Chap. 1 Basic Eqn. of Linear Elasticity

- Structural analysis ... evaluation of deformations and stresses arising within a solid object under the action of applied loads
  - if time is not explicitly considered as an independent variable
    → the analysis is said to be static
    → Otherwise, structural dynamic analysis or structural dynamics
- Under the assumption of { small deformation / linearly elastic material behavior

  three-dimensional formulation → a set of 15 linear 1st-order PDE involving
  - displacement field (3 components)
  - stress " (6 ")
  - strain " (6 ")

  ⇒ simpler, 2-D formulations { plane stress problem / plane strain
- For most situations not possible to develop analytical solutions
  ⇒ analysis of structural components -- bars, beams, plates, shells

1.1 The concept of stress

1.1.1 The state of stress at a point

- State of stress in a solid body -- measure of intensity of forces acting within the solid
  - distribution of forces and moments appearing on the surface of the cut ... equipollent force $\underline{F}$, and couple $\underline{M}$

    Newton's 3rd law → a force and couple of equal magnitudes and opposite directions acting on the two faces created by the cut

(Fig. 1.1)

- a small surface of area A_n located at point P on the surface generated by the cut → equipollent force $\underline{F}_n$. couple $\underline{M}_n$
  - limiting process of area → concept of "stress vector"

$$\underline{\tau}_n = \lim_{dA_n \to 0} \left(\frac{\underline{F}_n}{dA_n} \right) \qquad (1.1)$$

existence of limit : "fundamental assumption of continuum mechanics"

  - couple $\underline{M}_n \to 0$ as $dA_n \to 0$
    ∵ couple is the product of a differential element of force
    by " of moment arm
    → negligible, second order differential quantity
  - Total force acting on a differential element of area, dA_n

$$\underline{F}_n = dA_n \underline{\tau}_n \qquad (1.2)$$

Unit : force per unit area, N/m^2 or Pa

- Surface orientation, as defined by the normal to the surface, is kept constant during the limiting process

Fig. 1.2 ... three different cut and the resulting stress vector
first ... solid is cut at point P by a plane normal to axis $\bar{i}_1$;
differential element of surface with an area dA_1, stress vector $\underline{\tau}_1$
→ No reason that those three stress vectors should be identical.

- Component of each stress vectors acting on the three faces

$$\underline{\tau}_1 = \sigma_1 \bar{i}_1 + \tau_{12} \bar{i}_2 + \tau_{13} \bar{i}_3 \qquad (1.3a)$$

{ direct / normal } stress { shearing / shear } stress ... both act on the face normal to axis $\bar{i}_1$ in the dir. of $\bar{i}_2$ and $\bar{i}_3$

→ "engineering stress components"

Unit : force/area, Pa

"Positive" face ⋯ the outward normal to the face, i.e., the normal pointing away from the body, is in the same direction as the axis

→ sign convention (Fig. 1.3)

- 9 components of stress components → fully characterize the state of stress at P

FORCE ⋯ vector quantity, 3 components of the force vector (1st-order tensor)

STRESS ⋯ 9 quantities (2nd-order tensor)

↳ strain tensor
bending stiffness of a beam
mass moments of inertia

1.1.2 Volume equilibrium eqn.

- stress varies throughout a solid body

Fig 1.4 axial stress component at the negative face : σ_2

" at the positive face at coordinate $x_2 + dx_2$: $\sigma_2(x_2 + dx_2)$

if $\sigma_2(x_2)$ is an analytic function, using a Taylor series expansion

$$\sigma_2(x_2 + dx_2) = \sigma_2(x_2) + \left.\frac{\partial \sigma_2}{\partial x_2}\right|_{x_2} dx_2 + \dots \text{ h.o. terms in } dx_2$$

$$\sigma_2 + \frac{\partial \sigma_2}{\partial x_2} dx_2$$

- body forces $\underline{b}$ ⋯ gravity, inertial, electric, magnetic origin

$$\underline{b} = b_1 \bar{i}_1 + b_2 \bar{i}_2 + b_3 \bar{i}_3$$

Unit : force / volume, N/m³

i) Force equilibrium

dir. of axis $\bar{i}_1$

$$\begin{cases} \dfrac{\partial \sigma_1}{\partial x_1} + \dfrac{\partial \tau_{21}}{\partial x_2} + \dfrac{\partial \tau_{31}}{\partial x_3} + b_1 = 0 \qquad (1.4a) \\ \cdots \\ \cdots \end{cases}$$

must be satisfied at all points inside the body

- equilibrium should be enforced on the DEFORMED configuration (strictly) of the body. ← Unknown, unfortunately

"linear theory of elasticity" .. assumption that the displacements of the body under the applied loads are very small, and hence the difference between deformed and undeformed is very small.

ii) Moment equilibrium

about axis $\bar{i}_1$ → $\tau_{23} - \tau_{32} = 0$

→ "Principle of reciprocity of shear stress" (Fig. 1.5)

- only 6 independent components in 9 stress components

→ symmetry of the stress tensor (1.6)

1.1.3 Surface equilibrium eqn

- At the outer face of the body.

$\left\{ \begin{array}{l} \text{stress acting} \\ \text{inside the body} \end{array} \right\} \xleftrightarrow{\text{equilibrium}} \left\{ \begin{array}{l} \text{externally applied} \\ \text{surface tractions} \end{array} \right\} \underline{t}$

$\underline{t} = t_1 \bar{i}_1 + t_2 \bar{i}_2 + t_3 \bar{i}_3$

Fig 1.6 ... free body in the form of a differential tetrahedron bounded by
$\left\{ \begin{array}{l} \text{3 negative faces cut through the body in directions normal to axes } \bar{i}_1, \bar{i}_2, \bar{i}_3 \\ \text{a fourth face, ABC, of area } dA_n \end{array} \right.$

Unit normal to this element of area : $\bar{n} = n_1 \bar{i}_1 + n_2 \bar{i}_2 + n_3 \bar{i}_3$

"directional cosines"

$n_1 = \bar{n} \cdot \bar{i}_1 = \cos(\bar{n}, \bar{i}_1), \ldots$

- Force equilibrium along $\bar{i}_1$, and dividing by dA_n

$$t_1 = \sigma_1 n_1 + \tau_{21} n_2 + \tau_{31} n_3 \qquad (1.9a)$$

body force term vanishes since it is a h.o. differential term.

⇒ A body is said to be in equilibrium if eqns (1.4) are satisfied at all points inside the body and eqs (1.9) are satisfied at all points of its external surface.

1.2 Analysis of the state of stress at a point

$\underline{\tau}$ is fully defined once the stress components acting on three mutually orthogonal faces at a point are known.

