Naval Architecture & Ocean Engineering

Computer Aided Ship Design

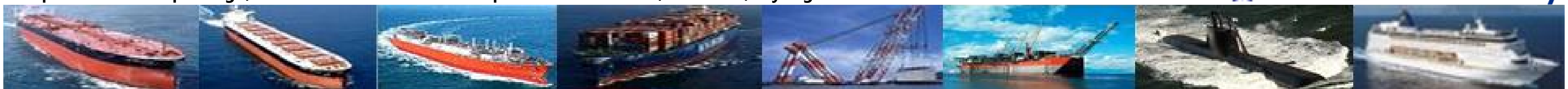
Part I. Optimization Method

Ch. 7 Constrained Nonlinear Optimization Method

September, 2013

Prof. Myung-Il Roh

Department of Naval Architecture and Ocean Engineering,
Seoul National University of College of Engineering



Ch. 7 Constrained Nonlinear Optimization Method

7.1 Quadratic Programming(QP)

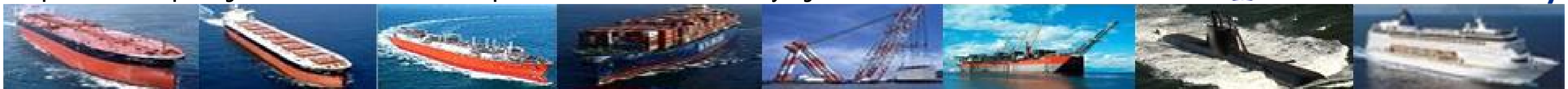
7.2 Sequential Linear Programming(SLP)

7.3 Sequential Quadratic Programming(SQP)

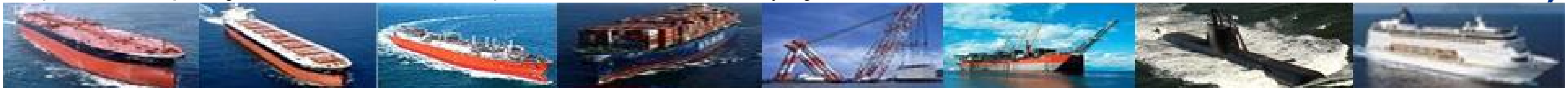
7.4 Determine the Search Direction of the Quadratic Programming Problem by Using the Simplex Method

7.5 Summary of the Sequential Quadratic Programming(SQP)

SYstem Design Laboratory



7.1 Quadratic Programming(QP)



[Review] 4.3: Finding the optimal solution for the quadratic objective function with linear inequality constraints problem by using the Kuhn-Tucker Necessary Condition, where xi are nonnegative (1/4)

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 \le 0$$

$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 \le 0$$

$$x_1 \ge 0, x_2 \ge 0$$

Minimum point: $\mathbf{x}^* = (\frac{4}{3}, \frac{4}{3}), f(\mathbf{x}^*) = \frac{2}{9}$

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 \le 0$$

$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 \le 0$$

$$-x_1 \le 0, -x_2 \le 0$$

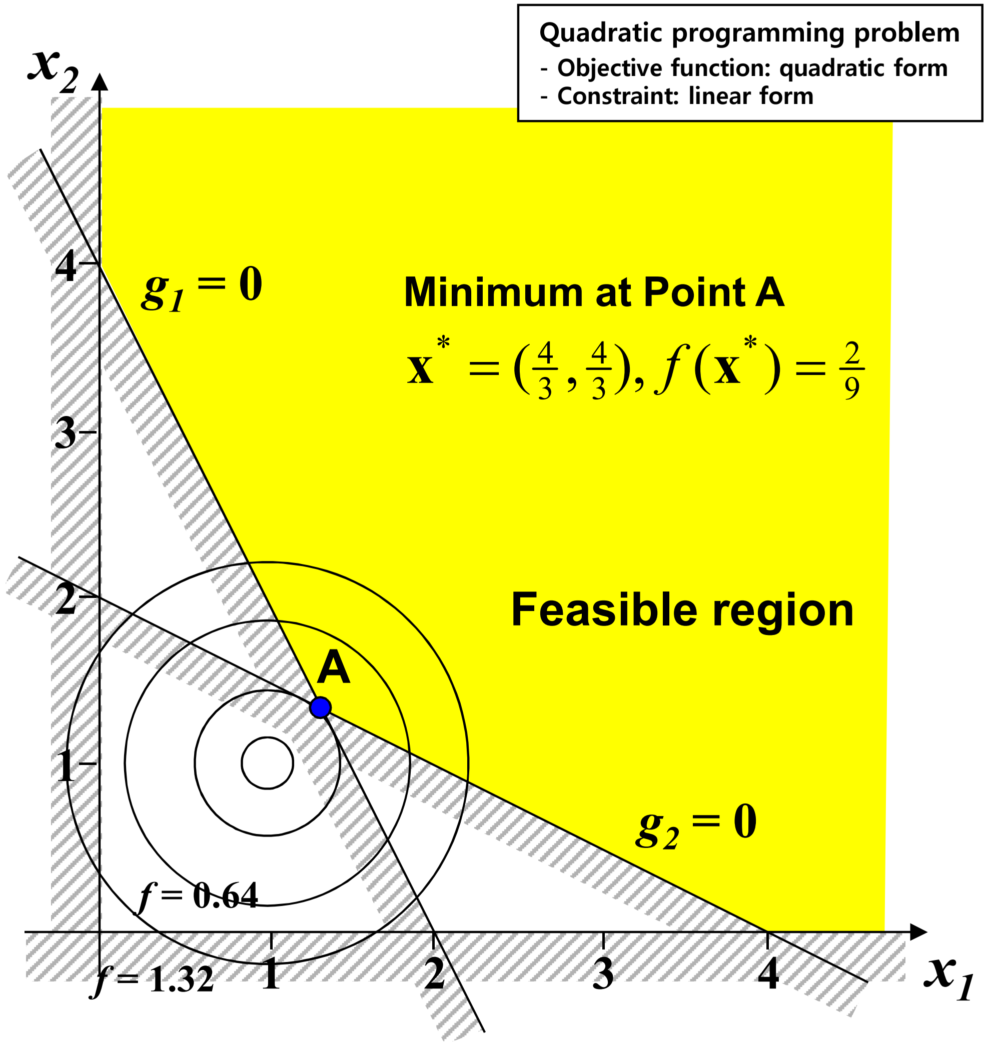
Inequality constraints are transformed to equality constraints by introducing the slack variable.

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 + s_1^2 = 0$$

$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 + s_2^2 = 0$$

$$-x_1 + \delta_1^2 = 0, -x_2 + \delta_2^2 = 0$$



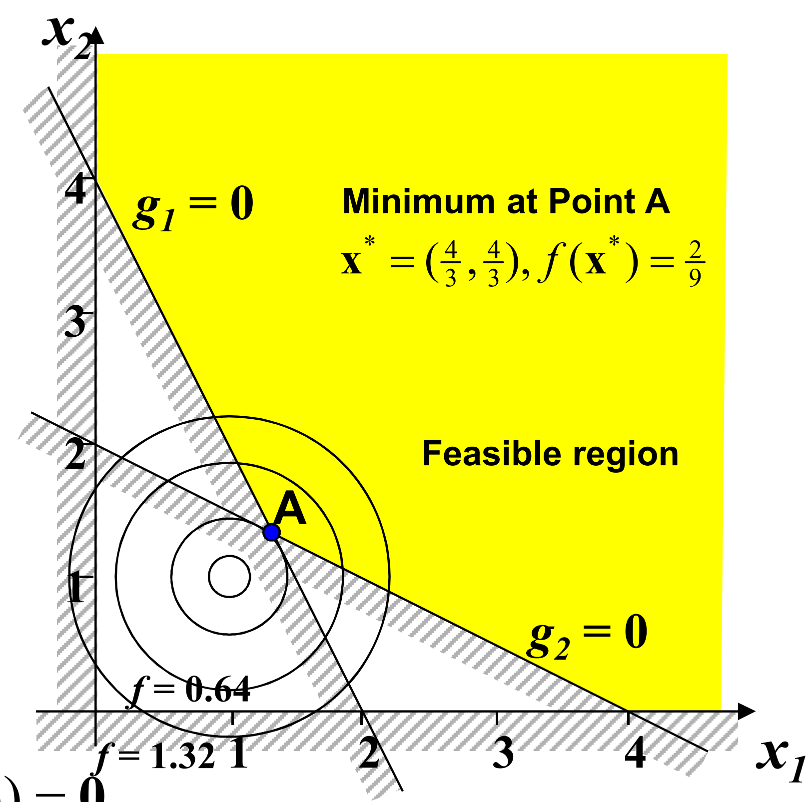
[Review] 4.3: Finding the optimal solution for the quadratic objective function with linear inequality constraints problem by using the Kuhn-Tucker Necessary Condition, where xi are nonnegative (2/4)

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 + s_1^2 = 0$$

$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 + s_2^2 = 0$$

$$-x_1 + \delta_1^2 = 0, -x_2 + \delta_2^2 = 0$$



Lagrange function

$$L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$+u_1(-2x_1 - x_2 + 4 + s_1^2)$$
$$+u_2(-x_1 - 2x_2 + 4 + s_2^2)$$
$$+\zeta_1(-x_1 + \delta_1^2) + \zeta_2(-x_2 + \delta_2^2)$$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta}) = 0$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2 - 2u_1 - u_2 - \zeta_1 = 0 \qquad \frac{\partial L}{\partial x_2} = 2x_2 - 2 - u_1 - 2u_2 - \zeta_2 = 0$$

$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1^2 = 0 \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_1^2 = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1 = 0 \qquad \frac{\partial L}{\partial s_2} = 2u_2 s_2 = 0$$

$$\frac{\partial L}{\partial \zeta_1} = \delta_1^2 - x_1 = 0 \qquad \frac{\partial L}{\partial \zeta_2} = \delta_2^2 - x_2 = 0 \qquad \frac{\partial L}{\partial \delta_1} = 2\zeta_1\delta_1 = 0 \qquad \frac{\partial L}{\partial \delta_1} = 2\zeta_1\delta_1 = 0$$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta})=\mathbf{0}$

$$\frac{\partial L}{\partial x_1}=2x_1-2-2u_1-u_2-\zeta_1=0 \qquad \frac{\partial L}{\partial x_2}=2x_2-2-u_1-2u_2-\zeta_2=0$$

$$\frac{\partial L}{\partial u_1}=-2x_1-x_2+4+s_1^2=0 \qquad \frac{\partial L}{\partial u_2}=-x_1-2x_2+4+s_2^2=0$$

$$\frac{\partial L}{\partial s_1}=2u_1s_1=0 \qquad \frac{\partial L}{\partial s_2}=2u_2s_2=0$$

$$\frac{\partial L}{\partial \zeta_1}=\delta_1^2-x_1=0\rightarrow\delta_1^2=x_1 \qquad \frac{\partial L}{\partial \zeta_2}=\delta_2^2-x_2=0\rightarrow\delta_2^2=x_2$$

Substitute **Substitute**

$$\frac{\partial L}{\partial \delta_1}=2\zeta_1\delta_1=0\rightarrow 2\zeta_1\delta_1^2=0 \qquad \frac{\partial L}{\partial \delta_1}=2\zeta_1\delta_1=0\rightarrow 2\zeta_2\delta_2^2=0$$

Multiply both sides by δ_1 Multiply both sides by δ_2

$$u_i,\zeta_i\ge 0,\ i=1,2$$

We eliminate two variables δ_1, δ_2 and two equations.

Reformulated Kuhn-Tucker necessary condition:

$$\frac{\partial L}{\partial x_1}=2x_1-2-2u_1-u_2-\zeta_1=0 \qquad \frac{\partial L}{\partial x_2}=2x_2-2-u_1-2u_2-\zeta_2=0$$

$$\frac{\partial L}{\partial u_1}=-2x_1-x_2+4+s_1^2=0 \qquad \frac{\partial L}{\partial u_2}=-x_1-2x_2+4+s_2^2=0$$

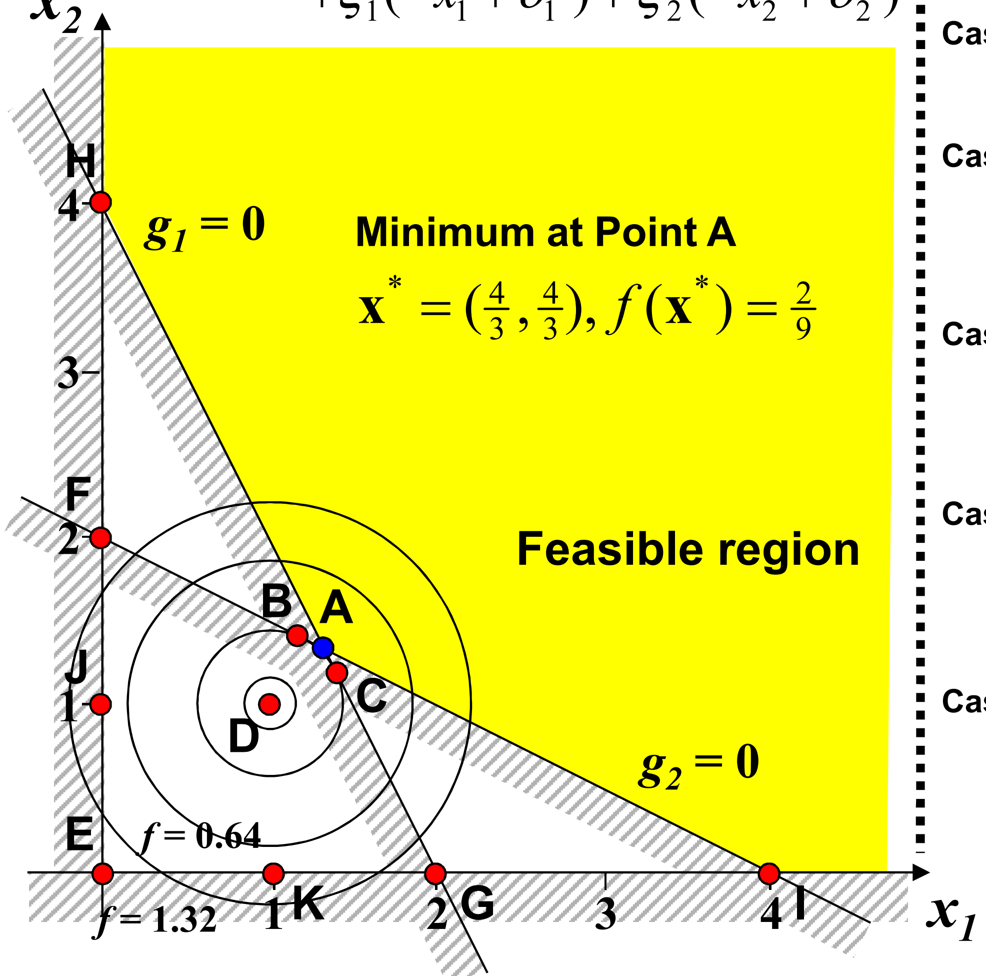
$$\frac{\partial L}{\partial s_1}=2u_1s_1=0 \qquad \frac{\partial L}{\partial s_2}=2u_2s_2=0$$

$$2\zeta_1x_1=0 \qquad 2\zeta_2x_2=0 \qquad u_i,\zeta_i,\delta_i\ge 0,\ i=1,2$$

[Review] 4.3: Finding the optimal solution for the quadratic objective function with linear inequality constraints problem by using the Kuhn-Tucker Necessary Condition, where xi are nonnegative (4/4)

Lagrangian function

$$L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$+u_1(-2x_1 - x_2 + 4 + s_1^2)$$
$$+u_2(-x_1 - 2x_2 + 4 + s_2^2)$$
$$+\zeta_1(-x_1 + \delta_1^2) + \zeta_2(-x_2 + \delta_2^2)$$



Case #1: $s_1=s_2=\zeta_1=\zeta_2=0$, (Point A)

$$x_1 = x_2 = \frac{4}{3}, u_1 = u_2 = \frac{2}{9}$$

Case #2: $u_1=s_2=\zeta_1=\zeta_2=0$, (Point B)

$$x_1 = \frac{6}{5}, x_2 = \frac{7}{5}, u_2 = \frac{2}{5}, s_1^2 = -\frac{1}{5}$$

It has to be nonnegative.

