Naval Architecture & Ocean Engineering

Computer Aided Ship Design

Part I. Optimization Method

Ch. 4 Optimality Condition Using Kuhn-Tucker Necessary Condition

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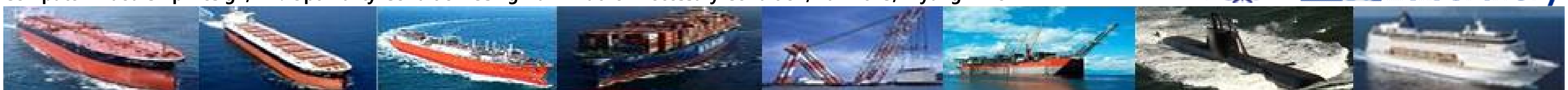

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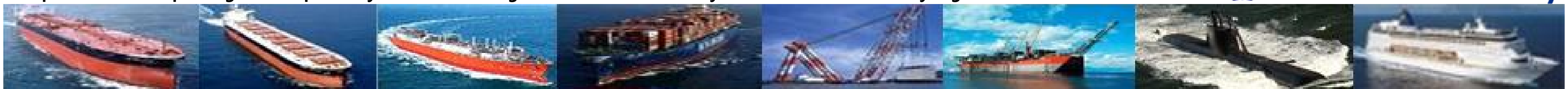
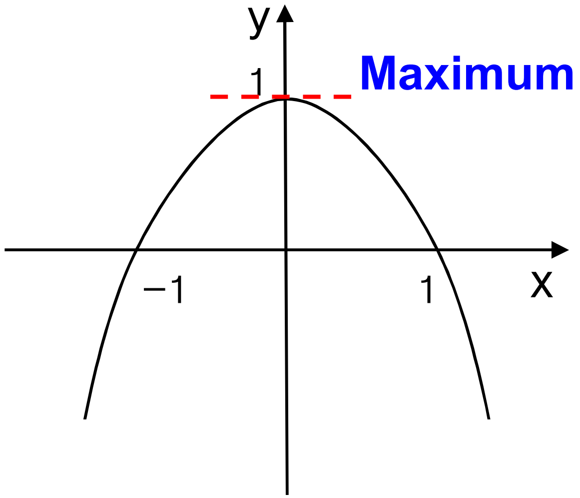
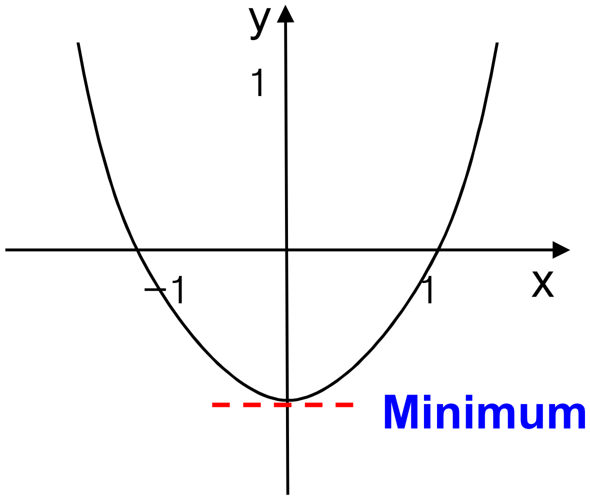


4.1 Optimal Solution Using Optimality Condition

Optimality Conditions for Function of Single Variable
- The Maximum and Minimum of the Function (Review of the Course of High School)

"수학의 정석"(Mathematics II) Review "6. Maximum, Minimum and Differentials" (p.104)





1) **Maximum value:** The increase of the value of the continuous function $f(x)$ is changed to the decrease of that at $x = x^*$.
2) **Minimum value:** The decrease of the value of the continuous function $f(x)$ is changed to the increase of that at $x = x^*$.

$$f'(x^*) = 0$$

(Necessary condition for $x = x^*$ to be a maximum or minimum)

Optimality Conditions for Function of Single Variable - First-Order Necessary Conditions

- First-order necessary condition for the function of a single variable: $f'(x^*) = 0$

Proof) The Taylor series expansion of $f(x)$ at the point x^* is as follows.

$$f(x) = f(x^*) + \frac{df(x^*)}{dx}(x - x^*) + \frac{1}{2}\frac{d^2 f(x^*)}{dx^2}(x - x^*)^2 + R$$

Remainder
: If the difference between x and x^* is small, the value of the remainder is also very small.

Let $x - x^* = d$, the equation is as follows.

$$f(x) = f(x^*) + f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$$

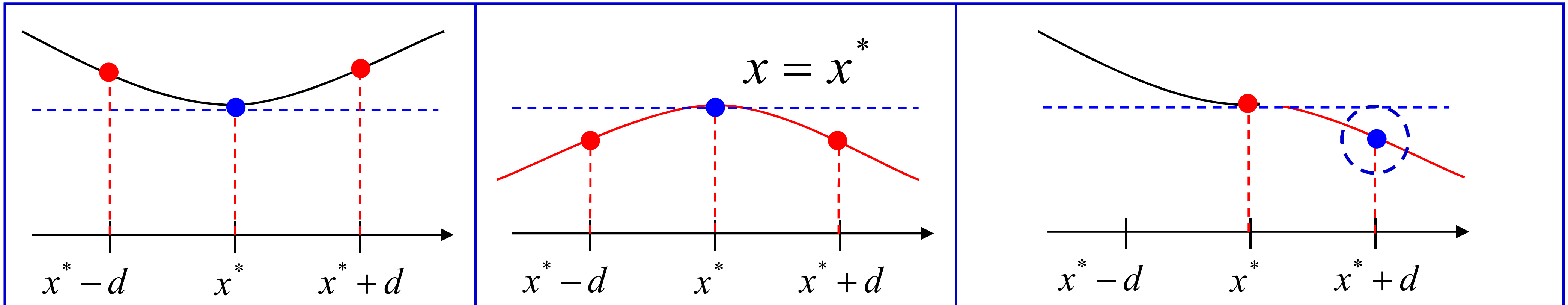
From this equation, the change in the function at x^*, i.e., $f(x) - f(x^*) = \Delta f(x)$ is given as

$$\Delta f(x) = f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$$

First-Order Necessary Conditions

$$\Delta f(x) = f(x) - f(x^*) = f'(x^*)d + \frac{1}{2} f''(x^*)d^2 + R$$

Δf must be positive, if x^* is a local minimum point.



$f(x^*)$ is minimum. | **$f(x^*)$ is maximum.** | **$f(x^*)$ is neither minimum nor maximum(inflection point).**

Since $d(= x - x^*)$ is small, the first-order term $f'(\mathbf{x}^*)d$ dominates other terms.

And the sign of the term $f'(\mathbf{x}^*)d$ is arbitrary.

Thus, the only way Δf can be positive regardless of the sign of d in the neighborhood of x^* is $f'(x^*) = 0$.

In the same way, Δf must be negative if x^* is a local maximum point. So, the only way Δf can be negative regardless of the sign of d in a neighborhood of x^* is $f'(x^*) = 0$.

Sufficient Conditions

$$\Delta f(x) = f(x) - f(x^*) = f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$$

- Now, we need a sufficient condition to determine which of the stationary points are actually minimum for the function.

Since stationary points satisfy the necessary condition $f'(x^*) = 0$, the change in function $\Delta f(x) = f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$ becomes as follows.

$$\Delta f(x) = \frac{1}{2}f''(x^*)d^2 + R$$

Since the second-order term dominates all other higher-order terms, the term can be positive for all $d \neq 0$, if

$$f''(x^*) > 0 \quad \text{(Sufficient condition)}$$

Summary

- First-order necessary condition

If x^* is a local minimum point, $f'(x^*) = 0$.

If $f'(x^*) = 0$, x^* is a stationary point(minimum, maximum, or inflection point).

- Sufficient condition

If x^* is a stationary point($f'(x^*) = 0$) and $f''(x^*) > 0$, x^* is a local minimum point.

[Review] Taylor Series Expansion for the Function of Two Variables

Taylor series expansion for the function of two variables $f(x_1, x_2)$ **at** (x_1^*, x_2^*)

$$f(x_1,x_2) = f(x_1^*,x_2^*) + \frac{\partial f}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f}{\partial x_2}(x_2 - x_2^*)$$

$$+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - x_2^*)^2\right) + R$$

Each term can be represented as follows:

$$\frac{\partial f}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f}{\partial x_2}(x_2 - x_2^*) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}^T \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix} = \underline{\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*)}$$

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - x_2^*)^2\right) = \frac{1}{2}\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x_1 - x_1^*) + \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_2 - x_2^*) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - x_1^*) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - x_2^*) \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} x_1 - x_1^* & x_2 - x_2^* \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

$$= \underline{\frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \mathbf{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*)}$$

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \underline{\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*)} + \underline{\frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \mathbf{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*)} + R$$

Element of the 2x2 Matrix

$$\left(\mathbf{x} = (x_1, x_2)^T, \ \mathbf{x}^* = (x_1^*, x_2^*)^T, \ \boxed{\mathbf{H} \in M_{2\times 2}} \right)$$

Optimality Conditions for Function of Several Variables

- Matrix form of the Taylor series expansion for the function of two variables

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x}-\mathbf{x}^*) + \frac{1}{2}\left(\mathbf{x}-\mathbf{x}^*\right)^T \mathbf{H}(\mathbf{x}^*)\left(\mathbf{x}-\mathbf{x}^*\right) + R$$

$$\left(\ \mathbf{x} = (x_1, x_2)^T,\ \mathbf{x}^* = (x_1^*, x_2^*)^T,\ \boxed{\mathbf{H} \in M_{2\times 2}}\ \right)$$

Element of the 2x2 Matrix

- Matrix form of the Taylor series expansion for the function of several variables

: It has the same form of the function of two variables.

$\mathbf{x},\ \mathbf{x}^*, \nabla f$: n dimension Vector

$\mathbf{H} \in M_{n\times n}$

- By defining $\mathbf{x}-\mathbf{x}^* = \mathbf{d}$, the Taylor series expansion for the function of the several variables is as follows.

$$f(\mathbf{x}^*+\mathbf{d}) = f(\mathbf{x}^*) + \boxed{\nabla f(\mathbf{x}^*)^T}\mathbf{d} + \boxed{\frac{1}{2}\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d}} + R$$

$$\boxed{\nabla f(\mathbf{x}^*)^T = 0,\ \frac{1}{2}\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0} \Rightarrow$$ Sufficient conditions for $\mathbf{x} = \mathbf{x}^*$ to be a local minimum

[Review] Hessian Matrix

- **Hessian matrix: Differentiating the gradient vector once again, we obtain a matrix of second partial derivatives for the function $f(x)$ called the Hessian matrix.**

That is, differentiating each component of the gradient vector with respect to $x_1, x_2, \cdots, x_n$, we obtain

$$\frac{\partial^2 f}{\partial \mathbf{x} \partial \mathbf{x}} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$$

: n-column Vector

- **Hessian matrix is denoted as H or $\nabla^2 f$.**

$$\mathbf{H} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right] \qquad (i = 1,2,\cdots,n \; ; \; j = 1,2,\cdots,n)$$

- **Property of the Hessian matrix**

$$\frac{\partial f}{\partial x_i \partial x_j} = \frac{\partial f}{\partial x_j \partial x_i}$$

Therefore, the Hessian matrix is always a symmetric matrix.

[Review] Quadratic Form

- Quadratic form: This is a special nonlinear function having only second-order terms.

Ex) $F(x_1, x_2, x_3) = \frac{1}{2}\left(2x_1^2 + 2x_1x_2 + 4x_1x_3 - 6x_2^2 - 4x_2x_3 + 5x_3^2\right)$

The quadratic form can be written in the following matrix notation.

$$F(x_1, x_2, x_3) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 1 & -6 & -2 \\ 2 & -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}\mathbf{x}^T \boxed{\mathbf{A}} \mathbf{x} \Leftrightarrow \frac{1}{2}\mathbf{d}^T \mathbf{H} \mathbf{d}$$

A: Symmetric matrix

- The element of symmetric matrix A is defined as follows(a_{ij}: element of the matrix A at (i, j)).
  1) The diagonal terms of the matrix are equal to the coefficient of the squared terms.

$$a_{ii} = \left(\text{coefficient of } x_i^2\right)$$

  2) The all terms except for diagonal terms(a_{ij}) are equal to a half of the coefficient of the $x_i x_j$.

$$a_{ij} = \left(\text{coefficient of } x_i x_j\right) \times \frac{1}{2}$$

Quadratic form may be either positive, negative, or zero for any X.

A symmetric matrix A is often referred to as **a positive definite** if the quadratic form associated with **A** is positive definite.

- **Form of a quadratic form**

1) Positive Definite

: $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ **for any x except for** $\mathbf{x} = 0$.

2) Positive Semidefinite

: $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ **for all x and there exists at least one** $\mathbf{x} \neq 0$ **with** $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$**.**

3) Negative Definite

: $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$ **for all x except for** $\mathbf{x} = 0$**.**

4) Negative Semidefinite

: $\mathbf{x}^T\mathbf{A}\mathbf{x} \leq 0$ **for all x and there exists at least one** $\mathbf{x} \neq 0$ **with** $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$**.**

5) Indefinite

: The quadratic form is positive for some vectors x and negative for others.

- **Use of the form of a quadratic form**

① Minimum condition for the function of the single variable

If x^* **is a stationary point(** $f'(x^*) = 0$ **)**
and $f''(x^*) > 0$, x^* **is a local minimum point.**

② Minimum condition for the function of the several variables

If $\mathbf{x}^*$ **is a stationary point(** $\nabla f(\mathbf{x}^*) = 0$ **)**
and $\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$**, i.e., the quadratic form is positive definite,** $\mathbf{x}^*$ **is a local minimum point.**

To be $\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$ **at** $\mathbf{x}^*$**,**
$\mathbf{H}(\mathbf{x}^*)$ **must be positive definite.**

Ref) KREYSZIG E., Advanced Engineering Mathematics, WILEY, 2006, 8.4. Eigenbasis. Diagonalization. Quadratic forms.

[Theorem] Methods for Checking Positive Definiteness or Semi-definiteness of a Quadratic Form or a Matrix

Let $\lambda_i, i = 1,\ldots,n$ be n eigenvalues of a symmetric $n \times n$ matrix $\mathbf{A}$ associated with the quadratic form $F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}$.

1) $F(\mathbf{x})$ is **positive definite** if and only if all eigenvalues of $\mathbf{A}$ are **strictly positive**, i.e.,

$$\lambda_i > 0, i = 1,\ldots,n$$

2) $F(\mathbf{x})$ is **positive semi-definite** if and only if all eigenvalues of $\mathbf{A}$ are **nonnegative**, i.e.,

$$\lambda_i \geq 0, i = 1,\ldots,n$$

3) $F(\mathbf{x})$ is **negative definite** if and only if all eigenvalues of $\mathbf{A}$ are **strictly negative**, i.e.,

$$\lambda_i < 0, i = 1,\ldots,n$$

4) $F(\mathbf{x})$ is **negative semi-definite** if and only if all eigenvalues of $\mathbf{A}$ are **nonpositive**, i.e.,

$$\lambda_i \leq 0, i = 1,\ldots,n$$

5) $F(\mathbf{x})$ is **indefinite** if some $\lambda_i < 0$ and some other $\lambda_i > 0$.

Eigenvalue of a Symmetric Matrix A Associated with the Quadratic Form

For a given matrix A, the eigenvalue problem is defined as $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$

, where λ is an eigenvalue and $\mathbf{V}$ is the corresponding eigenvector.

How to determine the eigenvalues:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \;\Rightarrow\; (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0 \;\Rightarrow\; \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Determine the eigenvalues and the form of the following matrix.

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\det\begin{bmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{bmatrix} = (2-\lambda)^2(8-\lambda) = 0$$

$$\therefore \lambda = 2(\text{equal root}),\ 8$$

Since all eigenvalues of A are positive, this matrix is positive definite.

[Summary] Optimality Conditions for Function of Several Variables

- The Taylor series expansion of $f(\mathbf{x})$, which is the function of n variables, gives

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} + R$$

- From this equation, the change in the function at $\mathbf{x}^*$, i.e.,$\Delta f(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}^*)$, is given as

$$\Delta f = \nabla f(\mathbf{x}^*)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} + R$$

- If we assume a local minimum is at $\mathbf{x}^*$, then Δf must be positive.

