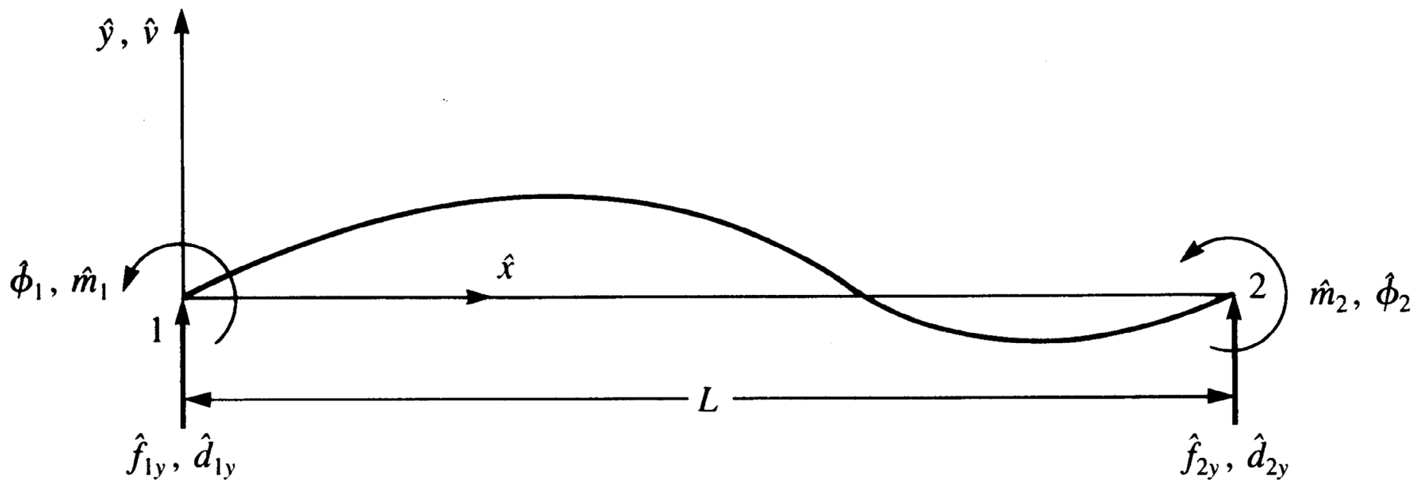
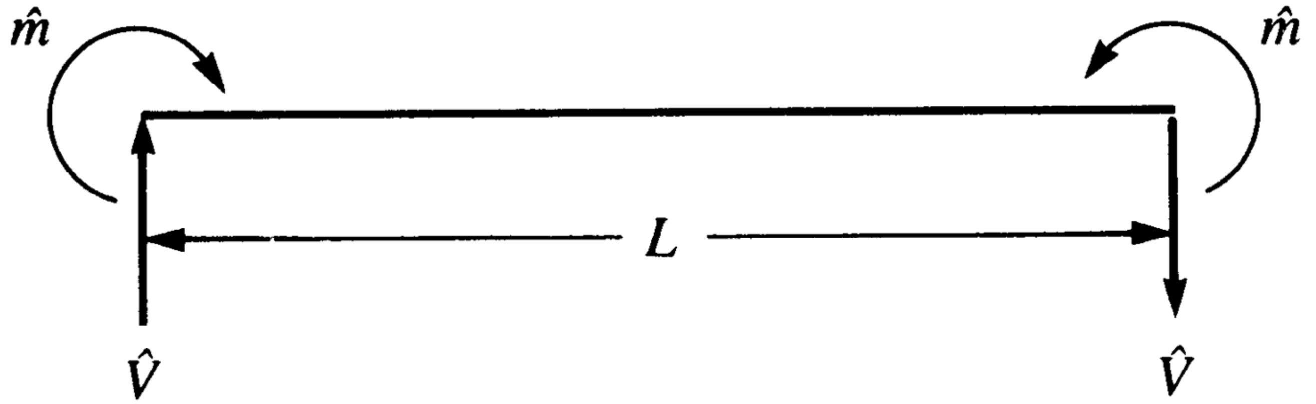
Beam Element



♦ Beam: A long thin structure that is subject to the vertical loads. A beam shows more evident bending deformation than the torsion and/or axial deformation.

♦ Bending strain is measured by the lateral deflection and the rotation -> Lateral deflection and rotation determine the number of DoF (Degree of Freedom).