Case #3: $u_2=s_1=\zeta_1=\zeta_2=0$, (Point C)

$$x_1 = \frac{7}{5}, x_2 = \frac{6}{5}, u_1 = \frac{2}{5}, s_2^2 = -\frac{1}{5}$$

It has to be nonnegative.

Case #4: $u_1=u_2=\zeta_1=\zeta_2=0$, (Point D)

$$x_1 = x_2 = 1, s_1^2 = s_2^2 = -1$$

It has to be nonnegative.

Case #5: $u_1=u_2=x_1=x_2=0$, (Point E)

$$x_1 = x_2 = 0, s_1^2 = s_2^2 = -4,$$

It has to be nonnegative.

$$\zeta_1 = \zeta_2 = -2$$

Case #6: $u_1=s_2=x_1=x_2=0$, (Point E)

$$x_1 = x_2 = 0, s_1^2 = -4,$$

It has to be nonnegative.

$$-x_1 - 2x_2 + 4 + s_2^2 \neq 0$$

The constraint is violated.

Case #7: $u_2=s_1=x_1=x_2=0$, (Point E)

$$x_1 = x_2 = 0, s_2^2 = -4,$$

It has to be nonnegative.

$$-2x_1 - x_2 + 4 + s_1^2 \neq 0$$

The constraint is violated.

Case #8: $s_1=s_2=x_1=x_2=0$, (Point E)

$$x_1 = x_2 = 0, -2x_1 - x_2 + 4 + s_1^2 \neq 0,$$
$$-x_1 - 2x_2 + 4 + s_2^2 \neq 0$$

The constraint is violated.

Case #9: $u_1=s_2=\zeta_2=x_1=0$, (Point F)

$$x_1 = 0, x_2 = 2, u_2 = 1,$$
$$s_1^2 = -2, \zeta_1 = -3$$

It has to be nonnegative.

Case #10: $u_2=s_1=\zeta_1=x_2=0$, (Point G)

$$x_1 = 2, x_2 = 0, u_1 = 1, s_2^2 = -2,$$
$$\zeta_2 = -3$$

It has to be nonnegative.

Case #11: $s_1=s_2=\zeta_1=x_2=0$, (Point G)

$$x_1 = 2, x_2 = 0,$$

The constraint is violated.

$$-x_1 - 2x_2 + 4 + s_2^2 \neq 0$$

Case #12: $u_2=s_1=\zeta_2=x_1=0$, (Point H)

$$x_1 = 0, x_2 = 4, u_1 = 6,$$
$$s_2^2 = 4, \zeta_1 = -14$$

It has to be nonnegative.

Case #13: $s_1=s_2=\zeta_2=x_1=0$, (Point H)

$$x_1 = 0, x_2 = 4,$$

The constraint is violated.

$$-x_1 - 2x_2 + 4 + s_2^2 \neq 0$$

Case #14: $u_1=s_2=\zeta_1=x_2=0$, (Point I)

$$x_1 = 4, x_2 = 0, u_2 = 6,$$
$$s_1^2 = 4, \zeta_2 = -14$$

It has to be nonnegative.

Case #15: $u_1=u_2=\zeta_2=x_1=0$, (Point J)

$$x_1 = 0, x_2 = 1, s_1^2 = -3,$$
$$s_2^2 = -2, \zeta_1 = -2$$

It has to be nonnegative.

Case #16: $u_1=u_2=\zeta_1=x_2=0$, (Point K)

$$x_1 = 1, x_2 = 0, s_1^2 = -2,$$

It has to be nonnegative.

$$s_2^2 = -3, \zeta_2 = -2$$

Summary for Solution of QP Problem Using the Kuhn-Tucker Necessary Condition

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$

Subject to

$$-2x_1 - x_2 \le -4 \qquad\qquad -2x_1 - x_2 + 4 + s_1^2 = 0$$
$$-x_1 - 2x_2 \le -4 \quad \Rightarrow \quad -x_1 - 2x_2 + 4 + s_2^2 = 0$$
$$x_1 \ge 0, x_2 \ge 0 \qquad\qquad -x_1 + \delta_1^2 = 0,\ -x_2 + \delta_2^2 = 0$$

Lagrange function

$$L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$+u_1(-2x_1 - x_2 + 4 + s_1^2) + u_2(-x_1 - 2x_2 + 4 + s_2^2)$$
$$+\zeta_1(-x_1 + \delta_1^2) + \zeta_2(-x_2 + \delta_2^2)$$ **where,** $u_i, \zeta_i \ge 0$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta}) = 0$

$$\frac{\partial L}{\partial x_1} = -2 + 2x_1 - 2u_1 - u_2 - \zeta_1 = 0, \qquad \frac{\partial L}{\partial x_2} = -2 + 2x_2 - u_1 - 2u_2 - \zeta_2 = 0$$

$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1^2 = 0, \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_2^2 = 0 \qquad \frac{\partial L}{\partial \zeta_1} = -x_1 + \delta_1^2 = 0$$

$$\underbrace{\frac{\partial L}{\partial s_1} = 2u_1 s_1 = 0,}_{\text{Equation ①}} \quad \underbrace{\frac{\partial L}{\partial s_2} = 2u_2 s_2 = 0}_{\text{Equation ②}} \qquad \underbrace{\frac{\partial L}{\partial \delta_1} = 2\zeta_1 \delta_1 = 0,}_{\text{Equation ③}} \quad \underbrace{\frac{\partial L}{\partial \delta_2} = 2\zeta_2 \delta_2 = 0}_{\text{Equation ④}} \qquad \frac{\partial L}{\partial \zeta_2} = -x_2 + \delta_2^2 = 0$$

where $u_i, \zeta_i \ge 0$

Multiply both side of each equation ①, ② by s_1, s_2, respectively

Multiply both side of each equation ③, ④ by δ_1, δ_2, respectively

Solution Procedure of Quadratic Programming(QP) Problem
- Approximate the Original Problem as a Quadratic Programming Problem

Minimize $f(\mathbf{x}+\Delta\mathbf{x}) \cong f(\mathbf{x}) + \nabla f^T(\mathbf{x})\Delta\mathbf{x} + 0.5\Delta\mathbf{x}^T\mathbf{H}\Delta\mathbf{x}$

The second-order Taylor series expansion of the objective function

Subject to $h_j(\mathbf{x}+\Delta\mathbf{x}) \cong h_j(\mathbf{x}) + \nabla h_j^T(\mathbf{x})\Delta\mathbf{x} = 0;\ j = 1\ to\ p$

The first-order(linear) Taylor series expansion of the equality constraints

$g_j(\mathbf{x}+\Delta\mathbf{x}) \cong g_j(\mathbf{x}) + \nabla g_j^T(\mathbf{x})\Delta\mathbf{x} \le 0;\ j = 1\ to\ m$

The first-order(linear) Taylor series expansion of the inequality constraints

Define: $\bar{f} = f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x}),\ e_j = -h_j(\mathbf{x}),\ b_j = -g_j(\mathbf{x}),$

$c_i = \partial f(\mathbf{x})/\partial x_i,\ n_{ij} = \partial h_j(\mathbf{x})/\partial x_i,\ a_{ij} = \partial g_j(\mathbf{x})/\partial x_i,$

$d_i = \Delta x_i$

Matrix form

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$: Quadratic objective function

Subject to $\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$: Linear equality constraints

$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$: Linear inequality constraints

Solution Procedure of Quadratic Programming(QP) Problem - Construction of Lagrange Function

$$\textit{Minimize} \quad \bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$$

$$\textit{Subject to} \quad \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$$

$$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)} \;\Rightarrow\; \mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} - \mathbf{b}_{(m\times 1)} + \mathbf{s}_{(m\times 1)}{}^2 = \mathbf{0}$$

Lagrange Function

$$\begin{aligned} L = {} & \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)} \\ & + \mathbf{u}^T{}_{(1\times m)}\left(\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{s}_{(m\times 1)}{}^2 - \mathbf{b}_{(m\times 1)}\right) \\ & + \mathbf{v}^T{}_{(1\times p)}\left(\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} - \mathbf{e}_{(p\times 1)}\right) \end{aligned}$$

Solution Procedure of Quadratic Programming(QP) Problem
- Apply the K-T Necessary Condition to the Lagrange Function

Lagrange Function

$$L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u}) = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$$

$$+\mathbf{u}^T{}_{(1\times m)}(\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{s}_{(m\times 1)}{}^2 - \mathbf{b}_{(m\times 1)})$$

$$+\mathbf{v}^T{}_{(1\times p)}(\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} - \mathbf{e}_{(p\times 1)})$$

Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u}) = \mathbf{0}$

$$\frac{\partial L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u})}{\partial \mathbf{d}_{(n\times 1)}} = \mathbf{c}_{(n\times 1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times 1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times 1)} = \mathbf{0}$$

$$\frac{\partial L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u})}{\partial \mathbf{v}_{(p\times 1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} - \mathbf{e}_{(p\times 1)} = \mathbf{0}$$

$$\frac{\partial L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u})}{\partial \mathbf{u}_{(n\times 1)}} = \mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{s}_{(m\times 1)}{}^2 - \mathbf{b}_{(m\times 1)} = 0$$

$$\frac{\partial L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u})}{\partial s_i} = u_i s_i = 0, \quad i = 0 \ to \ m$$

Solution Procedure of Quadratic Programming(QP) Problem - Method 1: Direct Solving the Eqs. from the K-T Conditions

Optimization problem

$$\textit{Minimize} \quad f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$$

$$\textit{Subject to} \quad h_i(\mathbf{x}) = 0, \quad i = 1,\ldots,p \quad \text{Equality constraint}$$

$$g_i(\mathbf{x}) \le 0, \quad i = 1,\ldots,m \quad \text{Inequality constraint}$$

Definition of Lagrange function

$$L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x}) + \sum_{i=1}^{m} u_i (g_i(\mathbf{x}) + s_i^2)$$

v_i : Lagrange multiplier for the equality constraint(It is free in sign.)

u_i : Lagrange multiplier for the inequality constraint(Nonnegative)

s_i : Slack variable transforming an inequality constraint to an equality constraint

---

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i}{\partial x_j} + \sum_{i=1}^{m} u_i^* \frac{\partial g_i}{\partial x_j} = 0, \quad j = 1, \ \ldots \ , n$$

$$\frac{\partial L}{\partial v_i} = h_i(\mathbf{x}^*) = 0, \quad i = 1, \ \ldots \ , p$$

$$\frac{\partial L}{\partial u_i} = g_i(\mathbf{x}^*) + s_i^{*2} = 0, \quad i = 1, \ \ldots \ , \mathrm{m}$$

Linear indeterminate equations

$$\frac{\partial L}{\partial s_i} = u_i^* s_i^* = 0, \quad i = 1, \ \ldots \ , m$$

Nonlinear indeterminate equations

$$u_i^* \ge 0, \quad i = 1, \ \ldots \ , m$$

Method 1:
- Find the solutions which satisfy the nonlinear indeterminate equations.
- Check whether the solutions satisfy the linear indeterminate equations and determine the solution of this problem.
- Human can find the solution of this problem easily by using this method.

Solution Procedure of Quadratic Programming(QP) Problem
- Method 2: Formulate the Problem of the K-T Condition as a LP Problem (1/3)

Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d},\mathbf{v},\mathbf{u},\mathbf{s})=\mathbf{0}$

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times1)}}=\mathbf{c}_{(n\times1)}+\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times1)}+\mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times1)}+\mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times1)}=\mathbf{0},\quad \frac{\partial L}{\partial \mathbf{v}_{(p\times1)}}=\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times1)}-\mathbf{e}_{(p\times1)}=\mathbf{0}$$

$$\frac{\partial L}{\partial s_i}=u_i\boxed{s_i}=0,\quad i=0\ to\ m \cdots\cdots ①\qquad \frac{\partial L}{\partial \mathbf{u}_{(n\times1)}}=\mathbf{A}^T_{(m\times n)}\mathbf{d}_{(n\times1)}+\mathbf{s}_{(m\times1)}{}^2-\mathbf{b}_{(m\times1)}=0$$

Multiply s_i both side of the Equation ①

$$u_is_i=0 \Rightarrow u_is_i^2=0$$

Although the Equation ① is multiplied by s_i, the solution($u_i=0\ or\ s_i=0$) is not changed.

