ENGINEERING MATHEMATICS II

010.141

Byeng Dong Youn (윤병동)

CHAP. 8
Linear Algebra: Matrix Eigenvalue Problem

ABSTRACT OF CHAP. 8

- ***Linear algebra in Chaps. 7 and 8*** discusses the theory and application of vectors and matrices, mainly related to linear systems of equations, eigenvalue problems, and linear transformation.

- ***Chapter 8*** concerns the solutions of vector equations

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

where A is a given square matrix and vector x and scalar λ.

  - Eigenvalue problems are of greatest practical interest to the engineer, physicist, and mathematician.
  - Eigenvectors are transformed to themselves multiplied with a characteristic constant (eigenvalues).

CHAP. 8.1
EIGENVALUES, EIGENVECTORS

Eigenvalues and eigenvectors imply characteristic values and vectors of a matrix **A**.

SOME DEFINITIONS

Let A be an **n × n** matrix and consider

$$\mathbf{Ax} = \lambda \mathbf{x} \tag{1}$$

λ, such that (1) has solution $\mathbf{x} \neq 0$ is an **eigenvalue** or characteristic value of **A**.

x are **eigenvectors** or characteristic vectors of **A**.

- The **spectrum** of A is the set of eigenvalues of **A**;
- max| λ | is the **spectral radius** of **A**.
- The set of eigenvectors corresponding to λ (including 0) is the **eigenspace** of **A** for λ.

SOME DEFINITIONS (cont)

Homogeneous linear system in x_1, x_2

$\quad | \rightarrow$ **Cramer's theorem**

$D(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$D(\lambda)$ is the characteristic determinant, and

$D(\lambda) = 0$ is the characteristic equation

EXAMPLE 1

EXAMPLE 1 Determination of Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Solution:

(a) ***Eigenvalues***. These must be determined ***first***. Equation (1) is

$$\mathbf{Ax} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \text{in components,} \qquad \begin{aligned} -5x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 - 2x_2 &= \lambda x_2 . \end{aligned}$$

Transferring the terms on the right to the left, we get

$$\text{(2*)} \qquad \begin{aligned} (-5-\lambda)x_1 + \quad 2x_2 \quad &= 0 \\ 2x_1 \quad + (-2-\lambda)x_2 &= 0. \end{aligned}$$

This can be written in matrix notation

$$\text{(3*)} \qquad (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

because (1) is $\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{Ax} - \lambda\mathbf{Ix} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, which gives (3*) . We see that this is a ***homogeneous*** linear system. By Cramer's theorem in Sec. 7.7 it has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$ (an eigenvector of **A** we are looking for) if and only if its coefficient determinant is zero, that is,

$$\text{(4*)} \qquad D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0 .$$

We call $D(\lambda)$ the **characteristic determinant** or, if expanded, the **characteristic polynomial**, and $D(\lambda)$ = 0 the **characteristic equation** of **A**. The solutions of this quadratic equation are $\lambda_1 = -1$ and $\lambda_2 = -6$. These are the eigenvalues of **A**.

EXAMPLE 1

(b_1) ***Eigenvector of* A *corresponding to* λ_1.** This vector is obtained from (2*) with $\lambda = \lambda_1 = -1$, that is,

$$\begin{aligned} -4x_1 + 2x_2 &= 0 \\ 2x_1 - \ \ x_2 &= 0\,. \end{aligned}$$

A solution is $x_2 = 2x_1$, as we see from either of the two equations, so that we need only one of them. This determines an eigenvector corresponding to $\lambda_1 = -1$ up to a scalar multiple. If we choose $x_1 = 1$, we obtain the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \qquad \text{Check:} \qquad \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)\mathbf{x}_1 = \lambda_1\mathbf{x}_1\,.$$

(b_2) ***Eigenvector of* A *corresponding to* λ_2.** For $\lambda = \lambda_2 = -6$, equation (2*) becomes

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + 4x_2 &= 0\,. \end{aligned}$$

A solution is $x_2 = -x_1/2$ with arbitrary x_1. If we choose $x_1 = 2$, we get $x_2 = -1$. Thus an eigenvector of **A** corresponding to $\lambda_2 = -6$ is

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \qquad \text{Check:} \qquad \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} = (-6)\mathbf{x}_2 = \lambda_2\mathbf{x}_2\,.$$

THEOREMS

THEOREM 1

Eigenvalues

The eigenvalues of a square matrix $\mathbf{A}$ *are the roots of the characteristic equation* (4) *of* $\mathbf{A}$.

Hence an $n \times n$ *matrix has at least one eigenvalue and at most* n *numerically different eigenvalues.*

THEOREM 2

Eigenvectors, Eigenspace

If $\mathbf{w}$ *and* $\mathbf{x}$ *are eigenvectors of a matrix* $\mathbf{A}$ *corresponding to* ***the same*** *eigenvalue* λ, *so are* $\mathbf{w} + \mathbf{x}$ (provided $\mathbf{x} \neq -\mathbf{w}$) *and* $k\mathbf{x}$ *for any* $k \neq 0$.

