integration of complex function

Chap 16. Laurent Series. Residue Integration.

i) Cauchy's integral formula ① $f(z) = \frac{1}{2\pi i}\oint \frac{f(z^*)}{z^*-z}dz^*$

ii) Another method for evaluating complex integrals and certain real integral : residue integration.

Laurent Series generalized Taylor Series. → can't develop $f(z)$ in powers of $z-z_0$ when $f(z)$ is singular at z_0.

$\gg$ larger concept

→ Converges in a disk, positive powers of $z-z_0$ — But Laurent Series can.

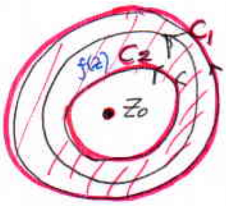
→ negative powers of $z-z_0$, converges in an annulus with center z_0.

→ We can represent a given function $f(z)$ that is analytic in an annulus.

①-1 $\oint_C f(z)\,dz = 0$.

Laurent's Theorem.

($f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2 with the center z_0 and the annulus between them)



(for path C between C_1 and C_2)

Then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

where $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$ and $b_n = \frac{1}{2\pi i}\oint_C (z^*-z_0)^{n-1} f(z^*)\,dz^*$ (n=1 인 경우 Residue integration)

alternatively $\left[f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^* \right]$

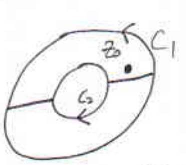
where ($n = 0, \pm 1, \pm 2, \pm 3 \ldots$)

* [principal part of $f(z)$ at z_0 : series (or finite term) of the negative powers]

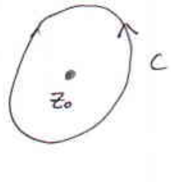
(proof : skip)

Proof) (a) nonnegative powers : Taylor Series.

from (3) in 14.3



[Cauchy's integral formula ①



$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$

Cauchy's integral Theorem]

$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)\,dz$ -

$f(z_0) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)dz}{z-z_0} + \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}dz$

$\therefore f(z) = g(z) + h(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z^*)}{z^*-z}dz^* - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z^*)}{z^*-z}dz^*$

· integrate counterclockwise over both C_1 and C_2.

z any point in the given annulus.

$\frac{1}{2\pi i}\oint_{C_1}\frac{f(z^*)}{z^*-z}dz^* = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

Taylor series with coefficients $a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$

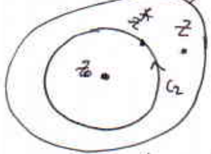
· C_1 can be replaced by C.

· z_0, the point where the integrand ~~(5)~~ is not analytic

(b) negative powers.

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad \text{where} \quad b_n = \frac{1}{2\pi i}\oint_c (z^*-z_0)^{n-1} f(z^*)\,dz^*$$

$$-\frac{1}{2\pi i}\oint_{C_2} \frac{f(z^*)}{z^*-z}\,dz^*$$ where z is in the annulus, outside of the path C_2.



$$\left|\frac{z^*-z_0}{z-z_0}\right| < 1$$

$$\therefore \frac{1}{z^*-z} \longrightarrow \text{develop in the powers of } \left|\frac{z^*-z_0}{z-z_0}\right|$$

$$\therefore \frac{1}{z^*-z} = \frac{1}{z^*-z_0-(z-z_0)} = \frac{-1}{(z-z_0)\left(1-\frac{z^*-z_0}{z-z_0}\right)} = \frac{-1}{z-z_0}\left(1+\frac{z^*-z_0}{z-z_0}+\left(\frac{z^*-z_0}{z-z_0}\right)^2+\cdots\left(\frac{z^*-z_0}{z-z_0}\right)^n\right)$$
$$-\frac{1}{z-z^*}\left(\frac{z^*-z_0}{z-z_0}\right)^{n+1}$$

multiplication $-f(z^*)/2\pi i$ integration over C_2

$$h(z) = -\frac{1}{2\pi i}\oint_{C_2} \frac{f(z^*)}{z^*-z}\,dz^*$$

$$= \frac{1}{2\pi i}\left\{\frac{1}{z-z_0}\oint_{C_2} f(z^*)\,dz^* + \frac{1}{(z-z_0)^2}\oint (z^*-z_0)f(z^*)\,dz^* + \cdots + \frac{1}{(z-z_0)^n}\oint_{C_2}(z^*-z_0)^{n-1}f(z^*)\,dz^*\right.$$
$$\left. + \frac{1}{(z-z_0)^{n+1}}\oint_{C_2}(z^*-z_0)^n f(z^*)\,dz^*\right\} + R_n^*(z)$$

Skip!