Transform Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d},\mathbf{v},\mathbf{u},\mathbf{s})=\mathbf{0}$

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times1)}}=\mathbf{c}_{(n\times1)}+\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times1)}+\mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times1)}+\mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times1)}=\mathbf{0},\quad \frac{\partial L}{\partial \mathbf{v}_{(p\times1)}}=\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times1)}-\mathbf{e}_{(p\times1)}=\mathbf{0}$$

$$\frac{\partial L}{\partial s_i}=u_i\boxed{s_i^2}=0,\quad i=0\ to\ m \cdots\cdots ①'\qquad \frac{\partial L}{\partial \mathbf{u}_{(n\times1)}}=\mathbf{A}^T_{(m\times n)}\mathbf{d}_{(n\times1)}+\mathbf{s}_{(m\times1)}{}^2-\mathbf{b}_{(m\times1)}=0$$

Solution Procedure of Quadratic Programming(QP) Problem
- Method 2: Formulate the Problem of the K-T Condition as a LP Problem (2/3)

Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times1)}} = \mathbf{c}_{(n\times1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times1)} = \mathbf{0}, \quad \frac{\partial L}{\partial \mathbf{v}_{(p\times1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times1)} - \mathbf{e}_{(p\times1)} = \mathbf{0}$$

$$\frac{\partial L}{\partial s_i} = u_i \boxed{s_i^2} = 0, \quad i = 0 \text{ to } m \cdots\cdots ①' \qquad \frac{\partial L}{\partial \mathbf{u}_{(n\times1)}} = \mathbf{A}^T_{(m\times n)}\mathbf{d}_{(n\times1)} + \boxed{\mathbf{s}_{(m\times1)}^2} - \mathbf{b}_{(m\times1)} = 0$$

Replace s_i^2 with s_i' (where $s_i' \geq 0$)

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times1)}} = \mathbf{c}_{(n\times1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times1)} = \mathbf{0}, \quad \frac{\partial L}{\partial \mathbf{v}_{(p\times1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times1)} - \mathbf{e}_{(p\times1)} = \mathbf{0}$$

$$\frac{\partial L}{\partial s_i} = u_i s_i' = 0, \ i = 0 \text{ to } m$$

Nonlinear indeterminate equations

$$\frac{\partial L}{\partial \mathbf{u}_{(n\times1)}} = \mathbf{A}^T_{(m\times n)}\mathbf{d}_{(n\times1)} + \mathbf{s}'_{(m\times1)} - \mathbf{b}_{(m\times1)} = 0$$

Linear indeterminate equations

Check whether the solution obtained from the linear indeterminate equations satisfies the nonlinear indeterminate equations and determine the solution.

Since these equations are linear in the variables $\mathbf{d}$, $\mathbf{s}'$, $\mathbf{u}$, $\mathbf{v}$, this problem is a linear programming problem only having the equality constraints.

where $u_i, s_i' \geq 0; \ i = 1 \text{ to } m$

Solution Procedure of Quadratic Programming(QP) Problem
- Method 2: Formulate the Problem of the K-T Condition as a LP Problem (3/3)

Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times 1)}} = \mathbf{c}_{(n\times 1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times 1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times 1)} = \mathbf{0},$$

$$\frac{\partial L}{\partial \mathbf{v}_{(p\times 1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} - \mathbf{e}_{(p\times 1)} = \mathbf{0}$$

$$\frac{\partial L}{\partial s_i} = u_i s_i' = 0,\ i = 0\ to\ m$$

Nonlinear indeterminate equations

$$\frac{\partial L}{\partial \mathbf{u}_{(n\times 1)}} = \mathbf{A}^T_{(m\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{s}'_{(m\times 1)} - \mathbf{b}_{(m\times 1)} = 0$$

Linear indeterminate equations

Check whether the solution obtained from the linear indeterminate equations satisfies the nonlinear indeterminate equations and determine the solution.

Since these equations are linear in the variables $\mathbf{d}$, $\mathbf{s}'$, $\mathbf{u}$, $\mathbf{v}$, this problem is a linear programming problem only having the equality constraints.

where $u_i, s_i' \ge 0;\ i = 1\ to\ m$

Since the design variables $\mathbf{d}_{(n\times 1)}$ are free in sign, we may decompose them as follows to use the Simplex method.

$$\mathbf{d}_{(n\times 1)} = \mathbf{d}^+_{(n\times 1)} - \mathbf{d}^-_{(n\times 1)},\ (d_i^+ \ge 0,\ d_i^- \ge 0; i = 1\ to\ n)$$

Also, the Lagrange multipliers $\mathbf{v}_{(p\times 1)}$ for the equality constraints are free in sign, we may decompose them as follows to use the Simplex method.

$$\mathbf{v}_{(p\times 1)} = \mathbf{y}_{(p\times 1)} - \mathbf{z}_{(p\times 1)},\ (y_i \ge 0,\ z_i \ge 0;\ i = 1\ \ to\ \ p)$$

Solution Procedure of Quadratic Programming(QP) Problem
- Method 2: Simplex Method for Solving Quadratic Programming Problem (1/2)

Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times 1)}} = \mathbf{c}_{(n\times 1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times 1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times 1)} = \mathbf{0},$$

$$\frac{\partial L}{\partial \mathbf{v}_{(p\times 1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} - \mathbf{e}_{(p\times 1)} = \mathbf{0}$$

$$\frac{\partial L}{\partial s_i} = u_i s_i' = 0, \; i = 0 \; to \; m$$

Nonlinear indeterminate equations

$$\frac{\partial L}{\partial \mathbf{u}_{(n\times 1)}} = \mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{s}'_{(m\times 1)} - \mathbf{b}_{(m\times 1)} = 0$$

Linear indeterminate equations

Check whether the solution obtained from the linear indeterminate equations satisfies the nonlinear indeterminate equations and determine the solution.

Since these equations are linear in the variables d, s', u, v, this problem is a linear programming problem only having the equality constraints.

where $u_i, s_i' \geq 0; \; i = 1 \; to \; m$

Because d and v are free in sign.

$$\mathbf{d}_{(n\times 1)} = \mathbf{d}^+_{(n\times 1)} - \mathbf{d}^-_{(n\times 1)}, \; (d_i^+ \geq 0, \; d_i^- \geq 0; i = 1 \; to \; n)$$

$$\mathbf{v}_{(p\times 1)} = \mathbf{y}_{(p\times 1)} - \mathbf{z}_{(p\times 1)}, \; (y_i \geq 0, \; z_i \geq 0; \; i = 1 \; to \; p)$$

Matrix From

$$\begin{bmatrix} \mathbf{H}_{(n\times n)} & -\mathbf{H}_{(n\times n)} & \mathbf{A}_{(n\times m)} & \mathbf{0}_{(n\times m)} & \mathbf{N}_{(n\times p)} & -\mathbf{N}_{(n\times p)} \\ \mathbf{A}^T{}_{(m\times n)} & -\mathbf{A}^T{}_{(m\times n)} & \mathbf{0}_{(m\times m)} & \mathbf{I}_{(m\times m)} & \mathbf{0}_{(m\times p)} & \mathbf{0}_{(m\times p)} \\ \mathbf{N}^T{}_{(p\times n)} & -\mathbf{N}^T{}_{(p\times n)} & \mathbf{0}_{(p\times m)} & \mathbf{0}_{(p\times m)} & \mathbf{0}_{(p\times p)} & \mathbf{0}_{(p\times p)} \end{bmatrix} \begin{bmatrix} \mathbf{d}^+_{(n\times 1)} \\ \mathbf{d}^-_{(n\times 1)} \\ \mathbf{u}_{(m\times 1)} \\ \mathbf{s}'_{(m\times 1)} \\ \mathbf{y}_{(p\times 1)} \\ \mathbf{z}_{(p\times 1)} \end{bmatrix} = \begin{bmatrix} -\mathbf{c}_{(n\times 1)} \\ \mathbf{b}_{(m\times 1)} \\ \mathbf{e}_{(p\times 1)} \end{bmatrix}$$

Introduce the artificial variables, define the artificial objective function, and solve the linear programming problem by using the Simplex method.

Solution Procedure of Quadratic Programming(QP) Problem
- Method 2: Simplex Method for Solving Quadratic Programming Problem (2/2)

Matrix From

Introduce the artificial variables, define the artificial objective function, and solve the linear programming problem by using the Simplex method.

$$\begin{bmatrix} \mathbf{H}_{(n\times n)} & -\mathbf{H}_{(n\times n)} & \mathbf{A}_{(n\times m)} & \mathbf{0}_{(n\times m)} & \mathbf{N}_{(n\times p)} & -\mathbf{N}_{(n\times p)} \\ \mathbf{A}^T_{(m\times n)} & -\mathbf{A}^T_{(m\times n)} & \mathbf{0}_{(m\times m)} & \mathbf{I}_{(m\times m)} & \mathbf{0}_{(m\times p)} & \mathbf{0}_{(m\times p)} \\ \mathbf{N}^T_{(p\times n)} & -\mathbf{N}^T_{(p\times n)} & \mathbf{0}_{(p\times m)} & \mathbf{0}_{(p\times m)} & \mathbf{0}_{(p\times p)} & \mathbf{0}_{(p\times p)} \end{bmatrix} \begin{bmatrix} \mathbf{d}^+_{(n\times 1)} \\ \mathbf{d}^-_{(n\times 1)} \\ \mathbf{u}_{(m\times 1)} \\ \mathbf{s}'_{(m\times 1)} \\ \mathbf{y}_{(p\times 1)} \\ \mathbf{z}_{(p\times 1)} \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{n+m+p} \end{bmatrix} = \begin{bmatrix} -\mathbf{c}_{(n\times 1)} \\ \mathbf{b}_{(m\times 1)} \\ \mathbf{e}_{(p\times 1)} \end{bmatrix}$$

Artificial variables

How to define the artificial objective function

1. Define an one equation by sum of all the equations from the 1^{st} row to $(n+m+p)$th row.
2. Define the sum of the all artificial variables($Y_1+Y_2+\ldots+Y_{n+m+p}$) as an objective function(w).

- Determine an initial basic feasible solution (to satisfy the artificial objective function(w) to be zero) for the linear programming problem by using the Simplex method (Phase 1).

- Check whether the initial basic feasible solutions satisfy the following nonlinear indeterminate equations and determine that as a solution.

$$\frac{\partial L}{\partial s_i} = u_i s_i' = 0, \ \ i = 0 \ to \ m$$

Solution Procedure of Quadratic Programming(QP) Problem
- Summary of Method 2 of Simplex Method for Solving Quadratic Programming Problem

Kuhn-Tucker Necessary Condition(Matrix form)

$$\mathbf{B}_{((n+m+p)\times(2n+2m+2p))}\mathbf{X}_{((2n+2m+2p)\times 1)} = \mathbf{D}_{((n+m+p)\times 1)}$$

Simplex Method for Solving Quadratic Programming Problem

1. The problem to solve the Kuhn-Tucker necessary condition is same with the problem having only the equality constraints(linear programming problem).

2. To solve the linear indeterminate equations, we introduce the artificial variables, define the artificial objective function, and determine the initial basic feasible solution by using the Simplex method.

$$\mathbf{B}_{((n+m+p)\times(2n+2m+2p))}\mathbf{X}_{((2n+2m+2p)\times 1)} + \mathbf{Y}_{((n+m+p)\times 1)} = \mathbf{D}_{((n+m+p)\times 1)}$$

If any of the elements in D is(are) negative, the corresponding equation must be multiplied by -1 to have a nonnegative element on the right side.

3. The artificial objective function is defined as follows.

$$w = \sum_{i=1}^{n+m+p} Y_i = \sum_{i=1}^{n+m+p} D_i - \sum_{j=1}^{2(n+m+p)} \sum_{i=1}^{n+m+p} B_{ij} X_j = w_0 + \sum_{j=1}^{2(n+m+p)} C_j X_j$$

where $\underbrace{C_j = -\sum_{i=1}^{n+m+p} B_{ij}}$, $\underbrace{w_0 = \sum_{i=1}^{n+m+p} D_i}$ **Initial value of the artificial objective function**

Add the elements of the *j* th column of the matrix B and change its sign.(Initial relative objective coefficient).

4. Solve the linear programming problem by using the Simplex and check whether the solution satisfies the following equation.

$$u_i s_i' = 0;\ i = 1\ to\ m$$ **: This equation is used to check whether the solution satisfies this equation.**

Solution Procedure of Quadratic Programming(QP) Problem
- Comparison between Method 1 and 2

Optimization problem

$$\textit{Minimize} \quad f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$$

$$\textit{Subject to} \quad h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \qquad \text{Equality constraint}$$

$$g_i(\mathbf{x}) = 0, \quad i = 1, \ldots, m \qquad \text{Inequality constraint}$$

Definition of Lagrange function

$$L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x}) + \sum_{i=1}^{m} u_i (g_i(\mathbf{x}) + s_i^2)$$

v_i : Lagrange multiplier for the equality constraint(It is free in sign.)

u_i : Lagrange multiplier for the inequality constraint(Nonnegative)

s_i : Slack variable transforming an inequality constraint to an equality constraint

---

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i}{\partial x_j} + \sum_{i=1}^{m} u_i^* \frac{\partial g_i}{\partial x_j} = 0, \quad j = 1, \ldots, n$$

$$\frac{\partial L}{\partial v_i} = h_i(\mathbf{x}^*) = 0, \quad i = 1, \ldots, p$$

$$\frac{\partial L}{\partial u_i} = g_i(\mathbf{x}^*) + s_i^{*2} = 0, \quad i = 1, \ldots, \mathrm{m}$$

Linear indeterminate equations

$$\frac{\partial L}{\partial s_i} = u_i^* s_i^* = 0, \quad i = 1, \ldots, m$$

Nonlinear indeterminate equations

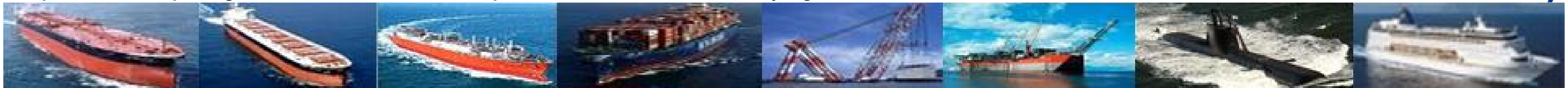
$$u_i^* \geq 0, \quad i = 1, \ldots, m$$

Method 1:
- Find the solutions to satisfy the nonlinear indeterminate equations.
- Check whether the solutions satisfy the linear indeterminate equations and determine the solution of this problem.
- Human can find the solution of this problem easily by using this method.

Method 2:
- Find the solutions to satisfy the linear indeterminate equations by using the Simplex method.
- Check whether the solutions satisfy the nonlinear indeterminate equations and determine the solution of this problem.
- Since the algorithm of this method is more systematical, this method is useful for the computational approach.

7.2 Sequential Linear Programming(SLP)

SYstem Design Laboratory



Sequential Linear Programming(SLP)

- Define the linear programming(LP) problem by linearizing the objective function and the constraints in the current design point.
- Compute the design change by solving the linear programming problem and obtain the improved design point.

$$\underbrace{\mathbf{x}^{(k+1)}}_{\text{Improved design point}} = \underbrace{\mathbf{x}^{(k)}}_{\text{Current design point}} + \underbrace{\mathbf{d}^{(k)}}_{\text{Design change obtained by solving the LP problem.}}$$

- This method is to find the optimal solution by solving the linear programming problem sequentially.