1) the first-order necessary condition:

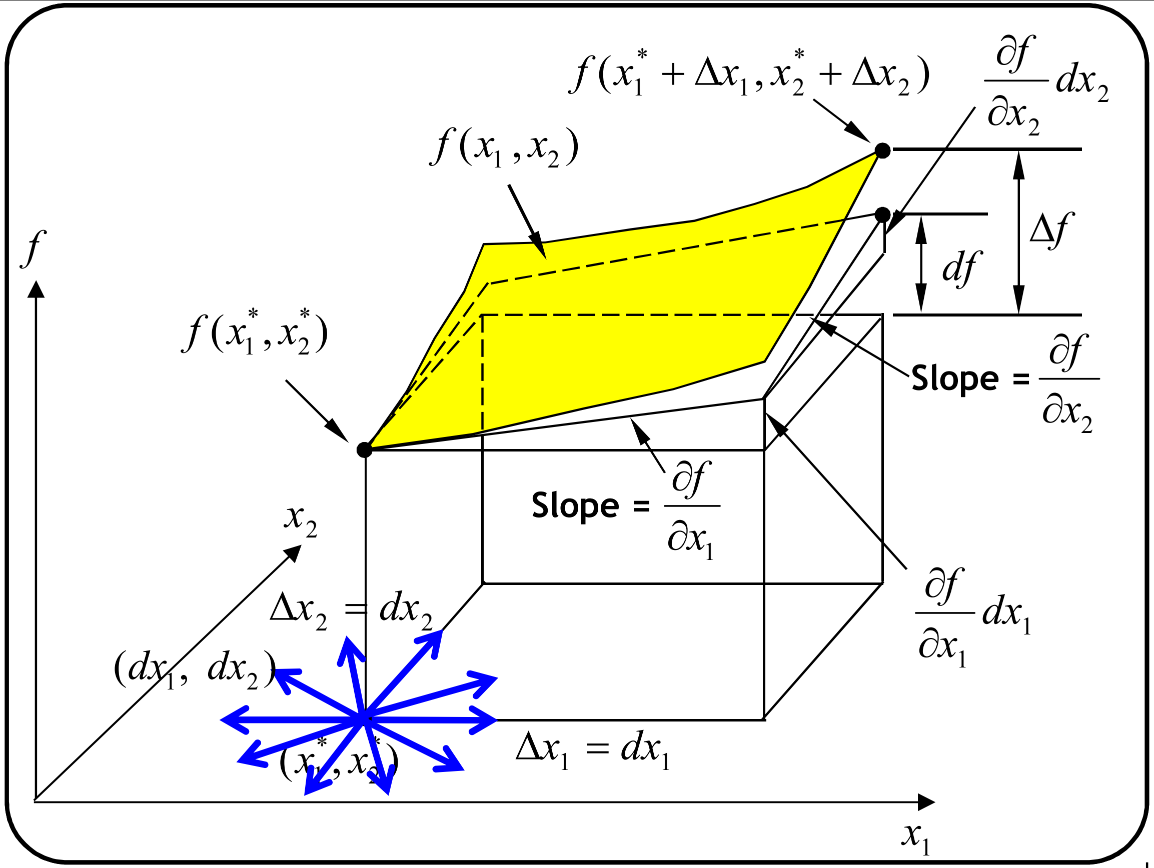
If $\nabla f(\mathbf{x}^*) = 0$, i.e., $\frac{\partial f(\mathbf{x}^*)}{\partial x_i} = 0$, $(i = 1, 2, \cdots n)$, x^* is a stationary point(minimum, maximum, or inflection point).

2) the sufficient condition:

If $\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$, then the stationary point $(\nabla f(\mathbf{x}^*)^T = 0 \Rightarrow \nabla f(\mathbf{x}^*) = 0)$ is a local minimum.

To be $\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$, $\mathbf{H}(\mathbf{x}^*)$ must be positive definite.

Necessary Condition for a Stationary Point: Total Derivative df = 0 ➡ grad f = 0



The change in the function f(x_1, x_2) can be expressed using Taylor expansion as follows

$$\Delta f = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}\Delta x_1^{\ 2} + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2^{\ 2}\right) + R$$

If $\Delta x_1 \to 0, \Delta x_2 \to 0$,

the first-order term $\frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$ dominates other terms.

Therefore, Δf can be approximated as $\Delta f \approx \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$.

The symbol *"d"* refers to the infinitesimal change. By definition of *"d"*, we can write the change in function f as follows:

the change of the function in x_2 direction

$$df = \boxed{\frac{\partial f}{\partial x_1}dx_1} + \boxed{\frac{\partial f}{\partial x_2}dx_2}$$

the change of the function in x_1 direction

If $df = 0$, then x^* is a stationary point.

To be $df = 0$ regardless of the sign of dx_1 and dx_2, $\partial f / \partial x_1$ and $\partial f / \partial x_2$ must be zero.

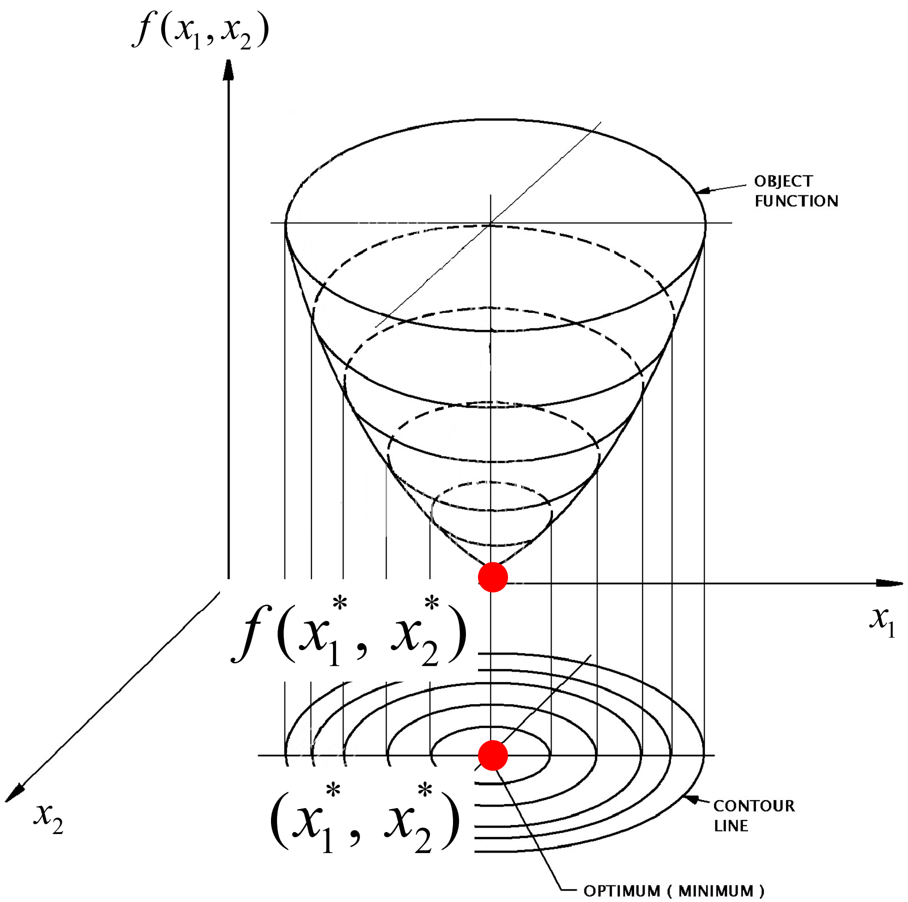
It means that the gradient of function f must be equal to zero.

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0 \Rightarrow \nabla f = 0$$

Definition of Stationary Point

Given: Minimize $f(x_1, x_2)$

Find: Stationary point (x_1^*, x_2^*)



The change in function(df) at the point(x_1^*, x_2^*) with the change in variables(dx_1, dx_2) is as follows.

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$$

The point at which the change in function(df) is zero is called **stationary point.**

It includes the minimum, maximum, and inflection(saddle) point.

Note: In the general engineering optimization problem, the optimum point ($\mathbf{x}$) is more important than the optimum value (f).

[Example] Principal dimensions of a ship (L, B, D, C_B) to minimize the shipbuilding cost is more important than the shipbuilding cost itself.

4.2 Lagrange Multiplier for Equality Constraints

Stationary Point for Unconstrained Optimization Problem

Given: $Minimize\ f(x_1, x_2, x_3)$

Find: Stationary point(x_1^*, x_2^*, x_3^*)

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$$

At the stationary point, the change in the function(df) is zero.
The gradient of the function at the stationary point must be zero, because the change in the function(df) can be only zero regardless of the sign of dx_1, dx_2, and dx_3.

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0$$

$$\Rightarrow \nabla f = 0$$

Because 'Minimize f' is formulated as an equation($df = 0$), the number of equations is equal to the number of unknown variables. (Determinate problem)

Stationary Point for Constrained Optimization Problem (1/3)

Problem: $Minimize \ f(x_1, x_2, x_3)$

$Subject \ to \ h(x_1, x_2, x_3) = 0$

Find: Stationary point (x_1^*, x_2^*, x_3^*)

Method:

1. Express h (equality constraint) as an explicit function of x_1.

2. Substitute x_1 into f and find the stationary point by using $df = 0$.

In many problem, it may not be possible to express h (equality constraint) as an explicit function of x_1.

Example) It is difficult to express the following equality constraint as an explicit function.

$$ex) \quad h(x_1, x_2, x_3) = \tan x_1 + \cos x_2 + e^{x_3} = 0$$

Is there any method to obtain the stationary point if the equality constraint can not be expressed as an explicit function?

Solution)

$df = 0$ at the stationary point.

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \quad \text{------} \ ①$$

Since $h(x_1,x_2,x_3)=0$, dh is also zero.

$$dh = \frac{\partial h}{\partial x_1}dx_1 + \frac{\partial h}{\partial x_2}dx_2 + \frac{\partial h}{\partial x_3}dx_3 = 0 \quad \text{------} \ ②$$

Since equation ① and ② are equal to zero, the following equation is always satisfied.

$$df + \lambda \cdot dh = 0$$, where λ is an undetermined coefficient 'Lagrange multiplier'.

Stationary Point for Constrained Optimization Problem (2/3)

Given: $Minimize\ f(x_1, x_2, x_3)$

$Subject\ to\ h(x_1, x_2, x_3) = 0$

Find: Stationary point(x_1^*, x_2^*, x_3^*)

① $$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0$$

② $$dh = \frac{\partial h}{\partial x_1}dx_1 + \frac{\partial h}{\partial x_2}dx_2 + \frac{\partial h}{\partial x_3}dx_3 = 0$$

Because of the equality constraint h, dx_1, dx_2, and dx_3 are not independent.

$$df + \lambda \cdot dh = 0$$

λ: **Undetermined Coefficient 'Lagrange multiplier'**

This equation can be rearranged as follows.

$$\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 + \lambda\left(\frac{\partial h}{\partial x_1}dx_1 + \frac{\partial h}{\partial x_2}dx_2 + \frac{\partial h}{\partial x_3}dx_3\right) = 0$$

$$\left(\frac{\partial f}{\partial x_1} + \lambda\frac{\partial h}{\partial x_1}\right)dx_1 + \left(\frac{\partial f}{\partial x_2} + \lambda\frac{\partial h}{\partial x_2}\right)dx_2 + \left(\frac{\partial f}{\partial x_3} + \lambda\frac{\partial h}{\partial x_3}\right)dx_3 = 0$$

Stationary Point for Constrained Optimization Problem (3/3)

Given: $Minimize\ f(x_1, x_2, x_3)$

$Subject\ to\ h(x_1, x_2, x_3) = 0$

Find: Stationary point(x_1^*, x_2^*, x_3^*)

① $df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0$

② $dh = \frac{\partial h}{\partial x_1}dx_1 + \frac{\partial h}{\partial x_2}dx_2 + \frac{\partial h}{\partial x_3}dx_3 = 0$

Because of the equality constraint h, dx_1, dx_2, and dx_3 are not independent.

$$df + \lambda \cdot dh = 0$$

λ: **Undetermined Coefficient 'Lagrange multiplier'**

$$\left(\frac{\partial f}{\partial x_1} + \lambda\frac{\partial h}{\partial x_1}\right)dx_1 + \left(\frac{\partial f}{\partial x_2} + \lambda\frac{\partial h}{\partial x_2}\right)dx_2 + \left(\frac{\partial f}{\partial x_3} + \lambda\frac{\partial h}{\partial x_3}\right)dx_3 = 0$$

If the dx_1, dx_2, and dx_3 were all independent of each other, all terms in the brackets will be zero. This however, is not the case because of the equality constraint h. Let's try to make the first term to be zero by determining a proper value of λ, so that the following equation is satisfied without considering the dx_1.

$$\left(\frac{\partial f}{\partial x_2} + \lambda\frac{\partial h}{\partial x_2}\right)dx_2 + \left(\frac{\partial f}{\partial x_3} + \lambda\frac{\partial h}{\partial x_3}\right)dx_3 = 0$$

Since dx_2 and dx_3 are independent, the terms in the brackets must be zero to satisfy the equation.

$$\therefore \left(\frac{\partial f}{\partial x_1} + \lambda\frac{\partial h}{\partial x_1}\right) = 0,\quad \left(\frac{\partial f}{\partial x_2} + \lambda\frac{\partial h}{\partial x_2}\right) = 0,\quad \left(\frac{\partial f}{\partial x_3} + \lambda\frac{\partial h}{\partial x_3}\right) = 0$$

Therefore, the point (x_1, x_2, x_3, λ) that satisfies the following equations is a stationary point.

$$\frac{\partial f}{\partial x_1} + \lambda\frac{\partial h}{\partial x_1} = 0,\qquad \frac{\partial f}{\partial x_2} + \lambda\frac{\partial h}{\partial x_2} = 0$$

$$\frac{\partial f}{\partial x_3} + \lambda\frac{\partial h}{\partial x_3} = 0,\qquad h(x_1, x_2, x_3) = 0$$

4 Unknown variables: (x_1, x_2, x_3, λ)

4 Equations

There exists an unique solution.

Lagrange Multiplier for Equality Constraints

The point(x_1, x_2, x_3, λ) that satisfies the following equations is a stationary point.

$$\frac{\partial f}{\partial x_1} + \lambda \frac{\partial h}{\partial x_1} = 0, \qquad \frac{\partial f}{\partial x_2} + \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\frac{\partial f}{\partial x_3} + \lambda \frac{\partial h}{\partial x_3} = 0, \qquad h(x_1, x_2, x_3) = 0$$

It is convenient to write these equations in terms of a Lagrange function, L, defined as

$$L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda h(x_1, x_2, x_3)$$

$$\nabla L(x_1, x_2, x_3, \lambda) = 0$$

Constrained optimization problem is transformed to an unconstrained optimization problem.

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda \frac{\partial h}{\partial x_1} = 0 \qquad \frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial x_3} = \frac{\partial f}{\partial x_3} + \lambda \frac{\partial h}{\partial x_3} = 0 \qquad \frac{\partial L}{\partial \lambda} = h(x_1, x_2, x_3) = 0$$

λ : Lagrange Multiplier
L : Lagrange Function

Optimization Problem

Minimize $f(x_1, x_2, x_3)$ ···· ①

Subject to $h_1(x_1, x_2, x_3) = 0$ ···· ②

$h_2(x_1, x_2, x_3) = 0$ ···· ③

Number of variables: 3

Number of equation: 2

Necessary condition that minimize f is $df = 0$.
$df = 0$ corresponds to Eq. ①′ as follows.

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \quad \cdots \text{①'}$$

Subject to $h_1(x_1, x_2, x_3) = 0$ ··· ②

$h_2(x_1, x_2, x_3) = 0$ ··· ③

Number of variables: 3
Number of equations: 3

➡ We can solve this.

Because 'Minimize f' is formulated as an equation($df = 0$), the number of equations is equal to the number of unknown variables. (Determinate problem)

Optimization Problem

Minimize $f(x_1, x_2, x_3)$ ···· ①

Subject to $h_1(x_1, x_2, x_3) = 0$ ···· ②

$h_2(x_1, x_2, x_3) = 0$ ···· ③

Number of variables: 3

Number of equation: 2

Necessary condition that minimize f is $df = 0$.
$df = 0$ corresponds to Eq. ①′ as follows.

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \quad \cdots ①'$$

Subject to $h_1(x_1, x_2, x_3) = 0$ ··· ②

$h_2(x_1, x_2, x_3) = 0$ ··· ③

To find the relationships among dx_1, dx_2, dx_3, we modify the equation ② and ③ to the form of total derivative dh_1, dh_2.

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \quad \cdots ①'$$

$$dh_1 = \frac{\partial h_1}{\partial x_1}dx_1 + \frac{\partial h_1}{\partial x_2}dx_2 + \frac{\partial h_1}{\partial x_3}dx_3 = 0 \quad \cdots ②'$$

$$dh_2 = \frac{\partial h_2}{\partial x_1}dx_1 + \frac{\partial h_2}{\partial x_2}dx_2 + \frac{\partial h_2}{\partial x_3}dx_3 = 0 \quad \cdots ③'$$

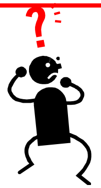
Optimization Problem

Minimize	$f(x_1, x_2, x_3)$	···· ①
Subject to	$h_1(x_1, x_2, x_3) = 0$	···· ②
	$h_2(x_1, x_2, x_3) = 0$	···· ③

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \quad \cdots \text{①'}$$

$$dh_1 = \frac{\partial h_1}{\partial x_1}dx_1 + \frac{\partial h_1}{\partial x_2}dx_2 + \frac{\partial h_1}{\partial x_3}dx_3 = 0 \quad \cdots \text{②'}$$

$$dh_2 = \frac{\partial h_2}{\partial x_1}dx_1 + \frac{\partial h_2}{\partial x_2}dx_2 + \frac{\partial h_2}{\partial x_3}dx_3 = 0 \quad \cdots \text{③'}$$



Are the equation ①', ②', and ③' differential equations with respect to f, h_1, h_2 ?
Answer: If the problem were given as follows:

- **Given:** $\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0,\ \frac{\partial h_1}{\partial x_1}dx_1 + \frac{\partial h_1}{\partial x_2}dx_2 + \frac{\partial h_1}{\partial x_3}dx_3 = 0,\ \frac{\partial h_2}{\partial x_1}dx_1 + \frac{\partial h_2}{\partial x_2}dx_2 + \frac{\partial h_2}{\partial x_3}dx_3 = 0,$

- **Find: Function f, h_1, h_2**

Then the equation ①', ②', and ③' would be differential equations.

