Aircraft Structures

CHAPER 7.
Torsion

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❖ Example of structural component which are designed to carry torsional loads

- Power of drive shaft
  - Solid or thin-walled circular cross-section

- Aircraft Wing
  - Needs to carry the bending and torsional moments generated by the aerodynamic forces

- ‘bar’ rather than ‘beam’

7.1 Torsion of circular cylinders

- **Fig.1**
  - Infinitely long, homogeneous, solid or hollow circular cylinder subjected to end torques Q_1

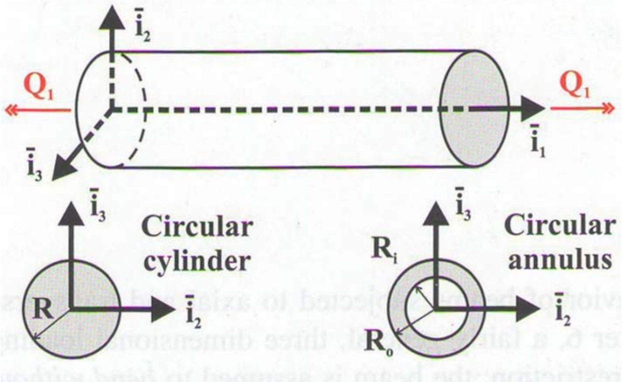


Fig. 7.1. Circular cylinder subjected to end torques.

- **2 types of symmetries**
  - ① Cylindrical symmetry about i_1 (Fig. 7.2)
  - ② Symmetric with regard to any plane, P, passing though axis i_1
    - Shear stress due to Q, must be of constant magnitude along circle C, and tangent to it
    - ⟶ loading is anti-symmetric with regard to P

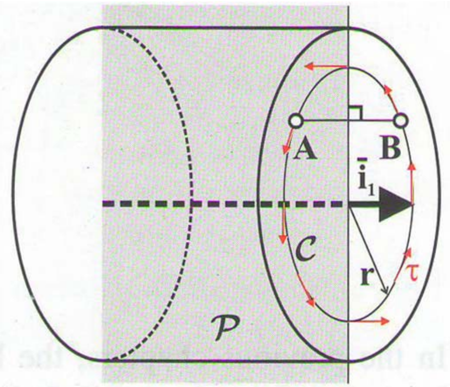


Fig. 7.2. A plane of symmetry, $\mathcal{P}$, of the circular cylinder.

7.1 Torsion of circular cylinders

- **Axial displacement at A and B, u_1^A and u_1^B**

① $u_1^A = u_1^B$
② $u_1^A = -u_1^B$

$$\left.\begin{array}{l} u_1^A = u_1^B \\ u_1^A = -u_1^B \end{array}\right\} u_1^A = u_1^B = 0$$

⟶ axial displacement must vanish
"the cross-section does not warp out-of plane"

- **Each axis "rotate about its own center like a rigid disk"**

7.1 Torsion of circular cylinders

7.1.1 Kinematic Description

- **Rotation angle** Φ_1
  - Rigid body rotation of each axis (Fig. 7.3)

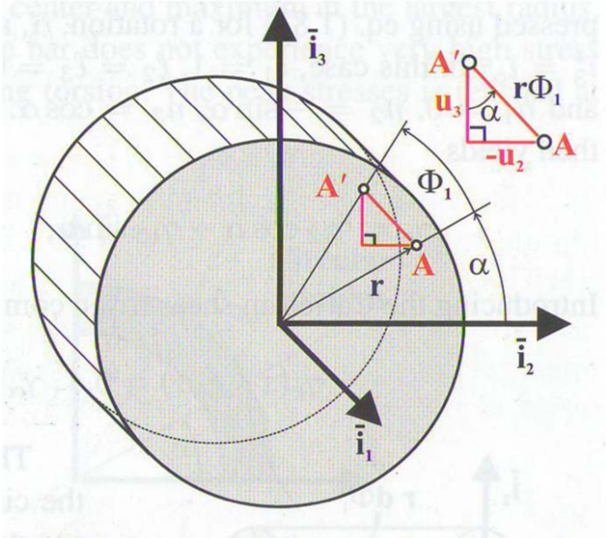


Fig. 7.3. In-plane displacements for a circular cylinder. The cross-section undergoes a rigid body rotation.

- **Sectional in-plane displacement field**

$$\left.\begin{aligned} u_2(x_1,r,\alpha) &= -r\Phi_1(x_1)\sin\alpha \\ u_3(x_1,r,\alpha) &= r\Phi_1(x_1)\cos\alpha \end{aligned}\right\} \quad (7.1)$$

- **Out-of-plane displacement field**
  - $u_1(x_1,x_2,x_3)=0$ (7.2)
  - $u_2(x_1,x_2,x_3)=-x_3\Phi_1(x_1)$
  - $u_3(x_1,x_2,x_3)=x_2\Phi_1(x_1)$

$$\left.\begin{aligned} u_2(x_1,x_2,x_3) &= -x_3\Phi_1(x_1) \\ u_3(x_1,x_2,x_3) &= x_2\Phi_1(x_1) \end{aligned}\right\} \quad (7.3) \quad \text{from Eq.(7.1)}$$

7.1 Torsion of circular cylinders

❖ Strain field

$$\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = 0 \quad (7.4)$$

$$\gamma_{23} = 0 \quad (7.5)$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -x_3\kappa_1(x_1), \gamma_{13} = x_2\kappa_1(x_1) \quad (7.6)$$

$$\kappa_1(x_1) = \frac{\partial \Phi_1}{\partial x_1} \qquad (7.7)$$ "section twist rate"

- To visualize the strain field, describe them in the polar coordinate (r, α)
  $\longrightarrow \gamma_{r1}$ and $\gamma_{\alpha 1}$, or simply γ_r and γ_α

7.1 Torsion of circular cylinders

- **Transformation between the Cartesian and the Polar strain component**

$$\gamma_\alpha = \gamma_{12}\cos\alpha + \gamma_{13}\sin\alpha, \quad \gamma_r = -\gamma_{12}\sin\alpha + \gamma_{13}\cos\alpha$$ (7.8) from Eq.(7.6)

$$\gamma_r(x_1, r, \alpha) = 0\ , \ \gamma_\alpha(x_1, r, \alpha) = r\kappa_1(x_1)$$ (7.9)

↳ circumferential shearing strain (Fig. 7.4)

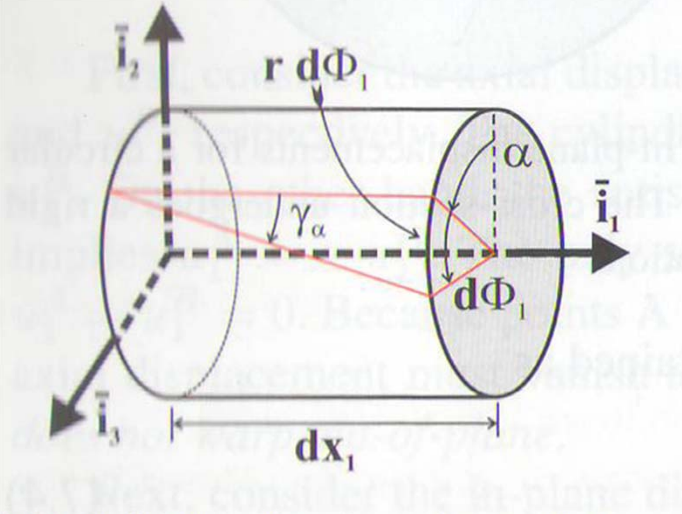


Fig. 7.4. Visualization of out-of-plane shear strain in polar coordinates.

