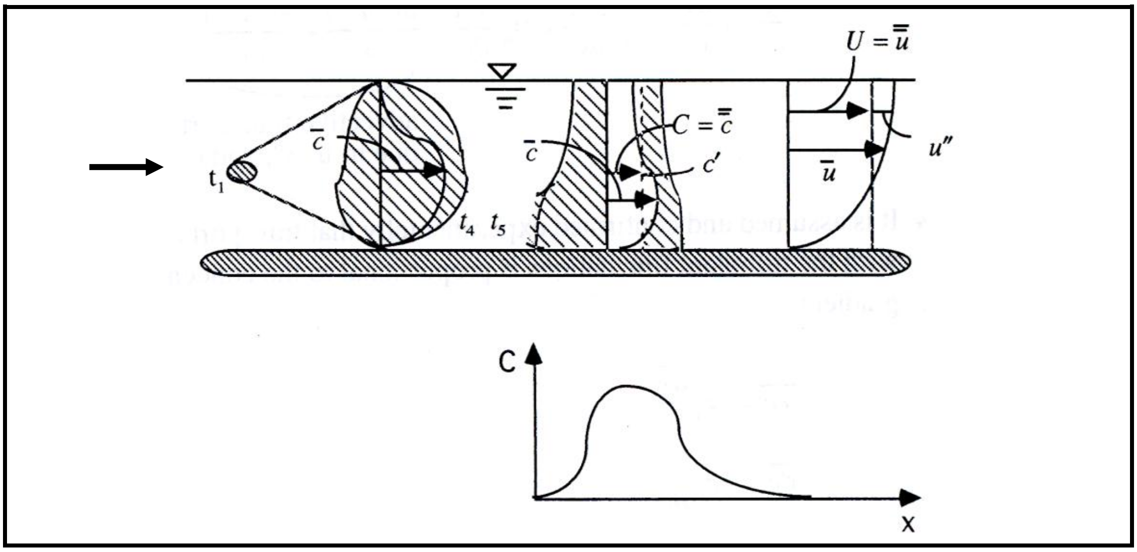
For 3-D flow:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = \frac{\partial}{\partial x}(\varepsilon_x \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y}(\varepsilon_y \frac{\partial c}{\partial y}) + \frac{\partial}{\partial z}(\varepsilon_z \frac{\partial c}{\partial z}) \qquad (3.5)$$

☞ Remember $\varepsilon_x \frac{\partial c}{\partial x}$, $\varepsilon_y \frac{\partial c}{\partial y}$, $\varepsilon_z \frac{\partial c}{\partial z}$ and are actually advective transport.

3.2.3 Longitudinal Dispersion

→ Ch. 4 Shear Flow Dispersion in Open Flows



After the tracer is essentially completely mixed both vertically and laterally, the primary variation of concentration is in just longitudinal direction.

→ one-dimensional equation

Far-field mixing

Decompose velocity and concentration into **cross-sectional mean** and deviation (fluctuation)

$$\bar{u} = U + u" \qquad \overline{\bar{u}"} = 0 \qquad \text{(c)}$$

$$\bar{c} = C + c" \qquad \overline{\bar{c}"} = 0$$

where U, C = cross-sectional average of the velocity and concentration

After substituting (c) into (3.3), integrating (average) Eq. (3.3) over the cross-sectional area yields

$$\overline{\overline{\frac{\partial (C+c")}{\partial t}}} + \overline{\overline{(U+u")\frac{\partial (C+c")}{\partial x}}} = \overline{\overline{(D_m+\varepsilon_x)\frac{\partial^2 (C+c")}{\partial x^2}}} + \overline{\overline{(D_m+\varepsilon_y)\frac{\partial^2 (C+c")}{\partial y^2}}}$$

3. Shear Flow Dispersion

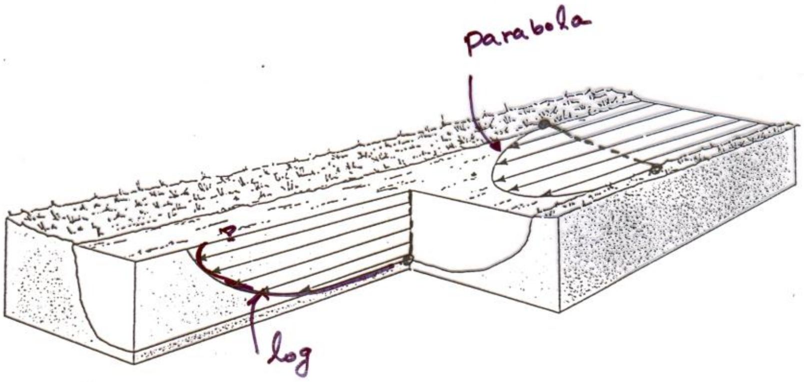
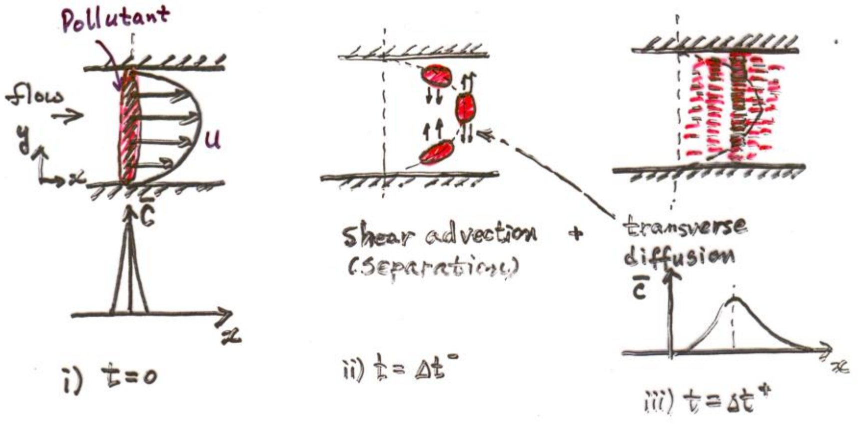