However, the function f, h_1, and h_2(equation ①, ②, ③) are given and differential quantities of dx_1, dx_2, and dx_3 are to find, the equation ①', ②', and ③' are algebraic equations for the variables x_1, x_2, and x_3.

Optimization Problem

Minimize $f(x_1, x_2, x_3)$ ···· ①

Subject to $h_1(x_1, x_2, x_3) = 0$ ···· ②

$h_2(x_1, x_2, x_3) = 0$ ···· ③

$$*df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \quad \cdots ①'$$

$$dh_1 = \frac{\partial h_1}{\partial x_1}dx_1 + \frac{\partial h_1}{\partial x_2}dx_2 + \frac{\partial h_1}{\partial x_3}dx_3 = 0 \quad \cdots ②'$$

$$dh_2 = \frac{\partial h_2}{\partial x_1}dx_1 + \frac{\partial h_2}{\partial x_2}dx_2 + \frac{\partial h_2}{\partial x_3}dx_3 = 0 \quad \cdots ③'$$

* Since dx_1, dx_2, and dx_3 are not independent because of the equality constraints h_1 and h_2.

We multiply the equation ②' and ③' by λ_1 and λ_2, respectively and add it to the equation ①':

$$df + \lambda_1 dh_1 + \lambda_2 dh_2 = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x_1} + \lambda_1\frac{\partial h_1}{\partial x_1} + \lambda_2\frac{\partial h_2}{\partial x_1}\right)}_{=0 \ \cdots ④}dx_1 + \underbrace{\left(\frac{\partial f}{\partial x_2} + \lambda_1\frac{\partial h_1}{\partial x_2} + \lambda_2\frac{\partial h_2}{\partial x_2}\right)}_{=0 \ \cdots ⑤}dx_2 + \underbrace{\left(\frac{\partial f}{\partial x_3} + \lambda_1\frac{\partial h_1}{\partial x_3} + \lambda_2\frac{\partial h_2}{\partial x_3}\right)}_{=0 \ \cdots ⑥}dx_3 = 0$$

➡ Determine λ_1, λ_2 so that the first term in the brackets becomes zero*. (to eliminate dx_1)

➡ Determine λ_1, λ_2 so that the second term in the brackets becomes zero*. (to eliminate dx_2)

➡ Since dx_3 is an independent variable

5 variables: $(x_1, x_2, x_3, \lambda_1, \lambda_2)$

5 equations: 2,3,4,5,6

There exists a unique solution.

[Summary] Stationary Point for a Constrained Optimization Problem
- Solution for a Constrained Optimization problem by using the Lagrange Multiplier (5/5)

The point$(x_1, x_2, x_3, \lambda_1, \lambda_2)$ that satisfies the following equations is a stationary point.

$$\frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial h_1}{\partial x_1} + \lambda_2 \frac{\partial h_2}{\partial x_1} = 0, \qquad \frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial h_1}{\partial x_2} + \lambda_2 \frac{\partial h_2}{\partial x_2} = 0$$

$$\frac{\partial f}{\partial x_3} + \lambda_1 \frac{\partial h_1}{\partial x_3} + \lambda_2 \frac{\partial h_2}{\partial x_3} = 0, h_1(x_1, x_2, x_3) = 0, \quad h_2(x_1, x_2, x_3) = 0$$

It is convenient to write these equations in terms of a Lagrange function, L, defined as

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = f(x_1, x_2, x_3) + \lambda_1 h_1(x_1, x_2, x_3) + \lambda_2 h_2(x_1, x_2, x_3)$$

$$\nabla L(x_1, x_2, x_3, \lambda_1, \lambda_2) = 0$$

λ : Lagrange Multiplier

L : Lagrange Function

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial h_1}{\partial x_1} + \lambda_2 \frac{\partial h_2}{\partial x_1} = 0 \text{ .... ④}$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial h_1}{\partial x_2} + \lambda_2 \frac{\partial h_2}{\partial x_2} = 0 \text{ .... ⑤}$$

$$\frac{\partial L}{\partial x_3} = \frac{\partial f}{\partial x_3} + \lambda_1 \frac{\partial h_1}{\partial x_3} + \lambda_2 \frac{\partial h_2}{\partial x_3} = 0 \text{ .... ⑥}$$

$$\frac{\partial L}{\partial \lambda_1} = h_1(x_1, x_2, x_3) = 0 \text{ .... ②}$$

$$\frac{\partial L}{\partial \lambda_2} = h_2(x_1, x_2, x_3) = 0 \text{ .... ③}$$

The Lagrange function gives us a simple way of formulating the equations that have to be satisfied at a stationary point.

[Example] Lagrange Multiplier for Equality Constraints - Quadratic Programming Problem (1/2)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Original Problem

Minimize $f(x_1, x_2) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$

Subject to $h(x_1, x_2) = x_1 + x_2 - 2 = 0$

Lagrange Function

Minimize
$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda h(x_1, x_2)$$
$$= (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + \lambda(x_1 + x_2 - 2)$$

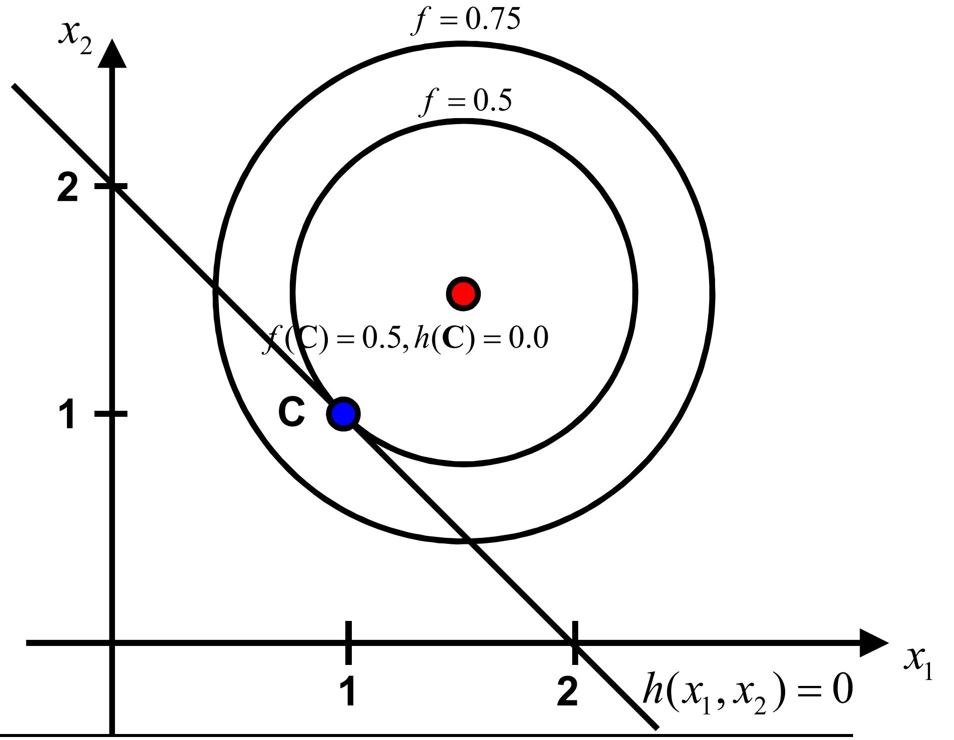
Necessary Condition: $\nabla L(x_1, x_2, \lambda) = 0$

$$\frac{\partial L}{\partial x_1} = 2(x_1 - 1.5) + \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 2(x_2 - 1.5) + \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 - 2 = 0$$

$\Rightarrow x_1^* = x_2^* = 1, \lambda^* = 1$ **(The point C is the stationary point.)**



[Example] Lagrange Multiplier for Equality Constraints - Quadratic Programming Problem (2/2)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Original Problem

Minimize $f(\mathbf{x}) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$

Subject to $h(\mathbf{x}) = x_1 + x_2 - 2 = 0$

Lagrange Function

Minimize $L(\mathbf{x},\ v) = f(\mathbf{x}) + vh(\mathbf{x})$

$$= (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + v(x_1 + x_2 - 2)$$

Necessary Condition: $\nabla L(\mathbf{x}^*, v^*) = 0$

$$\nabla f(\mathbf{x}^*) + v^* \nabla h(\mathbf{x}^*) = 0$$

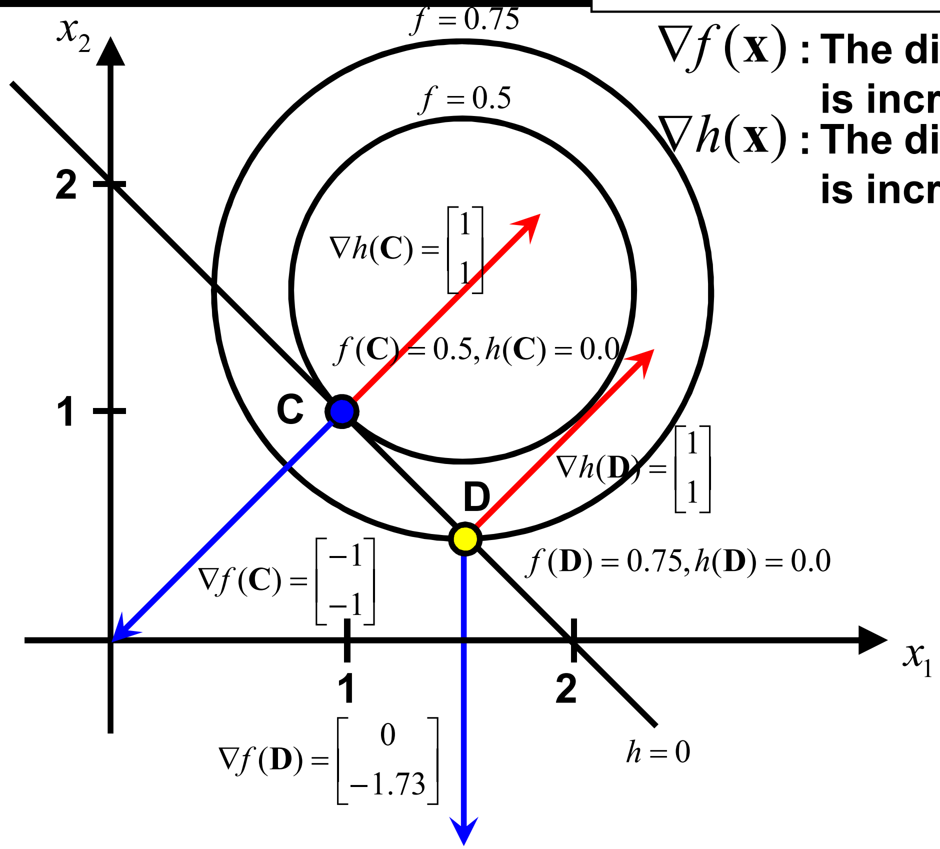
$$\therefore -\nabla f(\mathbf{x}^*) = v^* \nabla h(\mathbf{x}^*)$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2(x_1 - 1.5) \\ 2(x_2 - 1.5) \end{bmatrix},\quad \nabla h(\mathbf{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-2(x_1^* - 1.5) = v^*,\ -2(x_2^* - 1.5) = v^*$$

$$x_1^* + x_2^* - 2 = 0$$

$\Rightarrow x_1^* = x_2^* = 1, v^* = 1$ (The point C is the stationary point.)



$\nabla f(\mathbf{x})$: The direction where $f(\mathbf{x})$ is increased

$\nabla h(\mathbf{x})$: The direction where $h(\mathbf{x})$ is increased

At the candidate minimum C, the meaning of $-\nabla f(\mathbf{x}^*) = v^* \nabla h(\mathbf{x}^*)$ is

The gradient vector of the objective function and constraint are on the same line and proportional to each other, and the Lagrange multiplier v^* is the proportionality constant.

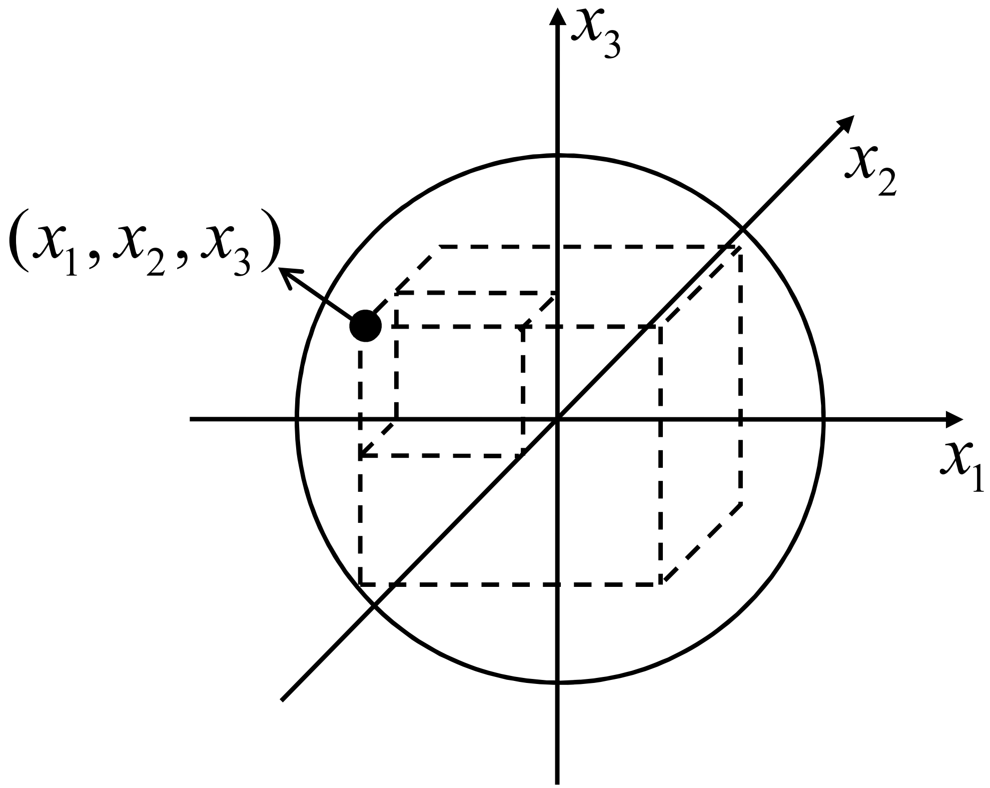
$$\nabla f(\mathbf{C}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\quad \nabla h(\mathbf{C}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad v^* = 1$$

But the point D is not a candidate minimum, because the gradient vector of the objective function and constraint are not on the same line.

[Example] Solving Nonlinear Constrained Optimization Problem by Using the Lagrange Multiplier (1/4)

☑ There is a sphere whose center is (0, 0, 0) and radius is c.

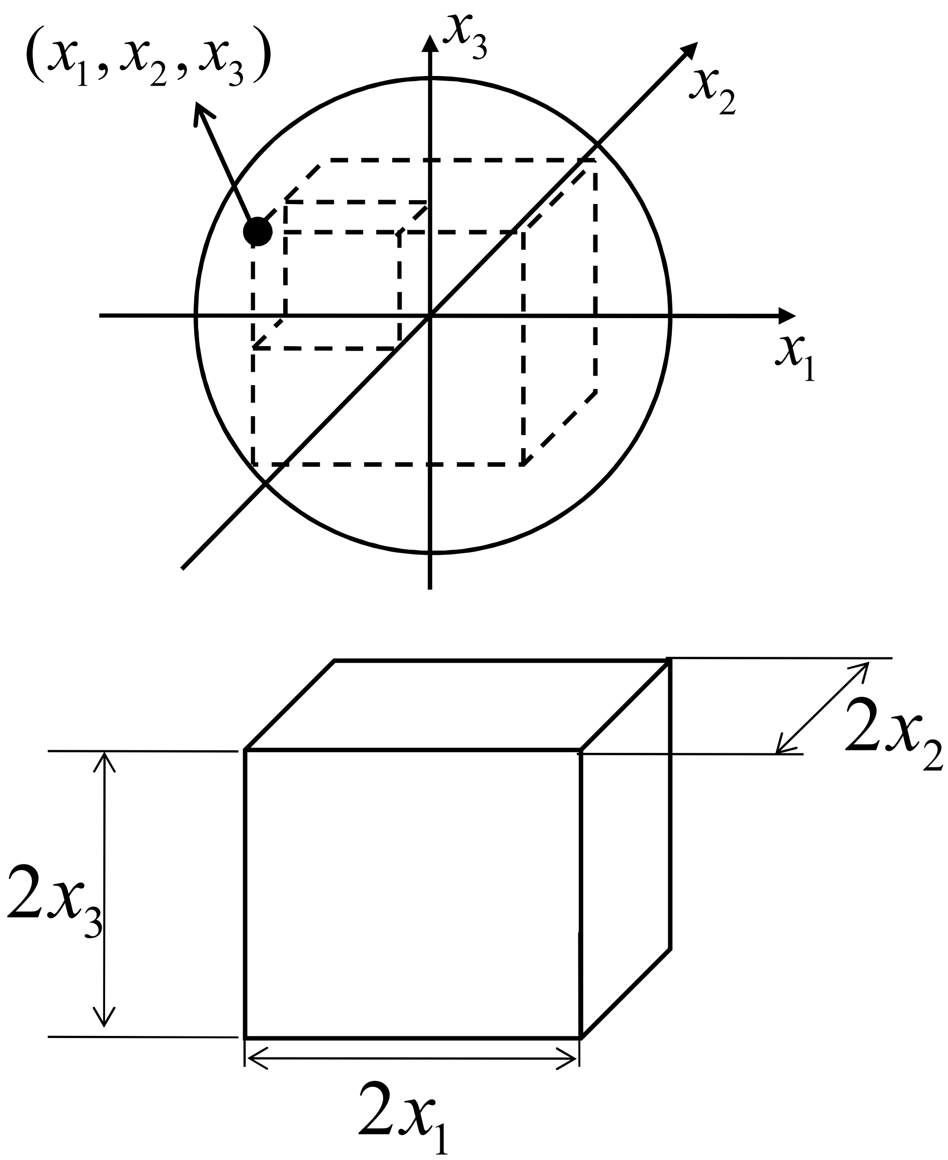
☑ Determine the maximum volume of the rectangular solid which is circumscribed* in the sphere.