7.1 Torsion of circular cylinders

7.1.2 The Strain Field

- **The only non-vanishing stress components**

$$\tau_{12} = -Gx_3\kappa_1(x_1),\ \ \tau_{13} = Gx_2\kappa_1(x_1) \quad (7.10)$$

using polar coordinate,

$$\tau_r(x_1,r,\alpha) = 0,\ \ \tau_\alpha(x_1,r,\alpha) = Gr\kappa_1(x_1) \quad (7.11)$$

↳radial ↳circumferential stress component

- **Distribution of the circumferential shear stress** (Fig. 7.5)

① Circumferential direction exists only, radial direction vanishes

② Varies linearly along the radial direction

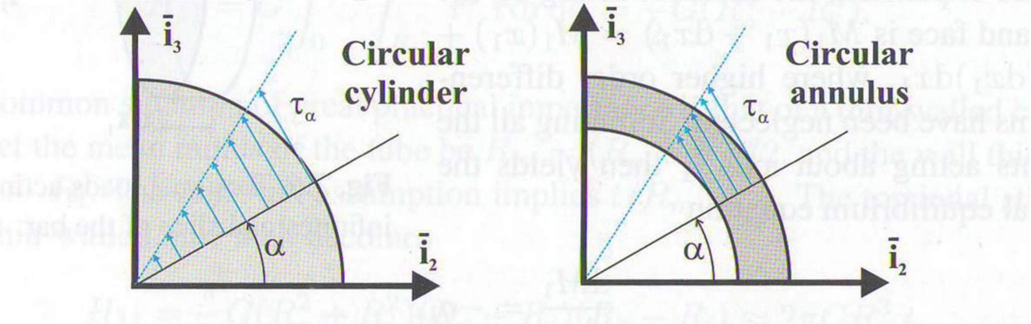


Fig. 7.5. Distribution of circumferential shearing stress over the cross-section.

7.1 Torsion of circular cylinders

7.1.3 Sectional Constitutive Law

- **Torque acting on the axis at a given span-wise location**

$$M_1(x_1) = \int_A \tau_\alpha r dA \quad (7.12)$$

$$M_1(x_1) = \int_A G r^2 \kappa_1(x_1) dA = \left[\int_A G r^2 dA\right] \kappa_1(x_1) = H_{11}\kappa_1(x_1) \quad (7.13)$$ from Eq. (7.11)

↳ torsional stiffness

$$H_{11} = \int_A G r^2 dA \quad (7.14)$$ for circular axis only

- Constitutive for the torsional behavior of the beam

- **If homogeneous material**

$H_{11} = GJ$, where $J = \int_A r^2 dA$: "area polar moment" for circular axis only

7.1 Torsion of circular cylinders

7.1.4 Equilibrium Equations

- Infinitesimal slice of the cylinder of length dx_1

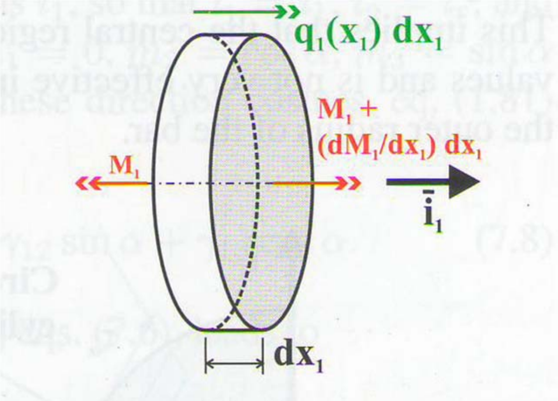


Fig. 7.6. Torsional loads acting on an infinitesimal slice of the bar.

- Torsional equilibrium equation

$$\frac{dM_1}{dx_1} = -q_1 \quad (7.15)$$

7.1 Torsion of circular cylinders

7.1.5 Governing Equations

❖ **Eq. (7.13) ⟶ (7.15) and recalling Eq. (7.7)**

$$\frac{d}{dx_1}\left[H_{11}\frac{d\Phi_1}{dx_1}\right]=-q_1 \qquad (7.16)$$

❖ **Boundary Condition**

① Fixed(clamped): $\Phi_1 = 0$

② Free(unloaded): $M_1 = 0 \rightarrow \kappa_1 = \dfrac{d\Phi_1}{dx_1} = 0$

③ subjected to a concentrated torque $Q_1 : M_1 = Q_1 \rightarrow H_{11}\dfrac{d\Phi_1}{dx_1} = Q_1$

7.1 Torsion of circular cylinders

7.1.6 Torsional Stiffness

- **If Homogeneous material**

$$H_{11} = G\int_{0}^{2\pi}\int_{R_i}^{R_0} r^2 r dr d\alpha = \frac{\pi}{2} G R^4 \qquad (7.17)$$

- **For a circular tube**

$$H_{11} = G\int_{0}^{2\pi}\int_{R_i}^{R_0} r^2 r dr d\alpha = \frac{\pi}{2} G(R_0^4 - R_i^4) \qquad (7.18)$$

- **For a thin-walled circular tube, mean radius**

$$H_{11} = \frac{\pi}{2} G(R_0^2 + R_i^2)(R_0 + R_i)(R_0 - R_i) \approx 2\pi G R_m^3 t \quad (7.19)$$

7.1 Torsion of circular cylinders

❖ **Thin-walled circular tube consisting of N concentric layer**

$$H_{11} = \frac{\pi}{2}\sum_{i=1}^{N} G^{[i]}\left[\left(R^{[i+1]}\right)^4 - \left(R^{[i]}\right)^4\right]$$

$$= 2\pi\sum_{i=1}^{N} G^{[i]}t^{[i]}\left(\frac{R^{[i+1]}+R^{[i]}}{2}\right)^3 \qquad (7.20)$$

- "weighted average" of the shear moduli of the various layer

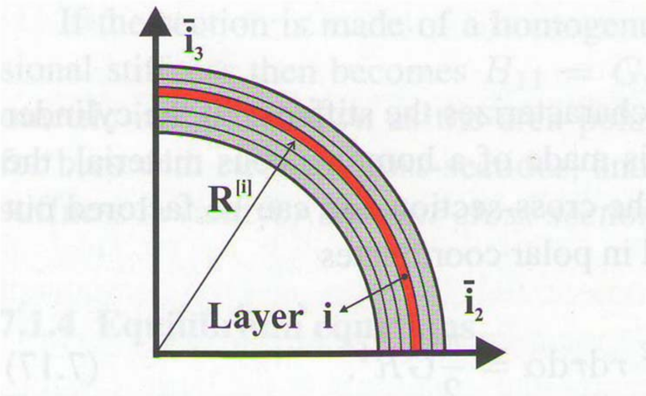


Fig. 7.7. Thin-walled tube made of layered materials.