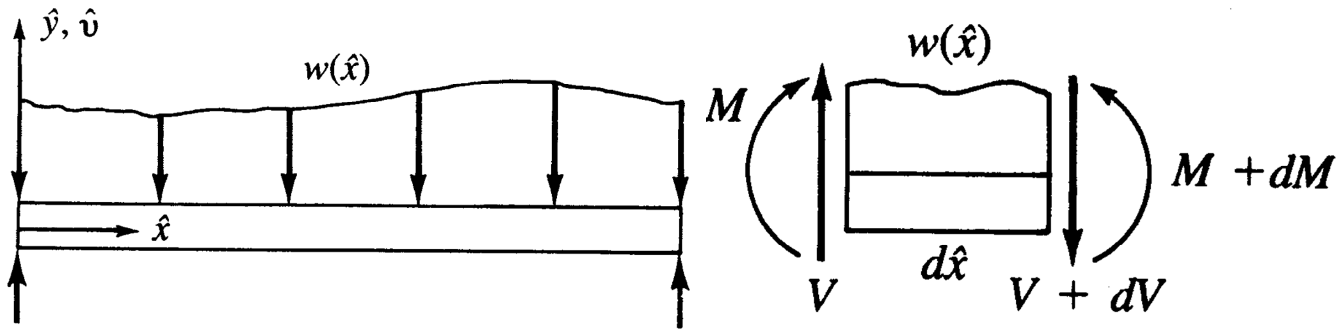


- Sign convention

  1. Positive bending moment: Anti-clock wise rotation.

  2. Positive load: $\hat{y}$-direction.

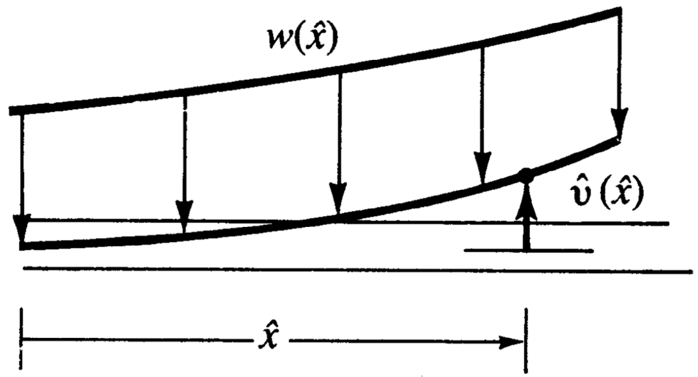
  3. Positive displacement: $\hat{y}$-direction.

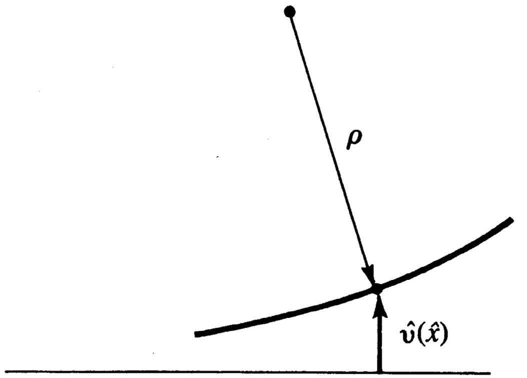


- The governing equation

$$-wd\hat{x} - dV = 0 \quad \text{or} \quad w = -\frac{dV}{d\hat{x}}$$

$$Vd\hat{x} - dM = 0 \quad \text{or} \quad V = \frac{dM}{d\hat{x}}$$





♦ Relation between beam curvature (k) and bending moment

$$k = \frac{1}{\rho} = \frac{M}{EI} \qquad \text{or} \qquad \frac{d^2\hat{v}}{d\hat{x}^2} = \frac{M}{EI}$$

♦ Curvature for small slope ($\theta = d\hat{v}/d\hat{x}$): $k = \dfrac{d^2\hat{v}}{d\hat{x}^2}$

ρ: Radius of the deflection curve, $\hat{v}$: Lateral displacement function along the $\hat{y}$-axis direction

E: Stiffness coefficient, I: Moment of inertia along the $\hat{z}$-axis

Solving the equation with M,

$$\frac{d^2}{d\hat{x}^2}\left(EI\frac{d^2\hat{v}}{d\hat{x}^2}\right) = -w(\hat{x})$$

When EI is constant, and force and moment are only applied at nodes,

$$EI\frac{d^4\hat{v}}{d\hat{x}^4} = 0$$

Step 1: To select beam element type

Step 2: To select displacement function

Assumption of lateral displacement

$$\hat{v}(\hat{x}) = a_1\hat{x}^3 + a_2\hat{x}^2 + a_3\hat{x} + a_4$$

- Complete 3-order displacement function is suitable, because it has four degree of freedom (one lateral displacement and one small rotation at each node)

- The function is proper, because it satisfies the fundamental differential equation of a beam.