* To draw a geometric figure around another figure so that the two are in contact but do not intersect.

[Example] Solving Nonlinear Constrained Optimization Problem by Using the Lagrange Multiplier (2/4)

☑ **Mathematical Modeling**



The volume of the rectangular sold f is

$$f(x_1, x_2, x_3) = 2x_1 \cdot 2x_2 \cdot 2x_3$$

Because the vertices of the rectangular solid are on the surface of the sphere,

$$h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - c^2 = 0$$

$cf)\ equation\ \ for\ \ a\ \ sphere: x^2 + y^2 + z^2 = r^2$

$$Maximize\ f(x_1, x_2, x_3) = 2x_1 \cdot 2x_2 \cdot 2x_3 = 8x_1x_2x_3$$

$Subject\ \ to$

$$h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - c^2 = 0$$

$$Minimize\ f(x_1, x_2, x_3) = -8x_1x_2x_3$$

$Subject\ \ to$

$$h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - c^2 = 0$$

[Example] Solving Nonlinear Constrained Optimization Problem by Using the Lagrange Multiplier (3/4)

☑ Solution(1/2)

$$\begin{aligned}
&\mathit{Minimize}\ f(x_1, x_2, x_3) = -8x_1x_2x_3 \\
&\mathit{Subject\ to} \\
&\qquad h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - c^2 = 0
\end{aligned}$$

Lagrange function for this problem is as follow.

$$\begin{aligned}
L(x_1, x_2, x_3, \lambda) &= f(x_1, x_2, x_3) + \lambda h(x_1, x_2, x_3) \\
&= -8x_1x_2x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - c^2)
\end{aligned}$$

$$\nabla L(x_1, x_2, x_3, \lambda) = 0 \quad \Rightarrow \quad
\begin{aligned}
&\frac{\partial L}{\partial x_1} = -8x_2x_3 + \lambda 2x_1 = 0 \qquad \frac{\partial L}{\partial x_2} = -8x_1x_3 + \lambda 2x_2 = 0 \\
&\frac{\partial L}{\partial x_3} = -8x_1x_2 + \lambda 2x_3 = 0 \qquad \frac{\partial L}{\partial \lambda} = x_1^2 + x_2^2 + x_3^2 - c^2 = 0
\end{aligned}$$

[Example] Solving Nonlinear Constrained Optimization Problem by Using the Lagrange Multiplier (4/4)

☑ Solution(2/2)

$$-8x_2x_3 + \lambda 2x_1 = 0 \quad \text{------- ①} \qquad \text{Equation ① X } x_1 \qquad -8x_1x_2x_3 + \lambda 2x_1^2 = 0 \qquad x_1^2 = \frac{4x_1x_2x_3}{\lambda}$$

$$-8x_1x_3 + \lambda 2x_2 = 0 \quad \text{------- ②} \qquad \text{Equation ② X } x_2 \qquad -8x_1x_2x_3 + \lambda 2x_2^2 = 0 \qquad x_2^2 = \frac{4x_1x_2x_3}{\lambda}$$

$$-8x_1x_2 + \lambda 2x_3 = 0 \quad \text{------- ③} \qquad \text{Equation ③ X } x_3 \qquad -8x_1x_2x_3 + \lambda 2x_2^2 = 0 \qquad x_3^2 = \frac{4x_1x_2x_3}{\lambda}$$

$$x_1^2 + x_2^2 + x_3^2 - c^2 = 0 \quad \text{------- ④}$$

Substitute these into the equation ④

$$\frac{4x_1x_2x_3}{\lambda} + \frac{4x_1x_2x_3}{\lambda} + \frac{4x_1x_2x_3}{\lambda} - c^2 = 0$$

$$\frac{12x_1x_2x_3}{\lambda} = c^2$$

$$\frac{12x_1x_2x_3}{c^2} = \lambda \quad \text{------- ⑤}$$

Substitute the equation ⑤ into the equation ①

$$-8x_2x_3 + \frac{12x_1x_2x_3}{c^2} 2x_1 = 0$$

$$-8x_2x_3 + \frac{24x_1^2x_2x_3}{c^2} = 0$$

$$-8x_2x_3\left(1 - \frac{3x_1^2}{c^2}\right) = 0$$

If x_2 or x_3 are zero 0, the volume of the rectangular solid is zero and the solution is trivial. Therefore,

$$1 - \frac{3x_1^2}{c^2} = 0$$

$$\frac{3x_1^2}{c^2} = 1$$

$$x_1^2 = \frac{c^2}{3}$$

$$x_1 = \pm\frac{c}{\sqrt{3}}$$

Because x_1 is a length, it must be positive.

x_2 and x_3 are found in the same way.

$$x_1 = \frac{c}{\sqrt{3}}, \quad x_2 = \frac{c}{\sqrt{3}}, \quad x_3 = \frac{c}{\sqrt{3}}$$

So, the maximum volume is

$$8x_1x_2x_3 = \frac{8c^3}{3\sqrt{3}}$$

[Summary] Constrained Optimization Method by Using the Lagrange Multiplier

☑Constrained Optimization Problem

$$\textit{Minimize} \quad f(\mathbf{x}) = f(x_1, x_2, \ \ldots \ , x_n)$$
$$\textit{Subject to} \quad h_i(\mathbf{x}) = 0, \quad i = 1, \ \ldots \ , p$$

- Determination of the propeller principal dimensions by using the Lagrange multiplier
- Determination of the principal dimension of a ship by using the Lagrange multiplier

☑Definition of the Lagrange function(L)

$$L(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x})$$
$$= f(\mathbf{x}) + \mathbf{v}^T \mathbf{h}(\mathbf{x})$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_p \end{bmatrix}$$

v_i are the Lagrange multipliers for the equality constraints and are free in sign, i.e., they can be positive, negative, or zero.

<Reason>

The solution does not change, even if the equality constraint is multiplied by the minus sign.

Comparison between Newton's Method and Method of Lagrange Multipliers

Newton' Method for Unconstrained Optimization Problem

Given: $\mathit{Minimize}\ \ f(\mathbf{x})$

Find: Local minimum design point

By defining $\mathbf{x}-\mathbf{x}^* = \mathbf{d}$, the Taylor series expansion for the function of multi variables is as follows.

$$f(\mathbf{x}^* + \mathbf{d}) = f(\mathbf{x}^*) + \boxed{\nabla f(\mathbf{x}^*)^T}\,\mathbf{d} + \boxed{\frac{1}{2}\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d}} + R$$

Necessary condition for $\mathbf{x} = \mathbf{x}^*$ to be a candidate local minimum (stationary point)

$$\boxed{\nabla f(\mathbf{x}^*)^T = 0}, \quad \frac{1}{2}\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$$

Sufficient conditions for $\mathbf{x} = \mathbf{x}^*$ to be a local minimum

Method of Lagrange Multipliers for Constrained Optimization Problem

Given: $\mathit{Minimize}\ \ f(\mathbf{x})$

$h(x_1, x_2, x_3) = 0$

Find: Local candidate minimum design point

$$\boxed{df + \lambda \cdot dh} = 0$$

λ: Undetermined Coefficient 'Lagrange multiplier'

Define Lagrange function, $L = df + \lambda \cdot dh$

Necessary condition for $\mathbf{x} = \mathbf{x}^*$ to be a candidate local minimum ➡ $\nabla L = 0$ (stationary point)

[Reference] Constrained Optimization Method for Candidate Minimum by Using the Lagrange Multiplier

Minimize $f(x_1, x_2)$, ***Subject to*** $h(x_1, x_2) = 0$

By using $h(x_1,\ x_2) = 0$, x_2 **can be expressed as the function of** x_1**, i.e.,** $f(x_1, x_2) = f(x_1, \phi(x_1))$

To determine the local candidate minimum of the function of the single variable,

$df(x_1, x_2)/dx_1 = 0$, **But, because** $df(x_1, x_2) = \dfrac{\partial f(x_1, x_2)}{\partial x_1} dx_1 + \dfrac{\partial f(x_1, x_2)}{\partial x_2} dx_2$, $\dfrac{df(x_1, x_2)}{dx_1} = \dfrac{\partial f(x_1, x_2)}{\partial x_1} + \dfrac{\partial f(x_1, x_2)}{\partial x_2} \dfrac{dx_2}{dx_1} = 0$

If we assume that $\mathbf{x}^* = (x_1^*,\ x_2^*)$ **is the local candidate minimum,**

$$\frac{\partial f(x_1^*, x_2^*)}{\partial x_1} + \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} \frac{d\phi(x_1)}{dx_1} = 0 \quad \cdots \textbf{ Equation (1)}$$

$x_2 = \phi(x_1)$ **is the explicit form, in general, it is impossible to represent the constraint as this from.**

Form the equality constraint $\boxed{h(x_1^*,\ x_2^*) = 0}$,

$$\Rightarrow \frac{dh(x_1^*, x_2^*)}{dx_1} = \frac{\partial h(x_1^*, x_2^*)}{\partial x_1} + \frac{\partial h(x_1^*, x_2^*)}{\partial x_2} \frac{d\phi(x_1)}{dx_1} = 0$$

$$\therefore \frac{d\phi(x_1)}{dx_1} = -\frac{\partial h(x_1^*, x_2^*)/\partial x_1}{\partial h(x_1^*, x_2^*)/\partial x_2} \quad \cdots \textbf{ Equation (2)}$$

Substitute the equation (2) into the equation (1)

$$\therefore \frac{\partial f(x_1^*, x_2^*)}{\partial x_1} - \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} \frac{\partial h(x_1^*, x_2^*)/\partial x_1}{\partial h(x_1^*, x_2^*)/\partial x_2} = 0 \quad \cdots \textbf{ Equation (3)}$$

If we assume that $\nu^* = -\dfrac{\partial f(x_1^*, x_2^*)/\partial x_2}{\partial h(x_1^*, x_2^*)/\partial x_2}$ $\cdots$ **Equation (4)**

The equation (3) becomes $\boxed{\dfrac{\partial f(x_1^*, x_2^*)}{\partial x_1} + \nu^* \dfrac{\partial h(x_1^*, x_2^*)}{\partial x_1} = 0}$

Equation (4) can be rearranged as follows.

$$\boxed{\frac{\partial f(x_1^*, x_2^*)}{\partial x_2} + \nu^* \frac{\partial h(x_1^*, x_2^*)}{\partial x_2} = 0}$$

In summary, for $\mathbf{x}^* = (x_1^*, x_2^*)$ **to become the local candidate minimum, the following three conditions have to be satisfied.**

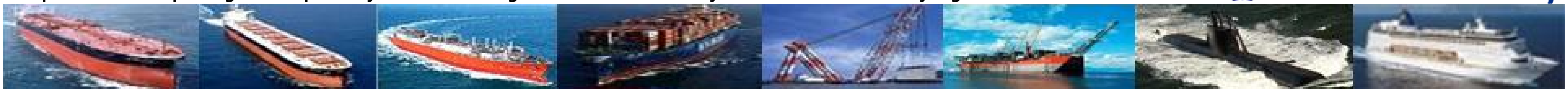
$$\boxed{h(x_1^*, x_2^*) = 0}$$

$$\boxed{\frac{\partial f(x_1^*, x_2^*)}{\partial x_1} + \nu^* \frac{\partial h(x_1^*, x_2^*)}{\partial x_1} = 0}$$

$$\boxed{\frac{\partial f(x_1^*, x_2^*)}{\partial x_2} + \nu^* \frac{\partial h(x_1^*, x_2^*)}{\partial x_2} = 0}$$

ν^* **is called the Lagrange multiplier.**

4.3 Kuhn-Tucker Necessary Condition for Inequality Constraints



Quadratic Programming Problem with Inequality Constraint

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

We can transform an inequality constraint to an equality constraint by adding a new variable, called the slack variable.

Original Problem

Minimize $f(\mathbf{x}) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$

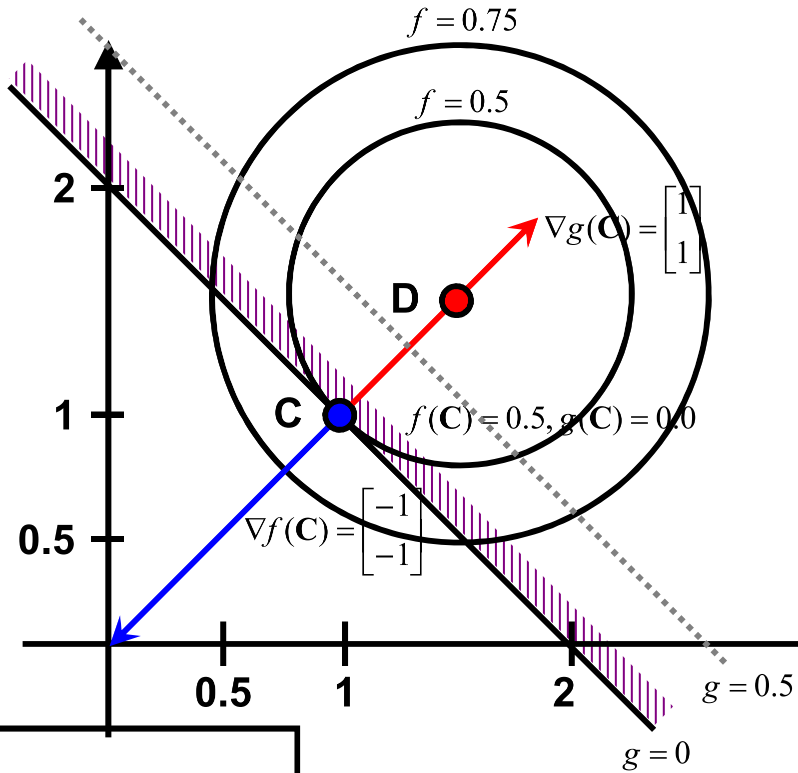
Subject to $g(\mathbf{x}) = x_1 + x_2 - 2 \le 0$

$\Rightarrow g(\mathbf{x}) + s^2 = x_1 + x_2 - 2 + \boxed{s^2} = 0$

Lagrange Function

Minimize $L(\mathbf{x}, u, s) = f(\mathbf{x}) + u[g(\mathbf{x}) + s^2]$

$$= (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + u(x_1 + x_2 - 2 + s^2)$$



Necessary Condition: $\nabla L(\mathbf{x}^*, u^*, s^*) = 0$

$\frac{\partial L}{\partial x_1} = 2(x_1 - 1.5) + u = 0, \quad \frac{\partial L}{\partial x_2} = 2(x_2 - 1.5) + u = 0$

$\frac{\partial L}{\partial u} = x_1 + x_2 - 2 + s^2 = 0,$

Linear indeterminate equation

$\frac{\partial L}{\partial s} = 2us = 0$

Nonlinear indeterminate equation

$u \ge 0$

(1) If s = 0, (Inequality constraint is transformed to the equality constraint.)

$x_1^* = x_2^* = 1, u^* = 1$ ➡ Candidate minimum point(The point C)

(2) If u = 0, (Inequality constraint is not active.)

$x_1^* = x_2^* = 1.5, u^* = 0, s^2 = -1$ (The point D: the constraint is violated.)

- At first, we find the solution for the nonlinear indeterminate equation of

$$u = 0 \text{ or } s = 0$$

- And substitute u=0 or s=0 into the linear indeterminate equation
- Then, solve the linear equation system.

Necessary Condition of Candidate Local Optimal Solution for the Inequality Constrained Problem (1/2)

[Ref] Lagrange function for the equality constrained problem

$$L(\mathbf{x},\mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h(\mathbf{x}) = f(\mathbf{x}) + \mathbf{v}^T \mathbf{h}(\mathbf{x})$$

v_i are the Lagrange multipliers for the equality constraints and are free in sign.

Inequality constraint

$$g_i(\mathbf{x}) \le 0, \quad i = 1, \ldots, m$$

To transform the inequality constraints to the equality constraints, the slack variables s_i^2 are introduced:

$$g_i(\mathbf{x}) + s_i^2 = 0, \quad i = 1, \ldots, m$$

Lagrange function for the inequality constrained problem

Since the inequality constraint can be transformed to the equality constraint by introducing the slack variable, the Lagrange function is defined as

$$L(\mathbf{x},\mathbf{u},\mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{m} u_i (g_i(\mathbf{x}) + s_i^2) = f(\mathbf{x}) + \mathbf{u}^T (\mathbf{g}(\mathbf{x}) + \mathbf{s}^2), \quad \underline{u_i \ge 0}$$

u_i are the Lagrange multiplier for the inequality constraints and have to be nonnegative.

s_i are the slack variables to transform the inequality constraints to the equality constraints.

