Mechanics and Design

Chapter 1

Vectors and Tensors

School of Mechanical and Aerospace Engineering

Seoul National University

- **Vectors, Vector Additions, etc.**

Convention:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Base vectors:

$$\mathbf{i},\quad \mathbf{j},\quad \mathbf{k} \quad \text{or} \quad \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$$

Indicial notation:

$$a_i = \mathbf{a} \cdot \mathbf{e_i}$$

$$\mathbf{a} = a_i\mathbf{e_i} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$$

- **Vectors, Vector Additions, etc.**

Summation Convention:

Repeated index

$$a_i a_i = a_1 a_1 + a_2 a_2 + a_3 a_3$$

$$a_{kk} = a_{11} + a_{22} + a_{33}$$

$$\begin{aligned} a_{ij} a_{ij} &= a_{11} a_{11} + a_{12} a_{12} + a_{13} a_{13} \\ &\quad + a_{21} a_{21} + a_{22} a_{22} + a_{23} a_{23} \\ &\quad + a_{31} a_{31} + a_{32} a_{32} + a_{33} a_{33} \end{aligned}$$

Scalar Product and Vector Product

Scalar Product

The scalar product is defined as the product of the two magnitudes times the cosine of the angle between the vectors

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta \qquad (1\text{-}1)$$

From the definition of eq. (1-1), we immediately have

$$(m\mathbf{a})\cdot(n\mathbf{b}) = mn(\mathbf{a}\cdot\mathbf{b})$$

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$$

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$$

The scalar product of two different unit base vectors defined above is zero, since $\cos(90°)=0$, that is,

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$$

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

Scalar Product and Vector Product

Then the scalar product becomes

$$\mathbf{a}\cdot\mathbf{b} = (a_x\mathbf{i} + a_y\mathbf{j} + a\mathbf{k})\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k})$$
$$= a_x b_x + a_y b_y + a_z b_z$$

or in an indicial notation, we have

$$\mathbf{a}\cdot\mathbf{b} = a_i b_i$$

Vector Product (or cross product)

$$\mathbf{c} = \mathbf{a}\times\mathbf{b}$$

defined as a vector c perpendicular to both **a** and **b** in the sense that makes **a**, **b**, **c** a right-handed system. Magnitude of vector **c** is given by

$$c = ab\sin(\theta)$$

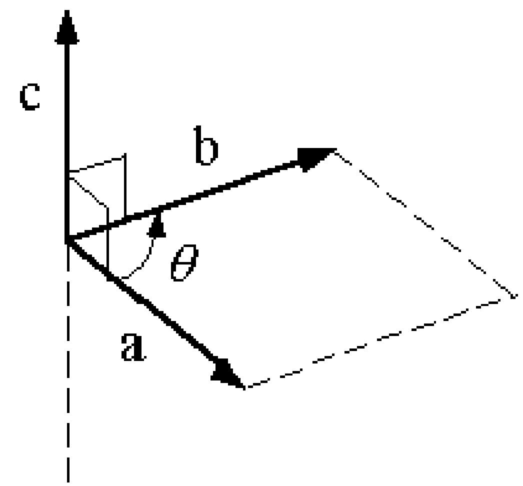
Scalar Product and Vector Product

In terms of components, vector product can be written as

$$\mathbf{a}\times\mathbf{b} = (a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})\times(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$



- Fig. 1.1 Definition of vector product

The vector product is distributive as

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$$

but it is not associative as

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$$

Scalar Product and Vector Product

Scalar Triple Product

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Permutation Symbol e_{mnr}

$$e_{mnr}=\begin{cases} 0 & \text{when any two indices are equal} \\ +1 & \text{when } (m,n,r) \text{ is even purmutation of } (1,2,3) \\ -1 & \text{when } (m,n,r) \text{ is odd purmutation of } (1,2,3) \end{cases}$$

where

even permutations are (1,2,3), (2,3,1), and (3,1,2)
odd permutations are (1,3,2), (2,1,3), and (3,2,1).

