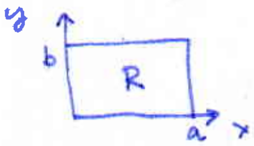
12.8 Modelling : Membrane . 2-D wave Eq'n .

HW. 12.1 ; 4. 7, 11, 22
12.3 ; 5, 9
12.6 ; 7, 12
12.7 ; 5, 6.

2013 / Due 10/31

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u}$$

12.9. Rectangular Membrane. Double Fourier Series.



$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$u = 0$ at time t : (boundary)

$u(x, y, 0) = f(x, y)$

$u_t(x, y, 0) = g(x, y)$

Step 1. 3 ODEs from the wave Eqn (1)

$u(x, y, t) = F(x, y)\, G(t)$, then $F(x, y) = H(x)\, Q(y)$

$F\ddot{G} = c^2(F_{xx}G + F_{yy}G)$ $\quad \frac{\ddot{G}}{c^2 G} = \frac{1}{F}(F_{xx} + F_{yy})$

only negative values lead to solution without being identically zero.

$$\therefore \frac{\ddot{G}}{c^2 G} = \frac{1}{F}(F_{xx} + F_{yy}) = -\nu^2$$

Time function $\ddot{G} + \lambda^2 G = 0$ where $\lambda = c\nu$

amplitude function $F_{xx} + F_{yy} + \nu^2 F = 0$, $\longrightarrow$ 2-D Helmholtz eq'n.

$F(x, y) = H(x)\, Q(y) \longrightarrow \frac{d^2H}{dx^2} Q = -\left(H\frac{d^2Q}{dy^2} + \nu^2 HQ\right)$ $\quad \frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right) = -k^2$

$\frac{d^2H}{dx^2} + k^2 H = 0$, and $\frac{d^2Q}{dy^2} + p^2 Q = 0$ where $p^2 = \nu^2 - k^2$

step 2. Satisfying the BC.

General solution $H = A\cos kx + B\sin kx$ $\quad Q = C\cos py + D\sin py$

BC. $u=0$ at the boundary $\begin{pmatrix} H(0)=H(a)=0 \\ A=0 \quad ka=m\pi \end{pmatrix}\begin{pmatrix} Q(0)=Q(b)=0 \\ C=0 \end{pmatrix}$

$k=\dfrac{m\pi}{a}$ (m integer) $\quad pb = n\pi \quad p=\dfrac{n\pi}{b}$

solution. $H_m(x) = \sin\dfrac{m\pi}{a}x \quad Q_n(y) = \sin\dfrac{n\pi}{b}y \quad \begin{pmatrix} m=1,2\ldots \\ n=1,2\ldots \end{pmatrix}$

$F_{mn}(x,y) = H_m(x)\,Q_n(y) = \sin\dfrac{m\pi x}{a}\sin\dfrac{n\pi}{a}y$

$p^2 = \nu^2 - k^2$ and $\lambda = c\nu$. $\quad \therefore \lambda = c\sqrt{p^2+k^2} = c\pi\sqrt{\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}}$

$k=\dfrac{m\pi}{a}, \quad p=\dfrac{n\pi}{b} \quad\quad \lambda = \lambda_{mn}$

General solution of $\ddot{G}+\lambda^2 G = 0$

$G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t$

eigenfunction이 0 이되는 것은 → eigenvalue (1-D.

eigenfunction 이 0 이되는 line → nodal line (in 2-D)

Eigenfunctions,

$$u_{mn}(x,y,t) = (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

(nodal line where $u_{mn}=0$ or does not move)

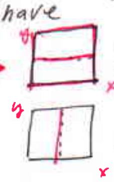
λ_{mn} = eigenvalues. $\quad$ frequency of $u_{mn} = \lambda_{mn}/2\pi$