Necessary Condition of Candidate Local Optimal Solution for the Inequality Constrained Problem (2/2)

Lagrange function for the inequality constrained problem

$$L(\mathbf{x},\mathbf{u},\mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{m} u_i (g_i(\mathbf{x}) + s_i^2) = f(\mathbf{x}) + \mathbf{u}^T (\mathbf{g}(\mathbf{x}) + \mathbf{s}^2)$$

u_i **are the Lagrange multiplier for the inequality constraints and have to be nonnegative.**

s_i **are the slack variables to transform the inequality constraints to the equality constraints.**

The necessary condition of the candidate local optimal solution for the inequality constrained problem

$$\nabla L(\mathbf{x}^*, \mathbf{u}^*, \mathbf{s}^*) = \mathbf{0}$$

⬇

$$\frac{\partial L}{\partial x_j} \equiv \frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} u_i^* \frac{\partial g_i}{\partial x_j} = 0, \qquad j = 1, \ldots, n$$

$$\frac{\partial L}{\partial u_i} \equiv g_i(\mathbf{x}^*) + s_i^{*2} = 0, \qquad i = 1, \ldots, \mathrm{m}$$

$$\frac{\partial L}{\partial s_i} \equiv u_i^* s_i^* = 0, \qquad i = 1, \ldots, m$$

$$u_i^* \geq 0, \qquad i = 1, \ldots, m$$

Kuhn-Tucker Necessary Condition for Inequality Constraints

Optimization Problem

$$\textit{Minimize} \quad f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$$

$$\textit{Subject to} \quad h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \qquad \text{Equality constraints}$$

$$g_i(\mathbf{x}) \le 0, \quad i = 1, \ldots, m \qquad \text{Inequality constraints}$$

Definition of the Lagrange function

$$L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x}) + \sum_{i=1}^{m} u_i (g_i(\mathbf{x}) + s_i^2)$$

$$= f(\mathbf{x}) + \mathbf{v}^T \mathbf{h}(\mathbf{x}) + \mathbf{u}^T (\mathbf{g}(\mathbf{x}) + \mathbf{s}^2)$$

v_i are the Lagrange multipliers for the equality constraints and are free in sign.

u_i are the Lagrange multiplier for the inequality constraints and have to be nonnegative.

s_i are the slack variables to transform the inequality constraints to the equality. constraints

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_j} \equiv \frac{\partial f}{\partial x_j} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i}{\partial x_j} + \sum_{i=1}^{m} u_i^* \frac{\partial g_i}{\partial x_j} = 0, \qquad j = 1, \ldots, n$$

$$\frac{\partial L}{\partial v_i} \equiv h_i(\mathbf{x}^*) = 0, \qquad i = 1, \ldots, p$$

$$\frac{\partial L}{\partial u_i} \equiv g_i(\mathbf{x}^*) + s_i^{*2} = 0, \qquad i = 1, \ldots, \mathrm{m}$$

$$\frac{\partial L}{\partial s_i} \equiv u_i^* s_i^* = 0, \qquad i = 1, \ldots, m$$

$$u_i^* \ge 0, \qquad i = 1, \ldots, m$$

The value of the objective function and gradient vectors have to be calculated a $\mathbf{x}^*$.

If x* is the candidate local minimum point, the equations from the Kuhn-Tucker necessary condition have to be satisfied.

Therefore, K.-T. condition can be used to find the candidate local minimum point for the equality and inequality constrained problem.

[Example] Nonlinear Constrained Optimization Problem #1 (1/2)

1

$$\textit{Minimize}\ \ f(\mathbf{x}) = x_1{}^2 + x_2{}^2 - 3x_1x_2$$

$$g(\mathbf{x}) = x_1{}^2 + x_2{}^2 - 6 \le 0$$

2

$$L(\mathbf{x}, u, s) = x_1{}^2 + x_2{}^2 - 3x_1x_2 + u(x_1{}^2 + x_2{}^2 - 6 + s^2)$$

3

$$\frac{\partial L}{\partial x_1} = 2x_1 - 3x_2 + 2ux_1 = 0 \text{ --------- ①}$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 3x_1 + 2ux_2 = 0 \text{ --------- ②}$$

$$\frac{\partial L}{\partial u} = x_1{}^2 + x_2{}^2 - 6 + s^2 = 0,\ s^2 \ge 0,\ u \ge 0 \text{ --------- ③}$$

$$\frac{\partial L}{\partial s} = \boxed{2us = 0}$$ ➡ **There are two cases.**

CASE #1: $u = 0$ **(The inequality constraint is considered as inactive at the solution point.)**

$$2x_1 - 3x_2 = 0$$
$$-3x_1 + 2x_2 = 0$$

➡ **Point A:** $x_1^* = 0,\ x_2^* = 0,\ f(x_1^*, x_2^*) = 0$

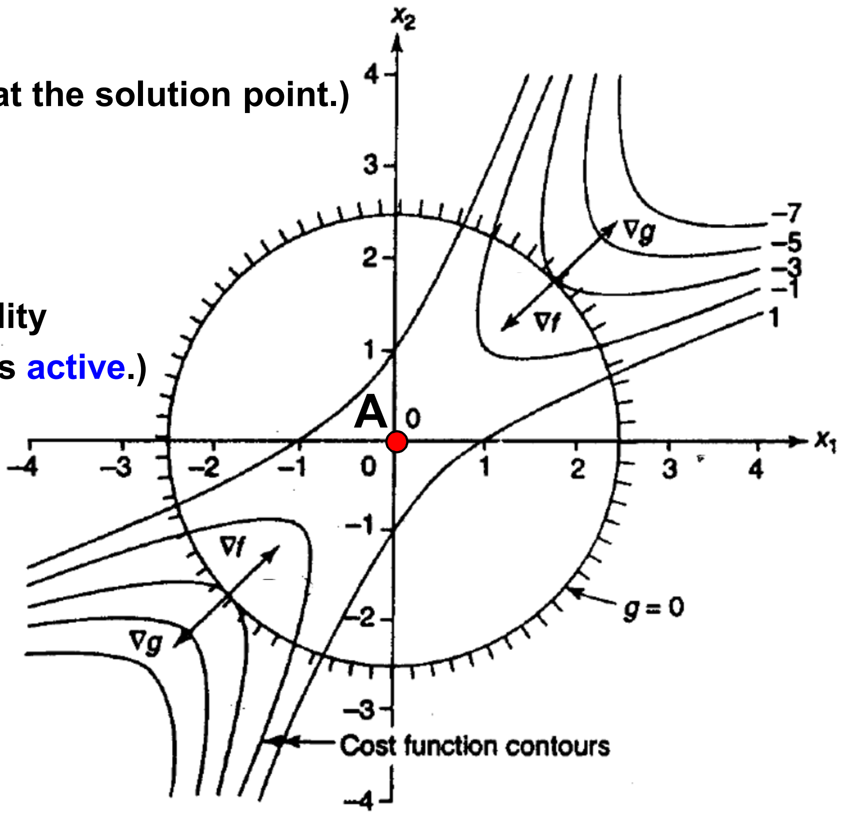
CASE #2: $s = 0$ **(The solution point is on the boundary of the inequality constraint. The inequality constraint is considered as active.)**

Rearrange the equation ① $\quad 2x_1 - 3x_2 + 2ux_1 = 0,\ \ u = -1 + \dfrac{3}{2}\dfrac{x_2}{x_1}$

Substitute u into the equation ② $\quad 2x_2 - 3x_1 + 2(-1 + \dfrac{3}{2}\dfrac{x_2}{x_1})x_2 = 0$

$$2x_2 - 3x_1 - 2x_2 + 3\frac{x_2^2}{x_1} = 0,\quad 3\frac{x_2^2}{x_1} = 3x_1,\quad x_2^2 = x_1^2$$

Substitute x_2 into the equation ③ $\quad 2x_1{}^2 = 6,\quad x_1 = \pm\sqrt{3}$



[Example] Nonlinear Constrained Optimization Problem #1 (2/2)

CASE #1: $u = 0$ **(The inequality constraint is considered as inactive at the solution point.)**

$$2x_1 - 3x_2 = 0$$
$$-3x_1 + 2x_2 = 0$$

➡ **Point A:** $x_1^* = 0,\ x_2^* = 0,\ f(x_1^*, x_2^*) = 0$

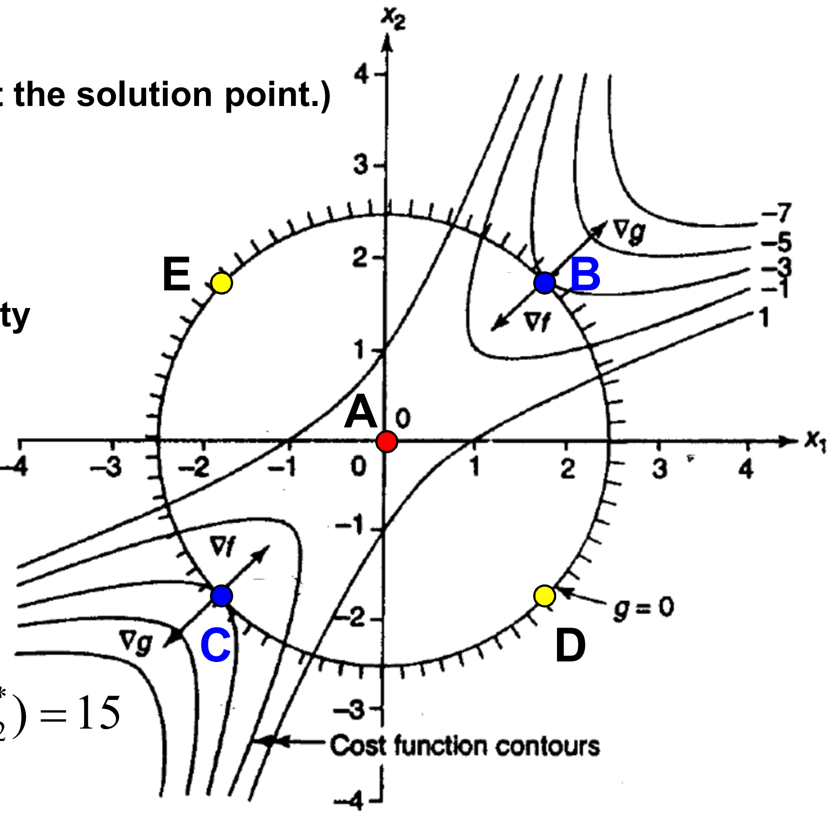
CASE #2: $s = 0$ **(The solution point is on the boundary of the inequality constraint.)**

$x_1 = x_2 = \sqrt{3},\ u = \dfrac{1}{2}$ ➡ **Point B:** $x_1^* = x_2^* = \sqrt{3}, f(x_1^*, x_2^*) = -3$

$x_1 = x_2 = -\sqrt{3},\ u = \dfrac{1}{2}$ ➡ **Point C:** $x_1^* = x_2^* = -\sqrt{3}, f(x_1^*, x_2^*) = -3$

$x_1 = -x_2 = \sqrt{3},\ u = -\dfrac{5}{2}$ ➡ **Point D:** $x_1^* = \sqrt{3}, x_2^* = -\sqrt{3}, f(x_1^*, x_2^*) = 15$

$x_1 = -x_2 = -\sqrt{3},\ u = -\dfrac{5}{2}$ ➡ **Point E:** $x_1^* = -\sqrt{3}, x_2^* = \sqrt{3}, f(x_1^*, x_2^*) = 15$



Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

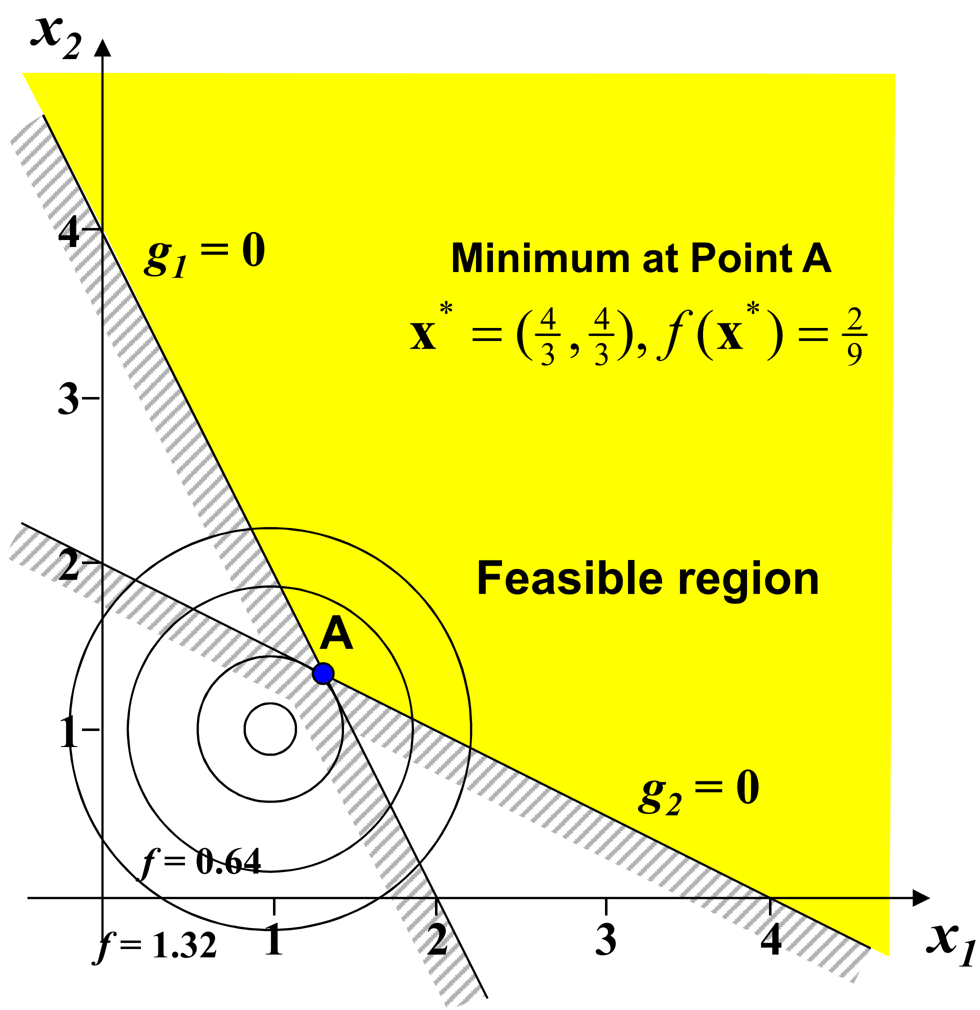
Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$

Subject to $g_1(\mathbf{x}) = -2x_1 - x_2 + 4 \le 0$

$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 \le 0$

Lagrange function

$$L(\mathbf{x}, \mathbf{u}, \mathbf{s}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$+u_1(-2x_1 - x_2 + 4 + s_1^2)$$
$$+u_2(-x_1 - 2x_2 + 4 + s_2^2)$$



Lagrange function

$$L(\mathbf{x},\mathbf{u},\mathbf{s}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$+u_1(-2x_1 - x_2 + 4 + s_1^2)$$
$$+u_2(-x_1 - 2x_2 + 4 + s_2^2)$$

$$\boxed{\begin{aligned} f(\mathbf{x}) &= x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2 \\ g_1(\mathbf{x}) &= -2x_1 - x_2 + 4 \le 0 \\ g_2(\mathbf{x}) &= -x_1 - 2x_2 + 4 \le 0 \end{aligned}}$$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{u},\mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2 - 2u_1 - u_2 = 0 \qquad \frac{\partial L}{\partial x_2} = 2x_2 - 2 - u_1 - 2u_2 = 0$$

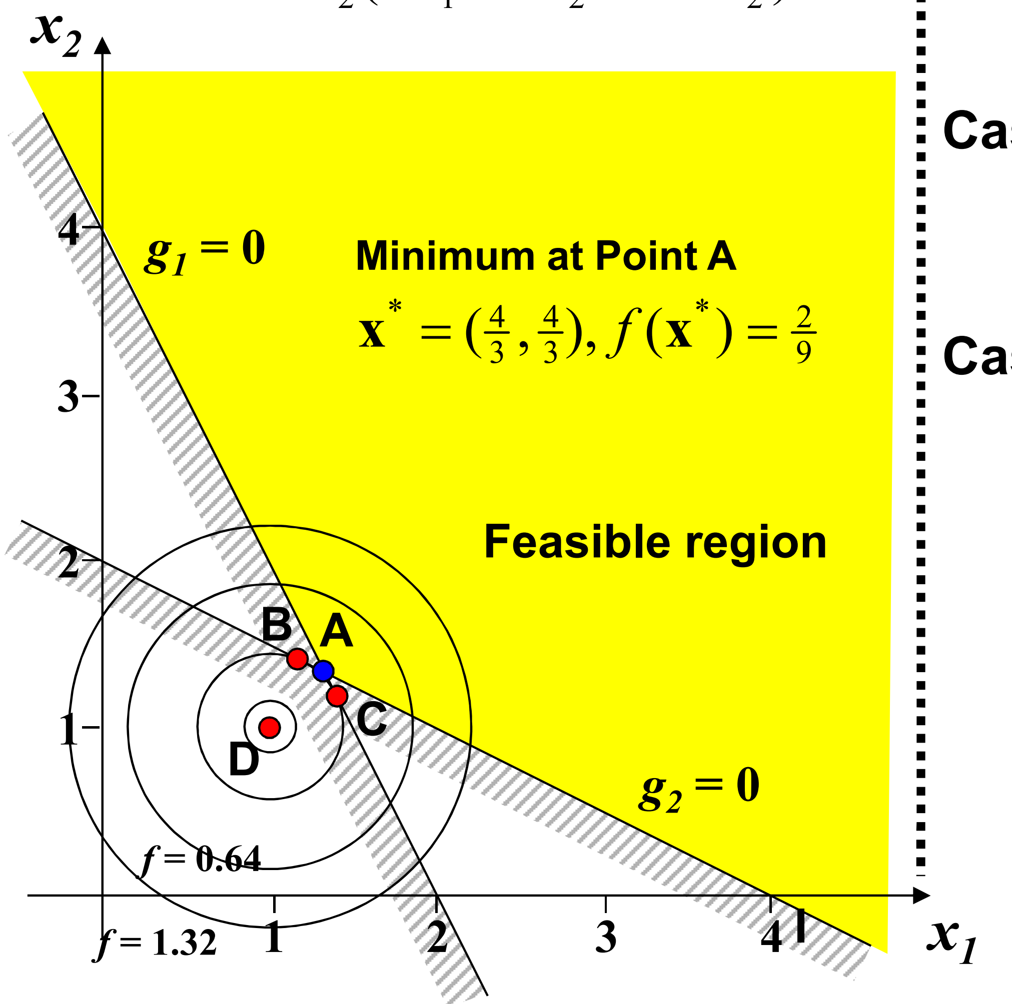
$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1^2 = 0 \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_2^2 = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1 = 0 \qquad \frac{\partial L}{\partial s_2} = 2u_2 s_2 = 0 \qquad u_i \ge 0,\ i = 1, 2$$

[Example] Nonlinear Constrained Optimization Problem #2
- Find the Optimal Solution for the Quadratic Programming Problem by using the Kuhn-Tucker Necessary Condition : xi are free in sign (3/3)

Lagrange function

$$L(\mathbf{x},\mathbf{u},\mathbf{s}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2 + u_1(-2x_1 - x_2 + 4 + s_1^2) + u_2(-x_1 - 2x_2 + 4 + s_2^2)$$



Case #1: $s_1 = s_2 = 0$

$$x_1 = x_2 = \frac{4}{3}, u_1 = u_2 = \frac{2}{9}$$ **(Minimum at Point A)**

Case #2: $u_1 = s_2 = 0$, (Point B)

$$x_1 = \frac{6}{5}, x_2 = \frac{7}{5}, u_2 = \frac{2}{5}, s_1^2 = -\frac{1}{5}$$

It has to be nonnegative(g_1).

