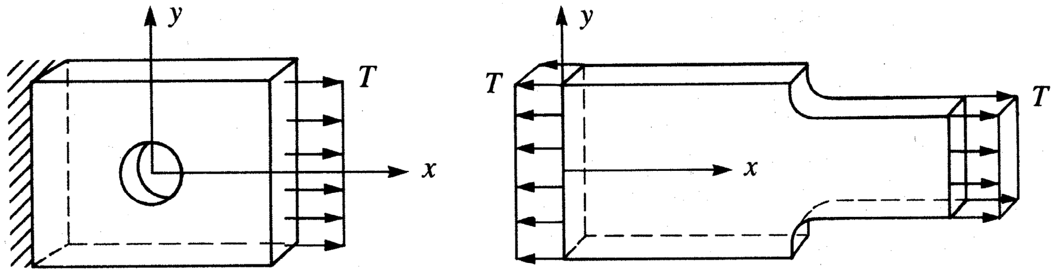
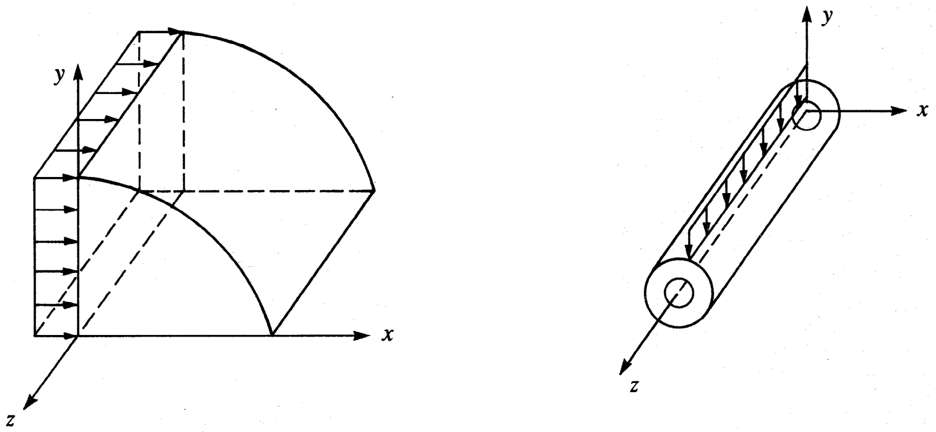
Plane stress and Plane strain

- Finite element in 2-D: Thin plate element required 2 coordinates
- Plane stress and plane strain problems
- Constant-strain triangular element
- Equilibrium equation in 2-D

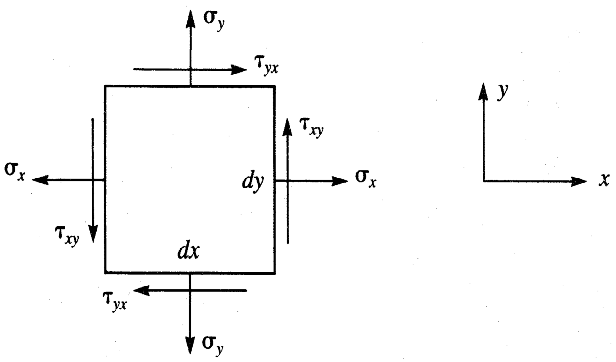
plane stress: The stress state when normal stress, which is perpendicular to the plane x-y, and shear stress are both zero.



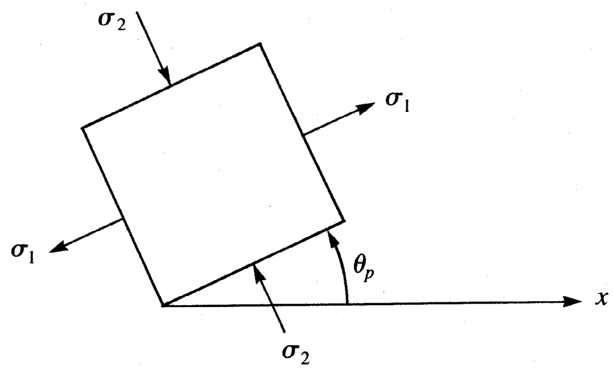
plane strain: The strain state when normal strain ε_z, which is perpendicular to the plane $x-y$, and shear strain γ_{xz}, γ_{yz} are both zero.