Ex1. for square membrane $a=b=1$. $\quad \lambda_{mn} = c\pi\sqrt{m^2+n^2} \rightarrow \lambda_{mn}=\lambda_{nm}$

for $m\neq n$. $\quad F_{mn} = \sin m\pi x \sin n\pi y \quad F_{nm} = \sin n\pi x \sin m\pi y \rightarrow \therefore F_{mn} \neq F_{nm}$

for example $F_{12} = \sin\pi x\sin 2\pi y \quad F_{21} = \sin 2\pi x \sin\pi y$

Which have → nodal line $y=1/2$



u_{12} $\quad$ nodal line $y=1/2$

u_{21} $\quad$ nodal line $x=1/2$

Hence corresponding solutions

$u_{12} = (B_{12}\cos c\pi\sqrt{5}t + B^*_{12}\sin c\pi\sqrt{5}t)F_{12}$

$u_{21} = (B_{21}\cos c\pi\sqrt{5}t + B^*_{21}\sin c\pi\sqrt{5}t)F_{21}$

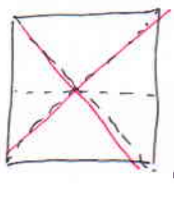
(if) Taking $B_{12}=1$, $\quad B^*_{12} = B^*_{21} = 0$ $\quad$ tentatively! (why?)

we obtain $\underline{u_{12}+u_{21}} = \cos c\pi\sqrt{5}t\,(\underbrace{F_{12}+B_{21}F_{21}})$

nodal line of this function = solution of the equation.

$F_{12}+B_{21}F_{21} = \sin\pi x\sin 2\pi y + B_{21}\sin 2\pi x\sin\pi y = 0.$

$\sin\pi x\sin\pi y\,(\cos\pi y + B_{21}\cos\pi x) = 0. \rightarrow$ solution depends on the value of B_{21}.



$B_{21}=-1 \rightarrow \cos\pi x = \cos\pi y \quad (x=y)$

$B_{21}=0$

$B_{21}=1 \quad (x=-y) \quad \left(\dfrac{\cos\pi x}{\cos\pi y}\right) = -1 \quad$ ~~cos πy~~ $\quad x=1 \; x=0 \quad y=0 \; y=1$

$F_{18}, F_{81}, F_{47}, F_{74} \Rightarrow \lambda_{18}=\lambda_{81}=\lambda_{47}=\lambda_{74}$

→ same eigenvalue.

Step 3. Solution of the Model (1).(2).(3) ~~By~~ Double Fourier Series

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

I.C. $u(x,y,0) = \boxed{f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}$

double Fourier Series. Setting ② $k_m(y) = \sum_{n=1}^{\infty} B_{mn}\sin\frac{n\pi y}{b}$

$\therefore f(x,y) = \sum_{m=1}^{\infty} k_m(y)\sin\frac{m\pi x}{a}$ ①

For fixed y Fourier ~~sine~~ series. $k_m(y) = \frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}dx$

$$B_{mn} = \frac{2}{b}\int_0^b k_m(y)\sin\frac{n\pi y}{b}.dy$$ ③

Generalized Euler formula.

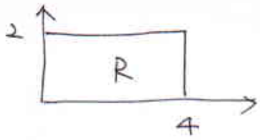
$$\therefore B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dx\,dy \quad \begin{cases} m = 1,2\ldots \\ n = 1,2\ldots\ldots \end{cases}$$

To determine B^*_{mn} $\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B^*_{mn}\lambda_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi}{b}y = g(x,y)$

in the same way as above.

$$B^*_{mn} = \frac{1}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}.dx\,dy$$

Ex 3.



Tension = 12.5 lb/ft.
density = 2.5 slug/ft^2.

$g(x,y)=0 \Rightarrow B_{mn}^* = 0$

$f(x,y) = 0.1(4x-x^2)(2y-y^2)$

$$B_{mn} = \frac{4}{4\cdot 2}\int_0^2\int_0^4 0.1(4x-x^2)(2y-y^2)\sin\frac{m\pi x}{4}\sin\frac{n\pi y}{2}\,dx\,dy.$$

$$= \frac{1}{20}\underbrace{\int_0^4 (4x-x^2)\sin\frac{m\pi x}{4}\,dx}_{\frac{256}{m^3\pi^3}\ (m\text{ odd})}\ \underbrace{\int_0^2 (2y-y^2)\sin\frac{n\pi y}{2}\,dy}_{\frac{32}{n^3\pi^3}\ (n\text{ odd})}.$$

even m,n $=0$.