1.2.1 Stress components acting on an arbitrary face

Fig 1.7 ... "Cauchy's tetrahedron" with a fourth face normal to unit vector $\bar{n}$ of arbitrary orientation

- Force equilibrium

$$\underline{\tau}_1 \, dA_1 + \underline{\tau}_2 \, dA_2 + \underline{\tau}_3 \, dA_3 = \underline{\tau}_n \, dA_n + \underline{b} \, dV$$

Dividing by dA_n and neglecting the body force term (since it is multiplied by a h.o. term)

$$\underline{\tau}_n = \underline{\tau}_1 n_1 + \underline{\tau}_2 n_2 + \underline{\tau}_3 n_3$$

- Expanding the 3 stress vectors,

$$\underline{\tau}_n = (\sigma_1 \bar{i}_1 + \tau_{12} \bar{i}_2 + \tau_{13} \bar{i}_3) n_1 + (\tau_{21} \bar{i}_1 + \sigma_2 \bar{i}_2 + \tau_{23} \bar{i}_3) n_2 \quad (1.10)$$
$$+ (\tau_{31} \bar{i}_1 + \tau_{32} \bar{i}_2 + \sigma_3 \bar{i}_3) n_3$$

- To determine the direct stress σ_n, project this vector eqn in the dir. of $\bar{n}$

$$\bar{n} \cdot \underline{\tau}_n = (\sigma_1 n_1 + \tau_{12} n_2 + \tau_{13} n_3) n_1 + (\tau_{21} n_1 + \sigma_2 n_2 + \tau_{23} n_3) n_2$$
$$+ (\tau_{31} n_1 + \tau_{32} n_2 + \sigma_3 n_3) n_3$$

$$\sigma_n = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 + 2\tau_{23} n_2 n_3 + 2\tau_{13} n_1 n_3 + 2\tau_{12} n_1 n_2 \quad (1.11)$$

- stress component acting in the plane of face ABC : τ_{ns}
  - by projecting Eq (1.10) along vector $\bar{s}$

$$\tau_{ns} = \sigma_1 n_1 s_1 + \sigma_2 n_2 s_2 + \sigma_3 n_3 s_3 + \tau_{12} (n_2 s_1 + n_1 s_2) + \tau_{13} (n_1 s_3 + n_3 s_1) \quad (1.12)$$
$$+ \tau_{23} (n_2 s_3 + n_3 s_2)$$

Eqn.s (1.11), (1.12) ... Once the stress components acting on 3 mutually orthogonal faces are known, the stress components on a face of arbitrary orientation can be readily computed.

- How much information is required to fully determine the state of stress at a point P of a solid?

  Complete definition of the state of stress at a point only requires knowledge of the stress vectors, or equivalently of the stress tensor components, acting on three mutually orthogonal faces

1.2.2 Principal stresses

- Is there a face orientation for which the stress vector is exactly normal to the face? Does a particular orientation, $\bar{n}$, exist for which the stress vector acting on this face consists solely of $\underline{\tau}_n = \sigma_p \bar{n}$, where σ_p is the yet unknown?

... Projecting Eq (1.10) along axes $\bar{i}_1, \bar{i}_2, \bar{i}_3$ → 3 scalar eqn.s

⇒ homogeneous system of linear eqn. for the unknown direction cosines

$$\begin{bmatrix} \sigma_1 - \sigma_p & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 - \sigma_p & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 - \sigma_p \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0 \qquad (1.13)$$

- Determinant of the system = 0, non-trivial sol. exists.

$$\hookrightarrow \quad \sigma_p^3 - I_1 \sigma_p^2 + I_2 \sigma_p - I_3 = 0 \qquad (1.14)$$

"stress invariants" (1.15)

sol. of Eq (1.14) ... "principal stress"

3 sol.s $\sigma_{p1}, \sigma_{p2}, \sigma_{p3}$ → non-trivial sol. for the direction cosines

"principal stress direction"

homogeneous eqn.s → arbitrary constant → enforcing the normality condition

$\bar{n}_1^2 + \bar{n}_2^2 + \bar{n}_3^2 = 1$

2.3 Rotation of stresses

- arbitrary basis : $\underline{I}^* = (\bar{i}_1^*, \bar{i}_2^*, \bar{i}_3^*)$ → $\left\{ \begin{array}{l} \sigma_1^*, \sigma_2^*, \sigma_3^* \\ \tau_{23}^*, \tau_{13}^*, \tau_{12}^* \end{array} \right.$

  - orientation of basis $\underline{I}^*$ relative to $\underline{I}$

    → matrix of direction cosine, or rotation matrix $\underline{\underline{R}}$ (A.36)

- Eq (1.11) → σ_1^* in terms of those resolved in axis $\underline{I}$

$$\sigma_1^* = \sigma_1 l_1^2 + \sigma_2 l_2^2 + \sigma_3 l_3^2 + 2\tau_{23} l_2 l_3 + 2\tau_{13} l_1 l_3 + 2\tau_{12} l_1 l_2 \qquad (1.18)$$

  l_1, l_2, l_3 : direction cosines of unit vector $\bar{i}_1^*$.

  Similar eqn's for σ_2^* ... m_1, m_2, m_3

  σ_3^* ... n_1, n_2, n_3

  Shear component : Eq (1.12) →

$$\tau_{12}^* = \sigma_1 l_1 m_1 + \sigma_2 l_2 m_2 + \sigma_3 l_3 m_3 + \tau_{12}(l_2 m_1 + l_1 m_2)$$
$$+ \tau_{13}(l_1 m_3 + l_3 m_1) + \tau_{23}(l_2 m_3 + l_3 m_2) \qquad (1.19)$$

- Compact matrix eqn.

$$\begin{bmatrix} \sigma_1^* & \tau_{12}^* & \tau_{13}^* \\ \tau_{21}^* & \sigma_2^* & \tau_{23}^* \\ \tau_{31}^* & \tau_{32}^* & \sigma_3^* \end{bmatrix} = \underline{\underline{R}}^T \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{bmatrix} \underline{\underline{R}} \qquad (1.20)$$

- "Stress invariant" ... invariant w.r.t. a change of coordinate system

  (1.21)

1.3 The state of plane stress

- All stress components acting along the direction of axis $\bar{i}_3$ are assumed to vanish or to be negligible → only non-vanishing components : $\sigma_1, \sigma_2, \tau_{12}$

  independent of x_3

  very thin plate or sheet subject to loads applied in its own plane (Fig. 1.11)

1.3.1 Equilibrium eqn.s

- Considerably simplified from the general, 3-D case → 2 remaining eqn.s

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + b_1 = 0 \; ; \quad \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + b_2 = 0 \qquad (1.26)$$

- surface tractions

$$t_1 = n_1 \sigma_1 + n_2 \tau_{21} \; , \quad t_2 = n_1 \tau_{12} + n_2 \sigma_2 \qquad (1.27)$$

∘ Fig. 1.11 : outer normal unit vector $\bar{n} = n_1 \bar{i}_1 + n_2 \bar{i}_2$,

$n = \cos\theta$, $n_2 = \sin\theta$, $n_3 = 0$

tangent " $\bar{s} = s_1 \bar{i}_1 + s_2 \bar{i}_2$

$s_1 = -\sin\theta$, $s_2 = \cos\theta$, $s_3 = 0$

Eq. (1.11) → $t_n = \cos^2\theta \, \sigma_1 + \sin^2\theta \, \sigma_2 + 2\sin\theta\cos\theta \, \tau_{12}$ (1.28)

(1.12) → $t_s = \sin\theta\cos\theta \, (\sigma_2 - \sigma_1) + (\cos^2\theta - \sin^2\theta) \tau_{12}$ (1.29)

1.3.2 Stress acting on an arbitrary face within the sheet

Fig 1.12 : 2-D version of Cauchy's tetrahedron (Fig. 1.7)

- Equilibrium of forces

$$\underline{T}_2 \, dx_1 + \underline{T}_1 \, dx_2 = \underline{T}_n \, ds + \underline{b} \, dx_1 dx_2 \frac{1}{2}$$

Dividing by ds,

$$\underline{T}_n = \underline{T}_1 n_1 + \underline{T}_2 n_2 - \underline{b} \, dx_1 dx_2 \frac{1}{2ds}$$

↑ neglected since multiplied by h.o. term

$$\underline{T}_n = (\sigma_1 \bar{i}_1 + \tau_{12} \bar{i}_2) \cos\theta + (\tau_{21} \bar{i}_1 + \sigma_2 \bar{i}_2) \sin\theta \qquad (1.30)$$

- Projecting in the dir. of unit vector $\bar{n}$ → σ_n

$$\sigma_n = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta + 2\tau_{12} \cos\theta \sin\theta \qquad (1.31)$$