- The function satisfies continuity of both displacement and slope at each node.

Representing $\hat{v}$ with functions of $\hat{d}_{1y}$, $\hat{d}_{2y}$, $\hat{\phi}_1$, $\hat{\phi}_2$

$$\hat{v}(0) = \hat{d}_{1y} = a_4$$

$$\frac{d\hat{v}(0)}{d\hat{v}} = \hat{\phi}_1 = a_3$$

$$\hat{v}(L) = \hat{d}_{2y} = a_1 L^3 + a_2 L^2 + a_3 L + a_4$$

$$\frac{d\hat{v}(L)}{d\hat{x}} = \hat{\phi}_2 = 3a_1L^2 + 2a_2L + a_3$$

Replacing a_1~a_4 with $\hat{d}_{1y}$, $\hat{d}_{2y}$, $\hat{\phi}_1$, $\hat{\phi}_2$

$$\hat{v} = \left[\frac{2}{L^3}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) + \frac{1}{L^2}\left(\hat{\phi}_1 + \hat{\phi}_2\right)\right]\hat{x}^3$$
$$+\left[-\frac{3}{L^2}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) - \frac{1}{L}\left(2\hat{\phi}_1 + \hat{\phi}_2\right)\right]\hat{x}^2 + \hat{\phi}_1\,\hat{x} + \hat{d}_{1y}$$

Representing it in matrix form, $\hat{v} = [N]\{\hat{d}\}$

$$\{\hat{d}\} = \begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} \qquad [N] = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}$$

$$N_1 = \frac{1}{L^3}\left(2\hat{x}^3 - 3\hat{x}^2 L + L^3\right) \qquad N_2 = \frac{1}{L^3}\left(\hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x}\,L^3\right)$$
$$N_3 = \frac{1}{L^3}\left(-2\hat{x}^3 + 3\hat{x}^2 L\right) \qquad N_4 = \frac{1}{L^3}\left(\hat{x}^3 L - \hat{x}^2 L^2\right)$$

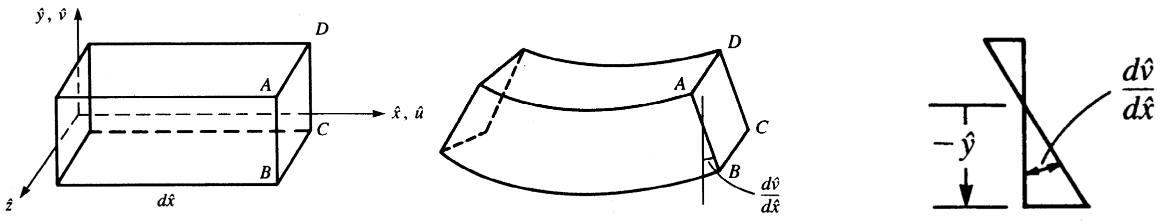
where N_1, N_2, N_3, N_4: shape functions of the beam element

Step 3: To define strain-displacement relation and stress-strain relation

Assume that the equation of strain-displacement relation is valid

$$\varepsilon_x(\hat{x},\hat{y}) = \frac{d\hat{u}}{d\hat{x}} \quad , \qquad \hat{u} = -\hat{y}\frac{d\hat{v}}{d\hat{x}} \quad \Rightarrow \quad \varepsilon_x(\hat{x},\hat{y}) = -\hat{y}\frac{d^2\hat{v}}{d\hat{x}^2}$$

Basic assumption: Cross-section of the beam sustains its shape after deformation by bending, and generally rotates by degree of $(d\hat{v}/d\hat{x})$.