$$\therefore B_{mn} = \frac{256\cdot 32}{20m^3n^3\pi^6} \approx \frac{0.426050}{m^3n^3} \quad (m \text{ and } n:\ \text{both odd})$$

$$u(x,y,t) = 0.426050\sum_{m,n\ \text{odd}}\sum \frac{1}{m^3n^3}\cos\left(\frac{\sqrt{5}\pi}{4}\sqrt{m^2+4n^2}\right)t\ \sin\frac{m\pi x}{4}\sin\frac{n\pi y}{2}$$

(m,n) $(1,1)$ $(1,3)$ $(3,1)$ $(3,3)$

$(1,3)$: two horizontal nodal line $y=\frac{2}{3},\frac{4}{3}$

$(3,1)$: two vertical nodal line $\left(x=\frac{4}{3},\frac{8}{3}\right)$

$(3,3)$: two vertical, two horizontal nodal line

12.10 Laplacian in Polar Coordinates. Circular Membrane. Fourier-Bessel Series

$$u_{tt} = c^2(u_{xx}+u_{yy})$$

transform Laplacian in the wave eq'n into polar coordinate

$x = r\cos\theta,\quad y = r\sin\theta$

$(\tan\theta)' = \frac{y}{x}$ $\quad \sec^2\theta\,\theta_x = \frac{y'x-y}{x^2}$

thus $r=\sqrt{x^2+y^2}$. $\quad \tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\frac{y}{x} \rightarrow \theta_x = \frac{1}{1+(\frac{y}{x})^2}\cdot\left(-\frac{y}{x^2}\right)$

$$u_x = \underbrace{u_r r_x + u_\theta\theta_x}_{\text{by Chain rule}}$$

$r_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$

$\theta_x = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}$

one more differentiation.

$$u_{xx} = (u_r r_x)_x + (u_\theta\theta_x)_x$$

$r_{xx} = \frac{r-xr_x}{r^2} = \frac{1}{r}-\frac{x^2}{r^3} = \frac{y^2}{r^3}$

$\theta_{xx} = -y\left(-\frac{2}{r^3}\right)r_x = \frac{2xy}{r^4}$

$$= (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x\theta_x + u_\theta\theta_{xx}$$

since $u(r,\theta)$

$$= (u_{rr}r_x + u_{r\theta}\theta_x)r_x + u_r r_{xx} + (u_{\theta r}r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta\theta_{xx}$$

$u_{r\theta} = u_{\theta r}$

$$u_{xx} = \frac{x^2}{r^2}u_{rr} - \frac{2xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + \frac{2xy}{r^4}u_\theta$$

Similarly
$$u_{yy} = \frac{y^2}{r^2}u_{rr} + \frac{2xy}{r^3}u_{r\theta} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{x^2}{r^3}u_r - \frac{2xy}{r^4}u_\theta.$$

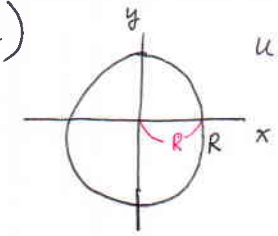
$$\frac{x^2+y^2}{r^2}u_{rr} + \frac{x^2+y^2}{r^4}u_{\theta\theta} + \frac{x^2+y^2}{r^3}u_r$$

$$u_{xx}+u_{yy}$$
$$\downarrow$$
$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

Laplacian of u in polar coordinates.

Circular membrane.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$



$u(t, r)$ so that $u_{\theta\theta}=0$ ⟨radially symmetric⟩

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

$u(R, t) = 0$ for all $t \geq 0$ → membrane is fixed at the boundary.

$u(r, 0) = f(r)$ initial deflection

$u_t(r, 0) = g(r)$ initial velocity

function of r, not on θ. ⇓ radially symmetric solutions $u(r, t)$

Step 1. 2 ODEs from the wave Eq'n

$u(r, t) = W(r)G(t)$

$$\frac{\ddot{G}}{c^2 G} = \frac{1}{W}\left(W'' + \frac{1}{r}W'\right) = -k^2$$

① $\ddot{G} + \lambda^2 G = 0$ where $\lambda = ck$

② $W'' + \frac{1}{r}W' + k^2 W = 0$ → we set $s = kr$, $\frac{1}{r} = \frac{k}{s}$