Ex1) Lauren series of $z^{-5}\sin z$ with center 0 (not analytic)

$$\sin z = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$$

$$\therefore z^{-5}\sin z = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n-4} = \underbrace{\frac{1}{z^4} - \frac{1}{6z^2}}_{\text{principal part}} + \frac{1}{120} - \frac{1}{5040}z^2 \cdots \qquad \underbrace{|z|>0}_{\text{annulus of convergence}}$$

(not analytic)

Ex2) $z^2 e^{1/z}$ with center 0.

$$e^z = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z^n}\right)$$

$$\therefore z^2 e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!}z^{2-n} = z^2 + z + \underbrace{\frac{1}{2} + \frac{1}{3!}\frac{1}{z} + \cdots}_{\text{principal part}} \qquad |z|>0.$$

principal part is infinite series

Ex3) $\frac{1}{1-z}$

ⓐ nonnegative power

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z|<1$$

ⓑ negative power

$$\frac{1}{1-z} = \frac{-1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \cdots \qquad |z|>1$$

$$\hookrightarrow \left|\frac{1}{z}\right| < 1$$

Ex 4. $\frac{1}{z^3 - z^4} = \frac{1}{z^3(1-z)}$ from Ex 3.

skip

if $|z|<1$ $\quad \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots$

if $|z|>1$ $\quad -\frac{1}{z^4} - \frac{1}{z^5} - \cdots$

Ex5. Taylor and Laurent series $f(z) = \frac{-2z+3}{z^2-3z+2}$ with center 0.

$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}$

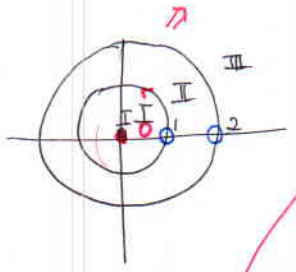
from Ex 3.

$-\frac{1}{z-1}$: $\sum_{n=0}^{\infty} z^n \quad |z|<1$ ✓

$\sum_{n=0}^{\infty} -\frac{1}{z^{n+1}} \quad |z|>1$ ✓

$-\frac{1}{z-2} = \frac{1}{2(1-\frac{1}{2}z)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \quad |z|<2$ ✓

$-\frac{1}{z-2} = -\frac{1}{z(1-\frac{2}{z})} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \quad |z|>2$ ✓



I : $|z|<1$

$\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n = \sum \left(1 + \frac{1}{2^{n+1}}\right) z^n$ //

II : $1<|z|<2$

$\sum_{n=0}^{\infty} \left(-\frac{1}{z^{n+1}} + \frac{1}{2^{n+1}} z^n \right) = \frac{1}{2} + \frac{1}{4} z + \frac{1}{8} z^2 \cdots \underbrace{- \frac{1}{z} - \frac{1}{z^2} - \cdots}_{b_n}$

III : $|z|>2$

$\sum \left(-\frac{1}{z^{n+1}} - \frac{2^n}{z^{n+1}} \right) = -\sum \left(1+2^n\right) \frac{1}{z^{n+1}}$

$\hookrightarrow$ all b_n

$\oint_C f(z)\,dz = 0$

analytic.

If $f(z)$ in Laurent theorem is analytic inside C_2,

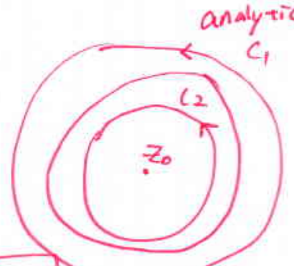
the coefficient $b_n = 0$ by Cauchy's integral theorem.

So that Laurent series becomes Taylor series.