Stress and strain in 2-D



Stresses in 2-D



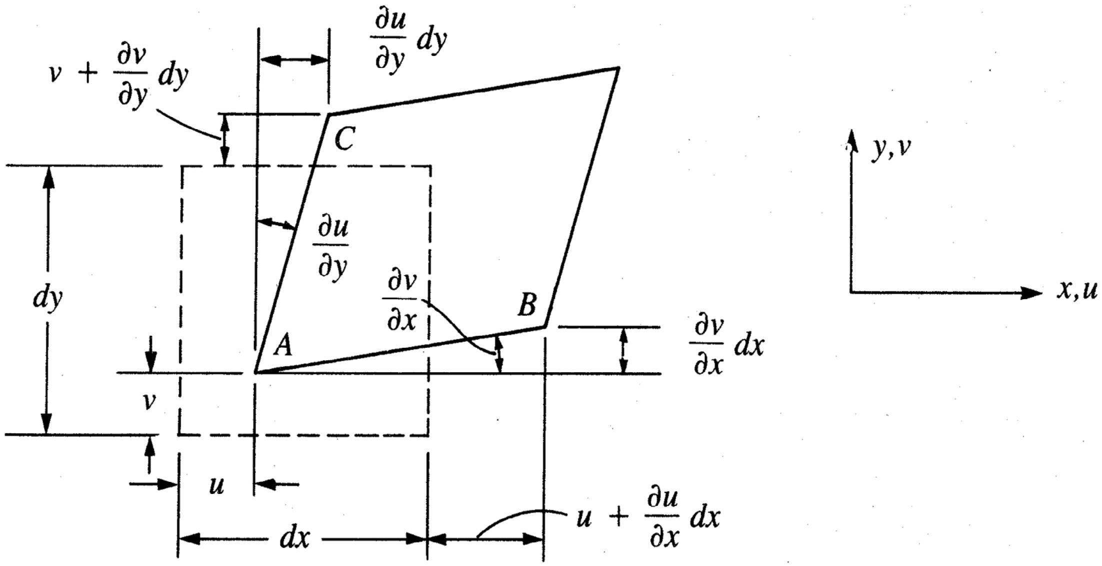
Principal stress and its direction

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\max}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\min}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



Displacement and rotation of plane element $x-y$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial \upsilon}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\varepsilon\}$$

Stress-strain matrix(or material composed matrix) of isotropic material for plane stress($\sigma_z = \tau_{xz} = \tau_{yz} = 0$)

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Stress-strain matrix(or material composed matrix) of isotropic material for plane deformation ($\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$)

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

General steps of formulation process for plane triangular element

Step 1: Determination of element type

Step 2: Determination of displacement function

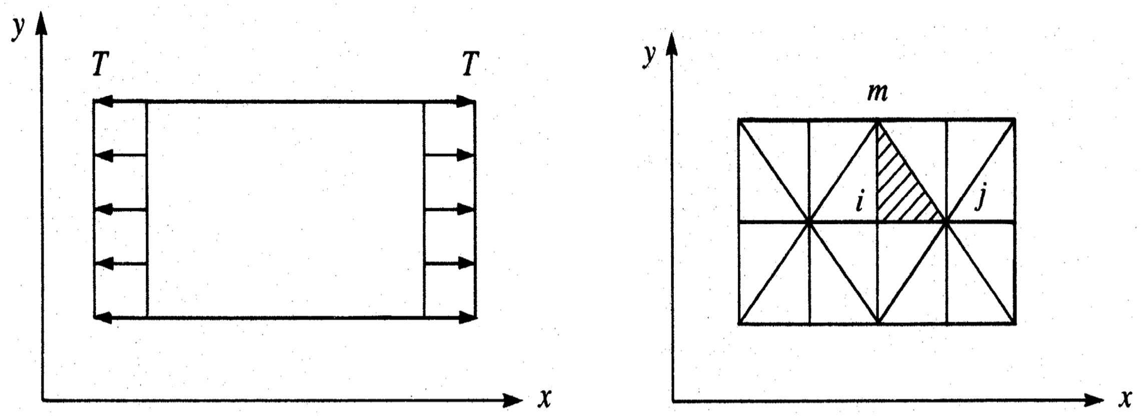
Step 3: Definition of relations deformation rate - strain and stress-strain

