Chap. 14. Complex Integration

14.1 Line Integral in the Complex Plane

$$\int_C f(z)\,dz \qquad \oint_C f(z)\,dz$$

closed path.

Theorem 1 $\displaystyle\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$ $\quad f(z) = F'(z)$

indefinite integration of analytic functions.

$$\int_0^{i+1} z^2\,dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3$$

integration by the use of the path.

Theorem 2) $\displaystyle\int_C f(z)\,dz = \int_a^b f[z(t)]\,\dot{z}(t)\,dt$

C: representation by $z = z(t)$, $a \le t \le b$

$z = x + iy$. $\quad \dot{z} = \dot{x} + i\dot{y}$

$$\int_a^b f(z(t))\,\dot{z}\,dt = \int_a^b \underbrace{(u+iv)}_{f(z)}(\dot{x}+i\dot{y})\,dt$$

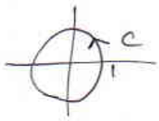
$\dot{x}\,dt = dx \quad \dot{y}\,dt = dy$

$$= \int_C [u\,dx - v\,dy + i(u\,dy + v\,dx)]$$

$$= \int_C [u\,dx - v\,dy] + i\int_C (u\,dy + v\,dx)$$

(A) represent the path C in the form. $z(t)$ $a \le t \le b$
(B) Calculate the derivative $\dot{z}(t) = \frac{dz}{dt}$
(C) Substitute $z(t)$ for every z in $f(z)$
(D) Integrate $f[z(t)]\,\dot{z}(t)$ over t

Ex5) $\displaystyle\oint_C \frac{dz}{z}$ $\quad$ C: unit circle



$|z| = 1$

a) $z(t) = \cos t + i\sin t = e^{it}$ $\quad 0 \le t \le 2\pi$

b) $\dot{z}(t) = ie^{it}$

c) $f(z(t)) = 1/z(t) = e^{-it}$

D) $\displaystyle\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it}\cdot ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i$

$$\therefore \oint_C \frac{dz}{z} = 2\pi i$$

Ex 6

$f(z) = (z-z_0)^m$ $\oint_C f(z) = ?$

$z(t) = z_0 + \rho(\cos t + i\sin t)$ $0 \le t \le 2\pi$

$= z_0 + \rho e^{it}$

$\therefore (z-z_0)^m = \rho^m e^{imt}$

$dz = i\rho e^{it} \cdot dt$

$$\oint_C (z-z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} \cdot i\rho e^{it} dt = i\rho^{m+1} \underbrace{\int_0^{2\pi} e^{i(m+1)t} dt}_{\text{Euler formula}}$$

$$= i\rho^{m+1}\left[\int_0^{2\pi} \cos(m+1)t\,dt + i\int_0^{2\pi} \sin(m+1)t\,dt\right]$$ Euler formula.

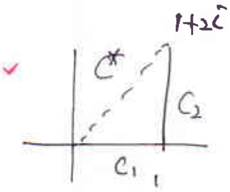
if $m=-1$ $\rho^{m+1} = 1$, $\cos 0 = 1$, $\sin 0 = 0$. $\longrightarrow 2\pi i$

if $m \neq -1$ 0 $\left(\because \int_0^{2\pi}\cos = \int_0^{2\pi}\sin = 0\right)$

$$\therefore \oint_C (z-z_0)^m dz = \begin{cases} 2\pi i & (m=-1) \\ 0 & (m \neq -1 \text{ and integer}) \end{cases}$$

Dependence on path. : Generally path dependence of complex line integral

Ex 7) $f(z) = \operatorname{Re} z = x$ ~~from~~ m 0 to $1+2i$



i) C^* : $z(t) = t + 2it$ $0 \le t \le 1$.