For example. $\left(\begin{array}{l} \text{Ex 5. I,} \\ \text{Ex 3. (a)} \end{array} \right)$

$b_n = \frac{1}{2\pi i} \oint_C (z^* - z_0)^{n-1} f(z^*)\,dz^*$

analytic $\quad$ analytic



$\boxed{\text{Closed domain}}$.

16.2 Singularity and Zeros. Infinity.

i) $f(z)$ is singular or has a singularity at a point $z=z_0$ if $f(z)$ is not analytic at $z=z_0$.

ii) isolated singularity if $z=z_0$ has a neighborhood without further singularities of $f(z)$

$\tan z$: isolated singularities at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

$\tan\frac{1}{z}$: nonisolated singularity. at $z=0$.

Isolated singularities of $f(z)$ at $z=z_0$ can be classified by the Laurent series

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

valid in the intermediate neighborhood of the singular point $z=z_0$ except at z_0 itself

$$0<|z-z_0|<R$$

analytic at $z=z_0$

i) If it has only finitely many terms.

$$\frac{b_1}{z-z_0}+\cdots+\frac{b_m}{(z-z_0)^m} \quad (b_m\neq 0)$$

[singularity of $f(z)$ at $z=z_0$: a pole with the order m.]

1st order pole : simple poles.

ii) infinitely many terms : $\boxed{\text{isolated essential singularity}}$

Ex 1) i) $f(z)=\frac{1}{z(z-2)^5}+\frac{3}{(z-2)^2}$ $\Rightarrow$ Simple pole at $z=0$, 5th order pole at $z=2$.

ii) $e^{1/z}=\sum\frac{1}{n!\,z^n}=1+\frac{1}{z}+\frac{1}{2!z^2}+\cdots$ isolated essential singularity at $z=0$.

$\sin\frac{1}{z}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{z^{2n+1}}=\frac{1}{z}-\frac{1}{3!z^3}+\frac{1}{5!z^5}-+\cdots$ 〃

iii) $z^{-5}\sin z=\frac{1}{z^4}-\frac{1}{6z^2}+\frac{1}{120}-\frac{1}{5040}z^2+-\cdots$: 4th order pole at $z=0$.

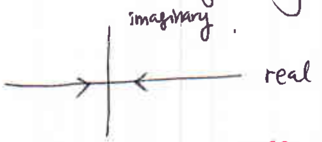
Ex 2) Behavior near a pole

$f(z)=\frac{1}{z^2}$ pole at $z=0$. as $z\to 0$, $|f(z)|\to\infty$

$\boxed{\text{Theorem 4) Poles}}$

(If $f(z)$ is analytic and has a pole at $z=z_0$, $|f(z)|\to\infty$ as $z\to z_0$ in any manner.)

Ex 3) $e^{1/z}$ essential singularity at $z=0$.



$$e^{1/z} = \sum \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

as $z \to 0^+$: ∞ e^{∞}

$z \to 0^-$: 0 $e^{-\infty}$

It takes on any given value $C = C_0 e^{i\alpha} \neq 0$ in an arbitrarily small ε-neighborhood of $z=0$.

Skip

Set $z = re^{i\theta}$

$$e^{1/z} = e^{(\cos\theta - i\sin\theta)/r} = C_0 e^{i\alpha}$$

$$\tan\theta = -\frac{\alpha}{\ln C_0}$$

$e^{\cos\theta/r} = C_0$ $\quad \cos\theta = r\ln C_0$ $\quad \cos^2\theta + \sin^2\theta = 1$

$\quad r^2 \ln^2 C_0 + \alpha^2 r^2 = 1$

$-\frac{\sin\theta}{r} = \alpha$ $\quad -\sin\theta = \alpha r$ $\quad r^2 = \frac{1}{(\ln C_0)^2 + \alpha^2}$

Hence r can be made arbitrarily small by adding multiples of 2π to α leaving C unaltered.

→ Picard's Theorem.

If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every value, (with at most one exceptional value,) in an arbitrarily small ε-neighborhood of z_0.

Removable singularities.

if $f(z)$ is not analytic at $z=z_0$. but can be made analytic there by assigning a suitable value $f(z_0)$

Ex. $f(z) = \frac{\sin z}{z}$ $\Rightarrow$ becomes analytic at $z=0$ if we define $f(0)=1$.