- " normal to $\bar{n}$ → τ_{ns}

$$\tau_{ns} = -\sigma_1 \cos\theta \sin\theta + \sigma_2 \sin\theta \cos\theta + \tau_{12} (\cos^2\theta - \sin^2\theta) \qquad (1.32)$$

→ knowledge of $\sigma_1, \sigma_2, \tau_{12}$ on 2 orthogonal faces allows computation of the stress components acting on a face with an arbitrary orientation

3.3 Principal stress

- simply write Eqn. (1.13) − (1.15) with $\sigma_3 = \tau_{23} = \tau_{13} = 0$

or, using Eq. (1.31) ... particular orientation θ_p, that maximizes (or minimizes)

$$\sigma_n \rightarrow \frac{d\sigma_n}{d\theta} = 0 \rightarrow \tan 2\theta_p = \frac{2\tau_{12}}{\sigma_1 - \sigma_2} = \frac{\sin 2\theta_p}{\cos 2\theta_p} \qquad (1.33)$$

2 sol.s θ_p and $\theta_p + \frac{\pi}{2}$ corresponding to 2 mutually orthogonal principal stress directions

(1.34)

$$\sin 2\theta_p = \frac{\tau_{12}}{\Delta}, \quad \cos 2\theta_p = \frac{(\sigma_1 - \sigma_2)}{2\Delta}, \quad \text{where } \Delta \text{ is determined by } \sin^2 2\theta_p + \cos^2 2\theta_p = 1$$

$$\Delta = \left[\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + (\tau_{12})^2 \right]^{1/2} \qquad (1.35)$$

→ unique sol. for θ_p

- Max / Min axial stress: "principal stress" by introducing Eq (1.34) into (1.31)

$$\sigma_{p1} = \frac{\sigma_1 + \sigma_2}{2} + \Delta \; ; \quad \sigma_{p2} = \frac{\sigma_1 + \sigma_2}{2} - \Delta \qquad (1.36)$$

where the shear stress vanishes.

- Max shear stress → θ_s → $\frac{d\tau_{ns}}{d\theta} = 0$ using Eq. (1.32) (1.37)

$$\rightarrow \tan 2\theta_s = -\frac{\sigma_1 - \sigma_2}{2\tau_{12}} = -\frac{1}{\tan 2\theta_p} \qquad (1.38)$$

2 sols. θ_s and $\theta_s + \frac{\pi}{2}$ corresponding to 2 mutually orthogonal faces

- Max shear stress $\tau_{max} = \Delta = \frac{\sigma_{p1} - \sigma_{p2}}{2}$ (1.40)

$$\theta_s = \theta_p - \frac{\pi}{4} \qquad (1.41)$$

- Max. shear stress occurs at a face inclined at a 45° angle w.r.t the principal stress directions

$$\sigma_{1s} = \sigma_{2s} = \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_{p1} + \sigma_{p2}}{2} \qquad (1.42)$$

1.3.4 Rotation of stresses

- Eq (1.31) → $\sigma_1^* = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta + 2\tau_{12} \sin\theta\cos\theta$ (1.45)

- (1.32) → $\tau_{12}^* = -\sigma_1 \sin\theta\cos\theta + \sigma_2 \sin\theta\cos\theta + \tau_{12}(\cos^2\theta - \sin^2\theta)$ (1.46)

- compact matrix form

$$\begin{Bmatrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \quad (1.47)$$

can be easily inverted by simply replacing θ by $-\theta$

- knowledge of the stress components $\sigma_1, \sigma_2, \tau_{12}$ on 2 orthogonal faces allows computation of those acting on a face with an arbitrary orientation

1.3.5 Special state of stresses

i) Hydrostatic stress state … $\sigma_{p1} = \sigma_{p2} = p$ "hydrostatic pressure"

$\tau_{12} = 0$ with any arbitrary orientation

ii) Pure shear state

… $\sigma_{p2} = -\sigma_{p1}$ (Fig. 1.13)

At the face inclined at a 45° angle w.r.t. the principal stress direction

$\tau_{12}^* = -\sigma_{p1}$; $\sigma_1^* = \sigma_2^* = 0$ (1.51)

iii) Stress state in thin-walled pressure vessels

Fig 1.14 .. cylindrical pressure vessel subjected to internal pressure p_i

2 in-plane stress components $\begin{cases} \sigma_a \text{ (axial dir.)} \\ \sigma_h \text{ (circumferential or "hoop" dir)} \end{cases}$

possibly a shear stress, τ_{ah}

— left right

- Axial force equilibrium $\sigma_a \pi R t = p_i \pi R^2/2$

area of cylinder cross-section in the dir. of axial

$\sigma_a = p_i R/2t$

tangential (hoop) dir. $2\sigma_h bt = p_i 2Rb$ ← internal area of cylinder projected to the tangential (hoop) dir.

$\sigma_h = p_i R/t$

$\tau_{ah} = 0$

1.3.6 Mohr's circle for plane stress

- σ_{p1}, σ_{p2} : principal stresses at a point

Eq. (1.49) → stresses acting on a face oriented at an angle θ w.r.t the principal stress directions

$$\sigma^* = \sigma_a + R\cos 2\theta \; ; \quad \tau^* = -R\sin 2\theta \qquad (1.52)$$

where $\sigma_a = (\sigma_{p1} + \sigma_{p2})/2$, $R = (\sigma_{p1} - \sigma_{p2})/2$

$$\Rightarrow \quad (\sigma^* - \sigma_a)^2 + (\tau^*)^2 = R^2 \qquad (1.53)$$

- eqn of a circle "Mohr's circle"

σ^* : horizontal axis, τ^* : vertical axis ("inverted")

center at a coord. σ_a on the horizontal axis, R : radius

… each point on Mohr's circle represents the state of stress acting at a face at a specific orientation

- observations

① @ P_1, $\sigma^* = \sigma_{p1}$, $\tau^* = 0$ … principal stress direction
P_2 " second " "

② @ E_1, $\theta = \frac{\pi}{4}$, $\tau^*_{max} = R = (\sigma_{p1} - \sigma_{p2})/2$ → Max. shear stress orientation
E_2

③ @ A_1, A_2, two faces oriented $90°$ apart, shear stresses are equal in magnitude and of opposite sign → principle of reciprocity

construction procedure

① first point A_1 @ (σ_1, τ_{12})

② second " A_2 @ $(\sigma_2, -\tau_{12})$, at a $90°$ angle counterclockwise w.r.t. the first point

③ straight line joining A_1 and A_2

④ stress component ⓐ on angle β ··· a new diameter $B_1 O B_2$ rotated 2β deg. from the ref. diameter $A_1 O A_2$

o Important features

① principal stress σ_{p1}, σ_{p2} → points P_1 and P_2, direct stress Max / Min
shear " $= 0$

② Max. shear stress ··· vertical line E_1, = radius, $\tau_{max} = (\sigma_{p1} - \sigma_{p2})/2$
direction · $45°$, since $P_1 O E_1 = 90°$

③ stress components acting on 2 mutually orthogonal faces ··· 2 diametrically opposite points on Mohr's circle

④ All the points on Mohr's circle represent the same state of stress at one point of the solid

1.3.7 Lamé's ellipse

(1.70) when selecting the principal stress direction,
→ $\underline{\tau}_n = \sigma_{1p} \cos\theta \, \bar{i}_1^* + \sigma_{2p} \sin\theta \, \bar{i}_2^*$

(x_1, x_2) : tip of the stress vector, $\underline{\tau}_n = x_1 \bar{i}_1^* + x_2 \bar{i}_2^*$

$x_1 = \sigma_{1p} \cos\theta, \quad x_2 = \sigma_{2p} \sin\theta$

Eliminating θ,
$$\left(\frac{x_1}{\sigma_{1p}}\right)^2 + \left(\frac{x_2}{\sigma_{2p}}\right)^2 = 1 \qquad (1.54)$$