Sequential Linear Programming(SLP)
- [Example] Problem with Inequality Constraints (1)

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_1 \le 0$$

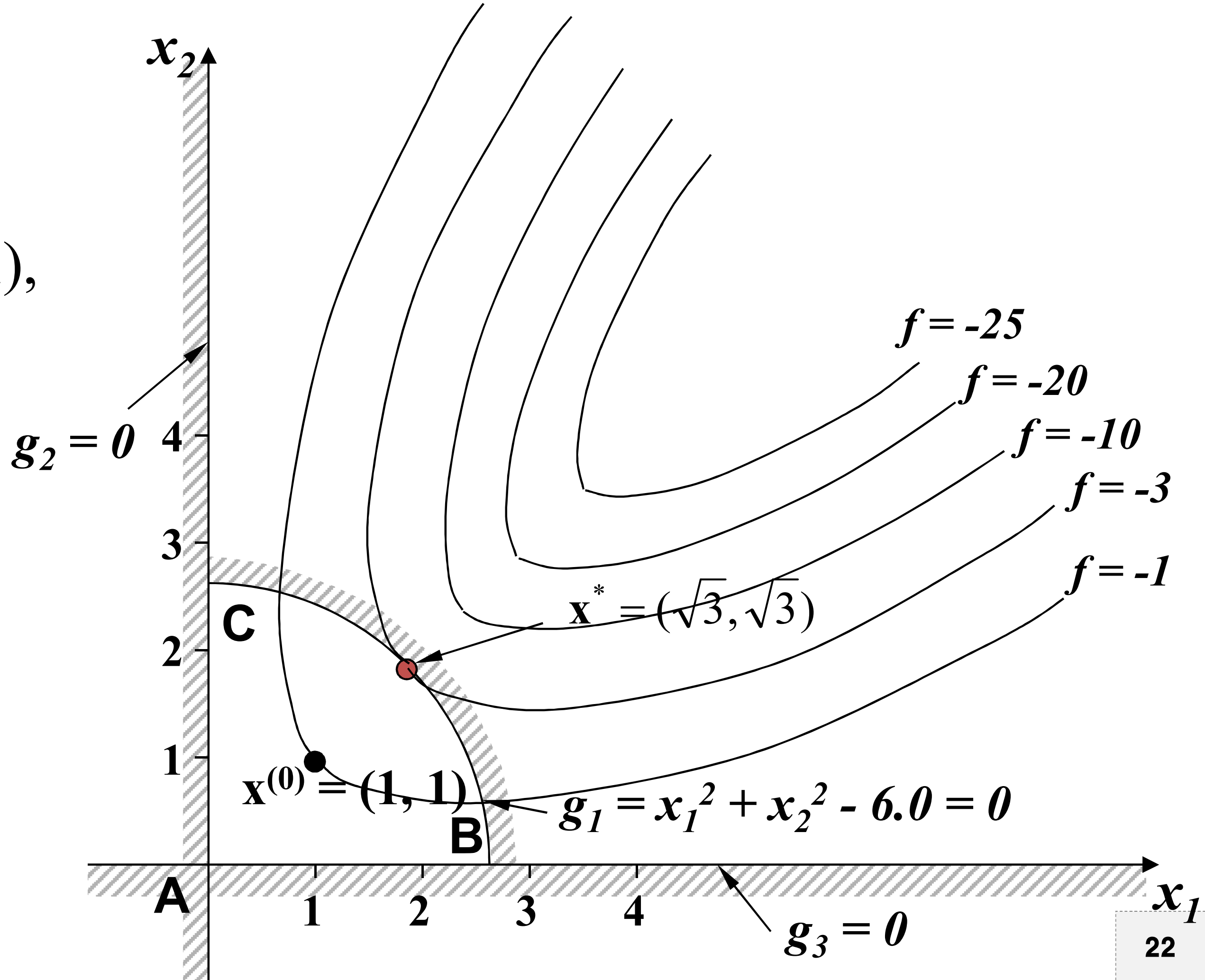
The starting design point: $\mathbf{x}^{(0)} = (1,1)$,

$\varepsilon_1 = \varepsilon_2 = 0.001$

Choose move limits such that a 15% design change is permissible.

The optimal solution:

$$\mathbf{x}^* = (\sqrt{3}, \sqrt{3}), f(\mathbf{x}^*) = -3$$



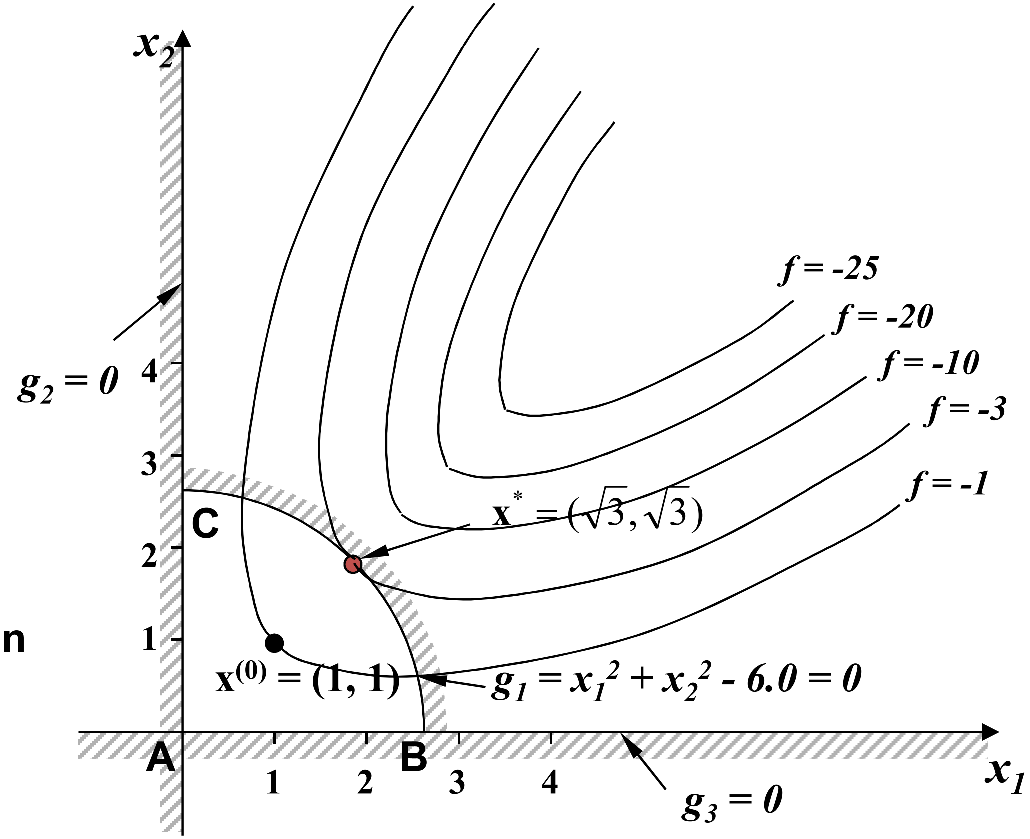
Sequential Linear Programming(SLP)
- [Example] Problem with Inequality Constraints (2)

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_1 \le 0$$



(1) Iteration 1(k = 0)

(i) Step 1

From the given point(starting point), the current design point is as follows.

$$\mathbf{x}^{(0)} = (1,1), \ \varepsilon_1 = \varepsilon_2 = 0.001$$

(ii) Step 2: Evaluate the objective and constraint function at the current design point.

$$f(1,1) = -1$$

$g_1(1,1) = -\frac{2}{3} < 0$ ➡ **Constraint is satisfied.**

$g_2(1,1) = -1 < 0$ ➡ **Constraint is satisfied.**

$g_3(1,1) = -1 < 0$ ➡ **Constraint is satisfied.**

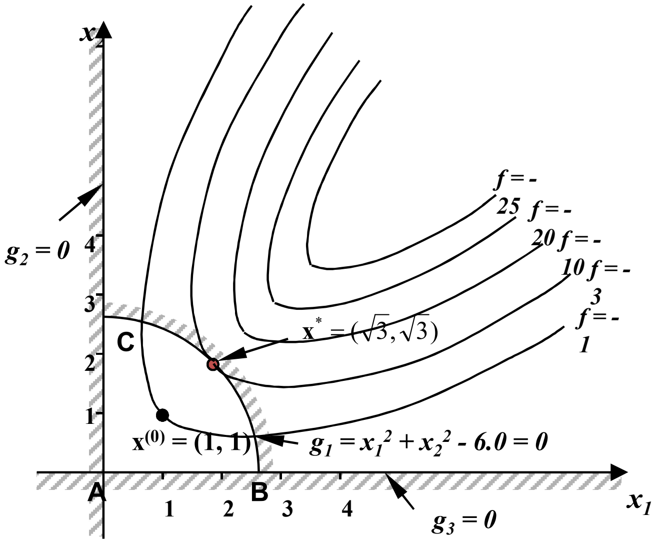
Sequential Linear Programming(SLP)
- [Example] Problem with Inequality Constraints (3)

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_2 \le 0$$



(1) Iteration 1(k = 0) $\quad \mathbf{x}^{(0)} = (1,1), \varepsilon_1 = \varepsilon_2 = 0.001$

(iii) Step 3: Define the LP problem(linearize the objective function).

$$\textit{Minimize:} \quad f(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) \cong f(\mathbf{x}^{(0)}) + \nabla f^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)}$$

The first-order(linear) Taylor series expansion of the objective function

$$\textit{Minimize:} \quad f(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) - f(\mathbf{x}^{(0)}) \cong \nabla f^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)}$$

$$\Delta\mathbf{x}^{(0)} = \mathbf{d}^{(0)}, \nabla f^T = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{d}^{(k)}$$

$$\mathbf{x}^{(k)} = \begin{bmatrix} x_1^{(k)} & x_2^{(k)} \end{bmatrix}^T$$

$$\mathbf{d}^{(k)} = \begin{bmatrix} d_1^{(k)} & d_2^{(k)} \end{bmatrix}^T = \begin{bmatrix} \Delta x_1^{(k)} & \Delta x_2^{(k)} \end{bmatrix}^T$$

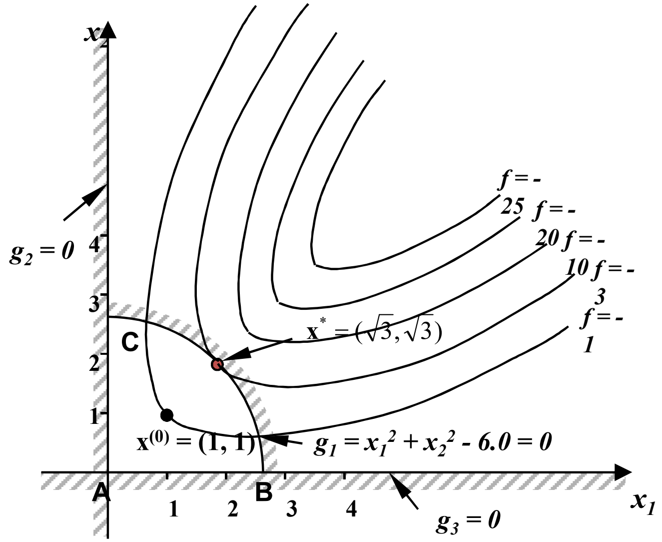
$$\textit{Minimize:} \quad f(\mathbf{x}^{(0)} + \mathbf{d}^{(0)}) - f(\mathbf{x}^{(0)}) \cong \begin{bmatrix} 2x_1 - 3x_2 & 2x_2 - 3x_1 \end{bmatrix}_{\mathbf{x}^{(0)}} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix}$$

$$\overline{f}(\mathbf{d}^{(0)}) \cong (2x_1^{(0)} - 3x_2^{(0)})d_1^{(0)} + (2x_2^{(0)} - 3x_1^{(0)})d_2^{(0)} \longleftarrow \textbf{Substitute} \quad \mathbf{x}^{(0)} = (1,1)$$

$$\boxed{\overline{f}(\mathbf{d}^{(0)}) \cong -d_1^{(0)} - d_2^{(0)}}$$

The linearized objective function

Sequential Linear Programming(SLP)
- [Example] Problem with Inequality Constraints (4)



$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_2 \le 0$$

(1) Iteration 1(k = 0) $\quad \mathbf{x}^{(0)} = (1,1), \varepsilon_1 = \varepsilon_2 = 0.001$

(iii) Step 3: Define the LP problem(linearize the constraints).

$$\textit{Subject to:} \; g_j(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) \Rightarrow g_j(\mathbf{x}^{(0)}) + \nabla g_j^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} \le 0; \; j = 1 \text{ to } m$$

The first-order(linear) Taylor series expansion of the constraints

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{d}^{(k)}$$

$$\nabla g_j^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} \le -g_j(\mathbf{x}^{(0)}); \; j = 1 \text{ to } m$$

$$\Delta\mathbf{x}^{(0)} = \mathbf{d}^{(0)}, \nabla g_j^T = \begin{bmatrix} \frac{\partial g_j}{\partial x_1} & \frac{\partial g_j}{\partial x_2} \end{bmatrix}, \; \nabla g_j^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} = \bar{g}_j(\Delta\mathbf{x}^{(0)}) = \bar{g}_j(\mathbf{d}^{(0)})$$

Subject to:

$$\bar{g}_1(\mathbf{d}^{(0)}) \Rightarrow \begin{bmatrix} \frac{1}{3}x_1^{(0)} & \frac{1}{3}x_2^{(0)} \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(\frac{1}{6}\left(x_1^{(0)}\right)^2 + \frac{1}{6}\left(x_2^{(0)}\right)^2 - 1.0 \right)$$

$$\bar{g}_2(\mathbf{d}^{(0)}) \Rightarrow \begin{bmatrix} -x_1^{(0)} & 0 \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(-x_1^{(0)}\right)$$

$$\bar{g}_3(\mathbf{d}^{(0)}) \Rightarrow \begin{bmatrix} 0 & -x_2^{(0)} \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(-x_2^{(0)}\right)$$

Substitute $\quad \mathbf{x}^{(0)} = (1,1)$

$$g_1(1,1) = -\tfrac{2}{3}$$

$$g_2(1,1) = -1$$

$$g_3(1,1) = -1$$

$$\boxed{\begin{aligned} \bar{g}_1(\mathbf{d}^{(0)}) &= \tfrac{1}{3}d_1^{(0)} + \tfrac{1}{3}d_2^{(0)} \le \frac{2}{3} \\ \bar{g}_2(\mathbf{d}^{(0)}) &= -d_1^{(0)} \le 1 \\ \bar{g}_3(\mathbf{d}^{(0)}) &= -d_2^{(0)} \le 1 \end{aligned}}$$

The linearized constraints

Sequential Linear Programming(SLP)
- [Example] Problem with Inequality Constraints (5)

(iv) Step 4: Solve LP problem for the design change($d^{(0)}$).