$\dot{z}(t) = 1+2i$

$f(z(t)) = x(t) = t$

$$\int_{C^*} \operatorname{Re} z \cdot dz = \int_0^1 t(1+2i)\,dt = \frac{1}{2}(1+2i) = \frac{1}{2} + i$$

ii) C_1: $z(t) = t$ $\dot{z}(t) = 1$ $f(z(t)) = x(t) = t$. $0 \le t \le 1$

C_2 : $z(t) = 1 + it$, $\dot{z}(t) = i$ $f(z) = x(t) = 1$ $0 \le t \le 2$

$$\int_C \operatorname{Re} z\, dz = \int_{C_1} {}_0^1 t\,dt + \int_{C_2} {}_0^2 i\,dt = \cancel{\tfrac{1}{2}} + \qquad = \frac{1}{2} + 2i$$

differ.

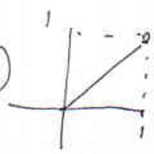
Bounds for integrals. ML-inequality.

$$\left|\int_C f(z)\,dz\right| \le ML.$$

L : length of C

M : a constant such that $|f(z)| \le M$.

Ex 8)



$\int_C z^2 \cdot dz$ $\sqrt{2} = L$. $|f(z)| = |z^2| \le 2$

14.2 Cauchy's Integral Theorem. (가장 중요한 part of this chapter)

시험 要

✓ simple closed path = a closed path that does not intersect or touch itself as show in Fig.

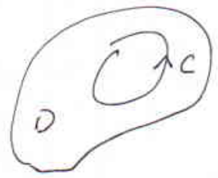
✓ Simply Connected domain D : vs multiply connected domain

vs. doubly, triply.

Theorem 1. Cauchy's Integral Theorem.

If $f(z)$ is analytic in a simply connected domain D, for every simple closed path C in D → also called contour ∴ contour integral

$$\oint_C f(z)\,dz = 0.$$



assumption

proof) From (8) in sec 14.1 " $f'(z)$ is continuous "

$$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx)$$

Green's Theorem ⇒ $\left(\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy = \oint_C (F_1 dx + F_2 dy)\right)$

$F_1 = u$, $-v = F_2$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

∵ analytic $f(z)$

은 $(=0$ ∵ Due to second Cauchy-Riemann equation)

$-v_x - u_y$ ($u_y = -v_x$).

in the same way

$$\oint_C (u\,dy + v\,dx) = \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) dx\,dy = 0$$

due to first C-R equ $u_x = v_y$

Ex 1) $\oint_C e^z dz = 0$ $\quad \oint_C \cos z\,dz = 0$ $\quad \oint_C z^n\,dz = 0$ for any closed path

since this functions are entire (or analytic for all z)

Ex 2) $\oint_C \sec z\,dz = 0$ $\quad \oint_C \frac{dz}{z^2+4} = 0.$ ⇒ c : unit circle,

$\oint_C \frac{1}{\cos z}$ ⇒ $\cos z = 0$ not analytic at $z = \pm\frac{\pi}{2}, \pm\frac{2\pi}{2} \cdots$, but outside C therefore 0

Ex 3) $\oint \frac{dz}{z^2+4}$ $z = \pm 2i$ analytic, but outside C ∴ $\oint \frac{dz}{z^2+4} = 0$

Ex 4) Nonanalytic function

$$\oint_C \bar{z}\,dz = \int_0^{2\pi} e^{-it}\,i e^{it}\,dt = 2\pi i$$

, not applied Cauchy's Integral Theorem

Ex 5) $\oint_C \frac{dz}{z^2} = \int_0^{2\pi} e^{-2it}\,i\,e^{it}\,dt = i\int_0^{2\pi} e^{-it}\,dt = - e^{-it}\Big|_0^{2\pi} = 0$

0 이라고 해서 analytic 이 아니다.

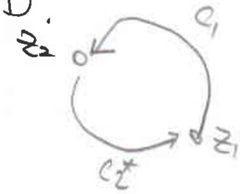
However, $\frac{1}{z^2}$ is not analytic at $z=0$.