Step 4: Derivation of element stiffness matrix and equation

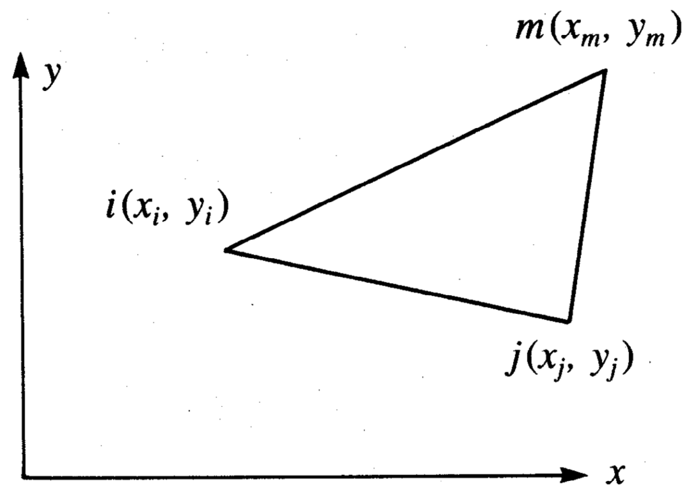
Step 5: Introduction a combination of element equation and boundary conditions for obtaining entire equations

Step 6: Calculation of nodal displacement

Step 7: Calculation of force(stress) in an element



Step 1: Determination of element type



$$\{d\} = \begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix} = \begin{Bmatrix} u_i \\ \upsilon_i \\ u_j \\ \upsilon_j \\ u_m \\ \upsilon_m \end{Bmatrix}$$

Considering a triangular element, the nodes i, j, m are notated in the anti clock wise direction.

The way to name the nodal numbers in an entire structure must be devised to avoid the element area comes to be negative.

Step 2: Determination of displacement function

$$u(x,y) = a_1 + a_2 x + a_3 y$$
$$\upsilon(x,y) = a_4 + a_5 x + a_6 y$$

Linear function gives a guarantee to satisfy the compatibility.

A general displacement function $\{\psi\}$ containing function u and υ can be expressed as below.

$$\{\psi\} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

Substitute nodal coordinates to the equation for obtaining the values of a.

Calculation of a_1, a_2, a_3 :

$$u_i = a_1 + a_2 x_i + a_3 y_i$$
$$u_j = a_1 + a_2 x_j + a_3 y_j$$
$$u_m = a_1 + a_2 x_m + a_3 y_m$$

or

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Solving a , $\{a\} = [x]^{-1}\{u\}$

Obtaining the inverse matrix of $[x]$,

$$[x]^{-1} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}$$

where $2A = x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)$: 2 times of triangle area

$$\alpha_i = x_j y_m - y_j x_m \qquad \alpha_j = y_i x_m - x_i y_m \qquad \alpha_m = x_i y_j - y_i x_j$$
$$\beta_i = y_j - y_m \qquad \beta_j = y_m - y_i \qquad \beta_m = y_i - y_j$$
$$\gamma_i = x_m - x_j \qquad \gamma_j = x_i - x_m \qquad \gamma_m = x_j - x_i$$

After calculation of $[x]^{-1}$, the equation $\{a\}=[x]^{-1}\{u\}$ can be expressed as an extended matrix form.

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

Similarly,

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} \upsilon_i \\ \upsilon_j \\ \upsilon_m \end{Bmatrix}$$

Derivation of displacement function $u(x,y)$ (υ can also be derived similarly)

$$\{u\} = [1 \quad x \quad y]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}[1 \quad x \quad y]\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

Arranging by depolyment:

$$u(x,y) = \frac{1}{24}\{(\alpha_i + \beta_i x + \gamma_i y)u_i + (\alpha_j + \beta_j x + \gamma_j y)u_j + (\alpha_m + \beta_m x + \gamma_m y)u_m\}$$