7.1 Torsion of circular cylinders

7.1.7 Measuring the Torsional Stiffness

- **Deformation of the test section**
  - Measured by the chevron strain gauge

    $\gamma_{12} = e_{+45} - e_{-45}$ (Fig. 7.8)

    $\kappa_1 = (e_{+45} - e_{-45}) / R$ (@ r=R)

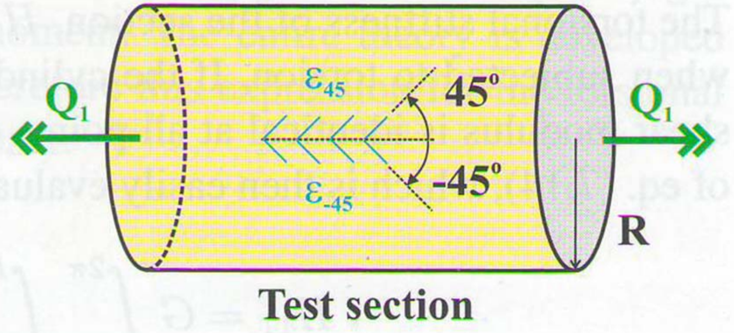


Fig. 7.8. Configuration of the test to determine the torsional stiffness.

- **Slope of θ_{3i}vs. κ_{1i}Curve → torsional stiffness**
  - Valid as long as the cylindrical symmetry is maintained

7.1 Torsion of circular cylinders

7.1.8 The Shear Stress Distribution

❖ **Local circumferential stress**

- Eq. (7.11) ⟶ (7.13)

$$\tau_\alpha = G\frac{M_1(x_1)}{H_{11}} r \quad (7.21)$$

- increases linearly from zero at the center to a max. value at the outer radius

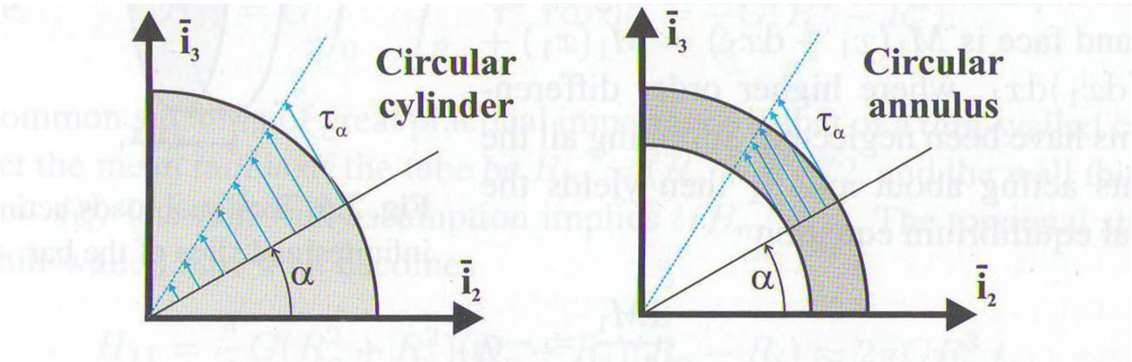


Fig. 7.5. Distribution of circumferential shearing stress over the cross-section.

7.1 Torsion of circular cylinders

- **Concentric layers of district material**

$$\tau_{\alpha}^{(i)} = G^{[i]} \frac{M_1}{H_{11}} r$$

  - which each layer, still linear distribution, but discontinuities at the interface

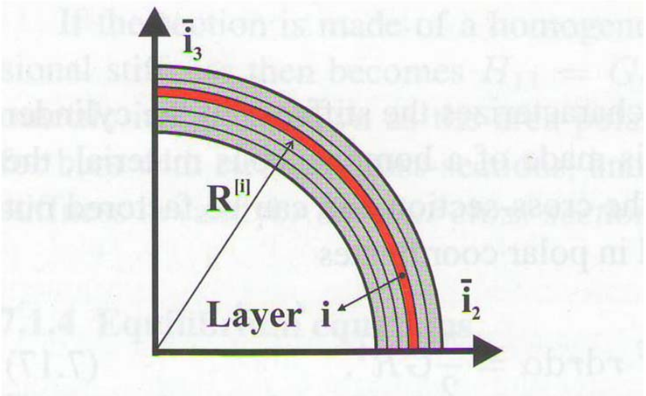


Fig. 7.7. Thin-walled tube made of layered materials.

- **Maximum shear stress for homogeneous material**

$$\tau_{\alpha}^{\max} = \frac{2M_1(x_1)}{\pi R^3} \quad (7.22)$$

- **Strength criterion**

$$\frac{GR}{H_{11}} \left| M_1^{\max} \right| \le \tau_{allow} \quad (7.26)$$

7.1 Torsion of circular cylinders

7.1.9 Rational Design of Cylinders under Torsion

- **Material near the center of the cylinder is not used efficiently since the shear stress becomes small**
  - Thin-walled tube is a far more efficient design

- **2 thin-walled tube of the same material, mass per unit span, but different mean radii R_m and R'_m**

① torsional stiffness:

$$\frac{H_{11}}{H_{11}^{'}} = \frac{(\mu/\rho)GR_m^2}{(\mu/\rho)GR_m^{'2}} = \left(\frac{R_m}{R_m^{'}}\right)^2 \quad (7.28)$$

② shear stress under the same torque

$$\frac{\tau_\alpha}{\tau'_\alpha} = \frac{GM_1R_m/H_{11}}{GM_1R'_m/H'_{11}} = \frac{R_m/H'_{11}}{R'_m/H_{11}} = \frac{R'_m}{R_m} \quad (7.29)$$

- inversely proportional to the mean radius

7.1 Torsion of circular cylinders

Large mean radius

- High H_{11} , lower $\max \tau$
- but in practice, limits "torsional buckling"

7.2 Torsion combined with axial forces and bending moments

- ❖ **What is the proper strength criterion to be used when both axial and shear stresses are acting simultaneously?**

1) Propeller shaft under torsion and thrust

- ➢ Torque M_1 and thrust N_1

$$\tau = \frac{2M_1}{\pi R^3}, \quad \sigma = \frac{N_1}{\pi R^2} \qquad (7.30)$$