→ eqn of ellipse with semi-axes equal to $|\sigma_{p1}|$ and $|\sigma_{2p}|$ (Fig. 1.17)

o Pure shear ··· ellipse → circle (Fig 1.18)

1.4 The concept of strain

· State of strain ··· characterization of the deformation in the neighborhood of a material point in a solid

at a given point P, located by a position vector $\underline{r} = x_1 \bar{i}_1 + x_2 \bar{i}_2 + x_3 \bar{i}_3$ (Fig 1.22)

small rectangular parallelepiped PQRST of differential size
"reference configuration," undeformed state

→ "deformed configuration" TQRST

∘ displacement vector $\underline{u}$ ... measure of how much a material point moves (1.56)

two parts { rigid body motion ... translation, rotation → does not produce strain
deformation or straining → strain-displacement relation

1.4.1 The state of strain at a point

∘ Material line PR in the ref conf → " in the deformed conf
assumed to be still straight, but

2 factors in the measure of state of strain / a parallelogram

{ stretching of a material line ... $\epsilon_1, \epsilon_2, \epsilon_3$
angular distortion between 2 material lines ... $\gamma_{23}, \gamma_{13}, \gamma_{12}$

Relative elongation or extensional strain

$$\epsilon_1 = \frac{\|PR\|_{def} - \|PR\|_{ref}}{\|PR\|_{ref}} \quad \cdots \text{ nondimensional quantity} \qquad (1.57)$$

$\|\cdot\|$: magnitude

$$\|PR\|_{ref} = \|dx_1 \bar{i}_1\| = dx_1$$

$$\|PR\|_{def} = \|dx_1 \bar{i}_1 + \underline{u}(x_1 + dx_1) - \underline{u}(x_1)\|$$

$$= \|dx_1 \bar{i}_1 + \bar{u}(x_1) + \frac{\partial \underline{u}}{\partial x_1} dx_1 - \underline{u}(x_1)\| = \|dx_1 \bar{i}_1 + \frac{\partial \underline{u}}{\partial x_1} dx_1\|$$

$$= \|\bar{i}_1 dx_1 + \left(\frac{\partial u_1}{\partial x_1}\bar{i}_1 + \frac{\partial u_2}{\partial x_1}\bar{i}_2 + \frac{\partial u_3}{\partial x_1}\bar{i}_3\right) dx_1\| \qquad (1.59)$$

$$= \sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2}\; dx_1$$

Then, $\epsilon_1 = \sqrt{\quad '' \quad} - 1$ (1.60)

- fundamental assumption of linear elasticity ... all displacement components remain very small so that all 2nd order terms can be neglected.

And, using the binomial expansion,

$$\epsilon_1 \simeq 1 + \frac{\partial u_1}{\partial x_1} - 1 = \frac{\partial u_1}{\partial x_1}, \qquad \text{"direct strains" or "axial strains"} \qquad (1.62)$$

$$\epsilon_2 = \frac{\partial u_2}{\partial x_2}, \qquad \epsilon_3 = \frac{\partial u_3}{\partial x_3} \qquad (1.63)$$

ii) Angular distortions or shear strains

γ_{23} between two material lines PT and PS, defined as the change of the initially right angle

$$\gamma_{23} = \langle TPS \rangle_{ref} - \langle TPS \rangle_{def} = \frac{\pi}{2} - \langle TPS \rangle_{def} \qquad (1.64)$$

$\langle \cdot\ \cdot \rangle$: angle between segments. ← non-dimensional quantities

$$\sin \gamma_{23} = \sin\left(\frac{\pi}{2} - \langle TPS \rangle_{def}\right) = \cos \langle TPS \rangle_{def} \qquad (1.65)$$

by law of cosine,

$$\|TS\|_{def}^2 = \|PT\|_{def}^2 + \|PS\|_{def}^2 - 2\underbrace{\cos \langle TPS \rangle_{def}}_{\sin \gamma_{23}} \|PT\|_{def} \|PS\|_{def} \qquad (1.66)$$

$$\gamma_{23} = \arcsin \frac{\|PT\|_{def}^2 + \|PS\|_{def}^2 - \|TS\|_{def}^2}{2\|PT\|_{def} \|PS\|_{def}} \qquad (1.67)$$

$$PT_{def} = \left(\bar{i}_3 + \frac{\partial u}{\partial x_3}\right) dx_3 = \underline{A}, \qquad PS_{def} = \left(\bar{i}_2 + \frac{\partial u}{\partial x_2}\right) dx_2 = \underline{B}$$

$$TS_{def} = PS_{def} - PT_{def} = \underline{B} - \underline{A}$$

Numerator $N = 2\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}\right) dx_2\, dx_3$

Denominator $D = 2\sqrt{\underline{A}\cdot\underline{A}}\sqrt{\underline{B}\cdot\underline{B}}$

- with the help of small displacement assumption,

$$N \simeq 2\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) dx_2\, dx_3$$

$$D \simeq 2\left(1 + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) dx_2\, dx_3$$

$$\Rightarrow \quad \gamma_{23} \simeq \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \qquad \text{"shearing strain"} \qquad (1.70)$$

$$\gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}, \qquad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \qquad \text{"shear strain"} \qquad (1.71)$$

- strain-displacement relationship, Eqs. (1.63), (1.70) ... under the small displacement assumption

large displacement → Eqns (1.60), (1.67) should be used

iii) Rigid body rotation

$$\omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \qquad (1.73a)$$

- rotation vector $\underline{\omega}^T = \{\omega_1, \omega_2, \omega_3\}$ ... the rotation of the solid about axes $\bar{i}_1, \bar{i}_2, \bar{i}_3$, respectively

1.4.2 The volumetric strain

- After deformation

$$V \simeq (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)dx_1 dx_2 dx_3 \simeq (1+\epsilon_1+\epsilon_2+\epsilon_3)dx_1 dx_2 dx_3 \qquad (1.74)$$

where h.o. strain quantities are neglected

- relative change in volume

$$e = \epsilon_1 + \epsilon_2 + \epsilon_3 \quad \text{"volumetric strain"} \qquad (1.75)$$

1.5 Analysis of the state of strain at a point

- arbitrary reference frame $\mathcal{I}^* = (\bar{i}_1^*, \bar{i}_2^*, \bar{i}_3^*)$

→ strain-displacement relationship in $\mathcal{I}^*$ (1.76), (1.77)

1.5.1 Rotation of strains

- chain rule

$$\epsilon_1^* = \frac{\partial u_1^*}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1}\frac{\partial x_1}{\partial x_1^*} + \frac{\partial u_1^*}{\partial x_2}\frac{\partial x_2}{\partial x_1^*} + \frac{\partial u_1^*}{\partial x_3}\frac{\partial x_3}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1}l_1 + \frac{\partial u_1^*}{\partial x_2}l_2 + \frac{\partial u_1^*}{\partial x_3}l_3 \qquad (1.78)$$

where Eq.(A.39) is used

- Next, u_1^* in terms of the components in $\mathcal{I}$

$$\epsilon_1^* = l_1\frac{\partial}{\partial x_1}(l_1u_1 + l_2u_2 + l_3u_3) + l_2\frac{\partial}{\partial x_2}(l_1u_1 + l_2u_2 + l_3u_3) + l_3\frac{\partial}{\partial x_3}(l_1u_1 + l_2u_2 + l_3u_3) \qquad (1.79)$$

using Eq (1.63) and (1.71)

$$\epsilon_1^* = \epsilon_1 l_1^2 + \epsilon_2 l_2^2 + \epsilon_3 l_3^2 + \gamma_{12} l_1 l_2 + \gamma_{13} l_1 l_3 + \gamma_{23} l_2 l_3$$

- similar eqn.s (1.80), (1.81)

$$\epsilon_{23} = \frac{\gamma_{23}}{2}, \quad \epsilon_{13} = \frac{\gamma_{13}}{2}, \quad \epsilon_{12} = \frac{\gamma_{12}}{2} \qquad (1.82)$$

tensor shear strain component

engineering shear strain comp.