As the same way

$$\upsilon(x,y) = \frac{1}{24}\{(\alpha_i + \beta_i x + \gamma_i y)\upsilon_i + (\alpha_j + \beta_j x + \gamma_j y)\upsilon_j + (\alpha_m + \beta_m x + \gamma_m y)\upsilon_m\}$$

Simple expression of u and υ :

$$u(x,y) = N_i u_i + N_j u_j + N_m u_m$$
$$\upsilon(x,y) = N_i \upsilon_i + N_j \upsilon_j + N_m \upsilon_m$$

where

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$
$$N_j = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y)$$
$$N_m = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$

$$\{\psi\} = \begin{Bmatrix} u(x,y) \\ \upsilon(x,y) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i \upsilon_i + N_j \upsilon_j + N_m \upsilon_m \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ \upsilon_i \\ u_j \\ \upsilon_j \\ u_m \\ \upsilon_m \end{Bmatrix}$$

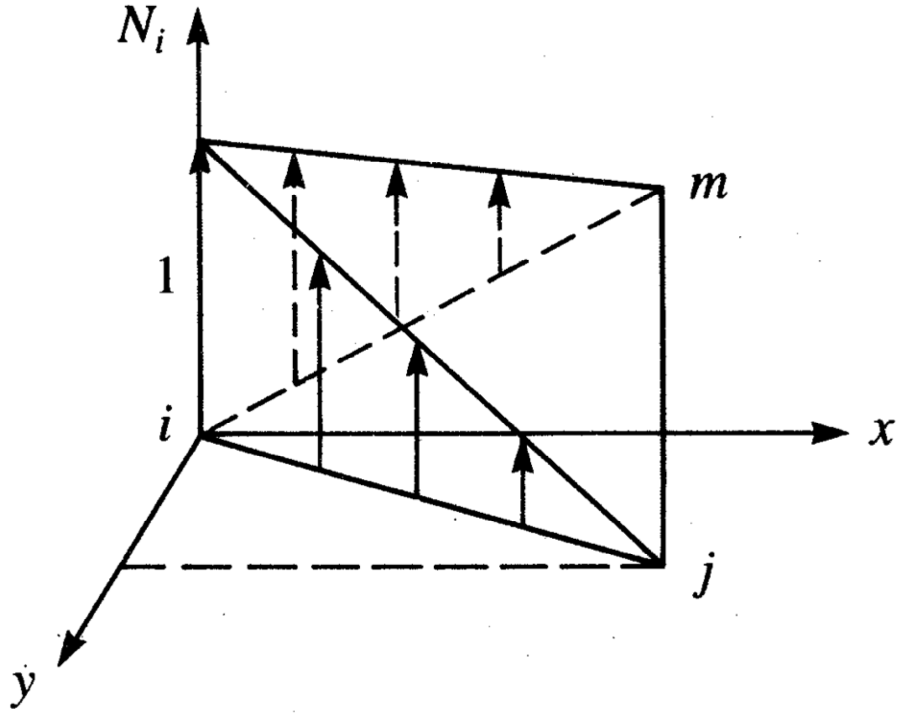
Making the equation be simple in a form of matrix, $\{\psi\} = [N]\{d\}$

where $[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$

The displacement function $\{\psi\}$ is represented with shape functions N_i, N_j , N_m and nodal displacement $\{d\}$.

Review of characteristics of shape function:

$N_i = 1$, $N_j = 0$, and $N_m = 0$ at nodes (x_i, y_i)



A change of N_i of general elements across the surface $x - y$

Step 3: Definition of relations deformation rate - strain and stress-strain

Deformation rate:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial \upsilon}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial \upsilon}{\partial x} \end{Bmatrix}$$

Calculation of partial differential terms

$$\frac{\partial u}{\partial x} = u_{,x} = \frac{\partial}{\partial x}(N_i u_i + N_j u_j + N_m u_m) = N_{i,x} u_i + N_{j,x} u_j + N_{m,x} u_m$$

$$N_{i,x} = \frac{1}{2A}\frac{\partial}{\partial x}(\alpha_i + \beta_i x + \gamma_i y) = \frac{\beta_i}{2A}, \qquad N_{j,x} = \frac{\beta_j}{2A}, \quad N_{m,x} = \frac{\beta_m}{2A}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{2A}(\beta_i u_i + \beta_j u_j + \beta_m u_m)$$

$$\frac{\partial \upsilon}{\partial y} = \frac{1}{2A}(\gamma_i \upsilon_i + \gamma_j \upsilon_j + \gamma_m \upsilon_m)$$