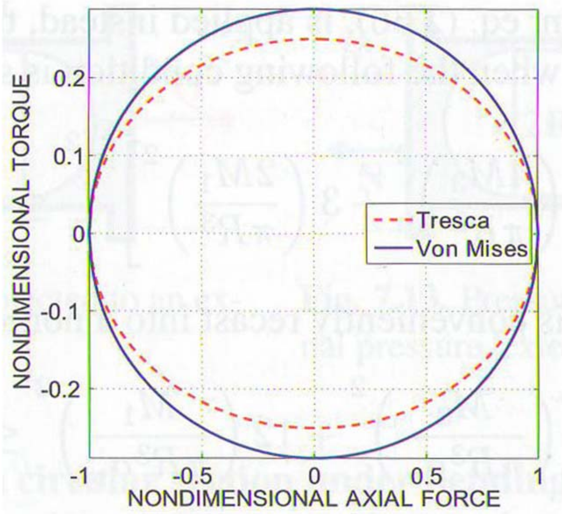
- ➢ Tresca’s criterion, Eq. (2.31)
  most stringent condition among 3

$$\left(\frac{N_1}{\pi R^2 \sigma_y}\right)^2 + 16\left(\frac{M_1}{\pi R^3 \sigma_y}\right)^2 = 1$$ ellipse in Fig. 7.10

- ➢ von Mises’ criterion, Eq.(2.36)

$$\left(\frac{N_1}{\pi R^2 \sigma_y}\right)^2 + 12\left(\frac{M_1}{\pi R^3 \sigma_y}\right)^2 \le 1$$ llipse in Fig. 7.10

Fig. 7.10



7.2 Torsion combined with axial forces and bending moments

2) Shaft under torsion and bending

- Bending moment M_3 and torque M_1

$$\sigma = \frac{4M_3 r}{\pi R^4}, \quad \tau = \frac{2M_1 r}{\pi R^4} \quad (7.31)$$

- Tresca's criterion

$$16\left(\frac{M_3}{\pi R^3 \sigma_y}\right)^2 + 16\left(\frac{M_1}{\pi R^3 \sigma_y}\right)^2 = 1$$

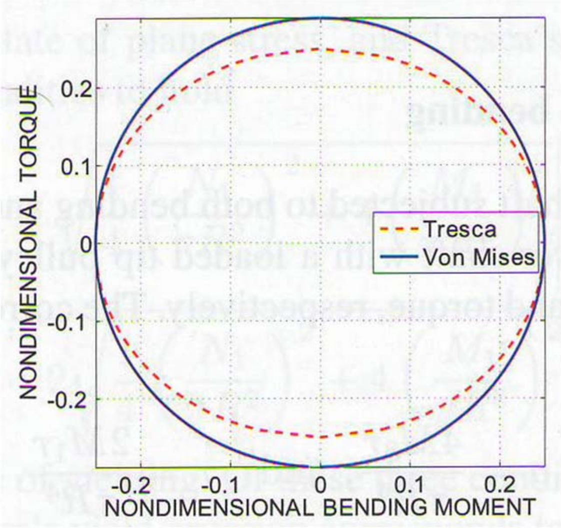
Fig. 7.11

- von Mises' criterion

$$16\left(\frac{M_3}{\pi R^3 \sigma_y}\right)^2 + 12\left(\frac{M_1}{\pi R^3 \sigma_y}\right)^2 \le 1$$

Fig. 7.11

Fig. 7.11



7.3 Torsion of Bar with Arbitrary Cross-Sections

7.3.1 Introduction

- Circular symmetry of the problem is not maintained any more

- At any point along the edge of the bar`s section,
  the shear stress must be tangent to the edge $\longrightarrow \tau_{13} = 0$
  but, non-zero τ_{13} is required from the circular symmetry

- Fewer symmetries than the circular cross section has.

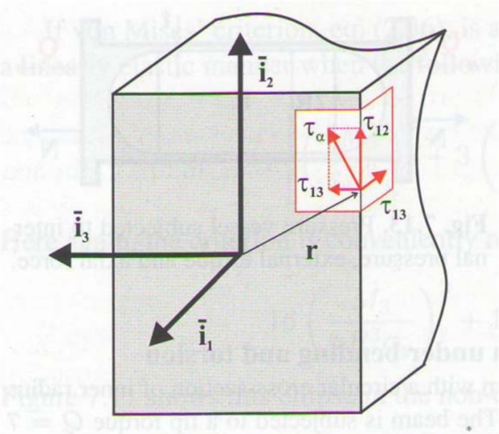


Fig. 7.14. Shearing stresses along the edge of a rectangular section.

7.3 Torsion of Bar with Arbitrary Cross-Sections

- Symmetry built planes ($\vec{i}_1$, $\vec{i}_2$) and ($\vec{i}_1$, $\vec{i}_3$) but, no circular symmetry

- Torsional loading and the resulting solution : anti-symmetry

$$\left.\begin{array}{lll} \text{with regard to } (\vec{i}_1,\ \vec{i}_2) & \longrightarrow & u_1^A = -u_1^B,\ \ u_1^C = -u_1^D \\ \text{with regard to } (\vec{i}_1,\ \vec{i}_3) & \longrightarrow & u_1^A = -u_1^D,\ \ u_1^B = -u_1^C \end{array}\right\}$$

Cross section will warp out-of-plane

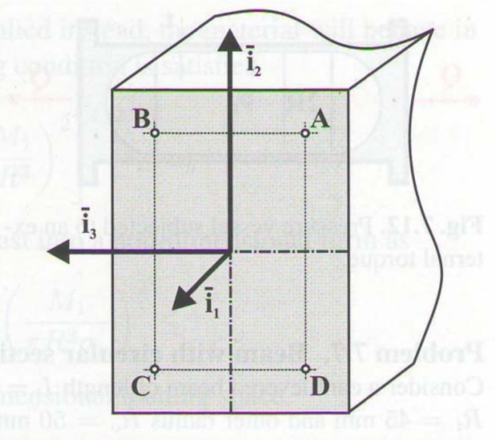


Fig. 7.15. Four points on a rectangular cross-section.