- compact matrix form

$$\begin{bmatrix} \epsilon_1^* & \epsilon_{12}^* & \epsilon_{13}^* \\ \epsilon_{12}^* & \epsilon_2^* & \epsilon_{23}^* \\ \epsilon_{13}^* & \epsilon_{23}^* & \epsilon_3^* \end{bmatrix} = \underline{\underline{R}}^T \begin{bmatrix} \epsilon_1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_2 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_3 \end{bmatrix} \underline{\underline{R}} \qquad (1.43)$$

1.5.2 Principal strains

- Is there a coordinate system $\mathcal{I}^*$ for which the shear strains vanish?

$$\begin{bmatrix} \epsilon_1^* & 0 & 0 \\ 0 & \epsilon_2^* & 0 \\ 0 & 0 & \epsilon_3^* \end{bmatrix} = \underline{\underline{R}}^T \begin{bmatrix} \epsilon_1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_2 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_3 \end{bmatrix} \underline{\underline{R}}$$

- Premultiplying $\underline{\underline{R}}$ and reversing the equality

$$\begin{bmatrix} \epsilon_1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_2 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_3 \end{bmatrix} \underline{\underline{R}} = \underline{\underline{R}} \begin{bmatrix} \epsilon_{p_1} & 0 & 0 \\ 0 & \epsilon_{p_2} & 0 \\ 0 & 0 & \epsilon_{p_3} \end{bmatrix}$$

where the orthogonality of $\underline{\underline{R}}$, Eq (A 37), is used.

- ϵ_{p_1}, ϵ_{p_2}, ϵ_{p_3} : sol. of 3 systems of 3 eqn.s

$$\begin{bmatrix} \epsilon_1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_2 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_3 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \epsilon_p \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

determinant of the system vanishes → non-trivial sol.

↳ cubic eqn : $\epsilon_p^3 - I_1 \epsilon_p^2 + I_2 \epsilon_p - I_3 = 0$ (1.45)

"strain invariant" (1.46)

3 sol. : ϵ_{p_1}, ϵ_{p_2}, ϵ_{p_3} → corresponding "principal strain direction"

→ homogeneous eqn. → arbitrary const. → normality condition

1.6 The state of plane strain

- displacement component along $\bar{i}_3$ is assumed to vanish, or to be negligible
- Example : a very long buried pipe aligned with $\bar{i}_3$ dir.

1.6.1 Strain-displacement relations for plane strain

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_2 = \frac{\partial u_2}{\partial x_2}, \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \qquad (1.47)$$

1.6.2 Rotation of strains

- chain rule

$$\epsilon_1^* = \frac{\partial u_1^*}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1}\frac{\partial x_1}{\partial x_1^*} + \frac{\partial u_1^*}{\partial x_2}\frac{\partial x_2}{\partial x_2^*} = \frac{\partial u_1^*}{\partial x_1}\cos\theta + \frac{\partial u_1^*}{\partial x_2}\sin\theta$$

Eq. (A.43)

- u_1^* in terms of those in I

$$\epsilon_1^* = \cos\theta \frac{\partial}{\partial x_1}(u_1\cos\theta + u_2\sin\theta) + \sin\theta\frac{\partial}{\partial x_2}(u_1\cos\theta + u_2\sin\theta) \quad (1.88)$$

- Then

$$\epsilon_1^* = \cos^2\theta\,\epsilon_1 + \sin^2\theta\,\epsilon_2 + \sin\theta\cos\theta\,\gamma_{12} \quad (1.89)$$

- Matrix form

$$\begin{Bmatrix} \epsilon_1^* \\ \epsilon_2^* \\ \epsilon_{12}^* \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & s^2 & 2s\cdot c \\ \sin^2\theta & c^2 & -2s\cdot c \\ -s\cdot c & s\cdot c & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_{12} \end{Bmatrix} \quad (1.91)$$

can be readily inverted by replacing θ by $-\theta$

1.6.3 Principal strains

- θ_p, in which the max. (or min) elongation occurs

$$\rightarrow \frac{d\epsilon_1^*}{d\theta} = 0 = -\frac{\epsilon_1 - \epsilon_2}{2}\,2\sin 2\theta_p + \frac{\gamma_{12}}{2}\,2\cos 2\theta_p = 0 \quad (1.95)$$

$$\tan 2\theta_p = \frac{\gamma_{12}/2}{(\epsilon_1 - \epsilon_2)/2} \quad (1.96)$$

2 sol.s … θ_{p1}, $\theta_{p2} = \theta_{p1} + \pi/2$ … 2 mutually orthogonal principal strain directions

$$\epsilon_{p1} = \frac{\epsilon_1 + \epsilon_2}{2} + \Delta, \quad \epsilon_{p2} = \frac{\epsilon_1 + \epsilon_2}{2} - \Delta \quad (1.99)$$

where shear strain vanishes

- The orientations of the $\left\{\begin{matrix}\text{principal stresses} \\ \text{〃 strains}\end{matrix}\right\}$ are not necessarily identical.

1.6.4 Mohr's circle for plane strain

- strains along a direction defined by angle θ w.r.t the principal strain direction

$$\epsilon^* = \epsilon_a + R\cos 2\theta, \quad \frac{\gamma^*}{2} = -R\sin 2\theta \quad (1.100)$$

where $\epsilon_a = (\epsilon_{p1} + \epsilon_{p2})/2$, $R = (\epsilon_{p1} - \epsilon_{p2})/2$

$$\Rightarrow (\epsilon^* - \epsilon_a)^2 + \left(\frac{\gamma^*}{2}\right)^2 = R^2 \quad \text{: Mohr's circle} \quad (1.101)$$

Fig. 1.23, positive angle θ ... counterclockwise dir.

shear strain .. positive downward

vertical axis .. strain tensor, $\gamma_{12}/2$

1.7 Measurement of strains

no practical experimental device for direct measurement of STRESS

--- indirect measurement of strain first → constitutive laws

i) strain gauges

- measurement of extensional strains on the body's external surface
  - very thin electric wire, or an etched foil pattern
  - extension .. wire's cross-section reduced by Poisson's effect, slightly increasing its electrical resistance

    compression ... " reduced resistance
  - wheatstone bridge ... accurate measurement

    "micro-strains" .. μ m/m $= 10^{-6}$ m/m

ii) Chevron strain gauges

Fig 1.24 ... e_{+45} and e_{-45}, experimentally measured relative elongations

Using Eq. (1.94a),

$$e_{+45} = \frac{\epsilon_1+\epsilon_2}{2} + \frac{\gamma_{12}}{2}$$

$$e_{-45} = \frac{\epsilon_1+\epsilon_2}{2} - \frac{\gamma_{12}}{2}$$

... Not sufficient to determine the strain state at the point

3 measurements would be required $\left\{ \begin{array}{l} \epsilon_1, \epsilon_2, \gamma_{12} \\ 2 \text{ principal strains \& dir.} \end{array} \right.$

However, can uniquely determine $\gamma_{12} = e_{+45} - e_{-45}$ (1.102)

iii) Strain gauge rosette

Fig. 1.25 ... 3 independent measurements, "delta rosette"

Eq. (1.94a) → $\epsilon_1 = e_1$, $\epsilon_2 = \frac{2}{3}(e_2 + e_3 - \frac{e_1}{2})$, $\gamma_{12} = \frac{2}{\sqrt{3}}(e_2 - e_3)$ (1.103)