Bending moment-lateral displacement relation and shear force-lateral displacement relation

$$\hat{m}(\hat{x}) = EI\frac{d^2\hat{v}}{d\hat{x}^2} \qquad \hat{V} = EI\frac{d^3\hat{v}}{d\hat{x}^3}$$

Step 4: To derive an element stiffness matrix and governing equations by direct stiffness method

♦ Element stiffness matrix and governing equations

$$\hat{f}_{1y} = \hat{V} = EI\frac{d^3\hat{v}(0)}{d\hat{x}^3} = \frac{EI}{L^3}\left(12\hat{d}_{1y} + 6L\hat{\phi}_1 - 12\hat{d}_{2y} + 6L\hat{\phi}_2\right)$$

$$\hat{m}_1 = -\hat{m} = -EI\frac{d^2\hat{v}(0)}{d\hat{x}^2} = \frac{EI}{L^3}\left(6L\hat{d}_{1y} + 4L^2\hat{\phi}_1 - 6\hat{d}_{2y} + 2L^2\hat{\phi}_2\right)$$

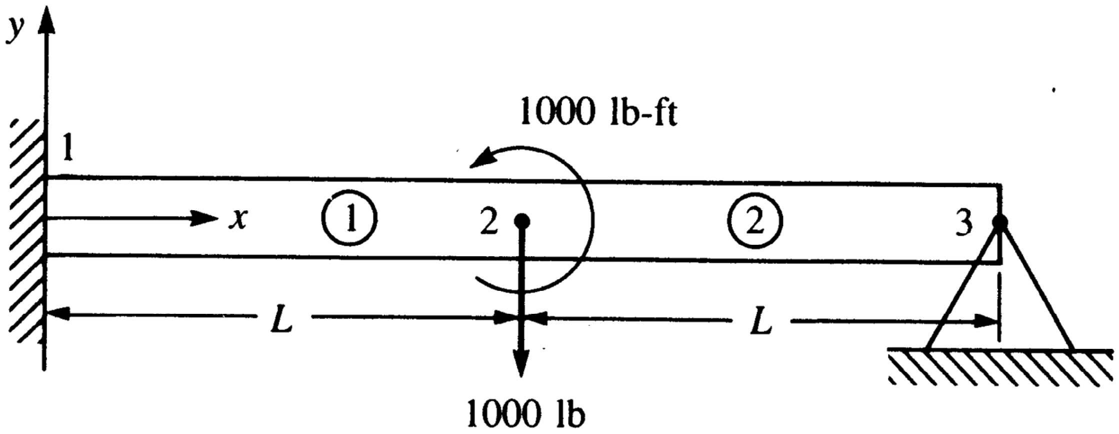
$$\hat{f}_{2y} = -\hat{V} = -EI\frac{d^3\hat{v}(L)}{d\hat{x}^3} = \frac{EI}{L^3}\left(-12\hat{d}_{1y} - 6L\hat{\phi}_1 + 12\hat{d}_{2y} - 6L\hat{\phi}_2\right)$$

$$\hat{m}_2 = \hat{m} = EI\frac{d^2\hat{v}(L)}{d\hat{x}^2} = \frac{EI}{L^3}\left(6L\hat{d}_{1y} + 2L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 4L^2\hat{\phi}_2\right)$$

\- Matrix Form

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} \qquad \underline{\hat{k}} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Step 5: To constitute a global stiffness matrix using boundary conditions



Assemble example

Assume EI of the beam element is constant.

$1000\ lb$ load and $1000\ lb-ft$ moment are applied at the center of the beam.

Assume load and moment were only applied at nodes.

Left end of the beam is fixed and right end is pin-connected.

The beam is divided into two elements (nodes 1, 2, and 3 as shown above figure).

$$\underline{k}^{(1)} = \frac{EI}{L^3}\begin{array}{cccc} d_{1y} & \phi_1 & d_{2y} & \phi_2 \\ \end{array}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\underline{k}^{(2)} = \frac{EI}{L^3}\begin{array}{cccc} d_{2y} & \phi_2 & d_{3y} & \phi_3 \\ \end{array}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

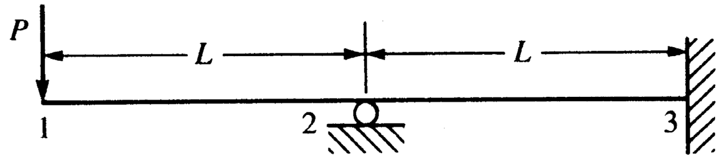
$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \times \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

Boundary conditions and constrains at the node 1(fixed) and node 3(pin-connected) are

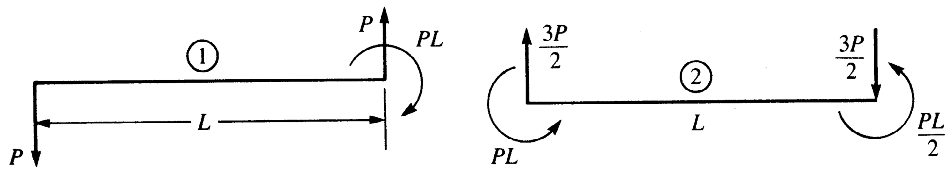
$$\phi_1 = 0 \qquad d_{1y} = 0 \qquad d_{3y} = 0$$