Zeros of analytic function $f(z)$ in a domain

: a $z=z_0$ in D such that $f(z_0)=0$.

A zero has order n $\longleftrightarrow$ $f, f', f'' \ldots f^{(n-1)}$ all 0 at $z=z_0$ but $f^{(n)}(z_0) \neq 0$.

ex) 2nd order zero, $f(z_0) = f'(z_0) = 0$, $f''(z_0) \neq 0$.

Ex 4. zeros.

$1+z^2$: simple zeros at $\pm i$

$(1-z^4)^2$: 2nd order zeros at $\pm 1, \pm i$ ✓

$\sin z$: simple zeros at $0, \pm\pi, \pm 2\pi$. ✓

$\sin^2 z$: 2nd order zeros at $0, \pm 2\pi, \pm 4\pi$

$2\sin z \cos z = \sin 2z$

Taylor Series at a zero.

at an nth-order zero $z=z_0$ of $f(z)$ $\rightarrow f'(z_0) \ldots f^{(n-1)}(z_0)=0$.

$\rightarrow a_0 \ldots a_{n-1}=0$ ✓

Theorem 3. zeros.

The zeros of an analytic function $f(z)$ ($\not\equiv 0$) are isolated

$\longrightarrow$ each of them has a neighborhood that contains no further zeros of $f(z)$

Theorem 4. Poles and zeros.

Let $f(z)$ be analytic at $z=z_0$ and have a zero of nth order at $z=z_0$.

Then $1/f(z)$ has a pole of nth order at $z=z_0$.

and so does $h(z)/f(z)$, provided that $h(z)$ is analytic at $z=z_0$ and $h(z_0)\neq 0$.

✓ Riemann Sphere. Point at infinity.

✓ Analytic or Singular at infinity.

investigate a function $f(z)$ for large $|z|$, set a $z=1/w$

a neighborhood of $w=0$.

$f(z)=f(1/w)\equiv g(w)$ in a neighborhood of $w=0$.

We define $f(z)$ to be analytic or singular at infinity

if $g(w)$ is analytic or singular, respectively at $w=0$.

$g(0)=\lim_{w\to 0} g(w)$

Ex 5) i) $f(z)=\frac{1}{z^2}$ analytic at infinity (since $g(w)=f(1/w)=w^2$ is analytic at $w=0$.

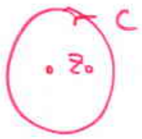
$\longrightarrow$ 2nd order zero at ∞

ii) $f(z)=z^3$ Singular at ∞ and 3rd order pole since $g(w)=f(1/w)=1/w^3$ has a pole at $w=0$.

iii) $f(z)=e^z$ essential singularity at ∞ since $e^{1/w}$ has an essential singularity at $w=0$.

16.3 Residue integration method.

Purpose of Cauchy's residue integration method
is the evaluation of integrals $\oint_C f(z)\,dz$



If $f(z)$ has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C and inside C.

Then $f(z)$ has Laurent series.

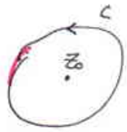
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

converges for all points near $z = z_0$ (except $z=z_0$ itself)

$0 < |z - z_0| < R$.

(b_1 = 1st negative power of $\frac{1}{z-z_0}$ of the Laurent series.) from Laurent's Theorem

$$b_1 = \frac{1}{2\pi i}\oint_C f(z)\,dz \quad \text{or} \quad \left(\operatorname*{Res}_{z=z_0} f(z)\right)$$



$$\oint_C f(z)\,dz = 2\pi i b_1$$

Ex1) $f(z) = z^{-4}\sin z$ counterclockwise around the unit circle C

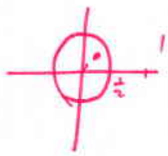
$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{1}{5!}z - \quad |z|>0$$

$f(z)$ has a pole of third order at $z=0$.

residue $b_1 = -\frac{1}{3!}$ $\quad \oint_C \frac{\sin z}{z^4}\,dz = 2\pi i b_1 = -\frac{\pi i}{3}$

Ex2) $f(z) = \dfrac{1}{z^3 - z^4}$ integrate clockwise around the circle $|z| = \frac{1}{2}$

clockwise $\to (-)$



singular at $z=0$ and $z=1$

$z=0 \to$ inside C

$z=1 \to$ outside C

$\therefore$ need the residue at $z=0$.

from Laurent series that converges for $0<|z|<1$

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \quad (0<|z|<1)$$

$$\therefore\ -2\pi i \left[\operatorname*{Res}_{z=0} f(z)\right] = -2\pi i$$

1.