Fig. 1.26 ... various arrangements of strain gauges

Chap 5 Euler-Bernoulli beam theory

- one of its dimensions much larger than the other two
  - civil engineering structure ... assembly or grid of beams with cross-sections having shapes such as T's or I's
  - machine parts ... beam-like structures lever arms, shafts, etc.
  - aeronautic structures ... wings, fuselages → can be treated as thin-walled beams
- "beam theory" ... important role, simple tool to analyze numerous structures
  valuable insight at a pre-design stage
- Euler-Bernoulli beam theory ... simplest, most useful
  - assumption: ① cross-section of the beam is infinitely rigid in its own plane
    → in-plane displacement field → { 2 rigid body translations, 1 " rotation }

    ② the cross-section is assumed to remain plane

    ③ " normal to the deformed axis

5.1 The Euler-Bernoulli assumptions

- Fig 5.1 ... "pure bending", beam deforms into a curve of constant curvature
  → a circle with center O, symmetric w.r.t. any plane perpendicular to its deformed axis
- Kinematic assumptions "Euler-Bernoulli" .. ① cross-section is infinitely rigid in its own plane

  ② " remains plane after deformation

  ③ " " normal to the deformed axis of the beam

  → valid for long, slender beams made of isotropic materials with solid cross-sections

5.2 Implications of the E-B assumptions

- $\left\{ \begin{array}{l} u_1(x_1, x_2, x_3) \\ u_2(\quad " \quad) \\ u_3(\quad " \quad) \end{array} \right\}$ displacement of an arbitrary point of the beam

  - E-B assumption ① → displacement field in the plane of X-S consists solely of 2 rigid body translations $\bar{u}_2(x_1)$, $\bar{u}_3(x_1)$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1), \qquad u_3(x_1, x_2, x_3) = \bar{u}_3(x_1) \qquad (5.1)$$

- E-B assumption ② → axial displacement field consists of $\begin{cases} \text{rigid body translation } \bar{u}_1(x_1) \\ 2 \text{ " rotation } \Phi_2(x_1) \\ \Phi_3(x_1) \end{cases}$ (Fig. 5.2)

$$u_1(x_1, x_2, x_3) = \bar{u}(x_1) + x_3\,\Phi_2(x_1) - x_2\,\Phi_3(x_1) \qquad (5.2)$$

- E-B assumption ③ → equality of $\begin{cases} \text{the slope of the beam} \\ \text{the rotation of the section} \end{cases}$ (Fig. 5.4)

$$\Phi_3 = \frac{d\bar{u}_2}{dx_1}, \qquad \Phi_2 = -\frac{d\bar{u}_3}{dx_1} \qquad (5.3)$$

consequence of the sign convention

- to eliminate the sectional rotation from the axial displacement field

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) - x_3\frac{d\bar{u}_3(x_1)}{dx_1} - x_2\frac{d\bar{u}_2(x_1)}{dx_1} \qquad (5.4a)$$

... important simplification of E-B : unknown displacements are functions of the span-wise coord, x_1, alone.

• Strain field

$$\epsilon_2 = 0, \quad \epsilon_3 = 0, \quad \gamma_{23} = 0 \qquad (5.5a) \leftarrow \text{E-B ①}$$

$$\gamma_{12} = 0, \qquad \gamma_{13} = 0 \qquad (\text{" } b) \leftarrow \text{" ③}$$

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1} = \underbrace{\frac{d\bar{u}_1(x_1)}{dx_1}}_{\bar{\epsilon}_1(x_1)} + x_3\underbrace{\frac{d^2\bar{u}_3(x_1)}{dx_1^2}}_{\kappa_2(x_1)} - x_2\underbrace{\frac{d^2\bar{u}_2(x_1)}{dx_1^2}}_{\kappa_3(x_1)} \qquad (\text{" } c)$$

with; sectional axial strain; sectional curvature about $\bar{i}_2$, $\bar{i}_3$ axes

$$\Rightarrow \epsilon_1(x_1, x_2, x_3) = \bar{\epsilon}_1(x_1) + x_3\,\kappa_2(x_1) - x_2\,\kappa_3(x_1) \qquad (5.7) \leftarrow \text{E-B ②}$$

- Assuming a strain field of the form Eqs (5.5a), (5.5b), (5.7)

... math. expression of the E-B assumptions

5.3 Stress resultants

◦ 3-D stress field → described in terms of sectional stresses called "stress resultants"

.. equipollent to specific components of the stress field

- 3 force resultants $\begin{cases} N_1(x_1) \text{ axial force} \\ V_2(x_1),\ V_3(x_1) \text{ transverse shearing forces} \end{cases}$

$$N_1(x_1) = \int_A \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.8)$$

$$V_2(x_1) = \int_A \tau_{12}(x_1, x_2, x_3)\, dA, \quad V_3(x_1) = \int_A \tau_{13}(x_1, x_2, x_3)\, dA \qquad (5.9)$$

- 2 moment resultants : $M_2(x_1)$, $M_3(x_1)$ bending moments

$$M_2(x_1) = \int_A x_3\, \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.10a)$$

$$M_3(x_1) = -\int_A x_2\, \sigma_1(x_1, x_2, x_3)\, dA \qquad (\ \ \text{b})$$

(+) equipollent bending moment about $\bar{i}_3$ (Fig. 5.5)

bending moments computed about point $P(x_{2p}, x_{3p})$

$$M_2^P(x_1) = \int_A (x_3 - x_{3p})\, \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.11a)$$

5.4 Beams subjected to axial loads

- distributed axial load $p_1(x_1)$ [N/m], concentrated axial load P_1 [N]
  - → axial displacement field $\bar{u}_1(x_1)$ ⇒ "bar" rather than "beam"

5.4.1 Kinematic description

- axial loads causes only axial displacement of the section

… Eq. (5.4) →

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) \qquad (5.12a)$$ … uniform over the c-s (Fig. 5.7)

$$u_2(\ \ ``\ \ ) = 0 \qquad (\ \ \text{b})$$

$$u_3(\ \ ``\ \ ) = 0 \qquad (\ \ \text{c})$$

axial strain field $\epsilon_1(\ \ ``\ \ ) = \bar{\epsilon}_1(x_1)$ (5.13)

5.4.2 Sectional constitutive law

- $\sigma_2 \ll \sigma_1$, $\sigma_3 \ll \sigma_1$ → transverse stress components $\simeq 0$, $\sigma_2 \simeq 0$, $\sigma_3 \simeq 0$
  - generalized Hooke's law → $\sigma_1(x_1, x_2, x_3) = E\,\epsilon_1(x_1, x_2, x_3)$ (5.14)
    ↖ at the "infinitesimal" level
- inconsistency in E-B beam theory …

  Eq. (5.12a) → $\epsilon_2 = 0$, $\epsilon_3 = 0$

  Hooke's law → if $\sigma_2 = \sigma_3 = 0$, then $\epsilon_2 = -\nu\sigma_1/E$, $\epsilon_3 = -\nu\sigma_1/E$

  (Poisson's effect) … very small effect, and assumed to vanish

Eq. (5.13) → (5.14) : $\sigma_1(x_1, x_2, x_3) = E\,\bar{\epsilon}_1(x_1)$ (5.15)

- axial force

$$N_1(x_1) = \int_A \sigma_1(x_1, x_2, x_3)\, dA = \Big[\underbrace{\int_A E\, dA}_{\text{axial stiffness}}\Big]\, \bar{\epsilon}_1(x_1) = S\, \bar{\epsilon}(x_1) \qquad (5.16)$$

$S = EA$ for homogeneous material

… constitutive law for the axial behaviour of the beam at the sectional level

5.4.3 Equilibrium eqn s

- Fig. 5.8 ... infinitesimal slice of the beam of length dx_1

force equilibrium in axial dir. $\rightarrow$ $\dfrac{dN_1}{dx_1} = -p_1$ (5.18)