Using the permutation symbol, the vector product can be represented by

$$\mathbf{a}\times\mathbf{b}=e_{pqr}a_q b_r \mathbf{i}_p$$

Scalar Product and Vector Product

Kronecker Delta δ_{ij}

$$\delta_{pq} = \begin{cases} 1 \text{ if } p = q \\ 0 \text{ if } p \neq q \end{cases}$$

Some examples are given below:

$$\delta_{ii} = 3$$

$$\delta_{ij}\delta_{ij} = \delta_{ii} = 3$$

$$u_i\delta_{ij} = u_i = u_j$$

$$T_{ij}\delta_{ij} = T_{ii}$$

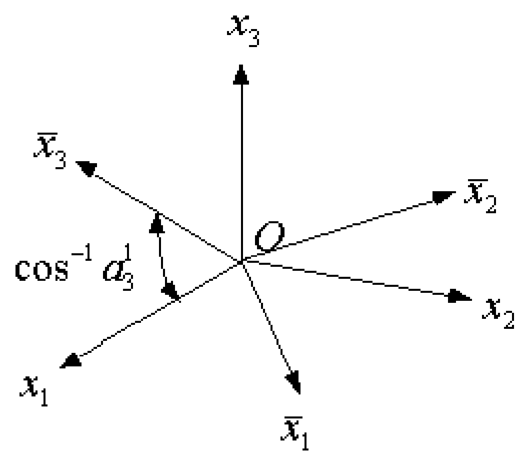
Rotation of Axes, etc

Change of Orthogonal Basis

- Vector is independent of a coordinate system.
- However, the components of a vector change when the coordinate system changes.

Let x_i and $\bar{x}_i$ be the two coordinate systems, as shown in Fig. 1-2, which have the same origins.
Also, let the orientation of the two coordinate systems is given by the direction cosines as

	$\bar{\mathbf{i}}_1$	$\bar{\mathbf{i}}_2$	$\bar{\mathbf{i}}_3$
$\mathbf{i}_1$	a_1^1	a_2^1	a_3^1
$\mathbf{i}_2$	a_1^2	a_2^2	a_3^2
$\mathbf{i}_3$	a_1^3	a_2^3	a_3^3



- Fig. 1.2 Coordinate transformation

Rotation of Axes, etc

Under this system,

$$\bar{\mathbf{i}}_{\mathrm{r}} = a_r^s \mathbf{i}_{\mathrm{s}} \quad \text{and} \quad \mathbf{i}_{\mathrm{s}} = a_r^s \bar{\mathbf{i}}_{\mathrm{r}}$$

Since the unit vectors are orthogonal, we have

$$\bar{\mathbf{i}}_{\mathrm{p}} \cdot \bar{\mathbf{i}}_{\mathrm{q}} = \delta_{pq} \quad \text{and} \quad \mathbf{i}_{\mathrm{p}} \cdot \mathbf{i}_{\mathrm{q}} = \delta_{pq}$$

From these

$$\bar{\mathbf{i}}_{\mathrm{p}} \cdot \bar{\mathbf{i}}_{\mathrm{q}} = a_p^s a_q^r \mathbf{i}_{\mathrm{s}} \cdot \mathbf{i}_{\mathrm{r}} = a_p^s a_q^r \delta_{rs} = a_p^s a_q^s = \delta_{pq}$$

$$\text{i.e.,} \quad a_p^s a_q^s = \delta_{pq}$$

Similarly, we get

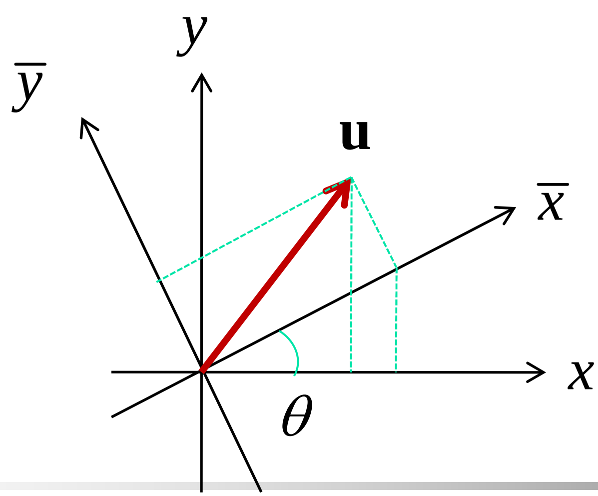
$$a_r^m a_r^n = \delta_{mn}$$

2D coodinate transformation

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix} \quad \text{or} \quad \mathbf{u} = A\bar{\mathbf{u}}$$

The inverse of the above transformation equation becomes

$$\begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad \text{or} \quad \bar{\mathbf{u}} = A^{-1}\mathbf{u} = A^T\mathbf{u}$$



Coordinate Transformation of Vector

Let, $$A = \begin{bmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{bmatrix}$$ Then, $$\mathbf{v} = A\overline{\mathbf{v}} \quad \text{or} \quad \overline{\mathbf{v}} = A^T\mathbf{v}$$

$$A = \begin{bmatrix} \overline{\mathbf{a}}_1 & \overline{\mathbf{a}}_2 & \overline{\mathbf{a}}_3 \end{bmatrix}$$