Likewise,

$$\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x} = \frac{1}{2A}(\gamma_i u_i + \beta_i \upsilon_i + \gamma_j u_j + \beta_j \upsilon_j + \gamma_m u_m + \beta_m \upsilon_m)$$

Summarizing the deformation rate equation

$$\{\varepsilon\} = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \begin{Bmatrix} u_i \\ \upsilon_i \\ u_j \\ \upsilon_j \\ u_m \\ \upsilon_m \end{Bmatrix} = [B]\{d\} = \begin{bmatrix} \underline{B}_i & \underline{B}_j & \underline{B}_m \end{bmatrix} \begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix}$$

where

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \qquad [B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix}$$

Strain is constant in an element, for matrix $\underline{B}$ regardless of x and y coordinates, and is influenced by only nodal coordinates in an element.

→ CST: constant-strain triangle

Relation of stress-strain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \rightarrow \quad \{\sigma\} = [D][B]\{d\}$$

Step 4: Derivation of element stiffness matrix and equation

Using minimum potential energy principle.

Total potential energy $$\pi_p = \pi_p(u_i, \upsilon_i, u_j, \ldots, \upsilon_m) = U + \Omega_b + \Omega_p + \Omega_s$$

Strain energy $$U = \frac{1}{2}\iiint_V \{\varepsilon\}^T \{\sigma\}\, dV = \frac{1}{2}\iiint_V \{\varepsilon\}^T [D]\{\varepsilon\}\, dV$$

Potential energy of body force $$\Omega_b = -\iiint_V \{\psi\}^T \{X\}\, dV$$

Potential energy of concentrated load $$\Omega_p = -\{d\}^T \{P\}$$

Potential energy of distributed load (or surface force) $$\Omega_S = -\iint_S \{\psi\}^T \{T\}\, dS$$

$$\therefore\ \pi_p = \frac{1}{2}\iiint_V \{d\}^T [B]^T [D][B]\{d\}\,dV - \iiint_V \{d\}^T [N]^T \{X\}\,dV$$

$$- \{d\}^T \{P\} - \iint_S \{d\}^T [N]^T \{T\}\,dS$$

$$= \frac{1}{2}\{d\}^T \iiint_V [B]^T [D][B]\,dV\{d\} - \{d\}^T \iiint_V [N]^T \{X\}\,dV$$

$$- \{d\}^T \{P\} - \{d\}^T \iint_S [N]^T \{T\}\,dS$$

$$= \frac{1}{2}\{d\}^T \iiint_V [B]^T [D][B]dV\{d\} - \{d\}^T \{f\}$$

where

$$\{f\} = \iiint_V [N]^T \{X\}\,dV + \{P\} + \iint_S [N]^T \{T\}\,dS \qquad (7.2.46)$$

Condition having pole values $\dfrac{\partial \pi_p}{\partial\{d\}} = \left[\iiint_V [B]^T [D][B]dV\right]\{d\} - \{f\} = 0$

$$\rightarrow\quad \iiint_V [B]^T [D][B]dV\{d\} = \{f\}$$

So, the element stiffness matrix is

$$[k] = \iiint_V [B]^T [D][B]\,dV$$

Case of an element having constant thickness t :

$$[k] = t\iint_A [B]^T [D][B]\,dx\,dy$$
$$= tA[B]^T [D][B]$$

Matrix $[k]$ is a 6×6 matrix, and the element equation is as below

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{16} \\ k_{21} & k_{22} & \cdots & k_{26} \\ \vdots & \vdots & & \vdots \\ k_{61} & k_{62} & \cdots & k_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ \upsilon_1 \\ u_2 \\ \upsilon_2 \\ u_3 \\ \upsilon_3 \end{Bmatrix}$$

Step 5: Introduction a combination of element equation and boundary conditions for obtaining a global coordinate system of equation.