7.3 Torsion of Bar with Arbitrary Cross-Sections

7.3.2 Saint-Venant's solution

1) Kinematic description

- Each cross section rotates like a rigid body, and warp out-of-plane
  → assumed displacement field

$$\left.\begin{aligned} u_1(x_1,x_2,x_3) &= \Psi(x_2,x_3)\kappa_1(x_1) \\ u_2(x_1,x_2,x_3) &= -x_3\Phi_1(x_1) \\ u_3(x_1,x_2,x_3) &= x_2\Phi_1(x_1) \end{aligned}\right\} \quad (7.32)$$

$\Psi(x_2,x_3)$:unknown warping function, will be determined by enforcing equilibrium equations for the resulting stress field

7.3 Torsion of Bar with Arbitrary Cross-Sections

2) The Strain field

- Eq.(7.32) ⟶ Eq. (1.63) and (7.71)

$$\left.\begin{array}{l} \varepsilon_1 = \Psi(x_2, x_3)\dfrac{d\kappa_1}{dx_1} = 0 \text{ due to "uniform torsion"} \\ \varepsilon_2 = 0,\ \varepsilon_3 = 0,\ \gamma_{23} = 0 \\ \gamma_{12} = \left(\dfrac{d\Psi}{dx_2} - x_3\right)\kappa_1,\quad \gamma_{13} = \left(\dfrac{d\Psi}{dx_3} + x_2\right)\kappa_1 \end{array}\right\} \quad (7.33)$$

3) The Stress field

$$\left.\begin{array}{l} \sigma_1 = 0,\ \sigma_2 = 0,\ \sigma_3 = 0,\ \tau_{23} = 0 \\ \tau_{12} = G\kappa_1\left(\dfrac{\partial\Psi}{\partial x_2} - x_3\right),\ \tau_{13} = G\kappa_1\left(\dfrac{\partial\Psi}{\partial x_3} + x_2\right) \end{array}\right\} \quad (7.34)$$

7.3 Torsion of Bar with Arbitrary Cross-Sections

4) Equilibrium equations

- Stress field must satisfy the general equilibrium equations.
  Eq.(1.4) at all point of the section.
  Neglecting body forces, the remaining equation is

$$\frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} = 0 \quad (7.35)$$

- Eq.(7.34c) ⟶ (7.35)

$$\frac{\partial^2 \Psi}{\partial x_2^2} + \frac{\partial^2 \Psi}{\partial x_3^2} = 0 \quad (7.36)$$

  the warping function must satisfy the PDE at all points of the cross section

7.3 Torsion of Bar with Arbitrary Cross-Sections

- Boundary condition: satisfaction of the equilibrium equations along the outer contour of the section (Fig. 7.16)
- Along the C, according to Fig. 7.14, $\tau_n = 0$ (7.37)

  τ_s does not necessarily vanish

  in terms of Cartesian components,

$$\tau_n = \tau_{12}\sin\beta + \tau_{13}\cos\beta = \tau_{12}\left(\frac{dx_3}{ds}\right) + \tau_{13}\left(-\frac{dx_2}{ds}\right) = 0 \qquad (7.38)$$

- Eq.(7.34c) → (7.38): $\left(\frac{\partial\Psi}{\partial x_2} - x_3\right)\frac{dx_3}{ds} - \left(\frac{\partial\Psi}{\partial x_3} + x_2\right)\frac{dx_2}{ds} = 0$ (7.39)

  Eq.(7.36): Laplace’s equations

  Eq.(7.39): rather complicated boundary condition

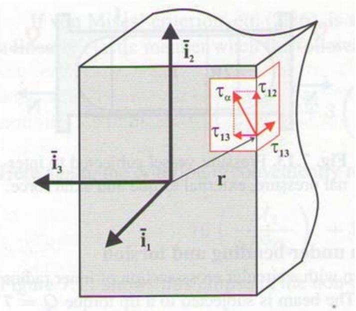


Fig. 7.14. Shearing stresses along the edge of a rectangular section.

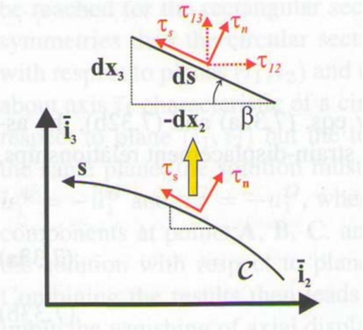


Fig. 7.16. Equilibrium condition along the outer contour $\mathcal{C}$.

7.3 Torsion of Bar with Arbitrary Cross-Sections

5) Prandtl's stress function

- Alternative formulation leading to simple boundary condition

  : stress function, $\phi(x_2, x_3)$

$$\tau_{12} = \frac{\partial \phi}{\partial x_3}, \tau_{13} = -\frac{\partial \phi}{\partial x_2} \qquad (7.41)$$

  automatically satisfies the load equilibrium equation, Eq.(7.35)

- By comparing Eq.(7.34c) and (7.41)

$$\tau_{12} = G\kappa_1 \left(\frac{\partial \Psi}{\partial x_2} - x_3\right) = \frac{\partial \phi}{\partial x_3}, \tau_{13} = G\kappa_1 \left(\frac{\partial \Psi}{\partial x_3} + x_2\right) = -\frac{\partial \phi}{\partial x_2} \qquad (7.42)$$

$$\underbrace{\overset{\uparrow}{\frac{\partial}{\partial x_3}[\ ]} \qquad\qquad\qquad \overset{\uparrow}{\frac{\partial}{\partial x_2}[\ ]}}_{(-)}$$

$$\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = -2G\kappa_1 \quad (7.43)$$

7.3 Torsion of Bar with Arbitrary Cross-Sections

❖ Boundary condition

- from Eq.(7.38), (7.41)

$$\tau_n = \frac{\partial \phi}{\partial x_3}\frac{dx_3}{ds} + \frac{\partial \phi}{\partial x_2}\frac{dx_2}{ds} = \frac{d\phi}{ds} = 0$$ (7.44)

↳ constant value may be chosen to vanish

: constant ϕ along C

- Eq.(7.43): Poisson's equation

- Eq.(7.44): much simpler boundary condition

7.3 Torsion of Bar with Arbitrary Cross-Sections

5) Sectional equilibrium

- Global equilibrium of the section
  - Resultant Shear force

$$V_2 = \int_A \tau_{12} dA = \int_{x_2} \int_{x_3} \frac{\partial \phi}{\partial x_3} dx_2 dx_3 = \int_{x_2} \left[\int_{x_3} \frac{\partial \phi}{\partial x_3} \right] dx_2 = 0$$

$$V_3 = 0$$

: no shear forces are applied

  - Total torque acting on the section

$$M_1 = \int_A (x_2 \tau_{13} - x_3 \tau_{12}) dA = \int_A (-x_2 \frac{\partial \phi}{\partial x_2} - x_3 \frac{\partial \phi}{\partial x_3}) dA \qquad (7.46)$$

Integrating by parts

$$M_1 = 2\int_A \phi dA - \int_{x_3} \left[x_2 \phi \right]_{x_2} dx_3 - \int_{x_2} \left[x_3 \phi \right]_{x_3} dx_2 \qquad (7.47)$$

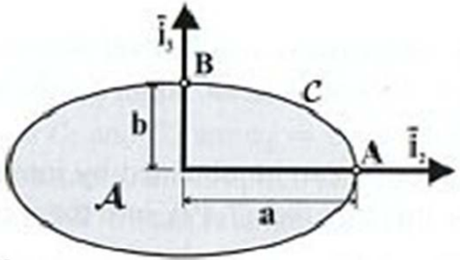
$$M_1 = 2\int_A \phi dA \qquad (7.48)$$

applied torque = 2 x "volume" under the stress function,

only valid for solid cross section bounded by a single curve otherwise, use Eq.(7.46)