Note that

$$A^T A = AA^T = 1$$

Second Order Tensors

The second order tensor may be expressed by tensor product or open product of two vectors. Let

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \qquad \text{and} \qquad \mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$$

$$\mathrm{T} = \mathbf{ab} = \mathbf{a} \otimes \mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \otimes (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)$$

$$= a_1 b_1 \ \mathbf{e}_1 \otimes \mathbf{e}_1 + a_1 b_2 \ \mathbf{e}_1 \otimes \mathbf{e}_2 + a_1 b_3 \ \mathbf{e}_1 \otimes \mathbf{e}_3$$

$$+ a_2 b_1 \ \mathbf{e}_2 \otimes \mathbf{e}_1 + a_2 b_2 \ \mathbf{e}_2 \otimes \mathbf{e}_2 + a_2 b_3 \ \mathbf{e}_2 \otimes \mathbf{e}_3$$

$$+ a_3 b_1 \ \mathbf{e}_3 \otimes \mathbf{e}_1 + a_3 b_2 \ \mathbf{e}_3 \otimes \mathbf{e}_2 + a_3 b_3 \ \mathbf{e}_3 \otimes \mathbf{e}_3$$

Or

$$\mathrm{T} = \quad T_{11} \ \mathbf{e}_1 \otimes \mathbf{e}_1 + T_{12} \ \mathbf{e}_1 \otimes \mathbf{e}_2 + T_{13} \mathbf{e}_1 \otimes \mathbf{e}_3 +$$

$$+ T_{21} \ \mathbf{e}_2 \otimes \mathbf{e}_1 + T_{22} \ \mathbf{e}_2 \otimes \mathbf{e}_2 + T_{23} \ \mathbf{e}_2 \otimes \mathbf{e}_3$$

$$+ T_{31} \ \mathbf{e}_3 \otimes \mathbf{e}_1 + T_{32} \ \mathbf{e}_3 \otimes \mathbf{e}_2 + T_{33} \ \mathbf{e}_3 \otimes \mathbf{e}_3$$

Here, we may consider $\mathbf{e}_i \otimes \mathbf{e}_j \neq \mathbf{e}_j \otimes \mathbf{e}_i$ as a base of the second order tensor, and T_{ij} is a component of the second order tensor T .

It can be seen that the second order tensor map a vector to another vector, that is,

$$\begin{aligned}\mathbf{u} = T\cdot\mathbf{v} = \ &(T_{11}\ \mathbf{e}_1\otimes\mathbf{e}_1 + T_{12}\ \mathbf{e}_1\otimes\mathbf{e}_2 + T_{13}\mathbf{e}_1\otimes\mathbf{e}_3 + \\ &+T_{21}\ \mathbf{e}_2\otimes\mathbf{e}_1 + T_{22}\ \mathbf{e}_2\otimes\mathbf{e}_2 + T_{23}\ \mathbf{e}_2\otimes\mathbf{e}_3 \\ &+T_{31}\ \mathbf{e}_3\otimes\mathbf{e}_1 + T_{32}\ \mathbf{e}_3\otimes\mathbf{e}_2 + T_{33}\ \mathbf{e}_3\otimes\mathbf{e}_3)\cdot(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\ =\ &(T_{11}v_1 + T_{12}v_2 + T_{13}v_3)\mathbf{e}_1 + (T_{21}v_1 + T_{22}v_2 + T_{23}v_3)\mathbf{e}_2 \\ &+(T_{31}v_1 + T_{32}v_2 + T_{33}v_3)\mathbf{e}_3 \\ =\ &T_{ij}v_j\mathbf{e}_i\end{aligned}$$

Symmetric Tensors and Skew tensors

Symmetric tensor ➡ $T_{ij} = T_{ji}$

Skew or Antisymmetric tensor ➡ $T_{ij} = -T_{ji}$

Rotation of axes, change of tensor components

Let $u_i = T_{ip} v_p$ and $\bar{u}_i = \bar{T}_{ip} \bar{v}_p$

Where u_i and $\bar{u}_i$ are the same vector but decomposed into two different coordinate systems, x_i and $\bar{x}_i$. The same applies to v_i and $\bar{v}_i$.