Formulas for residues

Simple poles at z_0 :

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z-z_0) f(z)$$

or

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Proof

① For a simple pole at $z=z_0$ $\longrightarrow$ $f(z)=\dfrac{p(z)}{q(z)}$ $\quad q(z)\to$ simple zero at $z=z_0$

$$f(z)=\frac{b_1}{z-z_0}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots \qquad (0<|z-z_0|<R)$$

$$\lim_{z\to z_0}(z-z_0)f(z)=b_1+\lim_{z\to z_0}(z-z_0)\left[a_0+a_1(z-z_0)\cdots\right]=b_1$$

② $q(z)=\overset{q(z_0)=0}{}(z-z_0)q'(z_0)+\dfrac{(z-z_0)^2}{2!}q''(z_0)+\cdots$ and set $f=p/q$

$$\operatorname*{Res}_{z=z_0}f(z)=\lim_{z\to z_0}(z-z_0)\frac{p(z)}{q(z)}=\lim_{z\to z_0}\frac{\cancel{(z-z_0)}\,p(z)}{\cancel{(z-z_0)}\left[q'(z_0)+(z-z_0)q''(z_0)/2+\cdots\right]}=\frac{p(z_0)}{q'(z_0)}$$

Ex3) Residue at a simple pole

$$f(z)=(9z+i)/(z^3+z) \qquad i$$

① $$\operatorname*{Res}_{z=i}\frac{9z+i}{z(z^2+1)}=\lim_{z\to i}\cancel{(z-i)}\frac{9z+i}{z(z+i)\cancel{(z-i)}}=\frac{10i}{-2}=-5i$$ ✓

or ② $$\operatorname*{Res}_{z=i}\frac{9z+i}{(z(z^2+1))'}=\lim_{z\to i}\frac{9z+i}{(3z^2+1)}=\frac{10i}{-2}=-5i$$ ✓

$f(z)=\dfrac{b_m}{(z-z_0)^m}+\cdots+\dfrac{b_1}{(z-z_0)}+a_0+a_1(z-z_0)+\cdots$

$(z-z_0)^m f(z)=b_m+b_{m-1}(z-z_0)\cdots+b_1(z-z_0)^{m-1}$

$b_1=\dfrac{1}{(m-1)!}g^{(m-1)}(z_0)$

coefficient of the power $(z-z_0)^{m-1}$ $=\dfrac{1}{(m-1)!}\dfrac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$

Poles of any order at z_0 / at mth-order pole at z_0

$$\operatorname*{Res}_{z=z_0}f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right\}$$ ✓

For 2nd order pole ($m=2$)

$$\operatorname*{Res}_{z=z_0}f(z)=\lim_{z\to z_0}\left\{(z-z_0)^2f(z)\right\}'$$

ex) $$f(z)=50z/(z^3+2z^2-7z+4)=50z/(z-1)^2(z+4)$$ 2nd order pole at $z=1$ ✓

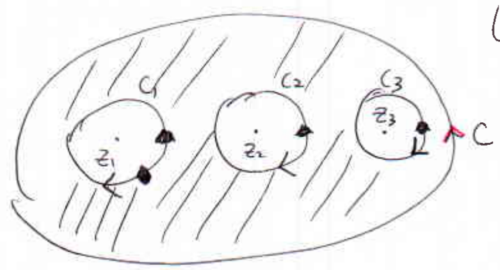
$$\operatorname*{Res}_{z=1}f(z)=\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2f(z)\right]=\lim_{z\to 1}\frac{d}{dz}\left(\frac{50z}{z+4}\right)=\frac{200}{5^2}=8$$

$$\frac{50(z+4)-50z}{(z+4)^2}=\frac{200}{(z+4)^2}$$

Several singularities inside the contour. Residue theorem.

$$\oint_C f(z)\,dz=2\pi i\sum_{j=1}^{k}\operatorname*{Res}_{z=z_j}f(z)$$

↳ singular point inside C



Cauchy's integral Theorem.