Example 7.1

❖ Torsion of an elliptical bar



- Curve $C:\left(\frac{x_2}{a}\right)^2+\left(\frac{x_3}{b}\right)^2=1$, A stress function of the following form is assumed

$$\phi=C_0\left[\left(\frac{x_2}{a}\right)^2+\left(\frac{x_3}{b}\right)^2-1\right],\quad C_0:\text{Unknown const.}$$

- Boundary condition, Eq.(7.45b) is clearly satisfied since $\phi=0$ along C.
- Substituting into the governing eqn., Eq.(7.45)

$$C_0\left(\frac{2}{a^2}+\frac{2}{b^2}\right)=-2G\kappa_1$$

$$C_0=\frac{-a^2b^2G\kappa_1}{a^2+b^2}$$

$$\phi=-\frac{a^2b^2}{a^2+b^2}\left[\left(\frac{x_2}{a}\right)^2+\left(\frac{x_3}{b}\right)^2-1\right]G\kappa_1 \qquad (7.49)$$

Example 7.1

Torsion of an elliptical bar

- Torque: Eq.(7.48)

$$M_1 = -\frac{2a^2b^2}{a^2+b^2} G\kappa_1 \int_A \left[\left(\frac{x_2}{a}\right)^2 + \left(\frac{x_3}{b}\right)^2 - 1 \right] dA = G\frac{\pi a^3 b^3}{a^2+b^2}\kappa_1 = H_{11}\kappa_1$$

- Torsional stiffness of the elliptical section

$$H_{11} = G\frac{\pi a^3 b^3}{a^2+b^2} \qquad (7.50)$$

- Stress fn. In terms of the applied torque

$$\phi = -\frac{M_1}{\pi ab}\left[\left(\frac{x_2}{a}\right)^2 + \left(\frac{x_3}{b}\right)^2 - 1 \right]$$

- Stress distribution: Eq.(7.41)

$$\tau_{12} = -\frac{2x_3}{\pi ab^3}M_1, \tau_{13} = \frac{2x_2}{\pi a^3 b}M_1$$

- Shear stress vector ··· Fig.7.18b, tangent to curve C.

$$|\tau_{\max}| = \frac{2M_1}{\pi ab^2}$$

(Fig. 7.18a)

Example 7.1

❖ Torsion of an elliptical bar

- Shear stress vector ··· Fig. 7.18b, tangent to curve C.
- Warping function ··· by integrating Eq.(7.42)

$$\frac{\partial \Psi}{\partial x_2} = -\frac{a^2 - b^2}{a^2 + b^2} x_3 , \frac{\partial \Psi}{\partial x_3} = -\frac{a^2 - b^2}{a^2 + b^2} x_2$$

$x_2 \downarrow$ Integrating w.r.t. $\downarrow x_3$

$$\underbrace{\Psi = -x_2 x_3 \frac{a^2 - b^2}{a^2 + b^2} + f(x_3) \quad \Psi = -x_2 x_3 \frac{a^2 - b^2}{a^2 + b^2} + g(x_2)}$$

Equal only if $f(x_3) = g(x_2) = 0$

$$\rightarrow \Psi = -\frac{a^2 - b^2}{a^2 + b^2} x_2 x_3$$

Eq.(7.32a) $\rightarrow u_1(x_2, x_3) = -x_1 \frac{a^2 - b^2}{a^2 + b^2} x_2 x_3$

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Example 7.1

Torsion of an elliptical bar

… 2 planes of symmetry, warping displacement antisymmetric w.r.t Z planes (Fig. 7.19)

- Circular section … a=b=R, warping fn.=0

7.3 Torsion of Bar with Arbitrary Cross-Sections

❖ Summary

- Bar of arbitrary cross section subjected to uniform torsion

- Stress distribution: Warping function Eq.(7.40)
  Stress function Eq.(7.45)

- Stress field: Eq.(7.34c) or Eq.(7.41)

  ⟶ exact solution although the displacement field is assumed as in Eq.(7.32)

7.3 Torsion of Bar with Arbitrary Cross-Sections

7.3.2 Saint-Venant's solution for a Rectangular X-S

- 2 Solution – approximation solution based on the co-location approach
  – exact solution based on the co-location approach

1) Approximate solution

- Rectangular cross section of width a, height b (Fig. 7.21)
  - Assumed stress function

$$\phi(\eta,\zeta)=C_0(\eta^2-\frac{1}{4})(\zeta^2-\frac{1}{4}) \quad , \quad \eta=\frac{x_2}{a}, \quad \zeta=\frac{x_3}{b}$$

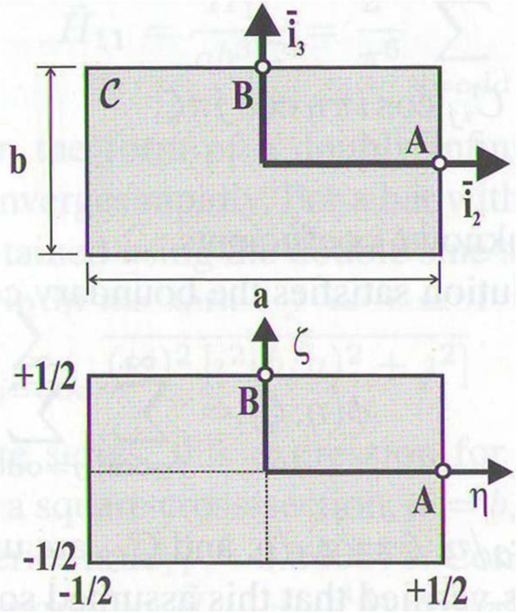
$$\phi(\eta=\pm 1/2,\zeta)=0,\phi(\eta,\zeta=\pm 1/2)=0$$

$\longrightarrow$ $\phi=0$ along the curve C

$\longrightarrow$ PDE, Eq(7.43):

$$2C_0\left(\zeta^2-\frac{1}{4}\right)\frac{1}{a^2}+2C_0\left(\eta^2-\frac{1}{4}\right)\frac{1}{b^2}=-2G\kappa_1$$

assumed solution does not satisfy PDE.