Then by the transformation of vector, we get

$$\bar{u}_i = a_i^j u_j = a_i^j T_{jq} v_q$$
$$= a_i^j T_{jq} a_p^q \bar{v}_p$$

Therefore,

$$\bar{T}_{ip} = a_i^j a_p^q T_{jq}$$

In matrix form, we have,

$$\bar{T} = A^T T A \quad \text{or,} \quad T = A \bar{T} A^T$$

Rotation of Axes, etc

Scalar product of two tensors

$$\mathrm{T}:\mathrm{U} = T_{ij}U_{ij}$$
$$\mathrm{T}\cdot\cdot\mathrm{U} = T_{ij}U_{ji}$$

If we list all the terms of tensor product, we get

$$\begin{aligned}\mathrm{T}:\mathrm{U} &= T_{ij}U_{ij}\\ &= T_{11}U_{11} + T_{12}U_{12} + T_{13}U_{13}\\ &\quad + T_{21}U_{21} + T_{22}U_{22} + T_{23}U_{23}\\ &\quad + T_{31}U_{31} + T_{32}U_{32} + T_{33}U_{33}\end{aligned}$$

The product of two second-order tensors

$$\mathrm{T}\cdot\mathrm{U}$$
$$(\mathrm{T}\cdot\mathrm{U})\cdot\mathrm{v} = \mathrm{T}\cdot(\mathrm{U}\cdot\mathrm{v})$$

If $\mathrm{P} = \mathrm{T}\cdot\mathrm{U}$, then $P_{ij} = T_{ik}U_{kj}$

■ Rotation of Axes, etc

The Trace

Definition : $tr(\mathrm{T}) = T_{kk}$

Note that

(1) $\mathrm{A}\cdot\cdot\mathrm{B} = tr(\mathrm{A}\cdot\mathrm{B})$

(2) $\mathrm{A}:\mathrm{B} = tr(\mathrm{A}\cdot\mathrm{B}^{T}) = tr(\mathrm{A}^{\mathrm{T}}\cdot\mathrm{B})$

(3) $tr(\mathrm{A}\cdot\mathrm{B}) = tr(\mathrm{B}\cdot\mathrm{A})$

(4) $tr(\mathrm{A}\cdot\mathrm{B}\cdot\mathrm{C}) = tr(\mathrm{B}\cdot\mathrm{C}\cdot\mathrm{A}) = tr(\mathrm{C}\cdot\mathrm{A}\cdot\mathrm{B})$ Cyclic property of trace

Proof >

(1) $\mathrm{A}\cdot\cdot\mathrm{B} = tr(\mathrm{A}\cdot\mathrm{B})$

Let, $\mathrm{P} = \mathrm{A}\cdot\mathrm{B}$, i.e., $P_{ij} = A_{ik}B_{kj}$

Then, $tr(\mathrm{P}) = P_{ii} = A_{ik}B_{ki} = \mathrm{A}\cdot\cdot\mathrm{B}$

(2) $\mathrm{A}:\mathrm{B} = tr(\mathrm{A}\cdot\mathrm{B}^{T}) = tr(\mathrm{A}^{\mathrm{T}}\cdot\mathrm{B})$

Let, $\mathrm{P} = \mathrm{A}\cdot\mathrm{B}^{\mathrm{T}}$, i.e., $P_{ij} = A_{ik}B_{jk}$

Then, $tr(\mathrm{A}\cdot\mathrm{B}^{T}) = P_{ii} = A_{ik}B_{ik} = \mathrm{A}:\mathrm{B}$

Review of Elementary Matrix Concepts

Eigenvectors and eigenvalues of a matrix

Linear transformation, $y = Mx$, associates, to each point $P(x_1, x_2, x_3)$, another point $Q(y_1, y_2, y_3)$. Also, it associates to any other point (rx_1, rx_2, rx_3) on the line OP , another point (ry_1, ry_2, ry_3) on the line OQ.
So we may consider the transformation to be a transformation of the line OP into the line OQ.

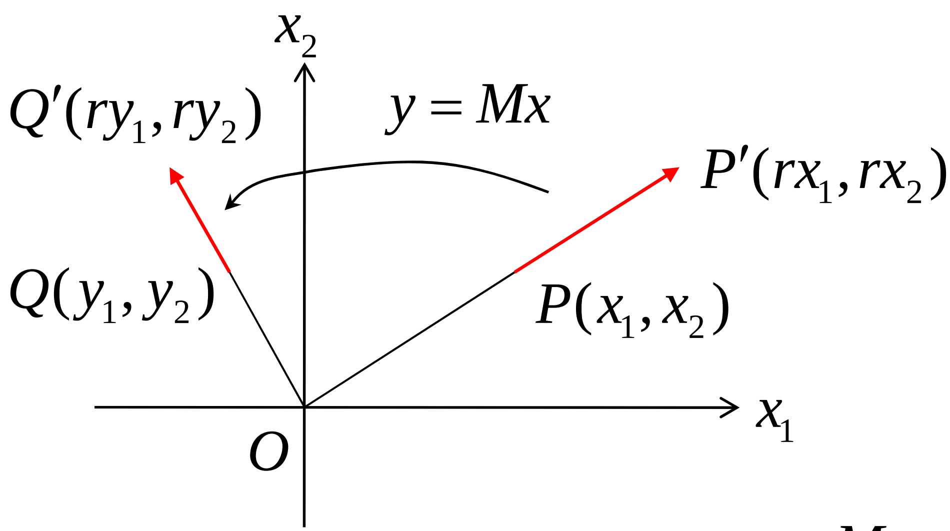
Now any line transformed into itself is called an eigenvector of the matrix M.
That is,