Hence the eigenvectors corresponding to one and the same eigenvalue λ *of* $\mathbf{A}$, *together with* $\mathbf{0}$, *form a vector space* (cf. Sec. 7.4), *called the* **eigenspace** *of* $\mathbf{A}$ *corresponding to that* λ.

PROOF

$\mathbf{Aw} = \lambda\mathbf{w}$ and $\mathbf{Ax} = \lambda\mathbf{x}$ *imply* $\mathbf{A}(\mathbf{w} + \mathbf{x}) = \mathbf{Aw} + \mathbf{Ax} = \lambda\mathbf{w} + \lambda\mathbf{x} = \lambda(\mathbf{w} + \mathbf{x})$ and $\mathbf{A}(k\mathbf{w}) = k(\mathbf{Aw}) = k(\lambda\mathbf{w}) = \lambda(k\mathbf{w})$; hence $\mathbf{A}(k\mathbf{w} + \ell\,\mathbf{x}) = \lambda(k\mathbf{w} + \ell\,\mathbf{x})$. ■

EXAMPLE 2

EXAMPLE 2 Multiple Eigenvalues

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution:

For our matrix, the characteristic determinant gives the characteristic equation

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0 .$$

The roots (eigenvalues of **A**) are $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$. To find eigenvectors, we apply the Gauss elimination (Sec. 7.3) to the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, first with $\lambda = 5$ and then with $\lambda = -3$. For $\lambda = 5$ the characteristic matrix is

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}. \quad \text{It row-reduces to} \quad \begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence it has rank 2. Choosing $x_3 = -1$ we have $x_2 = 2$ from $-\frac{24}{7}x_2 - \frac{48}{7}x_3 = 0$ and then $x_1 = 1$ from $-7x_1 + 2x_2 - 3x_3 = 0$. Hence an eigenvector of **A** coresponding to $\lambda = 5$ is $x_1 = [1 \quad 2 \quad -1]^T$.

EXAMPLE 2

For $\lambda = -3$ the characteristic matrix

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \quad \text{row-reduces} \quad \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence it has rank 1. From $x_1 + 2x_2 - 3x_3 = 0$ we have $x_1 = -2x_2 + 3x_3$. Choosing $x_2 = 1$, $x_3 = 0$ and $x_2 = 0$, $x_3 = 1$, we obtain two linearly independent eigenvectors of $\mathbf{A}$ corresponding to $\lambda = -3$ [as they must exist by (5), Sec. 7.5, with rank = 1 and $n = 3$],

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

MULTIPLICITY

The **algebraic multiplicity** is the order M_λ of λ in the **characteristic polynomial**.

The **geometric multiplicity** ($m\lambda$) of λ is the number of linearly independent vectors corresponding to λ.

$$\sum M_\lambda = n$$

$$m_\lambda \underset{12}{\leq} M_\lambda$$

$$\Delta_\lambda = M_\lambda - m_\lambda \quad (\text{defect of } \lambda)$$

Example 3

EXAMPLE 3 Algebraic Multiplicity, Geometric Multiplicity. Positive Defect

The characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{is} \quad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0 .$$

Hence $\lambda = 0$ is an eigenvalue of algebraic multiplicity $M_0 = 2$. But its geometric multiplicity is only $m_0 = 1$, since eigenvectors result from $-0x_1 + x_2 = 0$, hence $x_2 = 0$, in the form $[x_1 \quad 0]^{\mathsf{T}}$. Hence for $\lambda = 0$ the defect is $\Delta_0 = 1$.

Similarly, the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{is} \quad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0 .$$

Hence $\lambda = 3$ is an eigenvalue of algebraic multiplicity $M_3 = 2$, but its geometric multiplicity is only $m_3 = 1$, since eigenvectors result from $0x_1 + 2x_2 = 0$ in the form $[x_1 \quad 0]^{\mathsf{T}}$.

Example 4

EXAMPLE 4 Real Matrices with Complex Eigenvalues and Eigenvectors

Since real polynomials may have complex roots (which then occur in conjugate pairs), a real matrix may have complex eigenvalues and eigenvectors. For instance, the characteristic equation of the skew-symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{is} \quad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0\,.$$

It gives the eigenvalues $\lambda_1 = i\,(= \sqrt{-1})$, $\lambda_2 = -i$. Eigenvectors are obtained from $-ix_1 + x_2 = 0$ and $ix_1 + x_2 = 0$, respectively, and we can choose $x_1 = 1$ to get

$$\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

THEOREM 3

Eigenvalues of the Transpose

The transpose $\mathbf{A}^\mathsf{T}$ of a square matrix $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}$.

HOMEWORK IN 8.1

- HW1. Problem 6
- HW2. Problem 10
- HW3. Problem 30

CHAP. 8.2
SOME APPLICATIONS OF EIGENVALUE PROBLEMS

Range of applications of matrix eigenvalue problems.