Case #3: $u_2 = s_1 = 0$, (Point C)

$$x_1 = \frac{7}{5}, x_2 = \frac{6}{5}, u_1 = \frac{2}{5}, s_2^2 = -\frac{1}{5}$$

It has to be nonnegative(g_2).

Case #4: $u_1 = u_2 = 0$, (Point D)

$$x_1 = x_2 = 1, s_1^2 = s_2^2 = -1$$

It has to be nonnegative(g_2, g_2).

[Example] Nonlinear Constrained Optimization Problem #3
- Optimum Solution for the Case that x_i are"Nonnegative" (1/4)

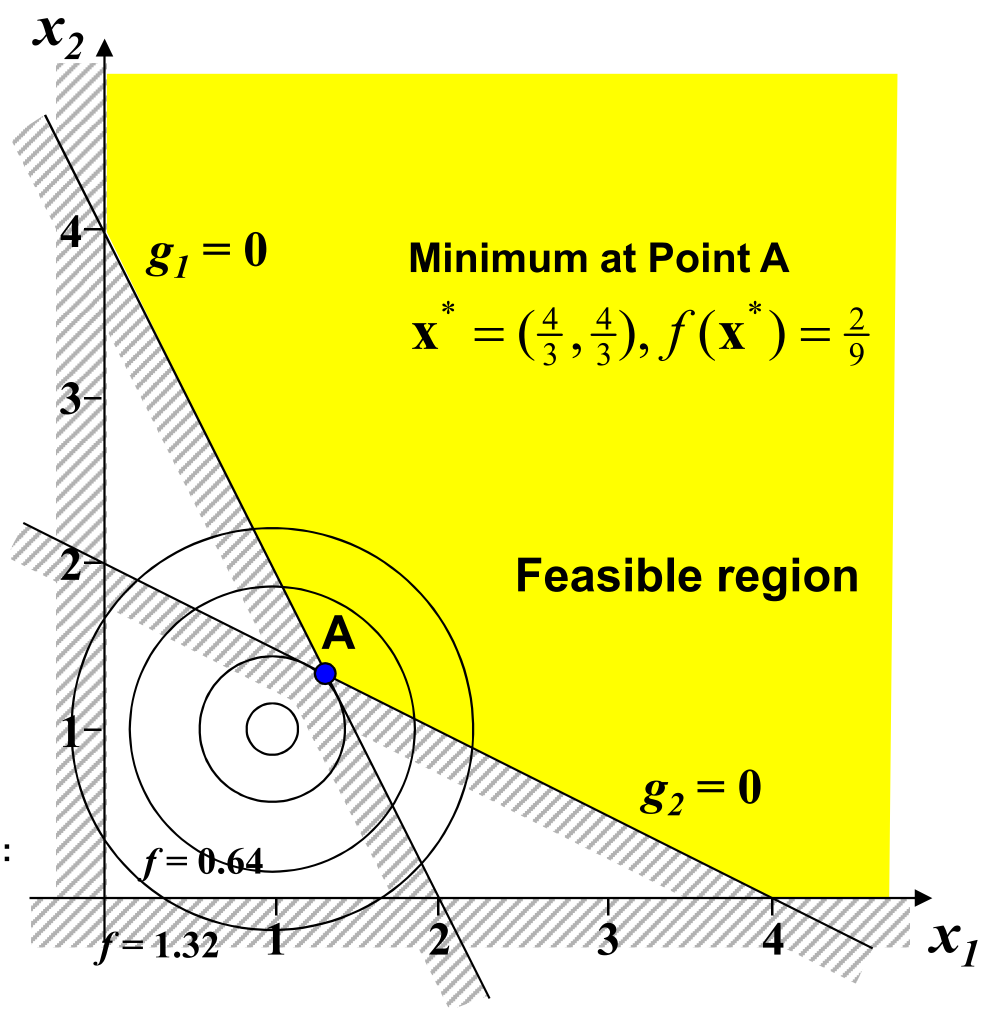
$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 \le 0$$
$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 \le 0$$
$$x_1 \ge 0, x_2 \ge 0$$

Minimum point: $\mathbf{x}^* = (\frac{4}{3}, \frac{4}{3}), f(\mathbf{x}^*) = \frac{2}{9}$

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 \le 0$$
$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 \le 0$$
$$-x_1 \le 0, -x_2 \le 0$$

Inequality constraints whose form are "≤":
Introduce the slack variable.

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 + s_1^2 = 0$$
$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 + s_2^2 = 0$$
$$-x_1 + \delta_1^2 = 0, -x_2 + \delta_2^2 = 0$$



[Example] Nonlinear Constrained Optimization Problem #3 - Optimum Solution for the Case that x_i are "Nonnegative" (2/4)

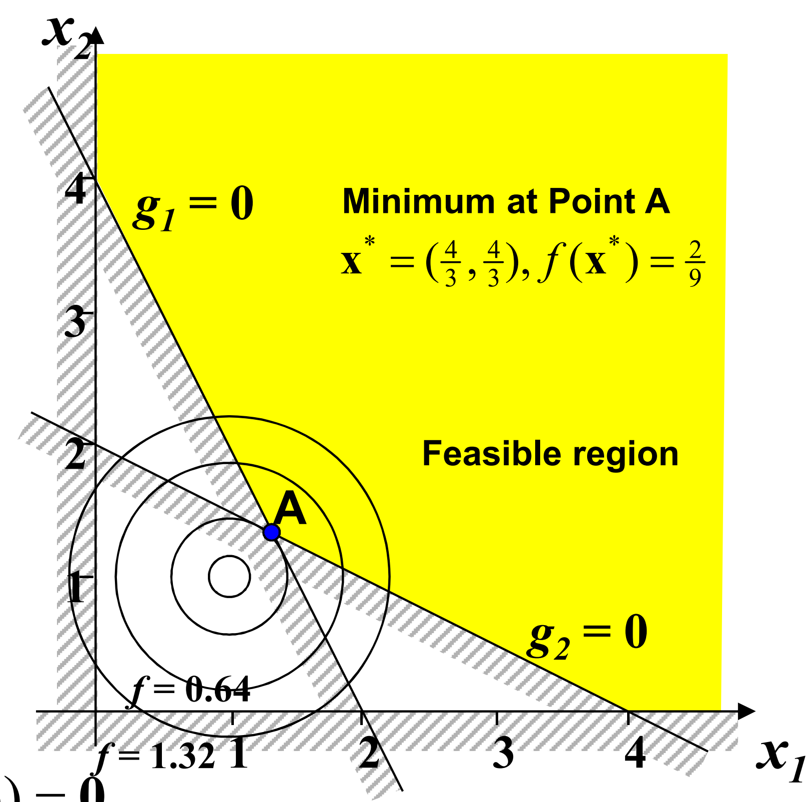
Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 + s_1^2 = 0$$

$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 + s_2^2 = 0$$

$$-x_1 + \delta_1^2 = 0, -x_2 + \delta_2^2 = 0$$



Lagrange function

$$L(\mathbf{x}, \mathbf{u}, \mathbf{s}, \boldsymbol{\zeta}, \boldsymbol{\delta}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$+u_1(-2x_1 - x_2 + 4 + s_1^2)$$
$$+u_2(-x_1 - 2x_2 + 4 + s_2^2)$$
$$+\zeta_1(-x_1 + \delta_1^2) + \zeta_2(-x_2 + \delta_2^2)$$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x}, \mathbf{u}, \mathbf{s}, \boldsymbol{\zeta}, \boldsymbol{\delta}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2 - 2u_1 - u_2 - \zeta_1 = 0 \qquad \frac{\partial L}{\partial x_2} = 2x_2 - 2 - u_1 - 2u_2 - \zeta_2 = 0$$

$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1^2 = 0 \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_1^2 = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1 = 0 \qquad \frac{\partial L}{\partial s_2} = 2u_2 s_2 = 0$$

$$\frac{\partial L}{\partial \zeta_1} = \delta_1^2 - x_1 = 0 \qquad \frac{\partial L}{\partial \zeta_2} = \delta_2^2 - x_2 = 0 \qquad \frac{\partial L}{\partial \delta_1} = 2\zeta_1 \delta_1 = 0 \qquad \frac{\partial L}{\partial \delta_1} = 2\zeta_1 \delta_1 = 0$$

[Example] Nonlinear Constrained Optimization Problem #3
- Optimum Solution for the Case that x_i are "Nonnegative" (3/4)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2 - 2u_1 - u_2 - \zeta_1 = 0 \qquad \frac{\partial L}{\partial x_2} = 2x_2 - 2 - u_1 - 2u_2 - \zeta_2 = 0$$

$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1^2 = 0 \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_2^2 = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1 = 0 \qquad \frac{\partial L}{\partial s_2} = 2u_2 s_2 = 0$$

$$\frac{\partial L}{\partial \zeta_1} = \delta_1^2 - x_1 = 0 \rightarrow \delta_1^2 = x_1 \qquad \frac{\partial L}{\partial \zeta_2} = \delta_2^2 - x_2 = 0 \rightarrow \delta_2^2 = x_2$$

Substitute **Substitute**

$$\frac{\partial L}{\partial \delta_1} = 2\zeta_1\delta_1 = 0 \rightarrow 2\zeta_1\delta_1^2 = 0 \qquad \frac{\partial L}{\partial \delta_1} = 2\zeta_1\delta_1 = 0 \rightarrow 2\zeta_2\delta_2^2 = 0$$

Multiply δ_1 **to the both sides.** **Multiply** δ_2 **to the both sides.**

$$u_i, \zeta_i \geq 0,\ i = 1, 2$$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2 - 2u_1 - u_2 - \zeta_1 = 0 \qquad \frac{\partial L}{\partial x_2} = 2x_2 - 2 - u_1 - 2u_2 - \zeta_2 = 0$$

$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1^2 = 0 \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_2^2 = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1 = 0 \qquad \frac{\partial L}{\partial s_2} = 2u_2 s_2 = 0$$

$$2\zeta_1 x_1 = 0 \qquad 2\zeta_2 x_2 = 0$$

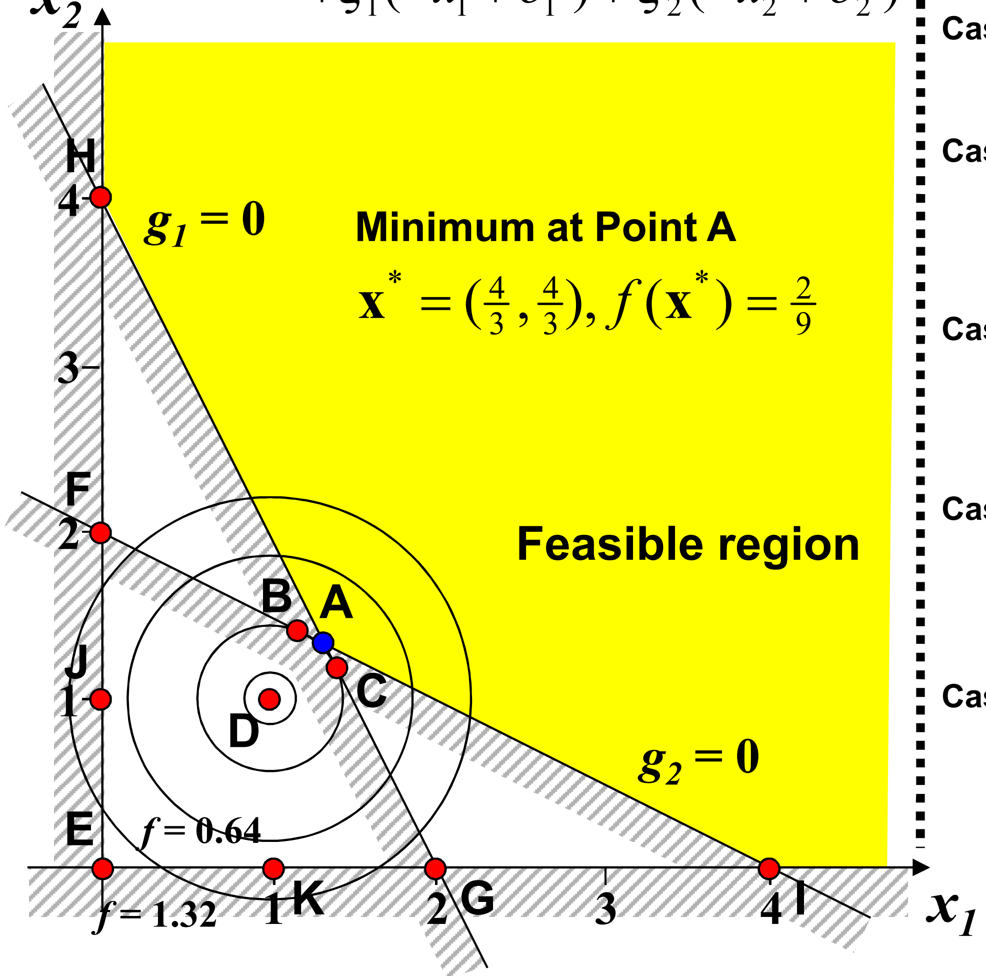
$$u_i, \zeta_i, \delta_i \geq 0,\ i = 1, 2$$

[Example] Nonlinear Constrained Optimization Problem #3 - Optimum Solution for the Case that x_i are "Nonnegative" (4/4)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Lagrangian function

$$L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$+u_1(-2x_1 - x_2 + 4 + s_1^2)$$
$$+u_2(-x_1 - 2x_2 + 4 + s_2^2)$$
$$+\zeta_1(-x_1 + \delta_1^2) + \zeta_2(-x_2 + \delta_2^2)$$



Case #1: $s_1=s_2=\zeta_1=\zeta_2=0$, (Point A)

$x_1 = x_2 = \frac{4}{3}, u_1 = u_2 = \frac{2}{9}$

Case #2: $u_1=s_2=\zeta_1=\zeta_2=0$, (Point B)

$x_1 = \frac{6}{5}, x_2 = \frac{7}{5}, u_2 = \frac{2}{5}, s_1^2 = -\frac{1}{5}$

It has to be nonnegative.

Case #3: $u_2=s_1=\zeta_1=\zeta_2=0$, (Point C)

$x_1 = \frac{7}{5}, x_2 = \frac{6}{5}, u_1 = \frac{2}{5}, s_2^2 = -\frac{1}{5}$

It has to be nonnegative.

Case #4: $u_1=u_2=\zeta_1=\zeta_2=0$, (Point D)

$x_1 = x_2 = 1, s_1^2 = s_2^2 = -1$

It has to be nonnegative.

Case #5: $u_1=u_2=x_1=x_2=0$, (Point E)

$x_1 = x_2 = 0, s_1^2 = s_2^2 = -4,$

It has to be nonnegative.

$\zeta_1 = \zeta_2 = -2$

Case #6: $u_1=s_2=x_1=x_2=0$, (Point E)

$x_1 = x_2 = 0, s_1^2 = -4,$

It has to be nonnegative.

$-x_1 - 2x_2 + 4 + s_2^2 \neq 0$

The constraint is violated.