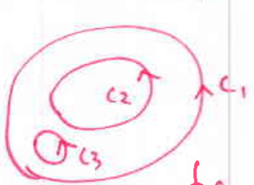
$$\oint_C f(z)dz+\oint_{C_1}f(z)dz+\oint_{C_2}f(z)dz+\cdots+\oint_{C_k}f(z)dz=0$$

C ↓ Counterclockwise $\qquad$ C_k's : clockwise.

$$\oint_C f(z)dz=\oint_{C_1}f(z)dz+\oint_{C_2}f(z)dz+\cdots+\oint_{C_k}f(z)dz$$

counterclockwise $=\oint_{C_j}f(z)dz=2\pi i\operatorname{Res}f(z)$

앞에서



$\oint_{C_1}=\oint_{C_2}$

$$\therefore\ \oint f(z)\,dz=2\pi i\sum_{j=1}^{k}\operatorname*{Res}_{z=z_j}f(z)$$

Ex5. $\oint_C \frac{4-3z}{z^2-z}\,dz$

4 cases

i) C encloses 0, 1 $\quad 2\pi i(-4+1) = -6\pi i$

ii) 0 $\quad 2\pi i(-4) = -8\pi i$

iii) 1 $\quad 2\pi i(1) = 2\pi i$

iv) 0 and 1 are outside. $\quad 0$.

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left.\frac{4-3z}{z-1}\right|_{z=0} = -4 \checkmark$$

$$\operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left.\frac{4-3z}{z}\right|_{z=1} = 1 \checkmark$$

Ex6. $\oint_C \frac{\tan z}{z^2-1}\,dz$ $\quad$ C : counterclockwise $|z| = \frac{3}{2}$

→ 분자에도 있을 수 있음.

① check singular points

② check inside or outside C

③ Count the ~~inside~~ singular points inside C.

✓ two possibilities

$\tan z : \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \cdots$ not analytic, but outside C.

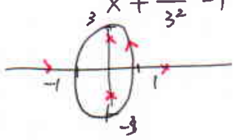
$z^2-1 = (z+1)(z-1)$

$$\oint_C \frac{\tan z}{z^2-1}\,dz = 2\pi i\left(\operatorname*{Res}_{z=1}\frac{\tan z}{z^2-1} + \operatorname*{Res}_{z=-1}\frac{\tan z}{z^2-1}\right) = 2\pi i\left(\left.\frac{\tan z}{2z}\right|_{z=1} + \left.\frac{\tan z}{2z}\right|_{z=-1}\right)$$

$$= 2\pi i \tan 1.$$

Ex 7. $\oint_C \left(\frac{z e^{\pi z}}{z^4-16} + z e^{\pi/z}\right) dz$ $\quad$ C : ellipse $4x^2+y^2 = 9$ counterclockwise.

$x^2 + \frac{y^2}{3^2} = 1$



1st term $\quad z^4-16=0 \quad \pm 2i, \pm 2$ → no interest.

$\pm 2i$: inside C

$$\operatorname*{Res}_{z=2i}\left(\frac{ze^{\pi z}}{z^4-16}\right) = \left.\frac{ze^{\pi z}}{4z^3}\right|_{z=2i} = -\frac{1}{16} \checkmark$$

$$\operatorname*{Res}_{z=-2i}\left(\frac{ze^{\pi z}}{z^4-16}\right) = \left.\frac{ze^{\pi z}}{4z^3}\right|_{z=-2i} = -\frac{1}{16} \checkmark$$

2nd term → essential singularity at $z=0$.