Eq. (1.4) · equilibrium condition for a differential element of a 3-D solid

(5.18) · " of a slice of the beam of differential length dx_1

5.4.4 Governing eqn s

- Eq. (5.16) $\rightarrow$ Eq. (5.18), and using Eq. (5.6)

$$\frac{d}{dx_1}\left[S\,\frac{d\bar{u}_1}{dx_1}\right] = -p_1(x_1) \qquad (5.19)$$

- 3 B.C. ·· ① fixed (clamped). $\bar{u}_1 = 0$

② free (unloaded): $N_1 = 0 \rightarrow \dfrac{d\bar{u}_1}{dx_1} = 0$

③ subjected to a concentrated load P_1: $N_1 = P_1 \rightarrow S\,\dfrac{d\bar{u}_1}{dx_1} = P_1$

5.4.5 The sectional axial stiffness

- homogeneous material · $S = EA$ (5.20)
- rectangular section of width b made of layered material of different moduli (Fig. 5.9)

$$S = \int_A E\, dA = \sum_{i=1}^{n} E^{[i]} \int_{A^{[i]}} dA^{[i]} = \sum_{i=1}^{n} E^{[i]}\, b\,(x_3^{[i+1]} - x_3^{[i]})$$

"weighted average" of the Young's modulus weighting factor: thickness

5.4.6 The axial stress distribution

- Eliminating the axial strain from Eq. (5.15) and (5.16)

$$\sigma_1(x_1, x_2, x_3) = \frac{E}{S}\, N_1(x_1) \qquad (5.21)$$

- homogeneous material

$$\sigma_1(x_1, x_2, x_3) = \frac{N_1(x_1)}{A} \qquad (5.22)$$

··· uniformly distributed over the section

- sections made of layers presenting different moduli

$$\sigma_1^{[i]}(x_1, x_2, x_3) = E^{[i]}\,\frac{N_1(x_1)}{S} \qquad (5.23)$$

··· stress in layer i is proportional to the modulus of the layer

Eq (5.13) → axial strain distribution is uniform over the section,

i.e., each layer is equally strained (Fig. 5.10)

- strength criterion

$$\frac{E}{S} |N_{1\,max}^{tens}| \le \sigma_{allow}^{ten}, \quad \frac{E}{S} |N_{1\,max}^{comp}| \le \sigma_{allow}^{comp} \qquad (5.24)$$

in case compressive, buckling failure mode may occur → Chap. 14

5.5 Beams subjected to transverse loads

- Fig 5.14 ... "transverse direction" distributed load, $p_2(x_1)$ [N/m]

  concentrated " P_2 [N]

  → bending moments, transverse shear forces, and {axial / transverse shearing} stresses

  will be generated

5.5.1 Kinematic description

- Assumption: transverse loads only cause {transverse displacement / curvature of the section}

  - General displacement field (Eq. (5.4)) →

$$u_1(x_1, x_2, x_3) = -x_2 \frac{d\bar{u}_2(x_1)}{dx_1} \qquad (5.29a)$$

$$u_2(\;''\;) = \bar{u}_2(x_1) \qquad (\;b)$$

$$u_3(\;''\;) = 0 \qquad (\;c)$$

→ Fig. 5.15 .. linear distribution of the axial displacement component over the x-s

- only non-vanishing strain component

$$\epsilon_1(x_1, x_2, x_3) = -x_2 \kappa_3(x_1) \qquad (5.30)$$

... linear distribution of the axial strain

5.5.2 Sectional constitutive law

- linearly elastic material, axial stress distribution

$$\sigma_1(x_1, x_2, x_3) = -E x_2 \kappa_3(x_1) \qquad (5.31)$$

- Sectional axial force, by Eq. (5.8),

$$N_1(x_1) = \int_A \sigma_1(x_1, x_2, x_3)\, dA = -\left[\int_A E x_2\, dA\right] \kappa_3(x_1) \qquad (5.32)$$

- axial force = 0 since subjected to transverse loads only

  $\kappa_3 \neq 0$, then $[\cdots] = 0$

$$\Rightarrow \quad x_{2c} = \frac{1}{S} \int_A E x_2 \, dA = \frac{S_2}{S} = 0 \qquad (5.33)$$

location of the "modulus-weighted centroid" of the x-s

- If homogeneous material,

$$x_{2c} = \frac{E \int_A x_2 dA}{E \int_A dA} = \frac{1}{A} \int_A x_2 \, dA = 0 \qquad (5.34)$$

... x_2 is simply the area center of the section

$\Rightarrow$ the axis system is located at the modulus-weight centroid / area center if homogeneous material

※ center of mass $x_{2m} = \dfrac{\rho \int_A x_2 dA}{\rho \int_A dA} = \dfrac{\int_A x_2 dA}{\int_A dA} = x_{2c}$, center of mass → 3 coincide

- Bending moment, by Eq (5.31)

$$M_3(x_1) = \left[\int_A E x_2^2 dA \right] \kappa_3(x_1) = H_{33}^c \, \kappa_3(x_1) \qquad (5.35)$$

"centroidal bending stiffness" about axis $\bar{i}_3$

... constitutive law for the bending behavior of the beam

bending moment & the curvature

bending stiffness (or "flexural rigidity")

$$\Rightarrow \quad M_3(x_1) = H_{33}^c \, \kappa_3(x_1) \qquad (5.37)$$

: "moment-curvature" relationship

5.3.3 Equilibrium eqn.s

∘ Fig 5.16 ... infinitesimal slice of the beam of length dx_1

$M_3(x_1)$, $V_2(x_1)$ acting at a face at location x_1,

@ $x_1 + dx_1$, evaluated using a Taylor series expansion, and h.o. terms ignored

→ 2 equilibrium eqn.s

vertical force → $\dfrac{dV_2}{dx_1} = -p_2(x_1)$ (5.38a)

moment about O → $\dfrac{dM_3}{dx_1} + V_2 = 0$ (5.38b)

→ $\dfrac{d^2 M_3}{dx_1^2} = p_2(x_1)$ (5.39)

5.5.4 Governing eqn.s

• Eq (5.37) → Eq (5.39), and recalling Eq (3.6) →

$$\frac{d^2}{dx_1^2} \left[H_{33}^c \frac{d^2 \bar{u}_2}{dx_1^2} \right] = p_2(x_1) \qquad (5.40)$$

4^{th} order DE

- 4 B.C. ... ① clamped end $\bar{u}_2 = 0$, $\dfrac{d\bar{u}_2}{dx_1} = 0$

② simply supported (pinned) : $\bar{u}_2 = 0, \quad \frac{d^2\bar{u}_2}{dx_1^2} = 0$

③ free (or unloaded) end : $\underbrace{\frac{d^2\bar{u}_2}{dx_1^2} = 0}_{\text{bending moment}}, \quad \underbrace{-\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right] = 0}_{\text{shear force}}$

④ end subjected to a concentrated transverse load P_2 : $P_2 = V_2 = -\frac{dM_3}{dx_1}$

$$\frac{d^2\bar{u}_2}{dx_1^2} = 0, \quad -\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right] = P_2$$

⑤ rectilinear spring (Fig. 5.17) : $-V_2(L) = k \cdot \bar{u}_2(L)$

$$\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right]_{x_1=L} - k\bar{u}_2(L) = 0, \quad \frac{d^2\bar{u}_2}{dx_1^2} = 0$$

sign convention: (+) when the spring is located at the left end

⑥ rotational spring (Fig. 5.18) : $-M_3(L) = k\Phi_3(L)$

$$H_{33}^c \left.\frac{d^2\bar{u}_2}{dx_1^2}\right|_{x_1=L} + k\frac{d\bar{u}_2}{dx_1} = 0, \quad -\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right] = 0$$