Figure 10.5
Variations in the velocity of flow in natural stream channels occur both horizontally and vertically. Friction reduces the velocity along the floor and sides of the channels. The maximum velocity in a straight channel is near the top and center of the channel.



By Reynolds rule

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = (D_m + \varepsilon_x)\frac{\partial^2 C}{\partial x^2} + (D_m + \varepsilon_y)\frac{\partial^2 C}{\partial y^2} - \frac{\partial\left(\overline{\overline{u"c"}}\right)}{\partial x} \qquad (3.6)$$

Then neglect $\frac{\partial^2 C}{\partial y^2}$ because after lateral mixing is completed, $\frac{\partial C}{\partial y} \approx 0;\ C = \bar{C} \neq f(y)$

Then, (3.6) becomes

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = (D_m + \varepsilon_x)\frac{\partial^2 C}{\partial x^2} + \frac{\partial\left(-\overline{\overline{u"c"}}\right)}{\partial x}$$

Taylor (1953, 1954) show that the advective transport associated with $u"$ is proportional to the longitudinal gradient of *C*.

$$-\overline{\overline{u"c"}} \propto \frac{\partial C}{\partial x}$$

$$-\overline{\overline{u"c"}} = K\frac{\partial C}{\partial x}$$

$$\frac{\partial}{\partial x}\left(-\overline{\overline{u"c"}}\right) = \frac{\partial}{\partial x}\left(K\frac{\partial C}{\partial x}\right)$$ → longitudinal dispersion

K = longitudinal dispersion coefficient

In turbulent uniform flow

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \left(D_m + \varepsilon_x + K\right)\frac{\partial^2 C}{\partial x^2}$$

$$(D_m + \varepsilon_x)\frac{\partial C}{\partial x} << -\overline{u"c"}$$

1% 99%

$$\boxed{\frac{\partial C}{\partial t}+U\frac{\partial C}{\partial x}=K\frac{\partial^2 C}{\partial x^2}} \tag{3.7}$$

→ 1-D Dispersion Equation

Because the velocity distribution influences u'' and the lateral diffusion plays a large role in determining the distribution of c''

→ both velocity distribution and lateral diffusion contribute to longitudinal dispersion.

- Limitation of Taylor's 1D model (Chatwin, 1970)

→ Taylor's model should be applied after initial period.

$$t>\frac{0.4h^2}{\varepsilon_t} \quad \rightarrow \text{ Taylor period}$$

$$x>\frac{0.4Uh^2}{\varepsilon_t}$$

3.2.4 Relative Importance of Dispersion

To investigate the relative importance of dispersion, use dimensionless term as

$$H=\frac{dispersion\ rate}{advective\ rate}=\frac{K\frac{\partial C}{\partial x}}{UC}=\frac{K}{U}\frac{1}{C}\frac{\partial C}{\partial x}=\frac{K}{U}\frac{\partial(\ln C)}{\partial x}$$

If $H<H_c\approx 0.01$ → dispersive transport may be neglected

3.2.5 Conclusion

1) Diffusion

= transport associated with **fluctuating components** of molecular action and with turbulent action

= transport in a given direction at a point in the flow due to the differences between the true advection in that direction and the time average of the advection in that direction

2) Dispersion

= transport associated with the deviations (variations) of the velocity across the flow section

= transport in a given direction due to the difference between the true advection in that direction and the spatial average of the advection in that direction

THE DIFFUSION AND DISPERSION SPECTRUM

	10^{-6} -	
	10^{-5} -	Molecular Diffusion in Water
	10^{-4} -	
	10^{-3} -	
	10^{-2} -	
	10^{-1} -	Molecular Diffusion Gases
$D_{ij}, cm^2/\sec$	1 -	
		Eddy Diffusion - Pipes
	10 -	
	100 -	Eddy Diffusion - Streams
	10000 -	
	10^{4} -	Longitudinal Dispersion - Streams
	10^{5} -	
	10^{6} -	Longitudinal Tidal Dispersion - Estuaries
	10^{7} -	
	10^{8} -	
	10^{9} -	
	10^{10} -	Eddy Diffusion - Atmosphere