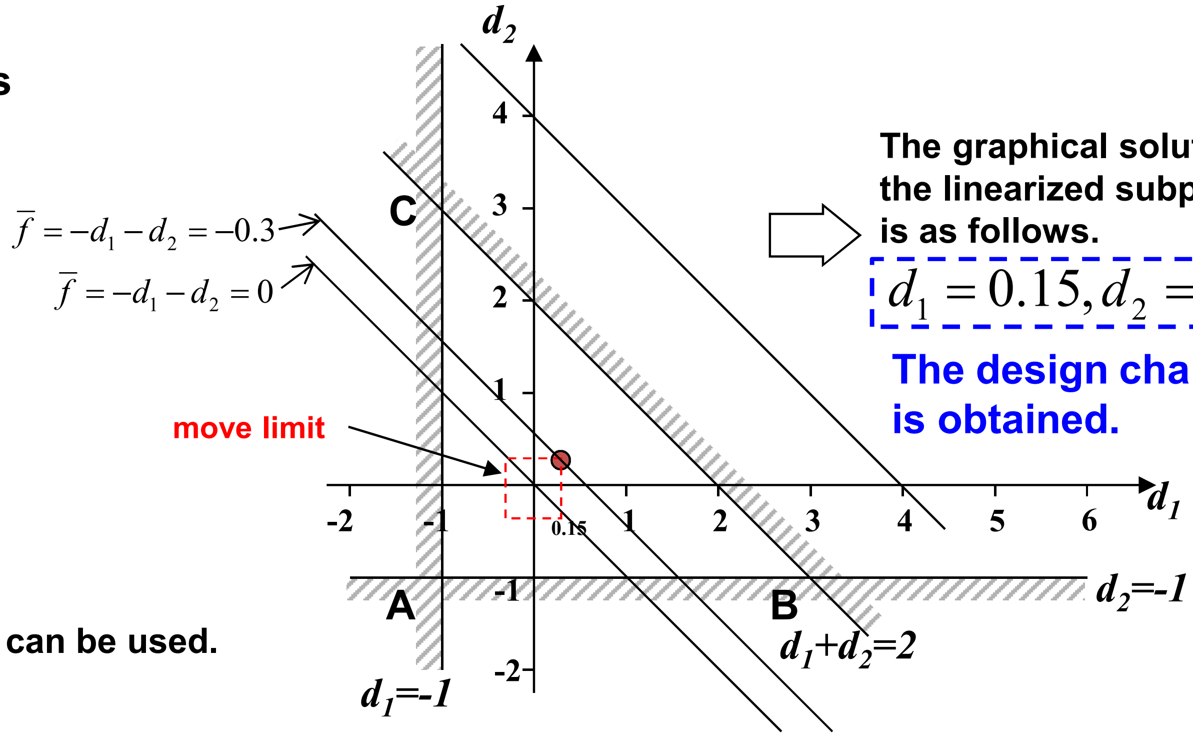
$$\text{Minimize} \quad \bar{f} = -d_1 - d_2$$
$$\text{Subject to} \quad \tfrac{1}{3}d_1 + \tfrac{1}{3}d_2 \le \tfrac{2}{3}$$
$$-d_1 \le 1$$
$$-d_2 \le 1$$
$$\boxed{\begin{aligned} -0.15 \le d_1 \le 0.15 \\ -0.15 \le d_2 \le 0.15 \end{aligned}}$$

Linearize the objective function and constraints.

$$f(1,1) = -1, g_1(1,1) = -\tfrac{2}{3},$$
$$g_2(1,1) = -1, g_3(1,1) = -1$$
$$\nabla f = (-1,-1), \nabla g_1 = (\tfrac{1}{3},\tfrac{1}{3}),$$
$$\nabla g_2 = (-1,0), \nabla g_3 = (0,-1)$$

$$\text{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$
$$\text{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_1 \le 0$$

Limits must be imposed on changes in design called move limit



The graphical solution for the linearized subproblem is as follows.

$$\boxed{d_1 = 0.15, d_2 = 0.15}$$

The design change is obtained.

To solve the problem, the Simplex method can be used.

Sequential Linear Programming(SLP) - [Example] Problem with Inequality Constraints (6)

(v) Step 5: Check for convergence by using the obtained design change $\mathbf{d}^{(0)}$.

$$\mathbf{d}^{(0)} = (d_1, d_2) = (0.15, 0.15)$$

Since $\left\|\mathbf{d}^{(0)}\right\| = \sqrt{0.15^2 + 0.15^2} = 0.212 > \varepsilon_2 (= 0.001)$**, the criterion for convergence is not satisfied.**

(vi) Step 6: Update the design point as $\mathbf{x}^{(k+1)} = \mathbf{x} + \mathbf{d}^{(k)}$. Set $k = k+1$ and go to Step 2.

$$\mathbf{x}^{(1)} = \mathbf{x}^{(1,1)} = \mathbf{x}^{(0)} + \mathbf{d}^{(0)} = (1,1) + (0.15, 0.15) = (1.15, 1.15)$$

$$k = k + 1 = 1$$

Summary of Sequential Linear Programming(SLP)

Minimize $f(\mathbf{x}^{(k)}+\Delta\mathbf{x}^{(k)}) \cong f(\mathbf{x}^{(k)})+\nabla f^T(\mathbf{x}^{(k)})\Delta\mathbf{x}^{(k)}$ **The first-order(linear) Taylor series expansion of the objective function**

Subject to $h_j(\mathbf{x}^{(k)}+\Delta\mathbf{x}^{(k)}) \cong h_j(\mathbf{x}^{(k)})+\nabla h_j^T(\mathbf{x}^{(k)})\Delta\mathbf{x}^{(k)} = 0;\ j=1\ to\ p$

The first-order(linear) Taylor series expansion of the equality constraints

$g_j(\mathbf{x}^{(k)}+\Delta\mathbf{x}^{(k)}) \cong g_j(\mathbf{x}^{(k)})+\nabla g_j^T(\mathbf{x}^{(k)})\Delta\mathbf{x}^{(k)} \le 0;\ j=1\ to\ m$

The first-order(linear) Taylor series expansion of the inequality constraints

Define: $\bar{f} = f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x}),\ e_j = -h_j(\mathbf{x}),\ b_j = -g_j(\mathbf{x}),$

$c_i = \partial f(\mathbf{x})/\partial x_i,\ n_{ij} = \partial h_j(\mathbf{x})/\partial x_i,\ a_{ij} = \partial g_j(\mathbf{x})/\partial x_i,$

$d_i = \Delta x_i$

Minimize $\bar{f} = \sum_{i=1}^{n} c_i d_i$

Subject to $\sum_{i=1}^{n} n_{ij} d_i = e_j;\ j=1\ to\ p$

$\sum_{i=1}^{n} a_{ij} d_i \le b_j;\ j=1\ to\ m$

where $d_{il} \le d_i \le d_{iu}\ (\Delta x_{il}^{(k)} \le \Delta x_i^{(k)} \le \Delta x_{iu}^{(k)})$

Matrix form

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)}$ **: Linearized objective function**

Subject to $\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$ **: Linearized equality constraint**

$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$ **: Linearized inequality constraint**

➡ **Linear Programming Problem**

➡ **It can be solved by using the Simplex method.**

Summary of the SLP Algorithm (1)

- **Step 1: Estimate a starting design point as $\mathbf{x}^{(0)}$. Set $k = 0$. Specify two small numbers, ε_1, ε_2(criterion for violating the constraints and convergence).**

- **Step 2: Evaluate objective and constraint function at current design point $\mathbf{x}^{(k)}$. Also evaluate the objective and constraint function gradients at the current design point.**

- **Step 3: Select the proper move limits $\Delta x_{il}^{(k)}$ and $\Delta x_{iu}^{(k)}$ as some fraction of the current design point. Define the linear programming problem.**

$$\Delta x_{il}{}^{(k)} \le \Delta x_i{}^{(k)} \le \Delta x_{iu}{}^{(k)}$$

Summary of the SLP Algorithm (2)

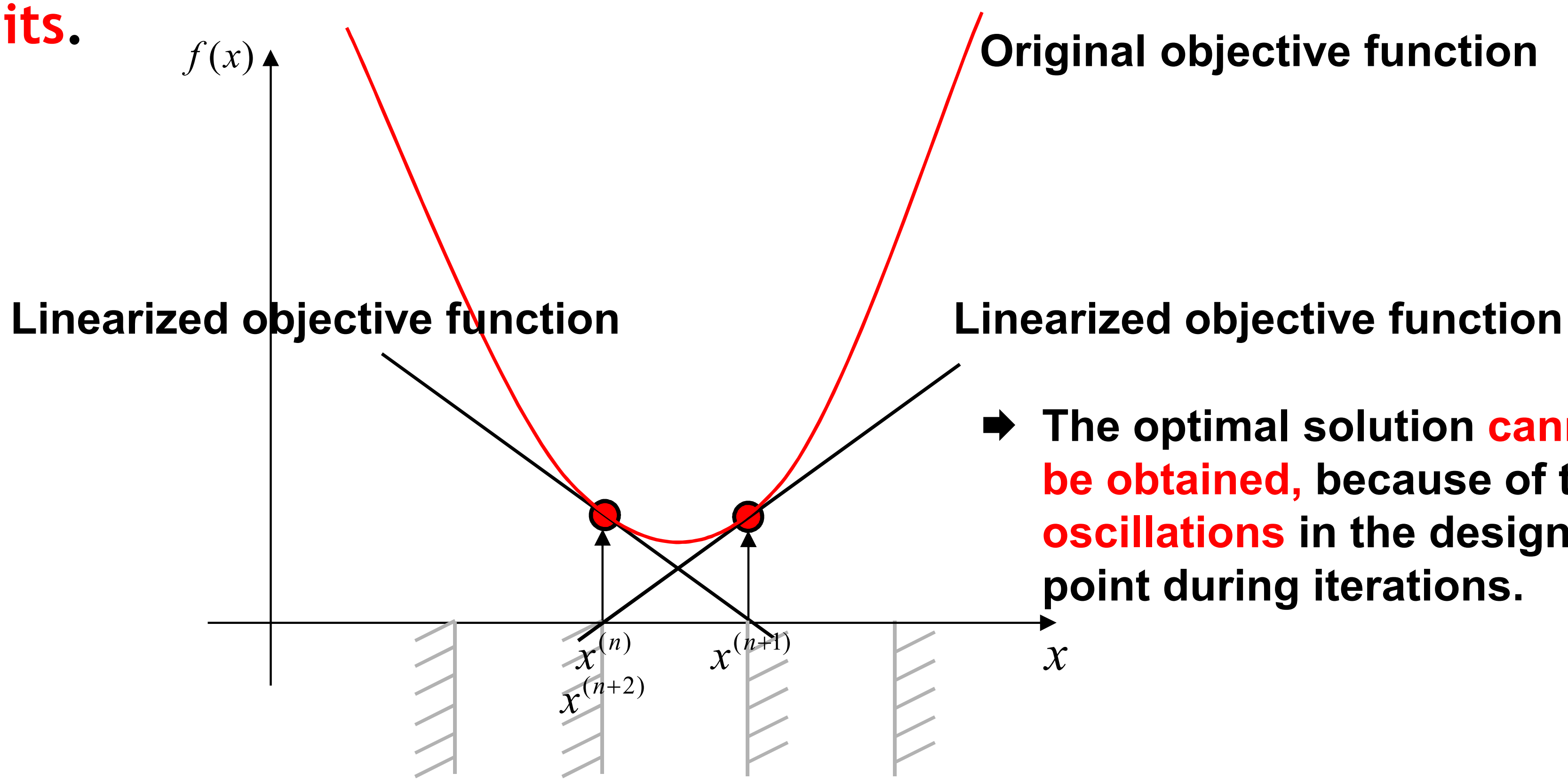
- Step 4: Solve the linear programming problem for $\mathbf{d}^{(k)}$ by using the Simplex method.

- Step 5: Check for convergence. If, $g_i \leq \varepsilon_1$ (i = 1 to m), $|h_i| \leq \varepsilon_1$ (i = 1 to p), and $\|\mathbf{d}^{(k)}\| \leq \varepsilon_2$, then stop and the current design point $\mathbf{x}^{(k)}$ is the optimal solution. Otherwise, continue.

- Step 6: Update the design point as $\mathbf{x}^{(k+1)} = \mathbf{x} + \Delta\mathbf{x}^{(k)}$, Set $k = k+1$ and go to Step 2.

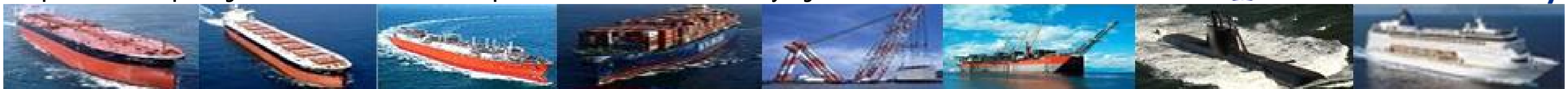
Limitations of SLP Method

- ☑ The move limits of the design variables should be defined by the user.
- ☑ If the move limits are too small, it take much time to find the optimal solution.
- ☑ If the move limits are too large, it can cause oscillations in the design point during iterations.
- ☑ Thus performance of the method depends heavily on selection of move limits.



➡ The optimal solution cannot be obtained, because of the oscillations in the design point during iterations.