EXAMPLE 1

EXAMPLE 1 Stretching of an Elastic Membrane

An elastic membrane in the x_1x_2-plane with boundary circle $x_1^2 + x_2^2 = 1$ (Fig. 158) is stretched so that a point P: (x_1, x_2) goes over into the point Q: (y_1, y_2) given by

(1) $$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \text{in components,} \qquad \begin{aligned} y_1 &= 5x_1 + 3x_2 \\ y_2 &= 3x_1 + 5x_2 . \end{aligned}$$

Find the **principal directions**, that is, the directions of the position vector $\mathbf{x}$ of P for which the direction of the position vector $\mathbf{y}$ of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

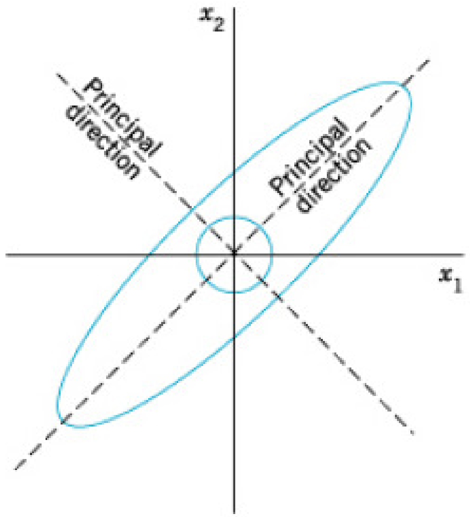

Fig. 158. Undeformed and deformed membrane in Example 1

EXAMPLE 1

The characteristic equation is

$$(3) \qquad \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = 0.$$

Its solutions are $\lambda_1 = 8$ and $\lambda_2 = 2$. These are the eigenvalues of our problem. For $\lambda = \lambda_1 = 8$, our system (2) becomes

$$\begin{aligned} -3x_1 + 3x_2 &= 0, \\ 3x_1 - 3x_2 &= 0. \end{aligned} \left| \begin{array}{l} \text{Solution } x_2 = x_1, \quad x_1 \text{ arbitrary,} \\ \text{for instance, } x_1 = x_2 = 1. \end{array} \right.$$

For $\lambda_2 = 2$, our system (2) becomes

$$\begin{aligned} 3x_1 + 3x_2 &= 0, \\ 3x_1 + 3x_2 &= 0. \end{aligned} \left| \begin{array}{l} \text{Solution } x_2 = -x_1, \quad x_1 \text{ arbitrary,} \\ \text{for instance, } x_1 = 1, x_2 = -1. \end{array} \right.$$

We thus obtain as eigenvectors of **A**, for instance, $[1 \quad 1]^T$ corresponding to λ_1 and $[1 \quad -1]^T$ corresponding to λ_2 (or a nonzero scalar multiple of these). These vectors make 45° and 135° angles with the positive x_1-direction. They give the principal directions, the answer to our problem. The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2, respectively; see Fig. 158.

EXAMPLE 2 – Markov Process

Suppose that the 2004 state of land use in a city of 60 mi^2 of built-up area is

C: Commercially Used 25% I: Industrially Used 20% R: Residentially Used 55%

$$\begin{array}{cccc} & \text{From C} & \text{From I} & \text{From R} & \\ \mathbf{A} = & \left[\begin{array}{c} 0.7 \\ 0.2 \\ 0.1 \end{array}\right. & \begin{array}{c} 0.1 \\ 0.9 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 0.2 \\ 0.8 \end{array}\right] & \begin{array}{c} \text{To C} \\ \text{To I} \\ \text{To R} \end{array} \end{array}$$

The eigenvalue problem can be used to identify the limit state of the process, in which the state vector **x** is reproduced under the multiplication by the stochastic matrix **A** governing the process, that is, **Ax=x**.

$$\mathbf{A}-\mathbf{I}=\begin{bmatrix} -0.3 & 0.1 & 0 \\ 0.2 & -0.1 & 0.2 \\ 0.1 & 0 & -0.2 \end{bmatrix} \Rightarrow$$

the limit state of the process

$$\mathbf{x}=[2 \quad 6 \quad 1]^T$$

EXAMPLE 4

EXAMPLE 4 Vibrating System of Two Masses on Two Springs (Fig. 159)