∴ condition that f be analytic in D is sufficient rather than necessary condition.

more more example in the back

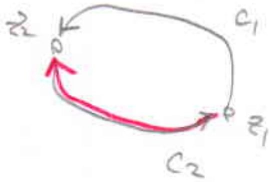
Independence of Path.

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.



$$\oint_C f(z) = \int_{C_1} f(z)\,dz + \int_{C_2^*} f(\,dz = 0,$$

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$



Principle of Deformation of Path.

(As long as our deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same ~~sa~~ value.)

Existence of Indefinite Integral.

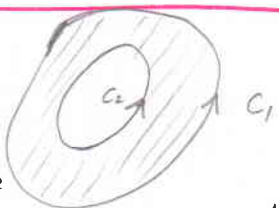
Theorem 3 ~~Ex.~~ If $f(z)$ is analytic in a simply connected domain

applied to multiply connected domains.

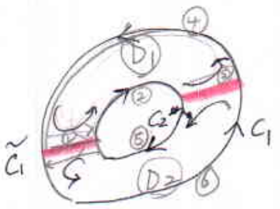
Cauchy's Integral Theorem for Multiply Connected domains.

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

both integral being taken counterclockwise



doubly connected domain



proof

$\int_{D_1} = 0$ $\qquad \int_{D_2} = 0.$

$① + ② + ③ + ④ = 0 \quad A \qquad ⑥ - ③ + ⑤ - ① = 0 \quad B$

$(A)+(B)$

$② + ④ + ⑤ + ⑥ = 0$

$C_1 = ④ + ⑥$

$-C_2 = ② + ⑤$

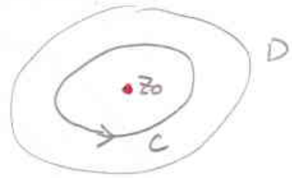
$\therefore C_1 - C_2 = 0$

$C_1 = C_2$ //

14.3 Cauchy's Integral formula. ← Cauchy's integral theorem.

(Theorem 1) Let $f(z)$ be analytic, in a simply connected domain, for any point z_0 in D and any closed path C in D and encloses z_0

$$\oint_C \frac{f(z)\,dz}{z-z_0} = 2\pi i\, f(z_0)$$ Cauchy's integral formula



or $$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz.$$

proof) $f(z) = f(z_0) + [f(z)-f(z_0)]$

$$\oint_C \frac{f(z)}{z-z_0}dz = f(z_0)\oint_C \frac{dz}{z-z_0} + \oint_C \underbrace{\frac{f(z)-f(z_0)}{z-z_0}}_{\text{analytic except } z=z_0} dz$$

$= f(z_0)\cdot 2\pi i$

앞에서.

$$\boxed{\oint_C \frac{dz}{z-0} = 2\pi i f(0) = 2\pi i}$$

0. 그러므로. C 보다 작은 Contour K of radius ρ and center z_0.

그러므로 an $\varepsilon>0$ being given, we can find a $\delta>0$ such that $|f(z)-f(z_0)|<\varepsilon$ for all z in the disk $|z-z_0|<\delta$. ($\rho<\delta$)

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\varepsilon}{\rho}$$

integration $\oint_K \left|\quad\right| dz < \frac{\varepsilon}{\rho}2\pi\rho = 2\pi\varepsilon$

for [redacted] $z_0=2$ is outside C. 0.

Ex 1. $$\oint_C \frac{e^z}{z-2}dz = 2\pi i\, e^z\Big|_{z=2} = 2\pi i\, e^2 \qquad f(z)=e^z$$

(C encloses 2) → $2\pi i f(z_0) = 2\pi i f(2)$

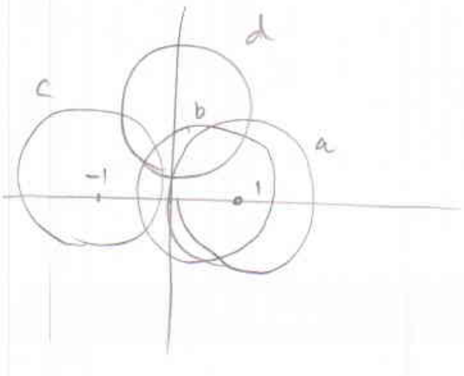
Ex 2 $$\oint_C \frac{z^3-6}{2z-i}dz = \oint_C \frac{\frac12 z^3-3}{z-\frac12 i}dz = 2\pi i f\left(\tfrac{i}{2}\right) = 2\pi i\left(\tfrac12\left(\tfrac12 i\right)^3-3\right)$$