(−) when at the left end

5.5.5 The sectional bending stiffness

∘ Homogeneous material,

$$H_{33}^c = E I_{33}^c \qquad (5.41)$$

$$I_{33}^c = \int_A x_2^2 \, dA \qquad (5.42)$$

: purely geometric quantity, the area second moment of the section computed about the area center

- rectangular section of width b made of layered materials (Fig 5.9)

$$H_{33}^c = \int_A E x_2^2 \, dA = \sum_{i=1}^{n} E^{[i]} \int_{A^{[i]}} x_2^2 \, dA^{[i]} = \frac{b}{3}\sum_{i=1}^{n} E^{[i]}\left[(x_2^{[i+1]})^3 - (x_2^{[i]})^3\right] \qquad (5.43)$$

"weighted average" of the Young's moduli

5.5.6 The axial stress distribution

∘ local axial stress ··· eliminating the curvature from Eq. (5.32) (5.37)

$$\sigma_1(x_1, x_2, x_3) = -E x_2 \frac{M_3(x_1)}{H_{33}^c} \qquad (5.44)$$

- homogeneous material $\quad \sigma_1(x_1, x_2, x_3) = -x_2 \frac{M_3(x_1)}{I_{33}} \qquad (5.45)$

∴ linearly distributed over the section, independent of Young's modulus

- various layers of materials

$$\sigma_1^{[i]}(x_1, x_2, x_3) = -E^{[i]} x_2 \frac{M_3(x_1)}{H_{33}^c} \qquad (5.46)$$

... axial STRAIN distribution is linear over the section ← Eq (5.30)

" stress " ... piecewise linear (Fig. 5.20)

- strength criterion

$$\frac{|x_2^{max}|}{H_{33}^c} E |M_3^{max}| \le \sigma_{allow}^{comp}, \quad \frac{|x_2^{min}|}{H_{33}^c} E |M_3^{max}| \le \sigma_{allow}^{ten}$$

↑ max (+) bending moment in the beam

- layers of various material

... must be computed at the $\begin{Bmatrix} top \\ bottom \end{Bmatrix}$ locations of each ply

§ 5.7 Rational design of beams under bending

- "Neutral axis" ... along axis $\bar{i}_3$, which passes through the section's centroid

- material located near the N. A. carries almost no stress

- " contributes little to the bending stiffness

⇒ Rational design ... removal of the material located at and near the N.A. and relocation away from that axis

- Fig. 5.21 ... $\begin{Bmatrix} rectangular \\ ideal \end{Bmatrix}$ section, same mass $m = bh\rho$

↖ a thin web would be used to keep the 2 flanges

- ratio of bending stiffness

$$\frac{H_{ideal}}{H_{rect}} = \frac{E \cdot 2\left[\frac{b(h/2)^3}{12} + \frac{bh}{2} d^2\right]}{E \frac{bh^3}{12}} = \frac{1}{4} + 12\left(\frac{d}{h}\right)^2$$

For $d/h = 10$, $[\quad] \simeq 1200$

- ratio of max. axial stress

$$\frac{\sigma_{rect}^{max}}{\sigma_{ideal}^{max}} = \frac{E \frac{h}{2} M_3 I_{ideal}}{I_{rect} E\left(d + \frac{h}{4}\right) M_3} = \frac{\frac{1}{4} + 12\left(\frac{d}{h}\right)^2}{\frac{1}{2} + 2\left(\frac{d}{h}\right)}$$

For $d/h = 10$, $[\quad] \simeq 6(d/h) = 60$

→ ideal section can carry a 60 times larger bending moment

- ideal section ⇒ "I beam", but prone to instabilities of web and flange buckling

5.6 Beams subjected to combined axial and transverse loads

Sec. 5.4, 5.5 ·· convenient to locate the origin of the axes system at the centroid of the beam's x-s.

5.6.1 Kinematic description

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) - (x_2 - x_{2c}) \frac{d\bar{u}_2(x_1)}{dx_1} \quad (5.73a)$$

↑ location of centroid

$$u_2(\;\;''\;\;) = \bar{u}_2(x_1) \quad (\;''\; b)$$

$$u_3(\;\;''\;\;) = 0 \quad (\;''\; c)$$

· strain field

$$\epsilon_1(x_1, x_2, x_3) = \bar{\epsilon}_1(x_1) - (x_2 - x_c)\,\kappa_3(x_1) \quad (5.74)$$

5.6.2 Sectional constitutive law

∘ axial stress distribution

$$\sigma_1(x_1, x_2, x_3) = E\bar{\epsilon}_1(x_1) - E(x_2 - x_{2c})\,\kappa_3(x_1) \quad (5.75)$$

- axial force

$$N_1 = \int_A [E\bar{\epsilon}_1(x_1) - E(x_2 - x_{2c})\,\kappa_3(x_1)]\,dA \quad \Rightarrow \quad N_1 = S\bar{\epsilon}_1$$

$$= \underbrace{\left[\int_A E\,dA\right]}_{S\text{ (axial stiffness)}} \bar{\epsilon}_1(x_1) - \underbrace{\left[\int_A E(x_2 - x_{2c})\,dA\right]}_{\int_A E x_2\,dA - x_{2c}\int_A E\,dA = S_2 - S x_{2c} = 0} \kappa_3(x_1)$$

- bending moment

$$M_3^c = -\int_A (x_2 - x_{2c})\,[E\bar{\epsilon}_1(x_1) - E(x_2 - x_{2c})\,\kappa_3(x_1)]\,dA \quad \Rightarrow M_3^c = H_{33}^c\,\kappa_3$$

$$= -\underbrace{\left[\int_A E(x_2 - x_{2c})\,dA\right]}_{=0} \bar{\epsilon}_1(x_1) + \underbrace{\left[\int_A E(x_2 - x_{2c})^2\,dA\right]}_{H_{33}^c\text{ (bending stiffness)}} \kappa_3(x_1)$$

⇒ "decoupled sectional constitutive law"

2 crucial steps { ① displacement field must be in the form of Eq. (5.73)
② bending moment must be evaluated w.r.t the centroid

∴ Thus, centroid plays a crucial role.

5.6.3 Equilibrium eqn.s

∘ Fig 5.47 ··· infinitesimal slice of the beam of length dx_1

- force equilibrium in horizontal dir.

$$\frac{dN_1}{dx_1} = -p_1 \quad \Rightarrow \quad (5.8)$$

- vertical equilibrium

$$\frac{dV_2}{dx_1} = -p_2 \quad \Rightarrow \quad (5.31a)$$

- equilibrium of moments about the centroid

$$\frac{dM_3}{dx_1} + V_2 = \underbrace{(x_{2a} - x_{2c})}_{\text{moment arm of the axial load w.r.t the centroid}} p_1 \qquad (5.77)$$

5.6.4 Governing eqns

$$\begin{cases} \frac{d}{dx_1}\left[S \frac{d\bar{u}_1}{dx_1}\right] = -p_1(x_1) & (5.78a) \Rightarrow (5.19) \\ \frac{d^2}{dx_1^2}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right] = p_2(x_1) + \frac{d}{dx_1}\left[(x_{2a}-x_{2c})p_1(x_1)\right] & (\ \ b) \end{cases}$$

$\Rightarrow$ almost similar to (5.40) except!

$\therefore$ "decoupled" eqns $\begin{cases} (5.78a) \rightarrow \bar{u}_1(x_1) \\ (\ \ b) \rightarrow \bar{u}_2(x_1) \end{cases}$ can be independently solved

If axial loads are applied @ centroid, extension and bending are "decoupled"

" not " " , " "coupled"