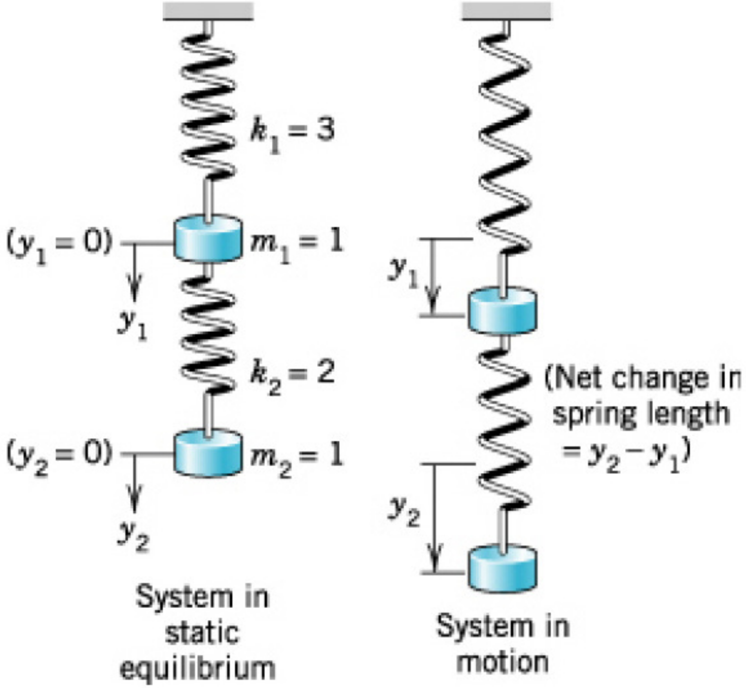

Fig. 159. Masses on springs in Example 4

$$y''_1 = -5y_1 + 2y_2$$
$$y''_2 = 2y_1 - 2y_2$$

$$\mathbf{y}'' = \begin{bmatrix} y''_1 \\ y''_2 \end{bmatrix} = \mathbf{A}\mathbf{y} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

EXAMPLE 4

EXAMPLE 4 Vibrating System of Two Masses on Two Springs (Fig. 159)

We try a vector solution of the form

(8) $$\mathbf{y} - \mathbf{x}e^{\omega t}.$$

This is suggested by a mechanical system of a single mass on a spring (Sec. 2.4), whose motion is given by exponential functions (and sines and cosines). Substitution into (7) gives

$$\omega^2 \mathbf{x} e^{\omega t} = \mathbf{A}\mathbf{x}e^{\omega t}.$$

Dividing by $e^{\omega t}$ and writing $\omega^2 = \lambda$, we see that our mechanical system leads to the eigenvalue problem

(9) $$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \qquad \text{where} \qquad \lambda = \omega^2.$$

From Example 1 in Sec. 8.1 we see that **A** has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$. Consequently, $\omega = \sqrt{-1} = \pm i$ and $\sqrt{-6} = \pm i\sqrt{6}$, respectively. Corresponding eigenvectors are

(10) $$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

From (8) we thus obtain the four complex solutions [see (10) , Sec. 2.2]

EXAMPLE 4

$$x_1 e^{\pm it} = x_1(\cos t \pm i \sin t),$$

$$x_2 e^{\pm i\sqrt{6}t} = x_2(\cos\sqrt{6}t \pm i \sin\sqrt{6}t).$$

By addition and subtraction (see Sec. 2.2) we get the four real solutions

$$\mathbf{x}_1 \cos t, \quad \mathbf{x}_1 \sin t, \quad \mathbf{x}_2 \cos\sqrt{6}t, \quad \mathbf{x}_2 \sin\sqrt{6}t .$$

A general solution is obtained by taking a linear combination of these,

$$y = x_1(a_1 \cos t + b_1 \sin t) + x_2(a_2 \cos\sqrt{6}t + b_2 \sin\sqrt{6}t)$$

with arbitrary constants a_1, b_1, a_2, b_2 (to which values can be assigned by prescribing initial displacement and initial velocity of each of the two masses). By (10) , the components of **y** are

$$y_1 = \ a_1 \cos t + b_1 \sin t + 2a_2 \cos\sqrt{6}t + 2b_2 \sin\sqrt{6}t$$

$$y_2 = 2a_1 \cos t + 2b_1 \sin t - a_2 \cos\sqrt{6}t - b_2 \sin\sqrt{6}t .$$

These functions describe harmonic oscillations of the two masses. Physically, this had to be expected because we have neglected damping.

HOMEWORK IN 8.2

- HW1. Problem 8
- HW2. Problem 19

CHAP. 8.3 SYMMETRIC, SKEW-SYMMETRIC, AND ORTHOGONAL MATRICES

Three classes of real square matrices frequently occurring in engineering applications.

SOME DEFINITIONS

DEFINITIONS

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A ***real*** square matrix $\mathbf{A} = [a_{jk}]$ is called

symmetric if transposition leaves it unchanged,

(1) $$\mathbf{A}^{\mathsf{T}} = \mathbf{A}, \qquad \text{thus} \qquad a_{kj} = a_{jk},$$

skew-symmetric if transposition gives the negative of **A**,

(2) $$\mathbf{A}^{\mathsf{T}} = -\mathbf{A}, \qquad \text{thus} \qquad a_{kj} = -a_{jk},$$

orthogonal if transposition gives the inverse of **A**,

(3) $$\mathbf{A}^{\mathsf{T}} = \mathbf{A}^{-1}.$$

Any real square matrix **A** may be written as the sum of a symmetric matrix **R** and a skew-symmetric matrix **S**, where

(4) $$\mathbf{R} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathsf{T}}) \qquad \text{and} \qquad \mathbf{S} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathsf{T}}).$$

SOME DEFINITIONS

EXAMPLE 2 Illustration of Formula (4)

$$\mathbf{A}=\begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} = \mathbf{R}+\mathbf{S}=\begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix}+\begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6.0 \\ 1.5 & 6.0 & 0 \end{bmatrix}$$