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right)$$

$$= z + \pi + \underbrace{\frac{\pi^2}{2}}_{b_1}\frac{1}{z} + \frac{\pi^3}{6}\frac{1}{z^2} + \cdots$$

$$2\pi i\left(-\frac{1}{16} - \frac{1}{16} + \frac{\pi^2}{2}\right)$$

16.4 Residue Integration of Real integrals.

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

substitution. $z = e^{i\theta}$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\frac{dz}{d\theta} = ie^{i\theta} = iz \quad \left(d\theta = \frac{dz}{iz}\right)$$

$$\therefore J = \oint_C f(z)\, \frac{dz}{iz}$$

C : unit circle.

Ex 1)

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad d\theta = \frac{dz}{iz}$$

$$\oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)} = \oint_C \frac{dz}{-\frac{i}{2}\left(z^2 - 2\sqrt{2}z + 1\right)} = -\frac{2}{i}\oint_C \frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}$$

$z = \sqrt{2}+1$, $\sqrt{2}-1$

↑ no interest ($\because$ outside C)

$$\operatorname{Res} \frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)} = \left.\frac{1}{z-\sqrt{2}-1}\right|_{z=\sqrt{2}-1} = -\frac{1}{2}$$

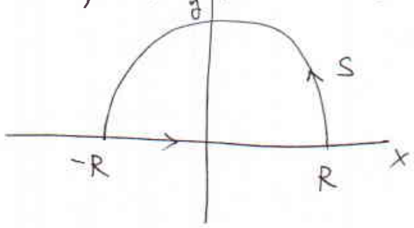
$$\therefore 2\pi i\left(-\frac{1}{2}\right)\left(-\frac{2}{i}\right) = 2\pi$$

$$\int_{-\infty}^{\infty} f(x)\, dx$$

: improper integral (not finite interval)

$$= \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx \Longrightarrow$$ (Cauchy principal value) of the integral.

$$= 2\pi i \sum \operatorname{Res} f(z).$$



$\oint_C f(z)\, dz$ around C. (upper plane)

$\oint$ Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper half-plane.

$$\oint_C f(z)\, dz = \underbrace{\int_S f(z)\, dz}_{z = Re^{i\theta},\ 0\le\theta\le\pi} + \int_{-R}^{R} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z)$$

Ex 2. $f(z)$ $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

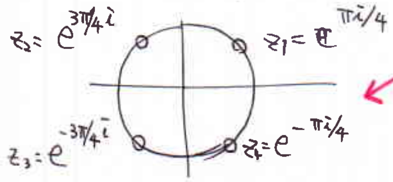
first $\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\int_0^\infty \frac{dx}{1+x^4}$. even function

$f(z) = \frac{1}{1+z^4}$ 4 simple poles.

$z^4 = -1$

$= \cos\pi + i\sin\pi$

$\left(\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$



only z_1, z_2 lies in the upper plane.

$$\operatorname*{Res}_{z=z_1} f(z) = \left(\frac{1}{(1+z^4)'}\right)_{z=z_1} = \frac{1}{4z^3}\bigg|_{z=z_1} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4}e^{\pi i/4}$$

$e^{\pi i}$

$$\operatorname*{Res}_{z=z_2} f(z) = \frac{1}{4z^3}\bigg|_{z=z_2} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4}e^{-\pi i/4}$$

$e^{3\pi/4 i}$

$$2\pi i\left(-\frac{1}{4}e^{\pi i/4} + \frac{1}{4}e^{-\pi i/4}\right) = -\frac{2\pi i}{4}\left(2i\sin\frac{\pi}{4}\right) = \pi\sin\frac{\pi}{2} = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Fourier integrals

$$\int_{-\infty}^{\infty} f(x)\cos sx\,dx \qquad \int_{-\infty}^{\infty} f(x)\sin sx\,dx \quad ?$$

$$\oint_c f(z)e^{isz}\,dz \qquad (s : \text{real positive})$$

$$\int_{-\infty}^{\infty} f(x)e^{isx}\,dx = (2\pi i)\left[\sum \operatorname{Res}\left[f(z)e^{isz}\right]\right] \rightarrow \boxed{\text{Real} + i\,\text{IM}}$$