7.3 Sequential Quadratic Programming(SQP)



Sequential Quadratic Programming(SQP)

- **Define the Quadratic programming(QP) problem by approximating quadratic form of the objective function and linear form of the constraints at the current design point.**
- **Compute the design change by solving the quadratic programming problem and obtain the improved design point.**

$$\underset{\uparrow \atop \text{Improved design point}}{\underline{\mathbf{x}^{(k+1)}}} = \underset{\uparrow \atop \text{Current design point}}{\underline{\mathbf{x}^{(k)}}} + \underset{\uparrow \atop \text{Design change obtained by solving the QP problem.}}{\underline{\mathbf{d}^{(k)}}}$$

- **This method is to find the optimal solution by solving the Quadratic programming problem sequentially.**

Formulation of the Quadratic Programming Problem to Determine the Search Direction

Minimize $f(\mathbf{x}+\Delta\mathbf{x}) \cong f(\mathbf{x}) + \nabla f^T(\mathbf{x})\Delta\mathbf{x} + 0.5\Delta\mathbf{x}^T\mathbf{H}\Delta\mathbf{x}$

The second-order Taylor series expansion of the objective function

Subject to $h_j(\mathbf{x}+\Delta\mathbf{x}) \cong h_j(\mathbf{x}) + \nabla h_j^T(\mathbf{x})\Delta\mathbf{x} = 0;\ j = 1\ to\ p$

The first-order(linear) Taylor series expansion of the equality constraints

$g_j(\mathbf{x}+\Delta\mathbf{x}) \cong g_j(\mathbf{x}) + \nabla g_j^T(\mathbf{x})\Delta\mathbf{x} \le 0;\ j = 1\ to\ m$

The first-order(linear) Taylor series expansion of the inequality constraints

Define: $\bar{f} = f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x}),\ e_j = -h_j(\mathbf{x}),\ b_j = -g_j(\mathbf{x}),$

$c_i = \partial f(\mathbf{x})/\partial x_i,\ n_{ij} = \partial h_j(\mathbf{x})/\partial x_i,\ a_{ij} = \partial g_j(\mathbf{x})/\partial x_i,$

$d_i = \Delta x_i$

Matrix form

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$ **: Quadratic objective function**

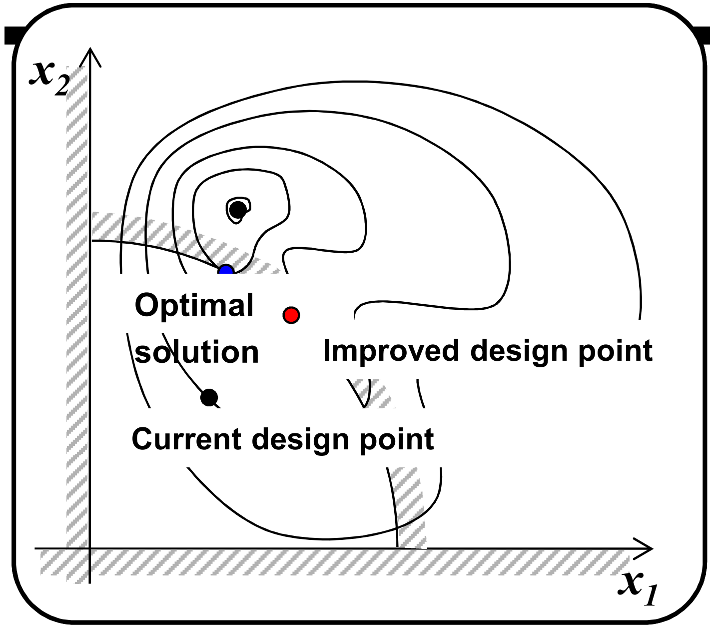
Subject to $\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$ **: Linear equality constraints**

$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$ **: Linear inequality constraints**

Algorithm of SQP

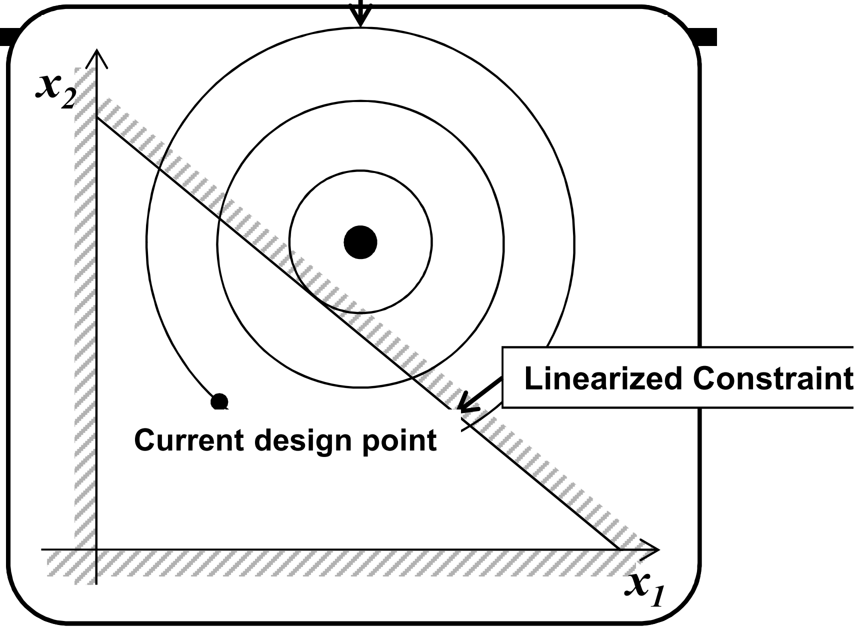
Objective function is approximated to the quadratic form.

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form



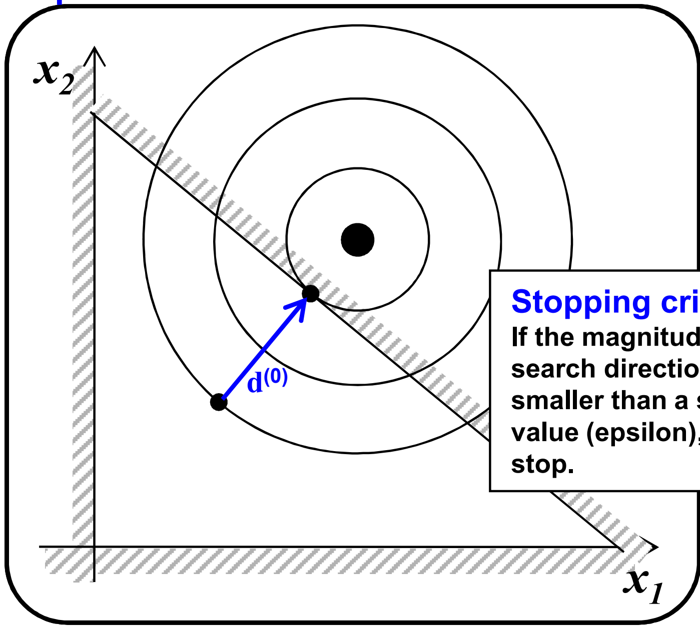
Step 1

Define the quadratic programming problem at the current point.



Step 2

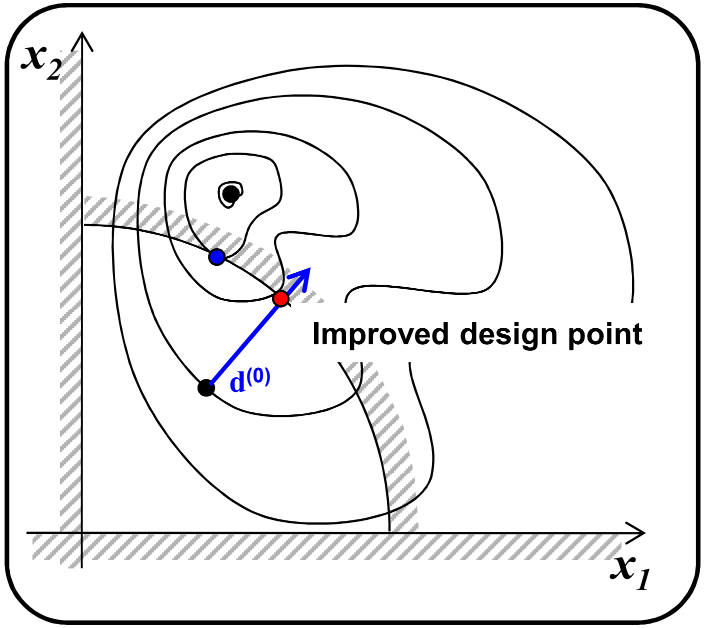
Calculate the search direction($d^{(0)}$) by solving the quadratic programming problem.



Stopping criteria
If the magnitude of the search direction $|d^{(0)}|$ is smaller than a small value (epsilon), then stop.

Step 3

Transform the constrained optimal design problem to the unconstrained problem by modifying the objective function which has added penalty for possible constraint violations to the current value of the objective function.
Then calculate the step size using the one dimensional search method, e.g., Golden section search method.



Go to the Step 1 at the improved design point.

Difference between Sequential Quadratic Programming(SQP) and CSD(Constrained Steepest Descent) Method

☑ Sequential Quadratic Programming(SQP)

- ① First, we define a quadratic programming problem for the objective function and constraints at the current design point, and find the search direction $\mathbf{d}^{(k)}$.
- ② We define the penalty function by adding a penalty for possible constraint violations to the current value of the objective function, and calculate the step size α_k to minimize the penalty function. For determination of the step size, one dimensional search method, e.g., Golden section search method can be used. And we determine the improved design point.
- ③ At the improved design point, we go to ①
- The method is to find the optimal solution by solving the quadratic programming problem sequentially.

☑ CSD(Constrained Steepest Descent) method

- This method is a kind of the SQP method.
- When defining the quadratic programming problem, the Hessian matrix is assumed to be equal to the identity Matrix.
- This method uses the Pshenichny's penalty function.

[Reference] If the Hessian matrix is equal to the Identity matrix, then the objective function is approximated as a centric circle form.

Define the QP problem

To find the search direction($\mathbf{d}^{(0)}$), we define the QP problem at current design point.

$$\textit{Minimize:}\ f(\mathbf{x}^{(0)}+\Delta\mathbf{x}^{(0)}) \cong f(\mathbf{x}^{(0)}) + \nabla f^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} + 0.5\Delta\mathbf{x}^{(0)T}\mathbf{H}\Delta\mathbf{x}^{(0)}$$

The second-order Taylor series expansion of the objective function

$$\textit{Minimize:}\ f(\mathbf{x}^{(0)}+\Delta\mathbf{x}^{(0)}) - f(\mathbf{x}^{(0)}) \cong \nabla f^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} + 0.5\Delta\mathbf{x}^{(0)T}\mathbf{H}\Delta\mathbf{x}^{(0)}$$

$$\Delta\mathbf{x}^{(0)} = \mathbf{d}^{(0)},\ \nabla f^T = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix},\ \mathbf{H} = \mathbf{I}$$

(In the CSD method, the Hessian matrix is assumed to be equal to the identity matrix.)

$$\textit{Minimize:}\ f(\mathbf{x}^{(0)}+\mathbf{d}^{(0)}) - f(\mathbf{x}^{(0)}) \cong \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} \end{bmatrix}_{\mathbf{x}^{(0)}} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} + 0.5(d_1^{(0)2} + d_2^{(0)2})$$

$$\bar{f}(\mathbf{d}^{(0)}) \cong \underbrace{\frac{\partial f(\mathbf{x}^{(0)})}{\partial x_1}}_{\text{constant}} d_1^{(0)} + \underbrace{\frac{\partial f(\mathbf{x}^{(0)})}{\partial x_2}}_{\text{constant}} d_2^{(0)} + 0.5(d_1^{(0)2} + d_2^{(0)2})$$

$$\bar{f}(\mathbf{d}^{(0)}) \cong \underline{c_1 d_1^{(0)} + c_2 d_2^{(0)} + 0.5(d_1^{(0)2} + d_2^{(0)2})}$$

It has the same form of the equation of circle.

Form of the equation of circle: $x_1^2 + x_2^2 + c_1 x_1 + c_2 x_2 + c_3 = 0$

Solution Procedure of SQP Using the Example
- Determination of the Search Direction (1/5) [Iteration 1]

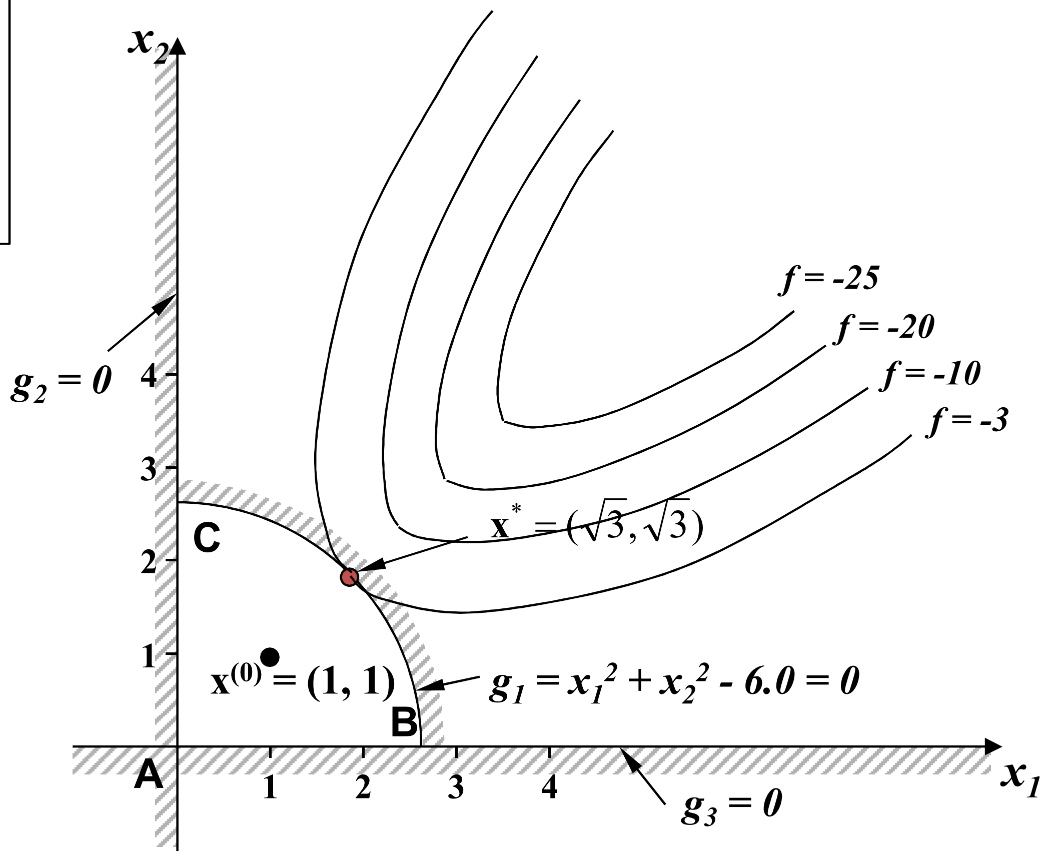
Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

Optimal solution: $\mathbf{x}^* = (\sqrt{3}, \sqrt{3}),\ f(\mathbf{x}^*) = -3$



Assume the starting point is $\mathbf{x}^{(0)} = (1,1)$.

(1) Iteration 1(*k* = 0)

(i) Step 1: Evaluate the objective function and constraints at the current design point.