THEOREM 1

Eigenvalues of Symmetric and Skew-Symmetric Matrices

a. *The eigenvalues of a symmetric matrix are real.*

b. *The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.*

ORTHOGONAL TRANSFORMATIONS

Orthogonal Transformations and Orthogonal Matrices

Orthogonal transformations are transformations

(5) $$y = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{A} \text{ is an orthogonal matrix.}$$

With each vector $\mathbf{x}$ in R^n such a transformation assigns a vector $\mathbf{y}$ in R^n. For instance, the plane rotation through an angle θ

(6) $$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an orthogonal transformation. It can be shown that any orthogonal transformation in the plane or in three-dimensional space is a **rotation** (possibly combined with a reflection in a straight line or a plane, respectively).

RELATED THEOREMS

THEOREM 2

Invariance of Inner Product

An orthogonal transformation preserves the value of the **inner product** *of vectors* **a** *and* **b** *in* R^n, *defined by*

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathsf{T}}\mathbf{b} = [a_1 \quad \cdots \quad a_n]\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}. \tag{7}$$

That is, for any **a** *and* **b** *in* R^n, *orthogonal* $n \times n$ *matrix* **A**, *and* $\mathbf{u} = \mathbf{Aa}, \mathbf{v} = \mathbf{Ab}$ *we have* $\mathbf{u} \bullet \mathbf{v} = \mathbf{a} \bullet \mathbf{b}$.

Hence the transformation also preserves the **length** *or* **norm** *of any vector* **a** *in* R^n *given by*

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\mathbf{a}^{\mathsf{T}}\mathbf{a}}. \tag{8}$$

THEOREM 3

Orthonormality of Column and Row Vectors

A real square matrix is orthogonal if and only if its column vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$ *(and also its row vectors) form an* **orthonormal system**, *that is,*

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^{\mathsf{T}}\mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases} \tag{10}$$

RELATED THEOREMS

PROOF

a. Let $\mathbf{A}$ be orthogonal. Then $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathsf{T}}\mathbf{A} = \mathbf{I}$, in terms of column vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$,

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathsf{T}}\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{\mathsf{T}} \\ \vdots \\ \mathbf{a}_n^{\mathsf{T}} \end{bmatrix} [\mathbf{a}_1 \cdots \mathbf{a}_n] = \begin{bmatrix} \mathbf{a}_1^{\mathsf{T}}\mathbf{a}_1 & \mathbf{a}_1^{\mathsf{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_1^{\mathsf{T}}\mathbf{a}_n \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{a}_n^{\mathsf{T}}\mathbf{a}_1 & \mathbf{a}_n^{\mathsf{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_n^{\mathsf{T}}\mathbf{a}_n \end{bmatrix}. \tag{11}$$

The last equality implies (10) , by the definition of the $n \times n$ unit matrix $\mathbf{I}$. From (3) it follows that the inverse of an orthogonal matrix is orthogonal (see CAS Experiment 20). Now the column vectors of $\mathbf{A}^{-1}(=\mathbf{A}^{\mathsf{T}})$ are the row vectors of $\mathbf{A}$. Hence the row vectors of $\mathbf{A}$ also form an orthonormal system.

b. Conversely, if the column vectors of $\mathbf{A}$ satisfy (10) , the off-diagonal entries in (11) must be 0 and the diagonal entries 1. Hence $\mathbf{A}^{\mathsf{T}}\mathbf{A} = \mathbf{I}$, as (11) shows. Similarly, $\mathbf{A}\mathbf{A}^{\mathsf{T}} = \mathbf{I}$. This implies $\mathbf{A}^{\mathsf{T}} = \mathbf{A}^{-1}$ because also $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ and the inverse is unique. Hence $\mathbf{A}$ is orthogonal. Similarly when the row vectors of $\mathbf{A}$ form an orthonormal system, by what has been said at the end of part (**a**).

RELATED THEOREMS

THEOREM 4

Determinant of an Orthogonal Matrix

The determinant of an orthogonal matrix has the value $+1$ *or* -1.

PROOF

From $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$ (Sec. 7.8, Theorem 4) and $\det \mathbf{A}^{\mathsf{T}} = \det \mathbf{A}$ (Sec. 7.7, Theorem 2d), we get for an orthogonal matrix

$$1 = \det \mathbf{I} = \det(\mathbf{AA}^{-1}) = \det(\mathbf{AA}^{\mathsf{T}}) = \det \mathbf{A} \det \mathbf{A}^{\mathsf{T}} = (\det \mathbf{A})^2 .$$

THEOREM 5

Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix $\mathbf{A}$ *are real or complex conjugates in pairs and have absolute value* 1.

PROOF

The first part of the statement holds for any real matrix $\mathbf{A}$ because its characteristic polynomial has real coefficients, so that its zeros (the eigenvalues of $\mathbf{A}$) must be as indicated. The claim that $|\lambda| = 1$ will be proved in Sec. 8.5.