Case #7: $u_2=s_1=x_1=x_2=0$, (Point E)

$x_1 = x_2 = 0, s_2^2 = -4,$

It has to be nonnegative.

$-2x_1 - x_2 + 4 + s_1^2 \neq 0$

The constraint is violated.

Case #8: $s_1=s_2=x_1=x_2=0$, (Point E)

$x_1 = x_2 = 0, -2x_1 - x_2 + 4 + s_1^2 \neq 0,$

$-x_1 - 2x_2 + 4 + s_2^2 \neq 0$

The constraint is violated.

Case #9: $u_1=s_2=\zeta_2=x_1=0$, (Point F)

$x_1 = 0, x_2 = 2, u_2 = 1,$

$s_1^2 = -2, \zeta_1 = -3$

It has to be nonnegative.

Case #10: $u_2=s_1=\zeta_1=x_2=0$, (Point G)

$x_1 = 2, x_2 = 0, u_1 = 1, s_2^2 = -2,$

$\zeta_2 = -3$

It has to be nonnegative.

Case #11: $s_1=s_2=\zeta_1=x_2=0$, (Point G)

$x_1 = 2, x_2 = 0,$

The constraint is violated.

$-x_1 - 2x_2 + 4 + s_2^2 \neq 0$

Case #12: $u_2=s_1=\zeta_2=x_1=0$, (Point H)

$x_1 = 0, x_2 = 4, u_1 = 6,$

$s_2^2 = 4, \zeta_1 = -14$

It has to be nonnegative.

Case #13: $s_1=s_2=\zeta_2=x_1=0$, (Point H)

$x_1 = 0, x_2 = 4,$

The constraint is violated.

$-x_1 - 2x_2 + 4 + s_2^2 \neq 0$

Case #14: $u_1=s_2=\zeta_1=x_2=0$, (Point I)

$x_1 = 4, x_2 = 0, u_2 = 6,$

$s_1^2 = 4, \zeta_2 = -14$

It has to be nonnegative.

Case #15: $u_1=u_2=\zeta_2=x_1=0$, (Point J)

$x_1 = 0, x_2 = 1, s_1^2 = -3,$

$s_2^2 = -2, \zeta_1 = -2$

It has to be nonnegative.

Case #16: $u_1=u_2=\zeta_1=x_2=0$, (Point K)

$x_1 = 1, x_2 = 0, s_1^2 = -2,$

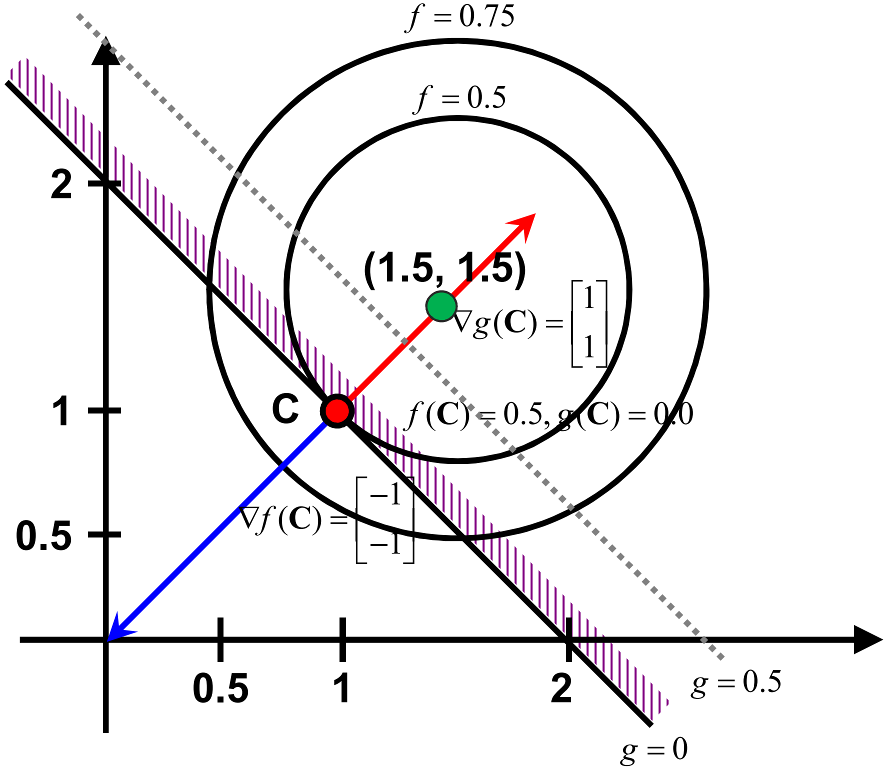
It has to be nonnegative.

$s_2^2 = -3, \zeta_2 = -2$

[Reference] The Reason Why Lagrange Multiplier for the Inequality Constraint has to be Positive

Original Problem

$$\textit{Minimize} \quad f(\mathbf{x}) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$
$$\textit{Subject to} \quad g(\mathbf{x}) = x_1 + x_2 - 2 \le 0$$



$$L(\mathbf{x}, u, s) = f(\mathbf{x}) + u(g(\mathbf{x}) + s^2)$$
$$\nabla L(\mathbf{x}, \mathbf{u}, \mathbf{s}) = \nabla f(\mathbf{x}) + u \nabla g(\mathbf{x}) = 0$$
$$-\nabla f(\mathbf{x}) = u \nabla g(\mathbf{x})$$

→ : ∇f

Direction of the gradients of the objective function

→ : ∇g

Direction of the gradients of the constraint

If $u > 0$, the gradients of the objective function and the constraint function point in the opposite direction:

$$-\nabla f \simeq \nabla g$$

In order to reduce the value of the objective function f, the design point has to move to the negative gradient direction.

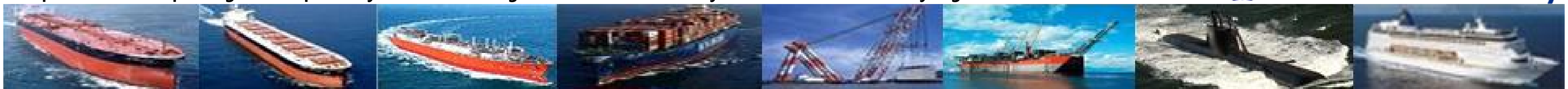
However, at the green point(1.5, 1.5), for example,

$$g(\mathbf{x}) = x_1 + x_2 - 2$$
$$= 1.5 + 1.5 - 2 = 1 \cancel{\le} 0 \quad \text{the constraint is violated.}$$

Therefore, this way, f cannot be reduced any further by moving to the negative gradient direction without violating the constraint.

That is, the point C is the optimal solution satisfying the constraint and minimizing the objective function.

[Appendix] Quadratic Programming Problem



[Example] Quadratic Programming Problem #1

Objective function: quadratic form
Constraint: linear form

Ex) Problem

$$\textit{Minimize}\quad f(x_1,x_2)=(x_1-1.5)^2+(x_2-1.5)^2$$
$$h(x_1,x_2)=x_1+x_2-2=0$$

Find: Local minimum point(x_1*, x_2*)

Method

Express h (equality constraint) as an explicit function of x_1.

Then substitute x_1 into f and find the stationary point by using $df = 0$.

Solution

Express x_2 as an explicit function of x_1,

$$x_2=-x_1+2=\Phi(x_1)$$

$$f(x_1,\Phi(x_1))=(x_1-1.5)^2+(-x_1+2-1.5)^2$$

This is an unconstrained optimization problem to determine the stationary point.

$$df=\frac{\partial f}{\partial x_1}dx_1=0\Rightarrow\frac{df}{dx_1}=0$$

$$\frac{df}{dx_1}=2(x_1-1.5)-2(-x_1+0.5)=0$$

$$\Rightarrow x_1=1$$

$$\Rightarrow x_2=-x_1+2=1$$

$$\frac{d^2f}{dx_1{}^2}=4>0$$

Local minimum point $\quad\therefore (x_1^*,\ x_2^*)=(1,\ 1)$

[Example] Solution for Quadratic Programming Problem #2

Given: $f(x_1, x_2, x_3) = x_1{}^2 + x_2{}^2 + x_3{}^2$

$$h(x_1, x_2, x_3) = x_1 + x_2 + x_3 + 1 = 0$$

Find: Stationary point (x_1^*, x_2^*, x_3^*)

Express h (equality constraint) as an explicit function of x_1.

$$x_1 = -x_2 - x_3 - 1$$

Substitute x_1 into the function f

$$f = (-x_2 - x_3 - 1)^2 + x_2{}^2 + x_3{}^2$$

$$= (x_2{}^2 + x_3{}^2 + 1 + 2x_2x_3 + 2x_2 + 2x_3) + x_2{}^2 + x_3{}^2$$

$$= 2x_2{}^2 + 2x_3{}^2 + 1 + 2x_2x_3 + 2x_2 + 2x_3$$

Determine the stationary point for the unconstrained optimization problem. $df = 0$

$$df = \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = 0$$

$$\therefore \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0$$

$$\frac{\partial f}{\partial x_2} = 4x_2 + 2x_3 + 2 = 0$$

$$\frac{\partial f}{\partial x_3} = 4x_3 + 2x_2 + 2 = 0$$

The solution of the equations are:

$$x_2 = -\frac{1}{3}, \; x_3 = -\frac{1}{3}$$

By substituting these value into the function f, w obtain

$$x_1 = -\frac{1}{3}$$

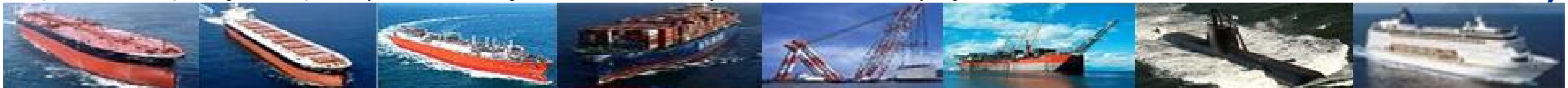
Thus, the stationary point is $\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$.

Naval Architecture & Ocean Engineering

[Appendix] Example of a Constrained Nonlinear Optimization Method by Using the Lagrange Multiplier

1. Determination of the Optimal Principal Dimensions of a Ship
2. Determination of the Optimal Principal Dimensions of a Propeller

SYstem Design Laboratory



- ***Given***: $DWT, V_{H.req}, D, T_s, T_d$
- ***Find***: L, B, C_B
  - **Hydrostatic equilibrium(Weight equation)**

$$L \cdot B \cdot T_s \cdot C_B \cdot \rho_{sw} \cdot C_\alpha = DWT_{given} + LWT(L, B, D, C_B)$$

$$= DWT_{given} + \boxed{C_s \cdot L^{1.6} \cdot (B+D)} + C_o \cdot L \cdot B + \boxed{C_{power} \cdot (L \cdot B \cdot T_d \cdot C_B)^{2/3} \cdot V^3} \quad ...(a)$$

Simplify ① $\rightarrow C_s{}' \cdot L^{2.0} \cdot (B+D)$

Simplify ② $\rightarrow C_{power}{}' \cdot (2 \cdot B \cdot T_d + 2 \cdot L \cdot T_d + L \cdot B) \cdot V^3$

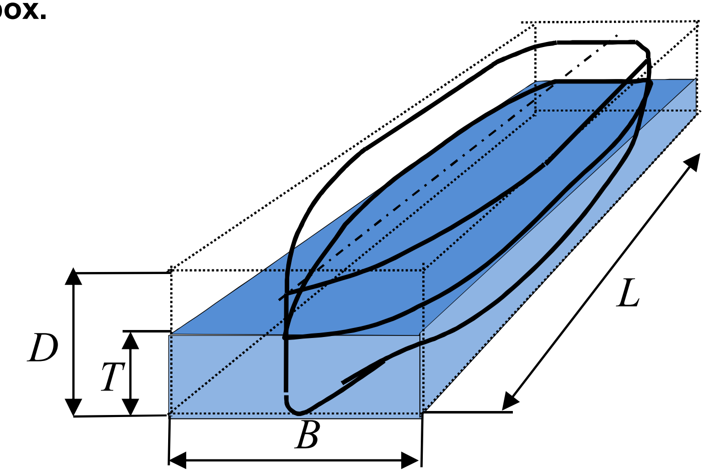
$(L \cdot B \cdot T_d \cdot C_B)^{2/3}$ **is (Volume)$^{2/3}$ and means the submerged area of the ship.**
So, we assume that the submerged area of the ship is equal to the submerged area of the rectangular box.

  - **Required cargo hold capacity(Volume equation)**

$$V_{H.req} = C_H \cdot L \cdot B \cdot D \quad ...(b)$$

  - **Recommended range of obesity coefficient considering maneuverability of a ship**

$$\frac{C_B}{(L/B)} < 0.15 \quad ...(c)$$



➡ **Indeterminate Equation: 3 variables(L, B, C_B), 2 equality constraints((a), (b))**

⇨ **It can be formulated as an optimization problem to minimize an objective function.**

- ***Given***: $DWT, V_{H.req}, D, T_s, T_d$
- ***Find***: L, B, C_B
- ***Minimize***: **Building Cost**

$$f(L,B,C_B) = C_{PS} \cdot C_s' \cdot L^{2.0} \cdot (B+D) + C_{PO} \cdot C_o \cdot L \cdot B + C_{PM} \cdot C_{power}' \cdot (2 \cdot B \cdot T_d + 2 \cdot L \cdot T_d + L \cdot B) \cdot V^3 \quad ...(d)$$

- ***Subject to***
  - **Hydrostatic equilibrium(Simplified weight equation)**

$$\begin{aligned} L \cdot B \cdot T_s \cdot C_B \cdot \rho_{sw} \cdot C_\alpha &= DWT_{given} + LWT(L,B,D,C_B) \\ &= DWT_{given} + C_s' \cdot L^{2.0} \cdot (B+D) + C_o \cdot L \cdot B + C_{power}' \cdot (2 \cdot B \cdot T_d + 2 \cdot L \cdot T_d + L \cdot B) \cdot V^3 \end{aligned} \quad ...(a')$$

$$V_{H.req} = C_H \cdot L \cdot B \cdot D \quad ...(b)$$

$$\frac{C_B}{(L/B)} < 0.15 \quad ...(c)$$

- **By introducing the Lagrange multipliers λ_1, λ_2, u, formulate the Lagrange function H.**

$$H(L,B,C_B,\lambda_1,\lambda_2,u,s)=f(L,B,C_B)+\lambda_1\cdot h_1(L,B,C_B)+\lambda_2\cdot h_2(L,B,D)+u\cdot g(L,B,C_B,s) \quad ...(e)$$

$$f(L,B,C_B)=C_{PS}\cdot C_s'\cdot L^2\cdot(B+D)+C_{PO}\cdot C_o\cdot L\cdot B+C_{PM}\cdot C_{power}'\cdot\{2\cdot(B+L)\cdot T_d+L\cdot B\}\cdot V^3$$

$$h_1(L,B,C_B)=L\cdot B\cdot T_s\cdot C_B\cdot\rho_{sw}\cdot C_\alpha-DWT_{given}-C_s'\cdot L^{2.0}\cdot(B+D)-C_o\cdot L\cdot B-C_{power}'\cdot\{2\cdot(B+L)\cdot T_d+L\cdot B\}\cdot V^3$$

$$h_2(L,B,D)=C_H\cdot L\cdot B\cdot D-V_{H.req}$$

$$g(L,B,C_B,s)=\frac{C_B}{(L/B)}-0.15+s^2$$

$$L\rightarrow x_1, B\rightarrow x_2, C_B\rightarrow x_3$$

$$H(x_1,x_2,x_3,\lambda_1,\lambda_2,u,s)$$

$$=C_{PS}\cdot C_s'\cdot x_1^2(x_2+D)+C_{PO}\cdot C_o\cdot x_1\cdot x_2+C_{PM}\cdot C_{power}'\cdot\{2\cdot(x_2+x_1)\cdot T_d+x_1\cdot x_2\}\cdot V^3$$

$$+\lambda_1\cdot[x_1\cdot x_2\cdot T_s\cdot x_3\cdot\rho_{sw}\cdot C_\alpha-DWT_{given}-C_s\cdot x_1^2\cdot(x_2+D)-C_o\cdot x_1\cdot x_2-C_{power}'\cdot\{2\cdot(x_2+x_1)\cdot T_d+x_1\cdot x_2\}\cdot V^3]$$

$$+\lambda_2\cdot\left(C_H\cdot x_1\cdot x_2\cdot D-V_{H.req}\right)$$

$$+u\cdot\left\{x_3/(x_1/x_2)-0.15+s^2\right\} \quad ...(f)$$

Example of a Constrained Nonlinear Optimization Method by Using the Lagrange Multiplier
- Determination of the Optimal Principal Dimensions of a Ship (4/5)

$L \to x_1, B \to x_2, C_B \to x_3$

$$H(x_1,x_2,x_3,\lambda_1,\lambda_2,u,s) = C_{PS}\cdot C_s{}'\cdot x_1{}^2(x_2+D) + C_{PO}\cdot C_o\cdot x_1\cdot x_2 + C_{PM}\cdot C_{power}{}'\cdot\{2\cdot(x_2+x_1)\cdot T_d + x_1\cdot x_2\}\cdot V^3$$
$$+\lambda_1\cdot[x_1\cdot x_2\cdot T_s\cdot x_3\cdot\rho_{sw}\cdot C_\alpha - DWT_{given} - C_s\cdot x_1{}^2\cdot(x_2+D) - C_o\cdot x_1\cdot x_2 - C_{power}{}'\cdot\{2\cdot(x_2+x_1)\cdot T_d + x_1\cdot x_2\}\cdot V^3]$$
$$+\lambda_2\cdot(C_H\cdot x_1\cdot x_2\cdot D - V_{H\_req}) + u\cdot\{x_3/(x_1/x_2) - 0.15 + s^2\} \quad ...(f)$$