$$[K]=\sum_{e=1}^{N}[k^{(e)}] \qquad \text{and} \qquad \{F\}=\sum_{e=1}^{N}\{f^{(e)}\}$$

$$\{F\}=[K]\{d\}$$

Step 6: Calculation of nodal displacement

Step 7: Calculation of force(stress) in an element

Transformation from the global coordinate system to the local coordinate system: (See Ch. 3)

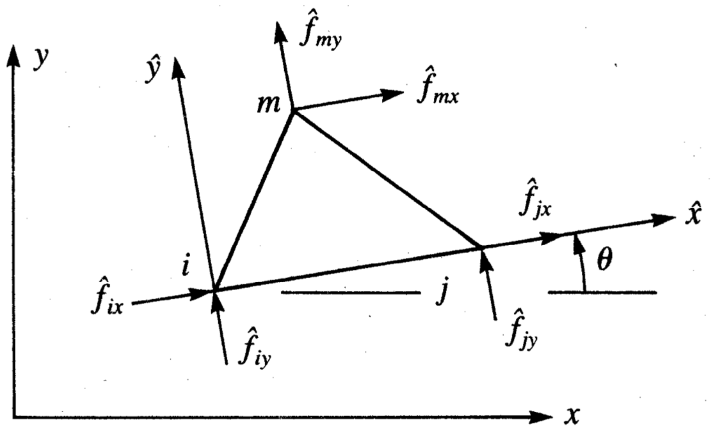
$$\underline{\hat{d}}=\underline{T}\,\underline{d} \qquad \underline{\hat{f}}=\underline{T}\,\underline{f} \qquad \underline{k}=\underline{T}^{T}\,\underline{\hat{k}}\,\underline{T}$$

Constant-strain triangle(CST) has 6 degrees of freedom.

$$\underline{T} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & -S & C & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix}$$

where

$$C = \cos\theta$$
$$S = \sin\theta$$

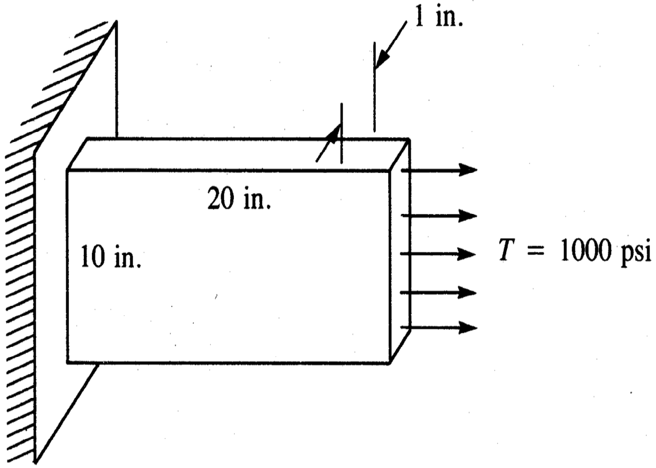


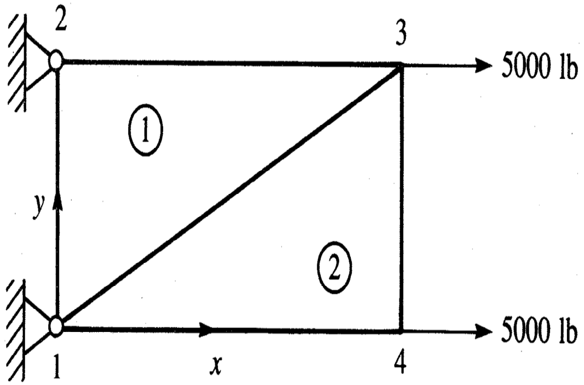
A triangular element with local coordinates system not along to the global coordinate system

Finite Element Method in a plane stress problem

Find nodal displacements and element stresses in the case of the thin plate(see below figure) under surface force.

thickness $t = 1$ in., $E = 30 \times 10^6$ psi, $\nu = 0.30$





(1) Discretization: Surface tension force is replaced by the following nodal loads.

$$F = \frac{1}{2}TA$$

$$F = \frac{1}{2}(1000\text{ psi})(1\text{ in.}\times 10\text{ in.})$$

$$F = 5000\text{ lb}$$

The global system of the governing equation is

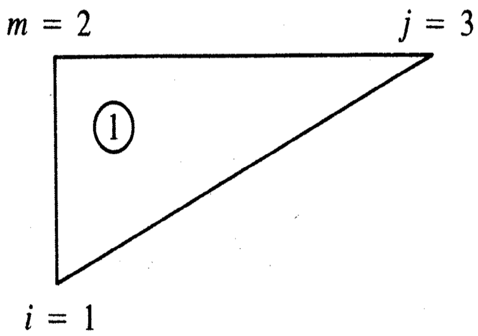
$$\{F\} = [K]\{d\} \quad \text{or} \quad \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = [K]\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix} = [K]\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

where $[K]$ is a 8×8 matrix.