↳ if C enclose $\frac12 i$ $\qquad f(z)=\frac{z^3}{2}-3 \qquad = \frac{\pi}{8} - 6\pi i$

Ex 3. $$\oint_C \left[f(z) = \frac{z^2+1}{z^2-1}\right] dz$$ where 4 C exists (Fig 358)

[not analytic at $z=1,-1$] →

(a) C : $|z-1|=1$ enclosed the point $z_0=1$, where $g(z)$ is not analytic



$$\oint_C \frac{\overbrace{(z^2+1)/z+1}^{f(z)}}{z-1}dz = 2\pi i f(1) = 2\pi i$$

(b) same

(c) $$\oint_C \frac{(z^2+1)/z-1}{z+1}dz = 2\pi i f(-1) = -2\pi i$$

(d) 0

14.4 Derivatives of Analytic Functions

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\cdot dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\, dz$$

...

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\cdot dz.$$

C: any simple closed path in D that encloses z_0 and whose full interior belongs to D

Cauchy's integral formula $\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\, dz}$

proof) $f'(z_0) = \lim_{\Delta z \to 0} \boxed{\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}}$

$$\to \frac{1}{2\pi i \Delta z}\left[\oint_C \frac{f(z)}{z-(z_0+\Delta z)}\, dz - \oint \frac{f(z)}{z-z_0}\, dz\right]$$

$$\frac{1}{2\pi i \Delta z}\oint_C \frac{\cancel{zf(z)} - z_0 f(z) - \cancel{zf(z)} + z_0 f(z) + \Delta z f(z)}{(z-z_0-\Delta z)(z-z_0)}\, dz$$

$$\frac{1}{2\pi i \cancel{\Delta z}}\oint_C \frac{\cancel{\Delta z} f(z)}{(z-z_0-\Delta z)(z-z_0)}\, dz$$

as we take $\Delta z \to 0$.

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\, dz \quad \cdots \text{ proof by induction}$$

Ex1) $\oint_C \frac{\cos z}{(z-\pi i)^2}\, dz = 2\pi i f'(\pi i) = -2\pi i \sin \pi i = -2\pi \sinh \pi$

$f(z)$, $-\sin \pi i$

any contour enclosing πi

Ex2) $\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\, dz$ for any contour enclosing $-i$

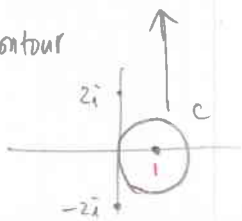
$\frac{2}{2!}\pi i f''(-i) = \pi i(-12-6) = -18\pi i.$

$f(z) = 4z^3 - 6z \qquad f''(z) = 12z^2 - 6$

Ex3) $\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\, dz = \oint_C \frac{f(z)}{(z-1)^2}\, dz$ where $f(z) = \frac{e^z}{z^2+4}$

$\therefore 2\pi i f'(1)$

for any contour



Cauchy's Inequality. Liouville's and Morera's Theorem.

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \le \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}$$

C : circle of radius ~~and~~

$|f(z)| \le M$

Liouville's Theorem.

If an entire function is bounded in an absolute value in the whole complex plane,

This function is constant.

Morera's Theorem.

If $f(z)$ is continuous in a simply connected domain D

if $\oint_C f(z)\,dz = 0$

for every closed path in D

then $f(z)$ is analytic in D.

14.1 26. 27

14.2 12. 22. 23. 25.

14.3 2. 13. 17

14.4

3. 4 10. 13