HOMEWORK IN 8.3

- HW1. Problem 10
- HW2. Problem 12
- HW3. Problem 14

CHAP. 8.4
EIGENBASES. DIAGONALIZATION. QUADRATIC FORMS

General properties of eigenvectors.

BASIS OF EIGENVECTORS

THEOREM 1

Basis of Eigenvectors

*If an $n \times n$ matrix $\mathbf{A}$ has n **distinct** eigenvalues, then $\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ for R^n.*

PROOF

All we have to show is that $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are linearly independent. Suppose they are not. Let r be the largest integer such that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is a linearly independent set. Then $r < n$ and the set $\{\mathbf{x}_1, \cdots, \mathbf{x}_r, \mathbf{x}_{r+1}\}$ is linearly dependent. Thus there are scalars $c_1, \cdots, c_{r+1}$, not all zero, such that

(2) $$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = \mathbf{0}$$

(see Sec. 7.4). Multiplying both sides by $\mathbf{A}$ and using $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, we obtain

(3) $$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = \mathbf{0}.$$

To get rid of the last term, we subtract λ_{r+1} times (2) from this, obtaining

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0.$$

Here $c_1(\lambda_1 - \lambda_{r+1}) = 0, \cdots, c_r(\lambda_r - \lambda_{r+1}) = 0$ since $\{x_1, \cdots, x_r\}$ is linearly independent. Hence $c_1 = \cdots = c_r = 0$, since all the eigenvalues are distinct. But with this, (2) reduces to $c_{r+1}\mathbf{x}_{r+1} = \mathbf{0}$, hence $c_{r+1} = 0$, since $\mathbf{x}_{r+1} \neq \mathbf{0}$ (an eigenvector!). This contradicts the fact that not all scalars in (2) are zero. Hence the conclusion of the theorem must hold.

BASIS OF EIGENVECTORS

EXAMPLE 1 Eigenbasis. Nondistinct Eigenvalues. Nonexistence

The matrix $\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ has a basis of eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to the eigenvalues $\lambda_1 = 8$, $\lambda_2 = 2$. (See Example 1 in Sec. 8.2.)

THEOREM 2

Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

EXAMPLE 2 Orthonormal Basis of Eigenvectors

The first matrix in Example 1 is symmetric, and an orthonormal basis of eigenvectors is $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T, \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$.

DIAGONALIZATION

Eigenbases also play a role in reducing a matrix $\mathbf{A}$ to a diagonal matrix whose entries are the eigenvalues of $\mathbf{A}$. This is done by a "similarity transformation," which is defined as follows (and will

DEFINITION **Similar Matrices. Similarity Transformation**

An $n \times n$ matrix $\hat{\mathbf{A}}$ is called **similar** to an $n \times n$ matrix $\mathbf{A}$ if

(4) $$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

for some (nonsingular!) $n \times n$ matrix $\mathbf{P}$. This transformation, which gives $\hat{\mathbf{A}}$ from $\mathbf{A}$, is called a **similarity transformation**.

THEOREM 3

Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ *is similar to* $\mathbf{A}$, then $\hat{\mathbf{A}}$ *has the same eigenvalues as* $\mathbf{A}$.

Furthermore, if $\mathbf{x}$ *is an eigenvector of* $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ *is an eigenvector of* $\hat{\mathbf{A}}$ *corresponding to the same eigenvalue.*

DIAGONALIZATION

PROOF From $\mathbf{Ax} = \lambda\mathbf{x}$ (λ an eigenvalue, $\mathbf{x} \neq \mathbf{0}$) we get $\mathbf{P}^{-1}\mathbf{Ax} = \lambda\mathbf{P}^{-1}\mathbf{x}$. Now $\mathbf{I} = \mathbf{PP}^{-1}$. By this ***"identity trick"*** the previous equation gives

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \mathbf{P}^{-1}\mathbf{APP}^{-1}\mathbf{x} = \hat{\mathbf{A}}(\mathbf{P}^{-1}\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x}\,.$$

Hence λ is an eigenvalue of $\hat{\mathbf{A}}$ and $\mathbf{P}^{-1}\mathbf{x}$ a corresponding eigenvector. Indeed, $\mathbf{P}^{-1}\mathbf{x} = \mathbf{0}$ would give $\mathbf{x} = \mathbf{Ix} = \mathbf{PP}^{-1}\mathbf{x} = \mathbf{P0} = \mathbf{0}$, contradicting $\mathbf{x} \neq \mathbf{0}$.