$f(1,1) = -1$

$g_1(1,1) = -\frac{2}{3} < 0$ ➡ **Constraint is satisfied.**

$g_2(1,1) = -1 < 0$ ➡ **Constraint is satisfied.**

$g_3(1,1) = -1 < 0$ ➡ **Constraint is satisfied.**

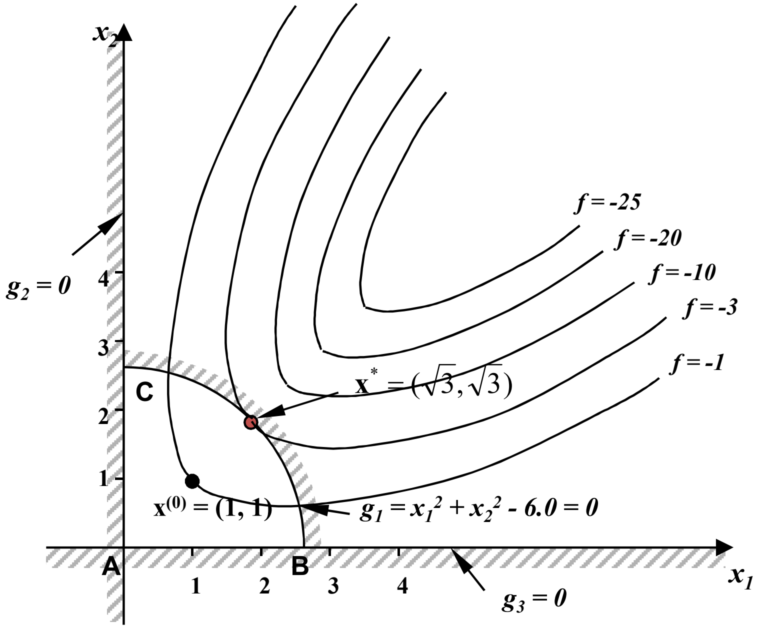
Solution Procedure of SQP Using the Example - Determination of the Search Direction (2/5)

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_2 \le 0$$



(1) Iteration 1(k = 0) $\quad \mathbf{x}^{(0)} = (1,1)$

(ii) Step 2: Define a QP problem(The objective function is approximated to the quadratic form.).

$$\textit{Minimize:}\ f(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) \cong f(\mathbf{x}^{(0)}) + \nabla f^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} + 0.5\Delta\mathbf{x}^{(0)T}\mathbf{H}\Delta\mathbf{x}^{(0)}$$

$$\textit{Minimize:}\ f(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) - f(\mathbf{x}^{(0)}) \cong \nabla f^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} + 0.5\Delta\mathbf{x}^{(0)T}\mathbf{H}\Delta\mathbf{x}^{(0)}$$

$$\Delta\mathbf{x}^{(0)} = \mathbf{d}^{(0)}, \nabla f^T = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}, \ \mathbf{H} = \mathbf{I}$$

$$\textit{Minimize:} \quad f(\mathbf{x}^{(0)} + \mathbf{d}^{(0)}) - f(\mathbf{x}^{(0)}) \cong \begin{bmatrix} 2x_1 - 3x_2 & 2x_2 - 3x_1 \end{bmatrix}_{\mathbf{x}^{(0)}} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} + 0.5(d_1^{(0)2} + d_2^{(0)2})$$

$$\overline{f}(\mathbf{d}^{(0)}) \cong (2x_1^{(0)} - 3x_2^{(0)})d_1^{(0)} + (2x_2^{(0)} - 3x_1^{(0)})d_2^{(0)} + 0.5(d_1^{(0)2} + d_2^{(0)2})$$

$$\overline{f}(\mathbf{d}^{(0)}) \cong \underline{-d_1^{(0)} - d_2^{(0)}} + \underline{0.5(d_1^{(0)2} + d_2^{(0)2})}$$

Objective function is approximated to the first order term.

Objective function is approximated to the second order term.

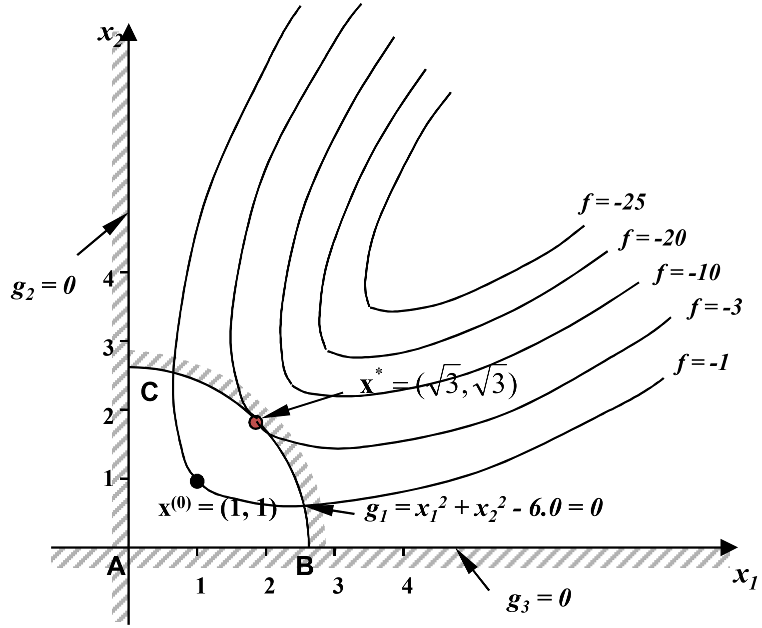
Solution Procedure of SQP Using the Example - Determination of the Search Direction (3/5)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$



(1) Iteration 1(k = 0) $\mathbf{x}^{(0)} = (1,1)$

Subject to: $g_j(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) \cong g_j(\mathbf{x}^{(0)}) + \nabla g_j^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} \le 0;\ j = 1 \text{ to } m$

The first-order(linear) Taylor series expansion of the constraints

$$\nabla g_j^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} \le -g_j(\mathbf{x}^{(0)});\ j = 1 \text{ to } m$$

$$\Delta\mathbf{x}^{(0)} = \mathbf{d}^{(0)},\ \nabla g_j^T = \begin{bmatrix} \frac{\partial g_j}{\partial x_1} & \frac{\partial g_j}{\partial x_2} \end{bmatrix}$$

Subject to:

$$\bar{g}_1(\mathbf{d}^{(0)}) : \begin{bmatrix} \frac{1}{3}x_1^{(0)} & \frac{1}{3}x_2^{(0)} \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(\frac{1}{6}\left(x_1^{(0)}\right)^2 + \frac{1}{6}\left(x_2^{(0)}\right)^2 - 1.0 \right)$$

$$\bar{g}_2(\mathbf{d}^{(0)}) : \begin{bmatrix} -x_1^{(0)} & 0 \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(-x_1^{(0)}\right)$$

$$\bar{g}_3(\mathbf{d}^{(0)}) : \begin{bmatrix} 0 & -x_2^{(0)} \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(-x_2^{(0)}\right)$$

Substitute $\mathbf{x}^{(0)} = (1,1)$

The constraints are linearized

$$\tfrac{1}{3}d_1^{(0)} + \tfrac{1}{3}d_2^{(0)} \le \frac{2}{3}$$

$$-d_1^{(0)} \le 1$$

$$-d_2^{(0)} \le 1$$

Solution Procedure of SQP Using the Example - Determination of the Search Direction (4/5)

(iii) Step 3: Solve the QP problem to find the search direction($\mathbf{d}^{(0)}$).

Constrained Optimal Design Problem (Original problem)

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_2 \le 0$$

①

$f(1,1) = -1, g_1(1,1) = -\frac{2}{3},$
$g_2(1,1) = -1, g_3(1,1) = -1$
$\nabla f = (-1,-1), \nabla g_1 = (\frac{1}{3}, \frac{1}{3}),$
$\nabla g_2 = (-1,0), \nabla g_3 = (0,-1)$

Quadratic Programming Problem

$$\textit{Minimize} \quad \bar{f} = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$$

$$\textit{Subject to} \quad \tfrac{1}{3}d_1 + \tfrac{1}{3}d_2 \le \tfrac{2}{3}$$

$$-d_1 \le 1$$

$$-d_2 \le 1$$

②

Lagrange function

$$L = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2) + u_1[\tfrac{1}{3}(d_1 + d_2 - 2) + s_1^2] + u_2(-d_1 - 1 + s_2^2) + u_3(-d_2 - 1 + s_3^2)$$

③

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{d}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$

$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$

$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}\left(d_1 + d_2 - 2\right) + s_1^2 = 0$$

$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2^2 = 0$$

$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3^2 = 0$$

$$\frac{\partial L}{\partial s_i} = u_i s_i = 0,\ u \ge 0,\ i = 1, 2, 3$$

The search direction is

$$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0, 0, 0),$$

$$\mathbf{s}^{(0)} = (s_1, s_2, s_3) = (0,\ 1.414,\ 1.414),$$

$$\boxed{\mathbf{d}^{(0)} = (d_1, d_2) = (1, 1)}$$

* The search direction also can be determined using the Simplex method.

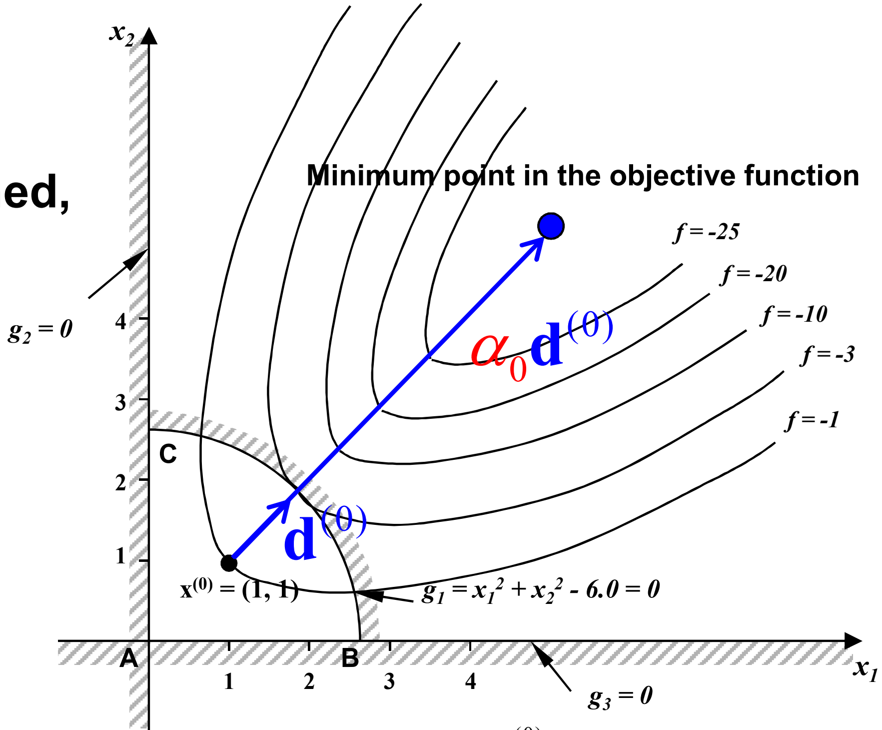
Solution Procedure of SQP Using the Example - Determination of the Search Direction (5/5)

$\boxed{\mathbf{d}^{(0)} = (d_1, d_2) = (1,1)}$ ← The search direction is determined.

(iv) Step 4: After the search direction($\mathbf{d}^{(0)}$) is determined, calculate the step size.

Calculate step size minimizing the value of the objective function along the search direction

$$\underbrace{\mathbf{x}^{(1)}}_{\text{Improved design point}} = \underbrace{\mathbf{x}^{(0)}}_{\text{Current design point}} + \alpha_0 \underbrace{\mathbf{d}^{(0)}}_{\text{Search direction obtained from the QP problem}}$$



$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$,

$\mathbf{d}^{(0)} = (d_1, d_2) = (1,1)$

Find α_k : *Minimize* $f\left(\mathbf{x}^{(1)}\right) = f\left(\underbrace{\mathbf{x}^{(0)}}_{\text{Given}} + \overbrace{\alpha_0}^{\text{Find}} \underbrace{\mathbf{d}^{(0)}}_{\text{Given}}\right) = f\left(\alpha_0\right)$

The improved design point can be found along the search direction by minimizing the objective function. However, it may violate the original constraints.

Therefore, a penalty function should be constructed by adding the penalty for possible constraint violations to the current value of the objective function.

By property of the nature, the objective function is decreased when the constraints is violated.

Solution Procedure of SQP Using the Example
- Definition of Penalty Function(Pshenichny's Descent Function) (1/2)

Penalty function(Pshenichny's descent function, $\Phi(\mathbf{x}^{(k)})$)

By adding a penalty for possible constraint violations to the current value of the objective function, the constrained optimal design problem is transformed into the unconstrained optimal design problem:

$$\Phi(\mathbf{x}^{(k)}) = f(\mathbf{x}^{(k)}) + R_k \cdot V(\mathbf{x}^{(k)})$$

where,

k: iteration number how many times the QP problem is defined approximately

$f(\mathbf{x}^{(k)})$: current(k^th iteration) value of the objective function

$V(\mathbf{x}^{(k)})$ is either the maximum constraint violation among all the constraints or zero.

$V(\mathbf{x}^{(k)})$ is nonnegative. If all the constraints are satisfied, the value of the $V(\mathbf{x}^{(k)})$ is zero.

$$V(\mathbf{x}^{(k)}) = \max\{0; |h_1|, |h_2|, \cdots, |h_p|; g_1, g_2, \cdots, g_m\}$$

where,

h_p: value of the equality constraint function at the design point $\mathbf{x}^{(k)}$

g_p: value of the inequality constraint function at the design point $\mathbf{x}^{(k)}$

R_k is a positive number called the penalty parameter.

$$R_k = \max\{\underline{R_0},\ \underline{r_k}\}$$

Initial value of R_k is specified by the user.

Summation of all the Lagrange multipliers

$$r_k = \sum_{i=1}^{p} |v_i^{(k)}| + \sum_{i=1}^{m} u_i^{(k)}$$

$v_i^{(k)}$: Lagrange multipliers for the equality constraints(free in sign)

$u_i^{(k)}$: Lagrange multiplier for the inequality constraints(nonnegative)

Solution Procedure of SQP Using the Example
- Definition of Penalty Function(Pshenichny's Descent Function) (2/2)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

(v) Step 5: Calculate the penalty parameter R_k(In this example, the initial penalty parameter is assumed as R_0=10.).

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$ **and** $r_k = \sum_{i=1}^{p} \left| v_i^{(k)} \right| + \sum_{i=1}^{m} u_i^{(k)}$

$$r_0 = \sum_{i=1}^{m} u_i^{(0)} = 0$$

Since this problem does not have the equality constraints, we do not consider the v_i.