- **To determine the stationary point(x_1, x_2, x_3) of the Lagrange function H(equation (f)), use the Kuhn-Tucker necessary condition: $\nabla H(x_1,x_2,x_3,\lambda_1,\lambda_2,u,s) = 0$.**

$$\frac{\partial H}{\partial x_1} = 2C_{PS}\cdot C_s{}'\cdot x_1\cdot(x_2+D) + C_{PO}\cdot C_o\cdot x_2 + C_{PM}\cdot C_{power}{}'\cdot(2\cdot T_d + x_2)\cdot V^3$$
$$+\lambda_1\cdot(x_2\cdot T_s\cdot x_3\cdot\rho_{sw}\cdot C_\alpha - [2\cdot C_s\cdot x_1\cdot(x_2+D) + C_o\cdot x_2 + C_{power}{}'\cdot(2\cdot T_d + x_2)\cdot V^3])$$
$$+\lambda_2\cdot(C_H\cdot x_2\cdot D) + u\cdot(-x_3\cdot x_2/x_1{}^2) = 0 \quad ...(1)$$

$$\frac{\partial H}{\partial x_2} = C_{PS}\cdot C_s{}'\cdot x_1{}^2 + C_{PO}\cdot C_o\cdot x_1 + C_{PM}\cdot C_{power}{}'\cdot(2\cdot T_d + x_1)\cdot V^3$$
$$+\lambda_1\cdot[x_1\cdot T_s\cdot x_3\cdot\rho_{sw}\cdot C_\alpha - C_s{}'\cdot x_1{}^2 - C_o\cdot x_1 - C_{power}{}'(2\cdot T_d + x_1)\cdot V^3]$$
$$+\lambda_2\cdot(C_H\cdot x_1\cdot D) + u\cdot(x_3/x_1) = 0 \quad ...(2)$$

- Determination of the Optimal Principal Dimensions of a Ship (5/5)

$L \rightarrow x_1, B \rightarrow x_2, C_B \rightarrow x_3$

$$H(x_1, x_2, x_3, \lambda_1, \lambda_2, u, s) = C_{PS} \cdot C_s' \cdot x_1^2 (x_2 + D) + C_{PO} \cdot C_o \cdot x_1 \cdot x_2 + C_{PM} \cdot C_{power}' \cdot \{2 \cdot (x_2 + x_1) \cdot T_d + x_1 \cdot x_2\} \cdot V^3$$

$$+ \lambda_1 \cdot [x_1 \cdot x_2 \cdot T_s \cdot x_3 \cdot \rho_{sw} \cdot C_\alpha - DWT_{given} - C_s \cdot x_1^2 \cdot (x_2 + D) - C_o \cdot x_1 \cdot x_2 - C_{power}' \cdot \{2 \cdot (x_2 + x_1) \cdot T_d + x_1 \cdot x_2\} \cdot V^3]$$

$$+ \lambda_2 \cdot (C_H \cdot x_1 \cdot x_2 \cdot D - V_{H\_req}) + u \cdot \{x_3 / (x_1 / x_2) - 0.15 + s^2\} \quad \ldots(f)$$

- **Kuhn-Tucker necessary condition** $\nabla H(x_1, x_2, x_3, \lambda_1, \lambda_2, u, s) = 0$.

$$\frac{\partial H}{\partial x_3} = \lambda_1 . x_1 \cdot x_2 \cdot T_s \cdot \rho_{sw} \cdot C_\alpha + u \cdot (x_2 / x_1) = 0 \quad \ldots(3)$$

$$\frac{\partial H}{\partial \lambda_1} = x_1 \cdot x_2 \cdot T_s \cdot x_3 \cdot \rho_{sw} \cdot C_\alpha - DWT_{given} - C_s \cdot x_1^2 \cdot (x_2 + D) - C_o \cdot x_1 \cdot x_2$$

$$- C_{power}' \cdot \{2 \cdot (x_2 + x_1) \cdot T_d + x_1 \cdot x_2\} \cdot V^3 \quad \cdots(4)$$

$$\frac{\partial H}{\partial \lambda_2} = C_H \cdot x_1 \cdot x_2 \cdot D - V_{H.req} = 0 \quad \ldots(5)$$

$$\frac{\partial H}{\partial u} = x_3 \cdot x_2 / x_1 - 0.15 + s^2 = 0 \quad \ldots(6)$$

$$\frac{\partial H}{\partial s} = 2 \cdot u \cdot s = 0, \quad (u \geq 0) \quad \ldots(7)$$

- $\nabla H(x_1, x_2, x_3, \lambda_1, \lambda_2, u, s)$: **Nonlinear simultaneous equation having the 7 variables((1)~(7)) and 7 equations**

➡ It can be solved by using a numerical method!

Given $P, n, A_E / A_O, V$

Find $J, P_i / D_P$

Maximize $\eta_O = \dfrac{J}{2\pi} \cdot \dfrac{K_T}{K_Q}$ → **Because K_T and K_Q are a function of J and P_i/D_p, the objective is also a function of J and P_i/D_p.**

Subject to $\dfrac{P}{2\pi n} = \rho \cdot n^2 \cdot {D_P}^5 \cdot K_Q$

: The propeller absorbs the torque delivered by Diesel Engine

Where,

$$J = \frac{V(1-w)}{n \cdot D_P}$$

$$K_T = f(J, P_i / D_P)$$

$$K_Q = f(J, P_i / D_P)$$

P: Delivered power to the propeller from the main engine, KW
n: Revolution per second, 1/sec
D_P: Propeller diameter, m
P_i: Propeller pitch, m
A_E/A_O: Expanded area ratio
V: Ship speed, m/s
η_O: Propeller efficiency(in open water)

➡ **Optimization problem having two unknown variables and one equality constraint**

$$\frac{P}{2\pi n} = \rho \cdot n^2 \cdot D_P^{\ 5} \cdot K_Q \quad \cdots\cdots \quad \text{(a)}$$

: The propeller absorbs the torque delivered by main engine

The constraint (a) is reformulated as follows:

$$C = \frac{K_Q}{J^5} = \frac{P \cdot n^2}{2\pi\rho \cdot V_A^{\ 5}}$$

$$G(J, P_i / D_P) = K_Q - C \cdot J^5 = 0 \quad \cdots\cdots \quad \text{(a')}$$

Propeller efficiency in open water η_0 is as follows.

$$F(J, P_i / D_P) = \eta_0 = \frac{J}{2\pi} \cdot \frac{K_T}{K_Q} \quad \cdots\cdots \quad \text{(b)}$$

The objective F is a function of J and P_i/D_p.

It is to determine the optimal principal dimensions(J and P_i/D_p) to maximize the propeller efficiency in open water satisfying the constraint (a').

$$G(J, P_i/D_P) = K_Q - C \cdot J^5 = 0 \quad \cdots\cdots \quad (a')$$

$$F(J, P_i/D_P) = \eta_0 = \frac{J}{2\pi} \cdot \frac{K_T}{K_Q} \quad \cdots\cdots \quad (b)$$

Introduce the Lagrange multiplier λ to the equation (a') and (b).

$$H(J, P_i/D_P, \lambda) = F(J, P_i/D_P) + \lambda G(J, P_i/D_P) \quad \cdots\cdots \quad (c)$$

Determine the value of the P_i/D_P and λ to maximize the value of the function H.

$$\frac{\partial H}{\partial J} = \frac{1}{2\pi}\left(\frac{K_T}{K_Q}\right) + \frac{J}{2\pi}\frac{\{(\frac{\partial K_T}{\partial J}) \cdot K_Q - (\frac{\partial K_Q}{\partial J}) \cdot K_T\}}{K_Q^{\ 2}} + \lambda\{(\frac{\partial K_Q}{\partial J}) - 5 \cdot C \cdot J^4\} = 0 \cdots (1)$$

$$\frac{\partial H}{\partial (P_i/D_P)} = \frac{J}{2\pi}\frac{\{(\frac{\partial K_T}{\partial P_i/D_P}) \cdot K_Q - (\frac{\partial K_Q}{\partial P_i/D_P}) \cdot K_T\}}{K_Q^{\ 2}} + \lambda(\frac{\partial K_Q}{\partial P_i/D_P}) = 0 \quad \cdots \quad (2)$$

$$\frac{\partial H}{\partial \lambda} = K_Q - C \cdot J^5 = 0 \quad \cdots\cdots\cdots\cdots \quad (3)$$

Eliminate λ in the equation (1), (2), and (3), and rearrange as follows.

$$\left(\frac{\partial K_Q}{\partial (P_i / D_P)}\right)\left\{J \cdot \left(\frac{\partial K_T}{\partial J}\right) - 4K_T\right\}$$

$$+\left(\frac{\partial K_T}{\partial (P_i / D_P)}\right)\left\{5K_Q - J \cdot \left(\frac{\partial K_Q}{\partial J}\right)\right\} = 0 \quad \cdots\cdots \quad (4)$$

$$K_Q - C \cdot J^5 = 0 \quad \cdots\cdots \quad (5)$$

By solving the nonlinear equation (4) and (5), we can determine J and P_i/D_p to maximize the propeller efficiency.

By definition $J = \frac{V(1-w)}{n \cdot D_P}$, if we have J we can find D_p. Then P_i is obtained from P_i/D_p.

Thus, we can find the propeller diameter(D_p) and pitch(P_i).

[Reference] Derivation of Eq. (4) from Eqs. (1)~(3) (1/3)

$$\frac{1}{2\pi}\left(\frac{K_T}{K_Q}\right)+\frac{J}{2\pi}\frac{\left\{\left(\frac{\partial K_T}{\partial J}\right)\cdot K_Q-\left(\frac{\partial K_Q}{\partial J}\right)\cdot K_T\right\}}{{K_Q}^2}+\lambda\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}=0 \quad \cdots\cdots \quad (1)$$

$$\frac{J}{2\pi}\frac{\left\{\left(\frac{\partial K_T}{\partial (P_i/D_P)}\right)\cdot K_Q-\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)\cdot K_T\right\}}{{K_Q}^2}+\lambda\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)=0 \quad \cdots\cdots \quad (2)$$

To eliminate λ, we calculate as follows.

$$\text{Eq. }(1)\times\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)-\text{Eq. }(2)\times\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}=0$$

$$\text{Eq. }(1)\times\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right):\ \frac{1}{2\pi}\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)\left(\frac{K_T}{K_Q}\right)+\frac{J}{2\pi}\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)\frac{\left\{\left(\frac{\partial K_T}{\partial J}\right)\cdot K_Q-\left(\frac{\partial K_Q}{\partial J}\right)\cdot K_T\right\}}{{K_Q}^2}+\cancel{\lambda\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}}=0$$

$$\text{Eq. }(2)\times\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}:\ \frac{J}{2\pi}\frac{\left\{\left(\frac{\partial K_T}{\partial (P_i/D_P)}\right)\cdot K_Q-\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)\cdot K_T\right\}}{{K_Q}^2}\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}+\cancel{\lambda\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}}=0$$

$$\text{Eq. }(1)\times\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)-\text{Eq. }(2)\times\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}$$

$$=\frac{1}{2\pi}\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)\left(\frac{K_T}{K_Q}\right)+\frac{J}{2\pi}\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)\frac{\left\{\left(\frac{\partial K_T}{\partial J}\right)\cdot K_Q-\left(\frac{\partial K_Q}{\partial J}\right)\cdot K_T\right\}}{{K_Q}^2}-\frac{J}{2\pi}\frac{\left\{\left(\frac{\partial K_T}{\partial (P_i/D_P)}\right)\cdot K_Q-\left(\frac{\partial K_Q}{\partial (P_i/D_P)}\right)\cdot K_T\right\}}{{K_Q}^2}\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}=0$$

[Reference] Derivation of Eq. (4) from Eqs. (1)~(3) (2/3)

$$\text{Eq. (1)}\times\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)-\text{Eq. (2)}\times\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}$$

$$=\frac{1}{2\pi}\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{K_T}{K_Q}\right)+\frac{J}{2\pi}\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\frac{\left\{\left(\frac{\partial K_T}{\partial J}\right)\cdot K_Q-\left(\frac{\partial K_Q}{\partial J}\right)\cdot K_T\right\}}{K_Q{}^2}-\frac{J}{2\pi}\frac{\left\{\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\cdot K_Q-\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\cdot K_T\right\}}{K_Q{}^2}\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}=0$$

Multiply 2π and the both side of the equation and rearrange the equation as follows.

$$\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{K_T}{K_Q}\right)+\frac{J}{K_Q{}^2}\left[\underline{\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left\{\left(\frac{\partial K_T}{\partial J}\right)\cdot K_Q-\left(\frac{\partial K_Q}{\partial J}\right)\cdot K_T\right\}-\left\{\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\cdot K_Q-\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\cdot K_T\right\}\left\{\left(\frac{\partial K_Q}{\partial J}\right)-5\cdot C\cdot J^4\right\}}\right]=0$$

The term underlined is rearranged as follows.

$$=\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_T}{\partial J}\right)\cdot K_Q-\cancel{\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_Q}{\partial J}\right)}\cdot K_T-\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_Q}{\partial J}\right)\cdot K_Q+\cancel{\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_Q}{\partial J}\right)}\cdot K_T+5\cdot\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\cdot K_Q\cdot C\cdot J^4-5\cdot\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\cdot K_T\cdot C\cdot J^4$$

$$=\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_T}{\partial J}\right)\cdot K_Q-\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_Q}{\partial J}\right)\cdot K_Q+5\cdot\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\cdot K_Q\cdot C\cdot J^4-5\cdot\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\cdot K_T\cdot C\cdot J^4$$

Substituting the rearranged term into the above equation.

$$\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{K_T}{K_Q}\right)+\frac{J}{K_Q{}^2}\left[\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_T}{\partial J}\right)\cdot K_Q-\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_Q}{\partial J}\right)\cdot K_Q+5\cdot\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\cdot K_Q\cdot C\cdot J^4-5\cdot\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\cdot K_T\cdot C\cdot J^4\right]=0$$

- Determination of the Optimal Principal Dimensions of a Propeller

[Reference] Derivation of Eq. (4) from Eqs. (1)~(3) (3/3)

$$\frac{\partial H}{\partial \lambda} = K_Q - C\cdot J^5 = 0 \quad \cdots\cdots \quad (3)$$

$$\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{K_T}{K_Q}\right)+\frac{J}{K_Q^2}\left[\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_T}{\partial J}\right)\cdot K_Q-\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_Q}{\partial J}\right)\cdot K_Q+5\cdot\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\cdot K_Q\cdot C\cdot J^4-5\cdot\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\cdot K_T\cdot C\cdot J^4\right]=0$$

Apply the distributive property.

$$\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{K_T}{K_Q}\right)+\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_T}{\partial J}\right)\cdot\frac{J}{K_Q}-\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_Q}{\partial J}\right)\cdot\frac{J}{K_Q}+5\cdot\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\cdot\cancel{\frac{C\cdot J^5}{K_Q}}-5\cdot\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\cdot\frac{K_T}{K_Q}\cdot\cancel{\frac{C\cdot J^5}{K_Q}}=0$$

By using Eq. (3) $\frac{C\cdot J^5}{K_Q}=1$

$$\underline{\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{K_T}{K_Q}\right)}+\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_T}{\partial J}\right)\cdot\frac{J}{K_Q}-\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_Q}{\partial J}\right)\cdot\frac{J}{K_Q}+5\cdot\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\underline{-5\cdot\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\cdot\frac{K_T}{K_Q}}=0$$

The underlined term is calculated as follows.

$$-4\cdot\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{K_T}{K_Q}\right)+\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_T}{\partial J}\right)\cdot\frac{J}{K_Q}-\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_Q}{\partial J}\right)\cdot\frac{J}{K_Q}+5\cdot\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)=0$$

Multiply K_Q and the both side of the equation.

$$-4\cdot\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)K_T+\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_T}{\partial J}\right)J-\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\left(\frac{\partial K_Q}{\partial J}\right)\cdot J+5\cdot K_Q\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)=0$$

Apply the distributive property. $\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right),\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)$

$$\left(\frac{\partial K_Q}{\partial(P_i/D_P)}\right)\left\{J\cdot\left(\frac{\partial K_T}{\partial J}\right)-4K_T\right\}+\left(\frac{\partial K_T}{\partial(P_i/D_P)}\right)\left\{5K_Q-J\cdot\left(\frac{\partial K_Q}{\partial J}\right)\right\}=0 \quad \cdots\cdots \quad (4)$$

[Reference] Linear Systems vs. Matrices

$$x_1 + 2x_2 + x_3 = 1$$
$$3x_1 - x_2 - x_3 = 2$$
$$2x_1 + 3x_2 - x_3 = -3$$

Row2 + Row1 x (-3)

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

Row2 + Row1 x (-3)

$$\begin{array}{rl} & 3x_1 - x_2 - x_3 = 2 \\ +) & -3x_1 - 6x_2 - 3x_3 = -3 \\ \hline & -7x_2 - 4x_3 = -1 \end{array}$$

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 - 7x_2 - 4x_3 = -1$$
$$2x_1 + 3x_2 - x_3 = -3$$

$$\begin{bmatrix} 0 & -7 & -4 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$