(2) A combination of stiffness matrix: $[k]=tA[B]^T[D][B]$

- Element 1

- Calculation of matrix $[B]$



$$[B]=\frac{1}{2A}\begin{bmatrix}\beta_i & 0 & \beta_j & 0 & \beta_m & 0\\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m\\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m\end{bmatrix}$$

where

$$\beta_i = y_j - y_m = 10-10=0$$
$$\beta_j = y_m - y_i = 10-0=10$$
$$\beta_m = y_i - y_j = 0-10=-10$$
$$\gamma_i = x_m - x_j = 0-20=-20$$
$$\gamma_j = x_i - x_m = 0-0=0$$
$$\gamma_m = x_j - x_i = 20-0=20$$

and

$$A=\frac{1}{2}bh$$
$$=\left(\frac{1}{2}\right)(20)(10)=100\text{ in.}^2$$

Then $[B]$ is

$$[B]=\frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}$$

- Matrix $[D]$ (plane stress)

$$[D]=\frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}=\frac{30(10^6)}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

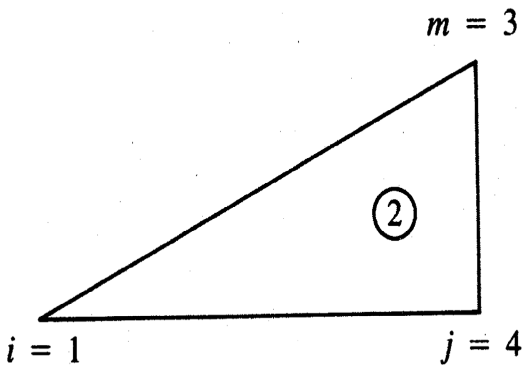
- Calculation of stiffness matrix

$i=1 \qquad j=3 \qquad m=2$

$$[k]=tA[B]^T[D][B]=\frac{75{,}000}{0.91}\begin{bmatrix} 140 & 0 & 0 & -70 & -140 & 70 \\ 0 & 400 & -60 & 0 & 60 & -400 \\ 0 & -60 & 100 & 0 & -100 & 60 \\ -70 & 0 & 0 & 35 & 70 & -35 \\ -140 & 60 & -100 & 70 & 240 & -130 \\ 70 & -400 & 60 & -35 & -130 & 435 \end{bmatrix}$$

- Element 2

- Calculation of matrix $[B]$



$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

where

$$\beta_i = y_j - y_m = 0 - 10 = -10$$
$$\beta_j = y_m - y_i = 10 - 0 = 10$$
$$\beta_m = y_i - y_j = 0 - 0 = 0$$
$$\gamma_i = x_m - x_j = 20 - 20 = 0$$
$$\gamma_j = x_i - x_m = 0 - 20 = -20$$
$$\gamma_m = x_j - x_i = 20 - 0 = 20$$

and

$$A = \frac{1}{2}bh = \left(\frac{1}{2}\right)(20)(10) = 100 \text{ in.}^2$$

Then matrix $[B]$ is

$$[B]=\frac{1}{200}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix}$$

- Matrix $[D]$ (plane stress)

$$[D]=\frac{30(10^6)}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

- Calculation of stiffness matrix

$$\begin{matrix} i=1 & & j=4 & & m=3 & \end{matrix}$$

$$[k]=\frac{75{,}000}{0.91}\begin{bmatrix} 100 & 0 & -100 & 60 & 0 & -60 \\ 0 & 35 & 70 & -35 & -70 & 0 \\ -100 & 70 & 240 & -130 & -140 & 60 \\ 60 & -35 & -130 & 435 & 70 & -400 \\ 0 & -70 & -140 & 70 & 140 & 0 \\ -60 & 0 & 60 & -400 & 0 & 400 \end{bmatrix}$$