(Fig. 7.21)

7.3 Torsion of Bar with Arbitrary Cross-Sections

- Approximate solution: “co-location method”, satisfy PDE only at a specific part of points of the cross section

  - PDE will be satisfied at the center, $\left(\eta,\zeta\right)=\left(0,0\right)$

$$-\frac{C_0}{2a^2}-\frac{C_0}{2b^2}=-2G\kappa_1\ ,\quad C_0=\frac{4G\kappa_1 a^2 b^2}{a^2+b^2}$$

  - Then, $\phi=\frac{4a^2b^2G\kappa_1}{a^2+b^2}(\eta^2-\frac{1}{4})(\zeta^2-\frac{1}{4})$

$M_1=2\int_A \phi dA$, torsional stiffness H_{11}

shear stress field τ_{12}, τ_{13}

7.3 Torsion of Bar with Arbitrary Cross-Sections

2) Open form exact solution using a Fourier series

- Fourier series expansion of the stress function

$$\phi(\eta,\zeta) = \sum_{i=odd}^{\infty} \sum_{j=odd}^{\infty} C_{ij} \cos i\pi\eta \cos j\pi\zeta$$

- Satisfaction of B.C. Eq.(7.45b): when $i, j = odd, \ \phi = 0$
  thus only odd values of i, j are included

- Governing PDE, Eq.(7.43)

$$\sum_{i=odd}^{\infty} \sum_{j=odd}^{\infty} C_{i,j} \left[\left(\frac{i\pi}{a} \right)^2 + \left(\frac{j\pi}{b} \right)^2 \right] \cos i\pi\eta \cos j\pi\zeta$$

- By using the orthogonality properties of cosine function

$$\sum_{i=odd}^{\infty} \sum_{j=odd}^{\infty} C_{i,j} \left[\left(\frac{i\pi}{a} \right)^2 + \left(\frac{j\pi}{b} \right)^2 \right] \left[\int_{-1/2}^{1/2} \cos m\pi\eta \cos i\pi\eta d\eta \right] \left[\int_{-1/2}^{1/2} \cos n\pi\zeta \cos j\pi\zeta d\zeta \right]$$

$$= -2G\kappa_1 \left[\int_{-1/2}^{1/2} \cos m\pi\eta d\eta \right] \left[\int_{-1/2}^{1/2} \cos n\pi\zeta d\zeta \right]$$

7.3 Torsion of Bar with Arbitrary Cross-Sections

- The bracket integrals vanish when $m \neq i$ on $n \neq j$. The remaining terms

$$C_{mn}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\frac{1}{4}=\frac{8}{mn\pi^2}\left(-1\right)^{\frac{m-1}{2}}\left(-1\right)^{\frac{n-1}{2}}G\kappa_1$$

- Then, $\phi(\eta,\zeta)=\dfrac{32G\kappa_1}{\pi^2}\displaystyle\sum_{i=odd}^{\infty}\sum_{j=odd}^{\infty}\frac{(-1)^{(i+j-2)/2}}{ij\left[(i\pi/a)^2+(j\pi/b)^2\right]}\cos i\pi\eta\cos j\pi\zeta$ (7.53)

- Externally applied torque
- Torsional stiffness
- Shear stress field: Although it is a doubly infinite series, it converges rapidly

  (1, 2 term) ⟶ Fig 7.22, 7.23

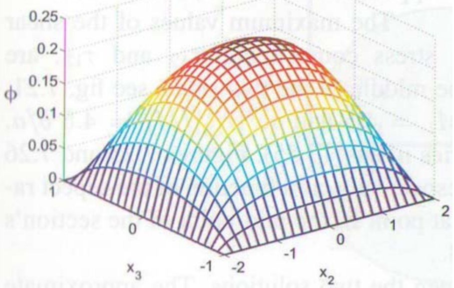


Fig. 7.22. Stress function, ϕ.

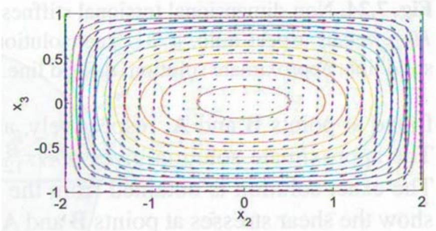


Fig. 7.23. Distribution of shear stress over cross-section. The arrows represent the shear stresses; the contours represent constant values of the stress function ϕ.

7.3 Torsion of Bar with Arbitrary Cross-Sections

3) Comparison of solution

- $\bar{H}_{11}$: Fig. 7.24
- Non-dimensional shear stress: Fig. 7.25, 7.26
- Large discrepancies, approximate solution is not good enough

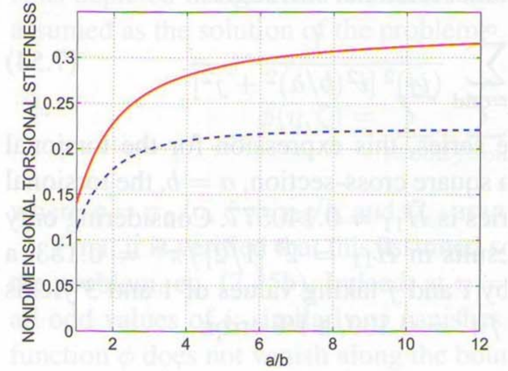


Fig. 7.24. Non-dimensional torsional stiffness, $\bar{H}_{11}$, versus aspect ratio, a/b. Exact solution: solid line; approximate solution: dashed line.

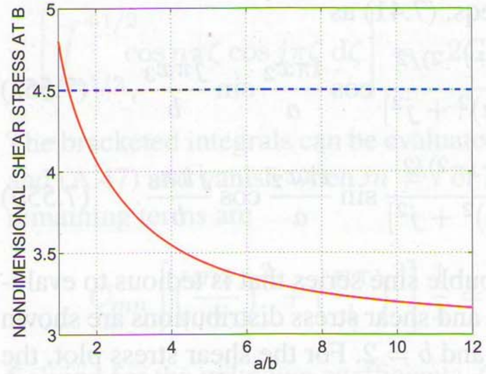


Fig. 7.25. Non-dimensional shear stress at point **B** versus aspect ratio a/b. Exact solution: solid line; approximate solution: dashed line.

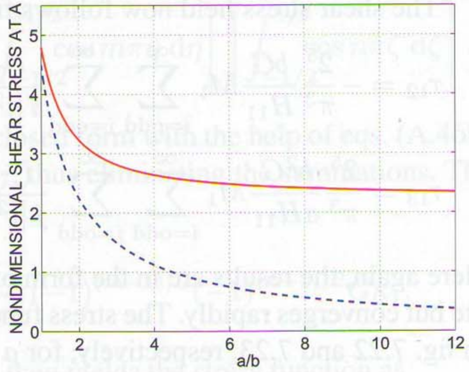


Fig. 7.26. Non-dimensional shear stress at point **A** versus aspect ratio a/b. Exact solution: solid line; approximate solution: dashed line.

7.4 Torsion of a thin rectangular Cross Section

- Fig. 7.28: $t \ll b$, assume that both stress function and associated shear stress distributions will be nearly constant along $\bar{i}_3$

$$\rightarrow \quad \frac{\partial \phi}{\partial x_3} \approx 0$$

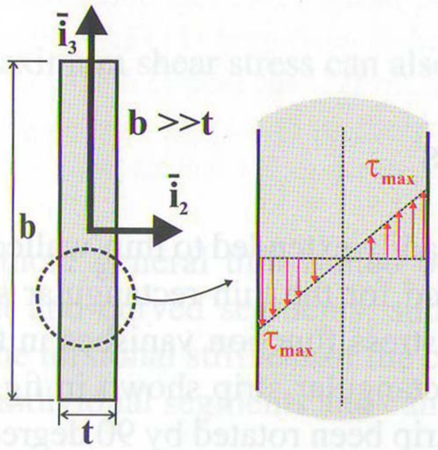


Fig. 7.28. Thin rectangular strip under torsion.