$$\mathbf{Mx} = \lambda \mathbf{x} = \lambda \mathbf{I}\, \mathbf{x}$$

A nontrivial solution will exist if and only if the determinant vanishes:

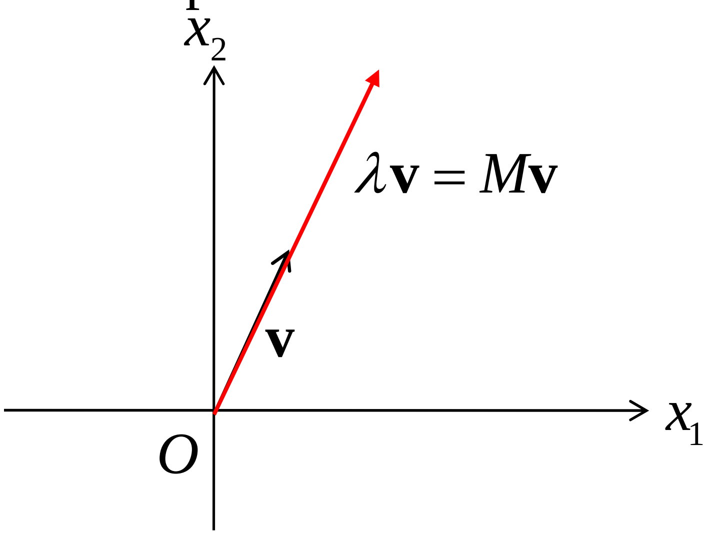
$$\left| (\mathbf{M} - \lambda \mathbf{I}) \right| = 0$$

Note that 3x3 determinant is expanded, it will be a cubic polynominal equation with real coefficients. The roots of this equation are called the eigenvalues of the matrix. Upon solving the equation, some of the roots could be complex numbers.



Linear Transformation

M maps $\mathbf{v}$ into the same vector $\mathbf{v}$.



Review of Elementary Matrix Concepts

A real symmetric matrix has only real eigenvalues.

If there were a complex root λ , then its complex conjugate $\bar{\lambda}$ is also a root. Therefore,

$$\mathbf{M}\mathbf{x} = \lambda \mathbf{x}$$

$$\bar{\mathbf{M}}\mathbf{x} = \bar{\lambda}\mathbf{x}$$

These equations can be written as

$$\bar{\mathbf{x}}^T \mathbf{M}\mathbf{x} = \lambda \bar{\mathbf{x}}^T \mathbf{x}$$

$$\mathbf{x}^T \mathbf{M}\bar{\mathbf{x}} = \bar{\lambda}\mathbf{x}^T \bar{\mathbf{x}}$$

Note that $\bar{\mathbf{x}}^T \mathbf{x} = x_k \bar{x}_k = \mathbf{x}^T \bar{\mathbf{x}}$ and since M is symmetric, we get

$$\bar{\mathbf{x}}^T \mathbf{M}\mathbf{x} = M_{ij}\bar{x}_i x_j$$

$$= M_{ji}\bar{x}_j x_i \quad \text{(by interchanging the dummy indices)}$$

$$= M_{ij} x_i \bar{x}_j = \mathbf{x}^T \mathbf{M}\bar{\mathbf{x}} \quad \text{(by symmetry of M)}$$

Review of Elementary Matrix Concepts

Subtracting, we get

$$(\lambda - \bar{\lambda})\bar{\mathbf{x}}^T \mathbf{x} = 0$$

Since x is nontrivial, $\bar{\mathbf{x}}^T \mathbf{x} \neq 0$. Therefore, we should have

$$\lambda = \bar{\lambda}$$

So that λ must be real.

We can obtain the eigenvector associated to each eigenvalue by substituting each eigenvalue into the matrix equation.

When eigenvalues are all distinct : ??
Two of the eigenvalues are equal : ??
All of the eigenvalues are equal : ??