Element 1:

$$
\begin{array}{c} \quad 1 \qquad\qquad 2 \qquad\qquad 3 \qquad\qquad 4 \end{array}
$$

$$
[k] = \frac{375{,}000}{0.91}\begin{bmatrix} 28 & 0 & -28 & 14 & 0 & -14 & 0 & 0 \\ 0 & 80 & 12 & -80 & -12 & 0 & 0 & 0 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -12 & -20 & 12 & 20 & 0 & 0 & 0 \\ -14 & 0 & 14 & -7 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Element 2:

$$
\begin{array}{c} \quad 1 \qquad\qquad 2 \qquad\qquad 3 \qquad\qquad 4 \end{array}
$$

$$
[k] = \frac{375{,}000}{0.91}\begin{bmatrix} 20 & 0 & 0 & 0 & 0 & -12 & -20 & 12 \\ 0 & 7 & 0 & 0 & -14 & 0 & 14 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -14 & 0 & 0 & 28 & 0 & -28 & 14 \\ -12 & 0 & 0 & 0 & 0 & 80 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix}
$$

(3) Calculation of displacement: Superpositioning element stiffness matrix, global system of stiffness matrix is obtained as below.

$$[K] = \frac{375{,}000}{0.91} \begin{array}{c} \begin{matrix} \quad 1 & \qquad\qquad 2 & \qquad\qquad 3 & \qquad\qquad 4 \end{matrix} \\ \begin{bmatrix} 48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\ 0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\ -26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix} \end{array}$$

Substituting $[K]$ to $\{F\} = [K]\{d\}$,

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = \frac{375{,}000}{0.91} \begin{bmatrix} 48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\ 0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\ -26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

Applying given boundary conditions with elimination of columns and rows.

$$\begin{Bmatrix} 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = \frac{375{,}000}{0.91} \begin{bmatrix} 48 & 0 & -28 & 14 \\ 0 & 87 & 12 & -80 \\ -28 & 12 & 48 & -26 \\ 14 & -80 & -26 & 87 \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

Transposing the displacement matrix to the left side

$$\begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix} = \frac{0.91}{375{,}000} \begin{bmatrix} 48 & 0 & -28 & 14 \\ 0 & 87 & 12 & -80 \\ -28 & 12 & 48 & -26 \\ 14 & -80 & -26 & 87 \end{bmatrix}^{-1} \begin{Bmatrix} 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 609.6 \\ 4.2 \\ 663.7 \\ 104.1 \end{Bmatrix} \times 10^{-6} \text{ in.}$$

The solution of 1-D beam under tension force is

$$\delta = \frac{PL}{AE} = \frac{(10{,}000)20}{10(30 \times 10^{6})} = 670 \times 10^{-6} \text{ in.}$$

Therefore, x-component of the displacement at nodes in the equation (7.5.27) of 2-D plane is quite accurate when the grid is considered as coarse grid.

(4) Stresses at each node: $\{\sigma\} = [D][B]\{d\}$

- Element 1

$$\{\sigma\} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \times \left(\frac{1}{2A}\right)\begin{bmatrix} \beta_1 & 0 & \beta_3 & 0 & \beta_2 & 0 \\ 0 & \gamma_1 & 0 & \gamma_3 & 0 & \gamma_2 \\ \gamma_1 & \beta_1 & \gamma_3 & \beta_3 & \gamma_2 & \beta_2 \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{3x} \\ d_{3y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

Calculating,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 1005 \\ 301 \\ 2.4 \end{Bmatrix} \text{psi}$$

- Element 2

$$\{\sigma\} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \times \left(\frac{1}{2A}\right)\begin{bmatrix} \beta_1 & 0 & \beta_4 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_4 & 0 & \gamma_4 \\ \gamma_1 & \beta_1 & \gamma_4 & \beta_4 & \gamma_3 & \beta_3 \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{4x} \\ d_{4y} \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

Calculating,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 995 \\ -1.2 \\ -2.4 \end{Bmatrix} \text{psi}$$