$W' = \frac{dW}{dr} = \frac{dW}{ds}\cdot\frac{ds}{dr} = \frac{dW}{ds}k$

$W'' = \frac{d^2W}{ds^2}k^2$

not same Bessel form

$\therefore k^2\frac{d^2W}{ds^2} + \frac{k}{s}\cdot k\frac{dW}{ds} + k^2 W = 0.$

⇓

$$\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W = 0$$

$s^2y'' + sy' + (s^2 - \nu^2)y = 0$ where $\nu = 0$ → Bessel's eq'n with parameter $\nu = 0$

∞ $W(r) = J_0(s) = J_0(kr)$

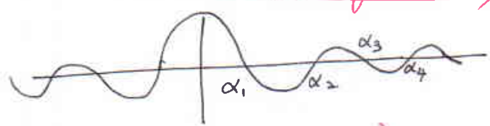
Step 2. Satisfying the BCs

(obtain the parameter k) ~~definite~~ finite deflection!

Two solution J_0, Y_0, $\because Y_0$ at 0 is ∞. we can't use it since solution must be finite. So, only J_0 →

$W(r) = J_0(s) = J_0(kr)$

Boundary condition at $r = R$. $J_0(kR) = 0$



$kR = \alpha_m$ thus $k = k_m = \frac{\alpha_m}{R}$ → from Table $\alpha_1 = 2.4048$, $\alpha_2 = 5.5201$, $\alpha_3 = 8.6537$ ...

$$W_m(r) = J_0(k_m r) = J_0\left(\alpha_m \frac{r}{R}\right)$$

$\therefore$ solutions that are zero on the boundary circle $r = R$

$\ddot{G} + \lambda G = 0.$ $\lambda = \lambda_m = ck_m = c\alpha_m/R$

$G_m(t) = A_m\cos\lambda_m t + B_m\sin\lambda_m t$

$U_m(r, t) = G_m W_m(r) = (A_m\cos\lambda_m t + B_m\sin\lambda_m t)\,J_0(k_m r)$

Eigenfunction with eigenvalue λ_m.

$m=1$ all the points of membrane move up (or down) at the same time

$m=2$; $W_2(r) = J_0(\alpha_2 r/R)$ is zero for $\alpha_2 r/R = \alpha_1$ thus $r = \alpha_1 R/\alpha_2$. Circle $r = \alpha_1 R/\alpha_2$

solution $U_m(r, t)$ has $m-1$ nodal lines, which are circle. nodal line → not moving.

Step 3. Solution of the Entire Problem.

$$u(r,t)=\sum_{m=1}^{\infty} W_m(r)G_m(t)=\sum_{m=1}^{\infty}(A_m\cos\lambda_m t+B_m\sin\lambda_m t)\,J_0\left(\frac{\alpha_m}{R}r\right)$$

initial condition: $u(r,0)=\sum_{m=1}^{\infty} A_m J_0\left(\frac{\alpha_m}{R}r\right)=f(r)$

A_m : coefficients of the Fourier-Bessel series. (p. 506-507)

$$A_m=\frac{2}{R^2 J_1^2(\alpha_m)}\int_0^R r f(r)\,J_0\left(\frac{\alpha_m}{R}r\right)\cdot dr$$

$f(x)=\sum_{m=1}^{\infty} a_m J_n(k_{n,m}x)$

$a_m=\frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})}\int_0^R x f(x) J_n(k_{n,m}x)dx$

B_m can be obtained in a similar fashion.

since $\|J_n(k_{n,m}x)\|^2=\int_0^R xJ_n^2(k_{n,m}x)\,dx=\frac{R^2}{2}J_{n+1}^2(k_{n,m}R)$

Ex. 1. vibrations of a circular drumhead

$R=1$ ft
$\rho=2$ slugs/ft^2
$T=8$ lf/ft.
$g(r)=0 \longrightarrow B_m=0$
$f(r)=1-r^2$

p. 507 Ex 3

$$A_m=\frac{2}{J_1^2(\alpha_m)}\int_0^1 r f(r)\,J_0\left(\frac{\alpha_m}{R}r\right)dr$$

$$=\frac{2}{J_1^2(\alpha_m)}\int_0^1 r(1-r^2)J_0(\alpha_m r)\,dr$$

$$=\frac{4\,J_2(\alpha_m)}{\alpha_m^2\,J_1^2(\alpha_m)}$$

$J_2(\alpha_m)=\frac{2}{\alpha_m}J_1(\alpha_m)-\cancel{J_0(\alpha_m)}$ (0)

$=\frac{2}{\alpha_m}J_1(\alpha_m)$

$$=\frac{8}{\alpha_m^3 J_1(\alpha_m)}$$

$\langle\, J_1(\alpha_m)$? $=-J_0'(\alpha_m)\,\rangle$

m $\quad\alpha_m\quad$ $J_1(\alpha_m)$ $\quad A_m=\frac{8}{\alpha_m^3 J_1(\alpha_m)}\longrightarrow$ Table.