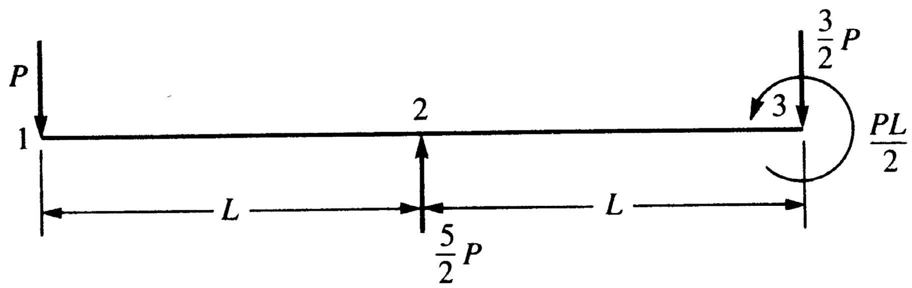
$$\begin{Bmatrix} -1000 \\ 1000 \\ 0 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ \phi_3 \end{Bmatrix} \tag{5.2.5}$$

Example: Beam analysis using direct stiffness method



Cantilever beam supported by roller at the center





Load P is applied at node 1.

Length: $2L$,Stiffness: EI

Constrains: (1) Roller at node 2, (2) Fixed at node 3.

- Global stiffness matrix

$$\underline{K} = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} d_{1y} & \phi_1 & d_{2y} & \phi_2 & d_{3y} & \phi_3 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 \\ & & 12+12 & -6L+6L & -12 & 6L \\ & & & 4L^2+4L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ & \mathit{Symmetry} & & & & 4L^2 \end{bmatrix} \end{array}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

Boundary conditions $d_{2y} = 0$ $d_{3y} = 0$ $\phi_3 = 0$

$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ \phi_2 \end{Bmatrix}$$

$$d_{1y} = -\frac{7PL^3}{12EI} \qquad \phi_1 = \frac{3PL^2}{4EI} \qquad \phi_2 = \frac{PL^2}{4EI}$$

◆ Substituting the obtained values to the final equation,

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -\dfrac{7PL^3}{12EI} \\ \dfrac{3PL^2}{4EI} \\ 0 \\ \dfrac{PL^2}{4EI} \\ 0 \\ 0 \end{Bmatrix}$$

$$F_{1y} = -P \qquad M_1 = 0 \qquad F_{2y} = \frac{5}{2}P$$

$$M_2 = 0 \qquad F_{3y} = -\frac{3}{2}P \qquad M_3 = \frac{1}{2}PL$$

$F_{1y} = -P$: Force at node 1

F_{2y} , F_{3y} , M_3 : Reacting forces and moment at nodes

M_1 , M_2 : Zero

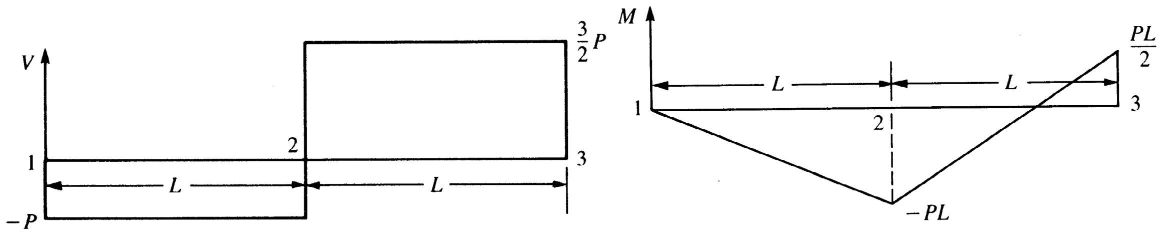
♦ Calculating local nodal loads

Force at the element 1.

When $\underline{\hat{f}} = \underline{\hat{k}}\underline{\hat{d}}$

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -\dfrac{7PL^3}{12EI} \\ \dfrac{3PL^2}{4EI} \\ 0 \\ \dfrac{PL^2}{4EI} \end{Bmatrix}$$

$$\hat{f}_{1y} = -P \qquad \hat{m}_1 = 0 \qquad \hat{f}_{2y} = P \qquad \hat{m}_2 = -PL$$



Shear moment curve

Homework: Distributed load

♦ Equivalent force