THEOREM 4

Diagonalization of a Matrix

If an $n \times n$ matrix $\mathbf{A}$ has a basis of eigenvectors, then

(5) $$\mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$$

is diagonal, with the eigenvalues of $\mathbf{A}$ as the entries on the main diagonal. Here $\mathbf{X}$ is the matrix with these eigenvectors as column vectors. Also,

(5*) $$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X} \qquad (m = 2, 3, \cdots)\,.$$

DIAGONALIZATION

Example: Diagonalize

$$\mathbf{A} = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

$$\lambda_1 = 3,\ \lambda_2 = -4,\ \lambda_3 = 0$$

$$\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} -3 & -4 & 0 \\ 9 & 4 & 0 \\ -3 & -12 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

QUADRATIC FORMS

Quadratic Forms. Transformation to Principal Axes

By definition, a **quadratic form** Q in the components $x_1, \cdots, x_n$ of a vector $\mathbf{x}$ is a sum of n^2 terms,

$$Q = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

$$\begin{aligned} = &a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\ &+ a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2 . \end{aligned} \tag{7}$$

$\mathbf{A} = [a_{jk}]$ is called the **coefficient matrix** of the form. We may assume that $\mathbf{A}$ is ***symmetric***,

EXAMPLE 5 Quadratic Form. Symmetric Coefficient Matrix

$$\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = [x_1 \quad x_2]\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2 = 3x_1^2 + 10x_1x_2 + 2x_2^2 .$$

Here 4 + 6 = 10 = 5 + 5. From the corresponding ***symmetric*** matrix $\mathbf{C} = [c_{jk}]$, where $c_{jk} = \frac{1}{2}(a_{jk} + a_{kj})$, thus $c_{11} = 3$, $c_{12} = c_{21} = 5$, $c_{22} = 2$, we get the same result; indeed,

$$\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x} = [x_1 \quad x_2]\begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2 = 3x_1^2 + 10x_1x_2 + 2x_2^2 .$$

QUADRATIC FORMS

By Theorem 2 the *symmetric* coefficient matrix **A** of (7) has an orthonormal basis of eigenvectors. Hence if we take these as column vectors, we obtain a matrix **X** that is orthogonal, so that $\mathbf{X}^{-1} = \mathbf{X}^{\mathsf{T}}$. From (5) we thus have $\mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1} = \mathbf{X}\mathbf{D}\mathbf{X}^{\mathsf{T}}$. Substitution into (7) gives

(8) $$Q = \mathbf{x}^{\mathsf{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathsf{T}}\mathbf{x}.$$

If we set $\mathbf{X}^{\mathsf{T}}\mathbf{x} = \mathbf{y}$, then, since $\mathbf{X}^{\mathsf{T}} = \mathbf{X}^{-1}$, we get

(9) $$\mathbf{x} = \mathbf{X}\mathbf{y}.$$

Furthermore, in (8) we have $\mathbf{x}^{\mathsf{T}}\mathbf{X} = (\mathbf{X}^{\mathsf{T}}\mathbf{x})^{\mathsf{T}} = \mathbf{y}^{\mathsf{T}}$ and $\mathbf{X}^{\mathsf{T}}\mathbf{x} = \mathbf{y}$, so that Q becomes simply

(10) $$Q = \mathbf{y}^{\mathsf{T}}\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

THEOREM 5

Principal Axes Theorem

The substitution (9) *transforms a quadratic form*

$$Q = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k \qquad (a_{kj} = a_{jk})$$

to the principal axes form or **canonical form** (10) , *where* $\lambda_1, \cdots, \lambda_n$ *are the* (*not necessarily distinct*) *eigenvalues of the* (*symmetric!*) *matrix* **A**, *and* **X** *is an orthogonal matrix with corresponding eigenvectors* $\mathbf{x}_1, \cdots, \mathbf{x}_n$, *respectively, as column vectors.*

HOMEWORK IN 8.4

- HW1. Problem 3
- HW2. Problem 7
- HW3. Problem 17
- HW4. Problem 22

CHAP. 8.5
COMPLEX MATRICES AND FORMS.

Encountered in some applications in quantum mechanics and wave propagations.

COMPLEX MATRICES: HERMITIAN, SKEW-HERMITIAN, UNITARY

Definitions:

A square matrix $A = [a_{jk}]$ is

- Hermitian if $\overline{A}^T = A$ — Symmetric
- Skew – Hermitian if $\overline{A}^T = -A$ ➡ Skew-symmetric
- Unitary if $\overline{A}^T = A^{-1}$ — Orthogonal

COMPLEX MATRICES: HERMITIAN, SKEW-HERMITIAN, UNITARY (cont)

- If A is hermitian $a_{jj} = \bar{a}_{jj} \rightarrow$ diagonal elements are real
- If A is skew - hermitian $a_{jj} = -\bar{a}_{jj} \rightarrow$ diagonal elements are pure imaginary
- If a hermitian matrix is real $\overline{A}^T = A^T = A \rightarrow$ symmetric
- If a skew - hermitian matrix is real $\overline{A}^T = A^T = -A \rightarrow$ skew - symmetric
- If a matrix is real and unitary, $\overline{A}^T = A^T = A^{-1} \rightarrow$ orthogonal

EIGENVALUES

Theorem:

- The eigenvalues of a Hermitian matrix are **real**.
- The eigenvalues of a skew-Hermitian matrix are **pure imaginary or 0**.
- The eigenvalues of a unitary matrix have **absolute value of "1"**.