Thus $f(r)=u(r,0)=\sum_{m=1}^{\infty}A_m J_0(\alpha_m r)=A_1J_0(\alpha_1 r)-\longrightarrow$

Since $\lambda_m=ck_m=c\,\alpha_m/R=2\alpha_m$

$$u(r,t)=\sum_{m=1}^{\infty}A_m\cos\lambda_m t)\,J_0(\alpha_m r)$$

Solution.

where $A_m=\frac{8}{\alpha_m^3 J_1(\alpha_m)}$

12.11 Laplace's Eq'n in Cylindrical and Spherical coordinates. Potential.

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$

solution : (u is called potential theory.

gravitation, electrostatics, steady state heat flow, fluid flow.

from 9.7. (gravitational potential $u(x.y.z) = \frac{c}{r}$ ↳ since $\nabla^2 u = 0$.)

Laplacian in cylindrical coordinates.

$$\underbrace{\underbrace{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}_{\text{in 2-D}} + \frac{\partial^2 u}{\partial z^2}}_{\text{in 3-D}}$$

$(x.y.z) \longrightarrow (r, \theta, z)$

$r\cos\theta = x$

$r\sin\theta = y$

$z = z$

or

$$\left(\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}. \right)$$

Laplacian in spherical coordinates.

$(x.y.z) \longrightarrow (r, \theta, \phi)$

$x = r\cos\theta\sin\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\phi$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \phi^2}$$

Or

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right]$$

BC problem in spherical coordinates. : ⟶ assumption $\frac{\partial^2 u}{\partial \theta^2} = 0$.

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right)\right] = 0$$

$u(r.\phi)$

$u(R.\phi) = f(\phi)$ → electrostatic potential on the surface.

$\lim_{r\to\infty} u(r.\phi) = 0$

Step 1. separating variables. $u(r.\phi) = G(r)H(\phi)$

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial(GH)}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\, G\frac{dH}{d\phi}\right)\right] = 0$$

$$= \frac{1}{r^2}\left[H\left[2r\frac{dG}{dr} + r^2\frac{d^2G}{dr^2}\right] + G\frac{1}{\sin\phi}\left[\cos\phi\frac{dH}{d\phi} + \sin\phi\frac{d^2H}{d\phi^2}\right]\right] = 0$$

$$\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = -\frac{1}{H\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) = k$$

if we set $k = n(n+1)$

① $\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = k$ ① $r^2\frac{d^2G}{dr^2} + 2r\frac{dG}{dr} = kG$ ⇒

$r^2G'' + 2rG' - n(n+1)G = 0$

What is that? Euler cauchy eq'n. $G = r^a$

$a(a-1) + 2a - n(n+1)$ $a^2+a-n(n+1)=0$

$a = n$ $\quad -n-1$

$G_n(r) = r^n \quad G_n^*(r) = \frac{1}{r^{n+1}}$

② and $\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) + kH = 0$.

$\frac{1}{\sin\phi}\frac{d}{d\phi} = -\frac{d}{dw}$

$\sin\phi\frac{d}{dw} = \frac{-d}{d\phi}$

Setting $\cos\phi = w \quad \sin^2\phi = 1-w^2$

$$\frac{1}{d\phi} = \frac{d}{dw}\cdot\frac{dw}{d\phi} = -\sin\phi\frac{d}{dw}$$

$$-\frac{d}{dw}\left(\sin\phi\cdot -\sin\phi\frac{dH}{dw}\right)$$

$$\frac{d}{dw}\left[(1-w^2)\frac{dH}{dw}\right] + n(n+1)H = 0$$

$$\rightarrow (1-w^2)\frac{d^2H}{dw^2} - 2w\frac{dH}{dw} + n(n+1)H = 0$$

$u(r.\phi) = A_n r^n P_n(\cos\phi)$. $\quad u_n^*(r.\phi) = \frac{B_n}{r^{n+1}}P_n(\cos\phi)$ $\quad \Leftarrow H = P_n(w) = P_n(\cos\phi) =$

two sequences of solution. $(n = 0, 1, 2 \ldots)$

Use of Fourier - Legendre Series

$\sum_{n=0}^{\infty} u_n(r,\phi)$ → (A_n^*, A_n ?) boundary condition.