- Governing Equation is from Eq.(7.43)

$$\frac{d^2\phi}{dx_2^2} = -2G\kappa_1 \quad (7.56)$$

$$\phi\left(x_2\right) = -G\kappa_1 x_2^2 + C_1 x_2 + C_2$$

- Boundary Condition Eq.(7.45b)

$$\phi(x_2 = \pm t/2) = 0 \rightarrow C_1 = 0, C_2 = G\kappa_1 t^2/4$$

$$\rightarrow \quad \phi(x_2) = -G\kappa_1(x_2^2 - \frac{t^2}{4})$$

7.4 Torsion of a thin rectangular Cross Section

- Resulting torque

$$M_1 = 2\int_A \phi dA = -2G\kappa_1 \int_{-t/2}^{t/2} (x_2^2 - \frac{t^2}{4}) b dx_2 = \frac{1}{3} G\kappa_1 b t^3$$

- Torsional stiffness

$$H_{11} = \frac{M_1}{\kappa_1} = \frac{1}{3} G b t^3 \qquad (7.58)$$

- Shear stress distribution

$$\tau_{12} = \frac{\partial \phi}{\partial x_3} = 0, \tau_{13} = -\frac{\partial \phi}{\partial x_2} = 2G\kappa_1 x_2 = \frac{6M_1}{bt^3} x_2 \qquad (7.59)$$

↳ R.H.S. of Fig. 7.28

7.4 Torsion of a thin rectangular Cross Section

- Warping function: Eq.(7.57) ⟶ (7.42)

$$\frac{\partial \Psi}{\partial x_2} = \frac{1}{G\kappa_1}\frac{\partial \phi}{\partial x_3} + x_3 = x_3, \quad \frac{\partial \Psi}{\partial x_3} = -\frac{1}{G\kappa_1}\frac{\partial \phi}{\partial x_2} - x_2 = x_2$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\underbrace{\Psi = x_2 x_3 + f\left(x_3\right) \qquad\qquad \Psi = x_2 x_3 + g\left(x_2\right)}_{\Psi = x_2 x_3}$$

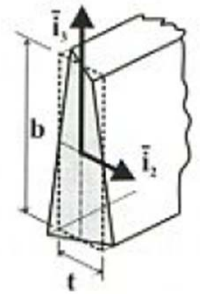


Fig. 7.29. Warping function for a thin rectangular strip.

- Axial displacement

$$u_1\left(x_2, x_3\right) = \Psi\left(x_2, x_3\right)\kappa_1 = \kappa_1 x_2 x_3 \quad (7.60)$$

anti-symmetric with regard to $\bar{i}_2$ and $\bar{i}_3$

7.5 Torsion of thin-walled open section

- Gradient of the stress function will vanish along the local tangent to the section’s thin wall: corresponding shear stress will be linear through the wall thickness

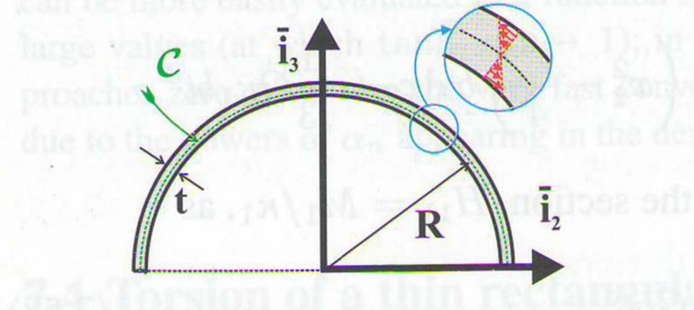


Fig. 7.30. Semi-circular thin-walled open section.

- Torsional stiffness: from Eq.(7.58) → $H_{11} = G\dfrac{lt^3}{3}$ (7.61)

- Shear stress: tangential shear stress, τ_s, only non-vanishing component,
  vary linearly from 0 at the middle to max.(+) and (-) at edges
  $\tau_s^{\max} = Gt\kappa_1$ (7.62)

7.5 Torsion of thin-walled open section

- More general thin-walled open section
  : multiple curved and straight sections (Fig. 7.31)

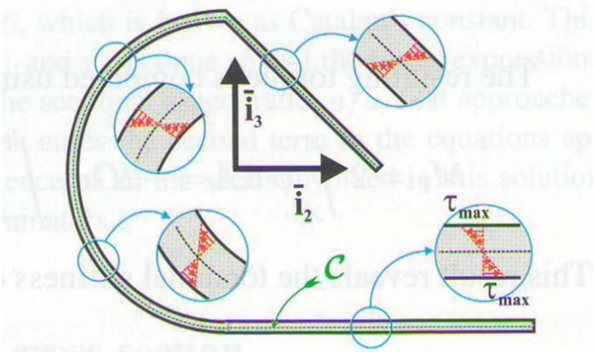


Fig. 7.31. Thin-walled open section composed of several curved.

- Torsional stiffness: sum of those corresponding to the individual segment

$$H_{11} = \sum_i H_{11}^{(i)} = \frac{1}{3} \sum_i G_i l_i t_i^3 \qquad (7.64)$$

- Max. shear stress

$$\tau_s^{max} = G t_{max} \frac{M_1}{H_{11}} \qquad (7.65)$$

- Warping: more complex, described in chap.8

Example 7.3

❖ Torsion of thin-walled section

- C-channel: torsional stiffness, by Eq. (7.64)

$$H_{11} = \frac{G}{3}\left(bt_f^3 + ht_w^3 + bt_f^3\right) = \frac{G}{3}\left(ht_w^3 + 2bt_f^3\right) \qquad (7.66)$$

- Tangential stress at the outer edge: by Eq. (7.62)

$$\tau_w = Gt_w\kappa_1 = Gt_w\frac{M_1}{H_{11}}, \tau_f = Gt_f\kappa_1 = Gt_f\frac{M_1}{H_{11}}$$

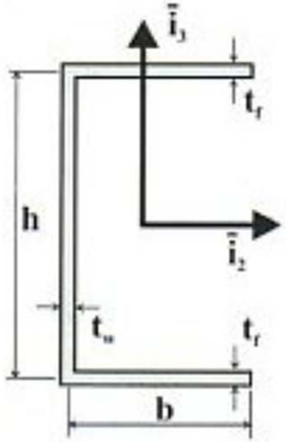


Fig. 7.32. A thin-walled C-channel section

- Max. shear stress exists in the segment featuring the max. thickness

Q & A