EIGENVALUES (cont)

Proof: Let λ be an eigenvalue of **A**, x be a corresponding eigenvector.

$$\mathbf{A}x = \lambda x$$

(a) Assume A is Hermitian

$$\bar{x}^T A x = \bar{x}^T \lambda x = \lambda \bar{x}^T x$$

$$\bar{x}^T x = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \bar{x}_1 x_1 + \cdots + \bar{x}_n x_n$$

$$= |x_1|^2 + \cdots + |x_n|^2$$

$$\neq 0 \qquad \text{since x } \neq 0$$

EIGENVALUES (cont)

$$\lambda = \frac{\bar{x}^T A x}{\bar{x}^T x}$$

λ is real if $\bar{x}^T A x$ is real

$$\underbrace{\bar{x}^T A x}_{\text{number}} = \left(\bar{x}^T A x\right)^T$$

$$= x^T A^T \bar{x}$$

$$= x^T \bar{A}\,\bar{x} = \left(\overline{\bar{x}^T A x}\right)$$

Hermitian : $\quad \bar{A}^T = A \quad$ or $\quad \bar{A} = A^T$

EIGENVALUES (cont)

(b) If A is skew-Hermitian

$$\lambda = \frac{\bar{x}^T A x}{\bar{x}^T x} \quad \text{since we made no use of propertry.}$$

$$\bar{x}^T A x = \left(\bar{x}^T A x\right)^T$$

$$= x^T A^T \bar{x}$$

$$= -x^T \bar{A}\,\bar{x} = -\left(\overline{\bar{x}^T A x}\right)$$

$$\bar{A}^T = -A$$

(c) If A is unitary

$$Ax = \lambda x \quad \text{and} \quad \left(\bar{A}\,\bar{x}\right)^T = \left(\overline{\lambda x}\right)^T = \bar{\lambda}\,\bar{x}^T$$

EIGENVALUES (cont)

Multiplying:

$$\left(\overline{\mathrm{A}}\,\overline{\mathrm{x}}\right)^{\mathrm{T}}\mathrm{Ax} = \overline{\lambda}\,\overline{\mathrm{x}}^{\mathrm{T}}\lambda\mathrm{x}$$

$$= \overline{\lambda}\lambda\,\overline{\mathrm{x}}^{\mathrm{T}}\mathrm{x}$$

$$= \left|\lambda\right|^{2}\overline{\mathrm{x}}^{\mathrm{T}}\mathrm{x}$$

$$\left(\overline{\mathrm{A}}\,\overline{\mathrm{x}}\right)^{\mathrm{T}}\mathrm{Ax} = \overline{\mathrm{x}}^{\mathrm{T}}\,\overline{\mathrm{A}}^{\mathrm{T}}\mathrm{Ax}$$

$$= \overline{\mathrm{x}}^{\mathrm{T}}\mathrm{A}^{-1}\mathrm{Ax} = \overline{\mathrm{x}}^{\mathrm{T}}\mathrm{x}$$

$$\left|\lambda\right|^{2} = 1$$

QUADRATIC FORM (cont)

If A is Hermitian or skew-Hermitian; the form is called **Hermitian** or **skew-Hermitian form**.

Theorem:

For every choice of x, the value of an **Hermitian form** is **real**, and the value of a **skew-Hermitian** form is **pure imaginary** or **0**.

Inner Product:

$$\mathrm{a} \cdot \mathrm{b} = \bar{\mathrm{a}}^{\mathrm{T}} \mathrm{b}$$

QUADRATIC FORM (cont)

Length or **Norm**

$$\|a\| = \sqrt{a \cdot a} = \sqrt{\bar{a}^T \cdot a} = \sqrt{\bar{a}_1 a_1 + \bar{a}_2 a_2 + \cdots + \bar{a}_n a_n}$$

$$= \sqrt{|a_1|^2 + \cdots + |a_n|^2}$$

Theorem:

A unitary transformation, y = Ax, A unitary, preserves the value of the inner product and the norm.

Proof:

$$u \cdot v = \bar{u}^T v = \left(\overline{A\,a}\right)^T (Ab) = \left(\overline{A\,a}\right)^T (Ab) = \bar{a}^T\, \bar{A}^T Ab$$

$$= \bar{a}^T A^{-1} Ab = \bar{a}^T b = a \cdot b$$

QUADRATIC FORM (cont)

Theorem:

A square matrix is unitary iff its column vectors (row vectors) form a unitary system, i.e.,

Proof:

$$a_j \cdot a_k = a_j^T a_k = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

Theorem:

The determinant of a unitary matrix has absolute value 1.

Proof:

$$1 = \det\left(\mathrm{A} \cdot \mathrm{A}^{-1}\right) = \det\left(\mathrm{A} \cdot \overline{\mathrm{A}}^{\mathrm{T}}\right) = \det \mathrm{A} \cdot \det\left(\overline{\mathrm{A}}^{\mathrm{T}}\right) = \det \mathrm{A} \cdot \det\left(\overline{\mathrm{A}}\right)$$

$$= \det \mathrm{A} \cdot \overline{\det \mathrm{A}} = \left| \det \mathrm{A} \right|^2$$

HOMEWORK IN 8.5

- HW1. Problem 5
- HW2. Problem 9
- HW3. Problem 14