Ⓐ interior problem. $0 \le r \le R$

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi)$$

$$u(R,\phi) = \sum_{n=0}^{\infty} \underline{A_n R^n} P_n(\cos\phi) = f(\phi)$$

Fourier- Legendre series. (p. 505)

$$f(x) = \sum_{n=0}^{\infty} a_m P_m(x) =$$

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x)\, dx$$

$$\|P_m\| = \sqrt{\int_{-1}^{1} P_m^2(x)\, dx} = \sqrt{\frac{2}{2m+1}}$$

since $\frac{1}{\|y_m\|^2}\int_a^b r f(x)\, y_m(x)\, dx$, ① ($r$)

$$A_n R^n = \frac{2n+1}{2}\int_{-1}^{1} \tilde{f}(w) P_n(w)\, dw$$

where $\tilde{f}(w)$ denotes $f(\phi)$ as a function of $w = \cos\phi$

$dw = -\sin\phi\, d\phi$

$-1 \le w \le 1$ → $\pi \le \phi \le 0$

$$A_n = \frac{2n+1}{2R^n}\int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi\, d\phi$$

Exterior problem : potential outside the sphere S. $r \ge R$

we can't use u_n since another B.C $\lim_{r\to\infty} u(r,\phi) = 0$

So, u_n^* can be used.

$$u(r,\phi) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\phi)$$

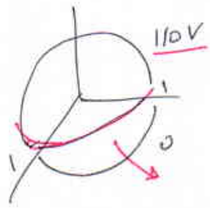
$$\left(\frac{B_n}{R^{n+1}}\right)_{at\ r=R} = \frac{2n+1}{2}\int_{-1}^{1} \tilde{f}(w) P_n(w)\, dw. \qquad w = \cos\phi, \quad dw = -\sin\phi\, d\phi.$$

$$\therefore B_n = \left(\frac{2n+1}{2}\right) R^{n+1}$$

$$B_n = \frac{2n+1}{2} R^{n+1}\int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi \cdot d\phi$$

Ex 1. spherical capacitor

: potential inside and outside a spherical capacitor consisting of two metallic hemisphere of radius 1 ft



B. C

$$f(\phi) = \begin{cases} 110 & \text{if } 0 \le \phi < \pi/2 \\ 0 & \text{if } \pi/2 < \phi \le \pi \end{cases}$$

$$A_n = \frac{2n+1}{2} \cdot 110 \int_0^{\pi} P_n(\cos\phi) \sin\phi\, d\phi$$

$$= \frac{2n+1}{2} \cdot 110 \int_0^{1} P_n(w)\, dw \Leftarrow \text{where } \cos\phi = w,$$

$$A_n = 55(2n+1)\sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}\int_0^1 w^{n-2m}\, dw$$

where $M = n/2$ for even n, $M = (n-1)/2$ for odd n.

$$\therefore A_n = \frac{55(2n+1)}{2^n}\sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{m!(n-m)!(n-2m+1)!} \Leftarrow \frac{1}{n-2m+1}$$

$n=0$	$A_0 = 55$
$n=1$	$A_1 = 165/2$
$n=2$	$A_2 = 0$

$n=3 \Rightarrow A_3 \quad -\frac{385}{8}$

interior

$$u(r,\phi) = 55 + \frac{165}{2} r P_1(\cos\phi) - \frac{385}{8} r^3 P_3(\cos\phi)$$

when $R=1$, $A_n = B_n$!!

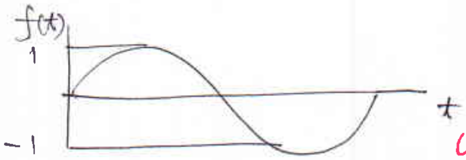
exterior $u(r,\phi) = \frac{55}{r} + \frac{165}{2r^2} P_1(\cos\phi) - \cdots$

12.12. Solution of PDEs by Laplace Transforms

① Take Laplace Transform with respect to ~~the~~ one of two variables, usually t.