where sum the residue of $f(z)e^{isz}$ at its poles in the upper half-plane

$$\int_{-\infty}^{\infty} f(x)\cos xs\,dx + i\int_{-\infty}^{\infty} f(x)\sin sx\,dx = -2\pi\sum \operatorname{Im}\operatorname{Res}\left[f(z)e^{isz}\right] + i\,2\pi\sum \operatorname{Re}\operatorname{Res}\left[f(z)e^{isz}\right]$$

$$\therefore \int_{-\infty}^{\infty} f(x)\cos sx\,dx = -2\pi\sum \operatorname{Im}\operatorname{Res}\left[f(z)e^{isz}\right]$$

$$\int_{-\infty}^{\infty} f(x)\sin sx\,dx = 2\pi\sum \operatorname{Re}\operatorname{Res}\left[f(z)e^{isz}\right].$$

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2}\,dx = \frac{\pi}{k} e^{-ks} \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2+x^2}\,dx = 0$$

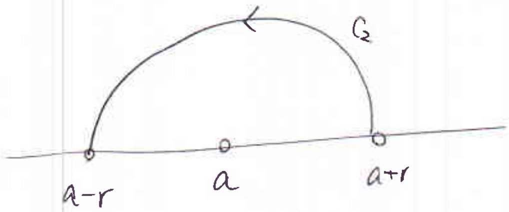
$e^{isz}\dfrac{1}{k^2+z^2}$ only one pole in the upper half-plane $z = ki$

$$\operatorname*{Res}_{z=ik} \frac{e^{isz}}{z^2+k^2} = \lim_{z\to ik} \frac{e^{isz}}{2z} = \frac{e^{-sk}}{2ik}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\overset{e^{isx}}{\cancel{\cos sx}}}{k^2+x^2}\,dx = 2\pi i \cdot \frac{e^{-ks}}{2ik} = \frac{\pi}{k} e^{-ks}$$

Since $\underline{e^{isx} = \cos sx + i \sin sx}$
$$\left(\begin{array}{l} \displaystyle\int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2}\,dx = \frac{\pi}{k} e^{-ks} \\ \displaystyle\int_{-\infty}^{\infty} \frac{\sin sx}{k^2+x^2}\,dx = 0 \end{array} \right.$$

Theorem 1. Simple poles on the Real axis

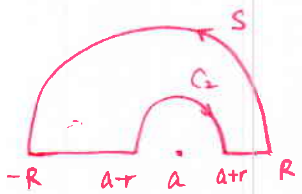


$f(z)$ has a simple pole at $z=a$ on the real axis

$$\lim_{r\to 0} \int_{C_2} f(z)\,dz = \underline{\pi i} \operatorname*{Res}_{z=a} f(z).$$

by definition of a simple pole

$$f(z) = \frac{b_1}{z-a} + g(z) \qquad b_1 = \operatorname*{Res}_{z=a} f(z)$$



$C_2 : z = a + re^{i\theta} \qquad 0 \le \theta \le \pi$

$$\int_{C_2} f(z)\,dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}} \cdot ire^{i\theta}\,d\theta + \int_{C_2} g(z)\,dz = b_1 \pi i + \boxed{\cancel{\int_{C_2} g(z)\,dz}}^{\,0}$$

$< \pi r M$

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx = \underbrace{2\pi i \sum \operatorname{Res} f(z)}_{\text{all poles in the upper half-plane}} + \underbrace{\pi i \sum \operatorname{Res} f(z)}_{\text{all poles on the real axis}}$$

Ex4.
$$\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)} = \int_{-\infty}^{\infty} \frac{dx}{(x-1)(x-2)(x+i)(x-i)}$$

$z=1,\ z=2$ on the real axis
$$\left(\begin{array}{ll} z=1 & \operatorname{Res} \dfrac{1}{(-1)\cdot 2} = -\dfrac{1}{2} \\ z=2 & \operatorname{Res} \dfrac{1}{5} = \dfrac{1}{5} \end{array} \right.$$

$z = i$ upper half-plane

$$z = i \qquad \frac{1}{(1-3i)(2i)} = \frac{1}{6+2i} = \frac{3-i}{30}$$

$$\therefore\ 2\pi i \left(\frac{3-i}{30} \right) + \pi i \left(-\frac{1}{2} + \frac{1}{5} \right) = \frac{\pi}{10}.$$