②

Ex 1. Semi-infinite string.



B. C.

$w(0,t) = f(t) = \begin{cases} \sin t & \text{if } 0 \le t \le 2\pi \\ 0 & \text{otherwise} \end{cases}$

wave equation. $\lim_{x \to \infty} w(x,t) = 0$

I. C

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \qquad \begin{pmatrix} w(x,0) = 0 \\ w_t(x,0) = 0 \end{pmatrix}$$

$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$

$$\left(\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)\right) \Rightarrow \mathcal{L}\left(\frac{\partial^2 w}{\partial t^2}\right) = s^2\mathcal{L}(w) - s\underbrace{w(x,0)}_{0} - \underbrace{w_t(x,0)}_{0} = c^2\mathcal{L}\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$\mathcal{L}\left(\frac{\partial^2 w}{\partial x^2}\right) = \int_0^\infty e^{-st} \frac{\partial^2 w}{\partial x^2}\, dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st} w\, dt = \frac{\partial^2}{\partial w^2}\mathcal{L}(w(x,t))$$

$\therefore$ $W(x,s) = \mathcal{L}(w(x,t))$

$$s^2 W = c^2 \frac{\partial^2 W}{\partial x^2} \quad \text{thus} \quad \frac{\partial^2 W}{\partial x^2} - \frac{s^2}{c^2}W = 0$$

2nd ODE

$$\therefore W(x,s) = \underline{A(s)}\, e^{sx/c} + \underline{B(s)}\, e^{-sx/c}$$

$c > 0$

$$W(0,s) = \mathcal{L}(w(0,t)) = \mathcal{L}(f(t)) = F(s)$$

$\left(\lim_{x \to \infty} w(x,t) = 0\right)$

$$\lim_{x\to\infty} W(x,s) = \lim_{x\to\infty}\int_0^\infty e^{-st} w(x,t)\, dt = \int_0^\infty e^{-st} \lim_{x\to\infty} w(x,t)\, dt = 0$$

Therefore $A(s) = 0$. since $c > 0$.

$$W(0,s) = F(s) = B(s)$$

$$\therefore \quad W(x,s) = F(s)\, e^{-sx/c}$$

s-shifting

inverse Transform.

unit function
$u(t) = 0$

$$w(x,t) = f\left(t - \frac{x}{c}\right) u\left(t - \frac{x}{c}\right).$$

since $f(x) = \sin x$

that is $W(x,t) = \sin\left(t - \frac{x}{c}\right)$ if $\frac{1}{c}x < t < \frac{x}{c} + 2\pi$

or

$ct > x > (t - 2\pi)c$

Traveling wave Fig 317

$0 < t - \frac{x}{c} < 2\pi$

(very similar to Fourier Transform method)

$$\left\{ \begin{array}{l} u_{tt} = c^2 \nabla^2 u \\ \left(\begin{array}{l} u_{tt} = c^2 u_{xx} \\ u_{tt} = c^2 (u_{xx} + u_{yy}) \end{array} \right) \\ u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r \right) \end{array} \right\}$$

$$\left(\begin{array}{l} u_t = c^2 u_{xx} \\ \nabla^2 u = 0 . \quad u_{xx} + u_{yy} = 0 \end{array} \right)$$

I.C. // B.C

Separating variable

$\hookrightarrow$ PDE $\xrightarrow{(2\sim3)}$ ODE

$\hookrightarrow$ B.C

Eigenfunction with corresponding eigenvalues.

$\hookrightarrow$ I.C.

Fourier series.

or

Fourier integral (not bounded)

$\boxed{\text{Fourier Transform \& Laplace Transform}}$

polar. $\Longrightarrow$ (Fourier - Bessel series
Bessel function

spherical $\Longrightarrow$ (Fourier - Legendre series
Legendre function.

H.W. Due 10/2~~5~~4 (before Exam

10. 12.

12.9 // 4. 11
12.10 // 3. 19.
12.11 // 17 ~~16~~. ~~17~~ 19
12.12 // 5. 7