Therefore, $R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$

$$\Phi(\mathbf{x}^{(k)}) = f(\mathbf{x}^{(k)}) + R_k \cdot V(\mathbf{x}^{(k)})$$

$$= x_1^2 + x_2^2 - 3x_1x_2 + 10 \cdot V(\mathbf{x}^{(k)}),$$ $V(\mathbf{x}^{(k)}) = \max\{0, g_1(\mathbf{x}^{(k)}), g_2(\mathbf{x}^{(k)}), g_3(\mathbf{x}^{(k)})\}$, **($k$=0)**

$g_1(\mathbf{x}^{(k)}) = \frac{1}{6}(x_1^{(k)})^2 + \frac{1}{6}(x_2^{(k)})^2 - 1.0$

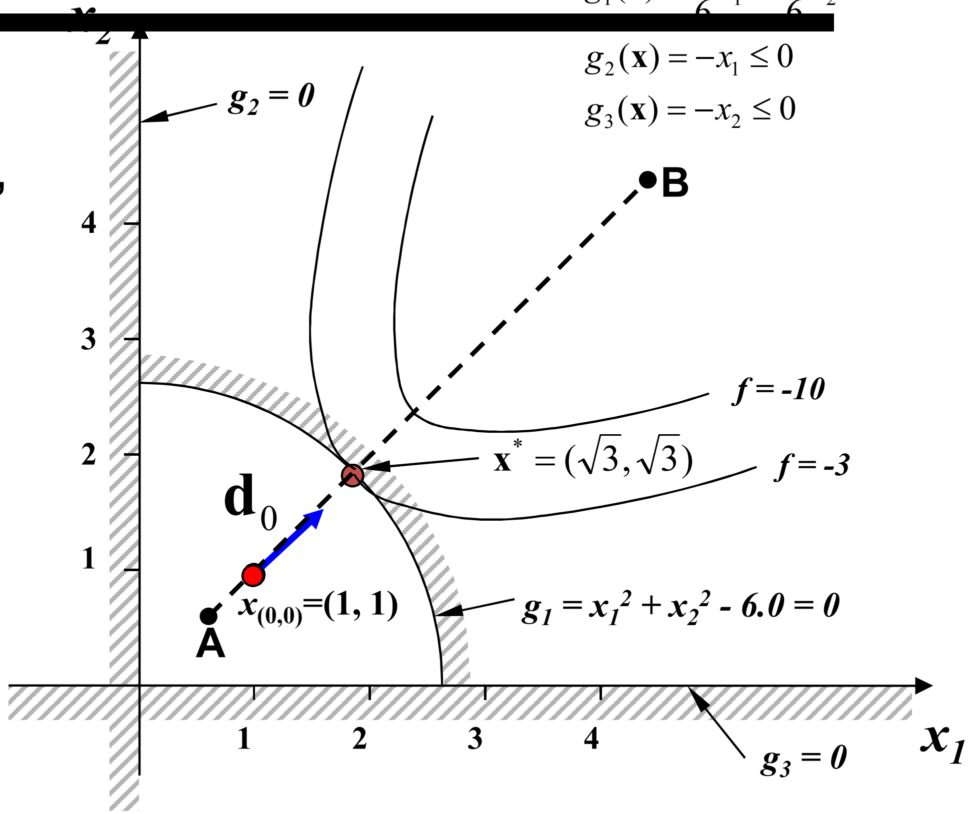
$g_2(\mathbf{x}^{(k)}) = -x_1^{(k)}$

$g_3(\mathbf{x}^{(k)}) = -x_2^{(k)}$

Solution Procedure of SQP Using the Example
- Determination of the Step Size

$$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$
$$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_2 \le 0$$

(vi) Step 6:
By using the one dimensional search method, e.g., Golden section search method, calculate the step size to minimize the penalty function along the search direction($d^{(0)}$), and determine the improved design point.



$$\Phi(\mathbf{x}^{(k)}) = f(\mathbf{x}^{(k)}) + R_k \cdot V(\mathbf{x}^{(k)})$$
$$= x_1^2 + x_2^2 - 3x_1x_2 + 10 \cdot V(\mathbf{x}^{(k)})$$

$$V(\mathbf{x}^{(k)}) = \max\{0, g_1(\mathbf{x}^{(k)}), g_2(\mathbf{x}^{(k)}), g_3(\mathbf{x}^{(k)})\}, \ (k\text{=0})$$

After the k-th search direction is found, one dimensional search for the step size is started.

$$\mathbf{x}^{(k,j)} = \underline{\mathbf{x}^{(k)}} + \alpha_{(k,j)}\underline{\mathbf{d}^{(k)}}$$

The iteration number k does not change during the one dimensional search for the step size.

$$\Phi\left(\mathbf{x}^{(k,j)}\right) = f\left(\mathbf{x}^{(k,j)}\right) + \underline{R_k} \cdot V\left(\mathbf{x}^{(k,j)}\right), V(\mathbf{x}^{(k,j)}) = \max\{0, g_1(\mathbf{x}^{(k,j)}), g_2(\mathbf{x}^{(k,j)}), g_3(\mathbf{x}^{(k,j)})\}$$

The iteration number k does not change during the one dimensional search method.

After completing the one dimensional search, *k* is changed to *k*+1:

$\mathbf{x}^{(k,j)}$ **is changed to** $\mathbf{x}^{(k+1)}$.

Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (1/6)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

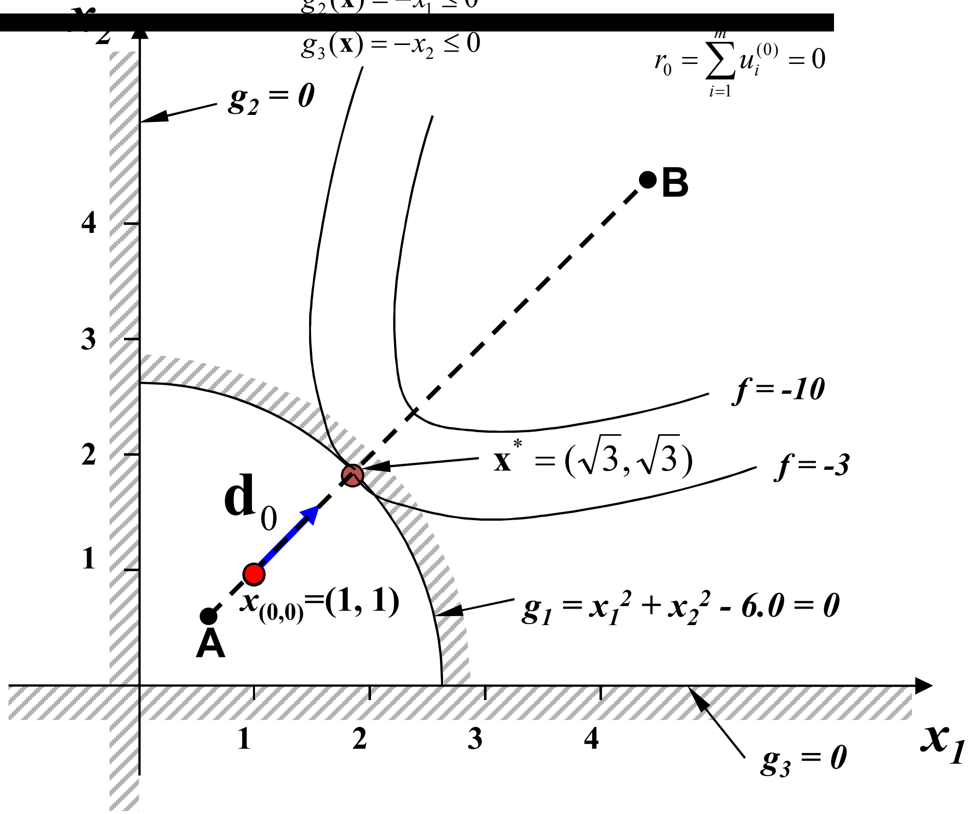
$g_3(\mathbf{x}) = -x_2 \le 0$

$R_0 = \max\{R_0, r_0\}$
$= \max\{10, 0\} = 10$

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$

$r_0 = \sum_{i=1}^{m} u_i^{(0)} = 0$

(vi) Step 6:



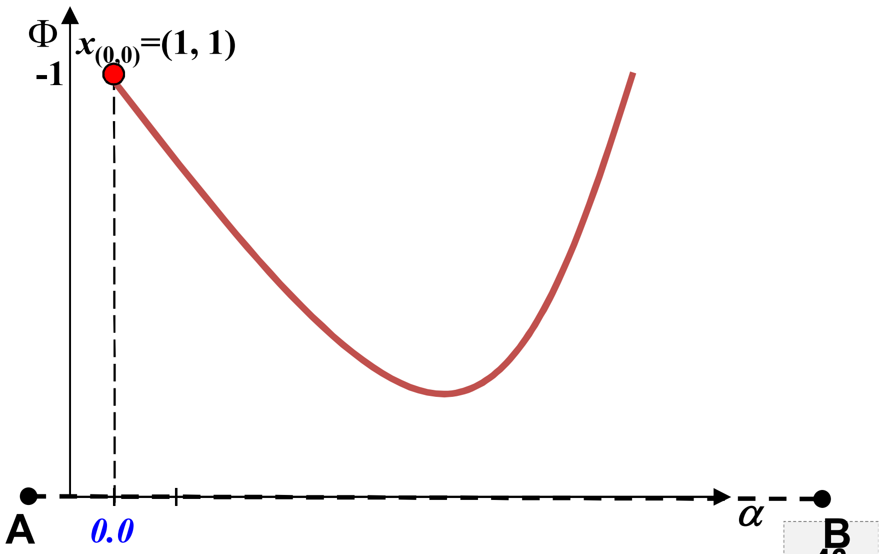
Search direction: $\mathbf{d}_0 = (1,1), \quad k = 0, \; j = 0$

When $\alpha_{(0,j)} = 0.0$

$$\mathbf{x}^{(0,j)} = \mathbf{x}^{(0)} + \alpha_{(0,j)} \cdot d^{(0)} = (1,1) + 0 \cdot (1,1) = (1,1)$$

$$\Phi\left(\mathbf{x}^{(0,j)}\right) = f\left(\mathbf{x}^{(0,j)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,j)}\right) = -1 + 10 \times 0 = -1$$

$$where,\ V(\mathbf{x}^{(0,j)}) = \max\{0, g_1(\mathbf{x}^{(0,j)}), g_2(\mathbf{x}^{(0,j)}), g_3(\mathbf{x}^{(0,j)})\}$$
$$= \max\{0, -\tfrac{2}{3}, -1, -1\} = 0$$

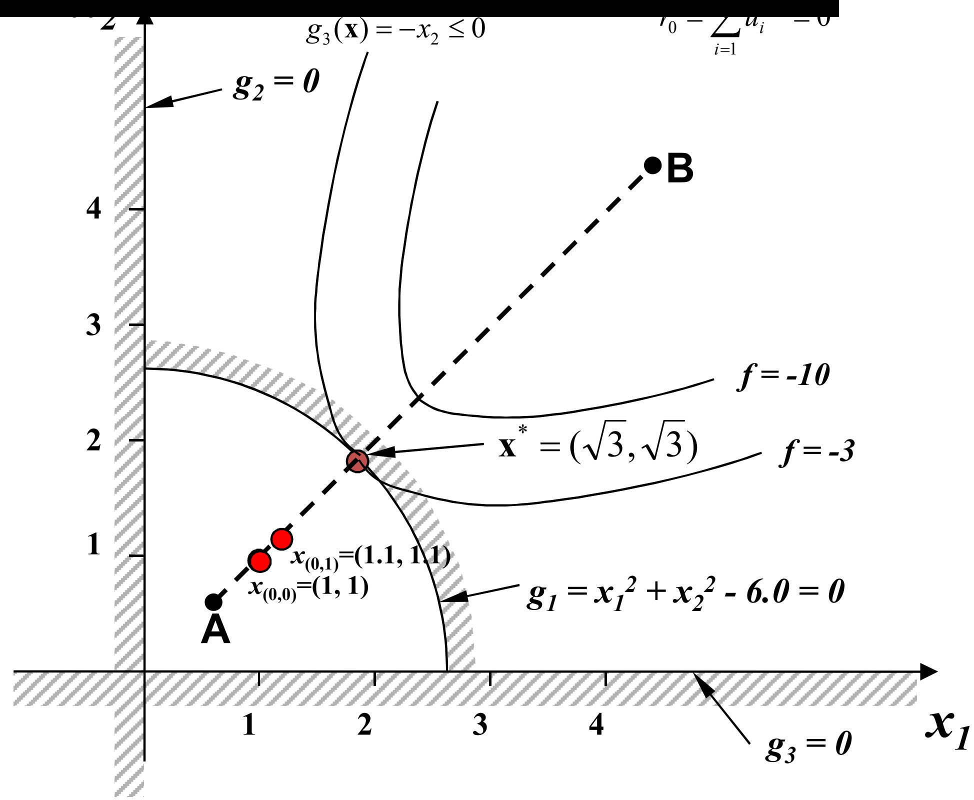


Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (2/6)

$$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$
$$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_2 \le 0$$

$$R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$$
$$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$$
$$r_0 = \sum_{i=1}^{m} u_i = 0$$

(vi) Step 6:



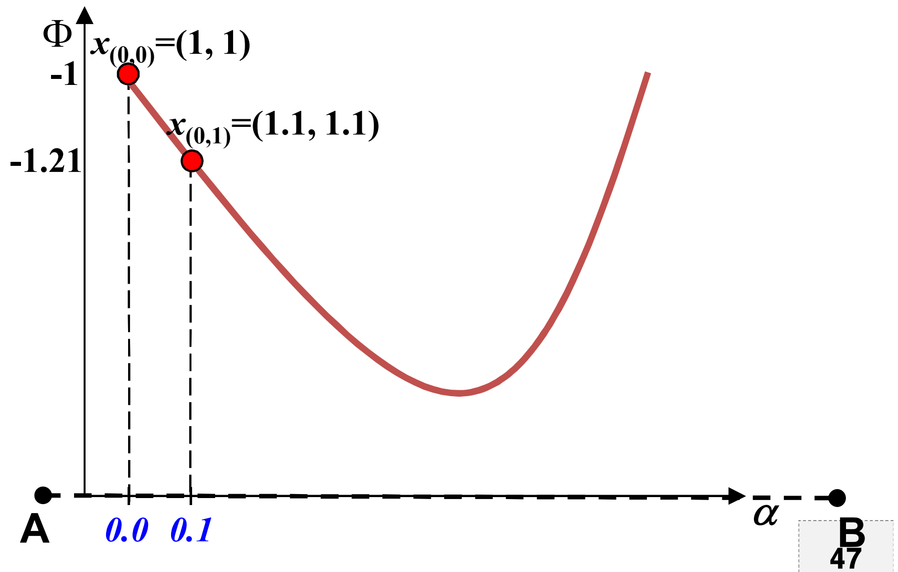
Search direction: $\mathbf{d}_0 = (1,1), \quad k = 0, \; j = 1$

Assume $\alpha_{(0,j)} = 0.1$*,

$$\mathbf{x}^{(0,j)} = \mathbf{x}^{(0)} + \alpha_{(0,j)} \cdot d^{(0)} = (1,1) + 0.1 \cdot (1,1) = (1.1, 1.1)$$

$$\Phi\left(\mathbf{x}^{(0,j)}\right) = f\left(\mathbf{x}^{(0,j)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,j)}\right) = -1.21 + 10 \times 0 = -1.21$$

$$where,\; V(\mathbf{x}^{(0,j)}) = \max\{0, g_1(\mathbf{x}^{(0,j)}), g_2(\mathbf{x}^{(0,j)}), g_3(\mathbf{x}^{(0,j)})\}$$
$$= \max\{0, -0.57, -1.1, -1.1\} = 0$$



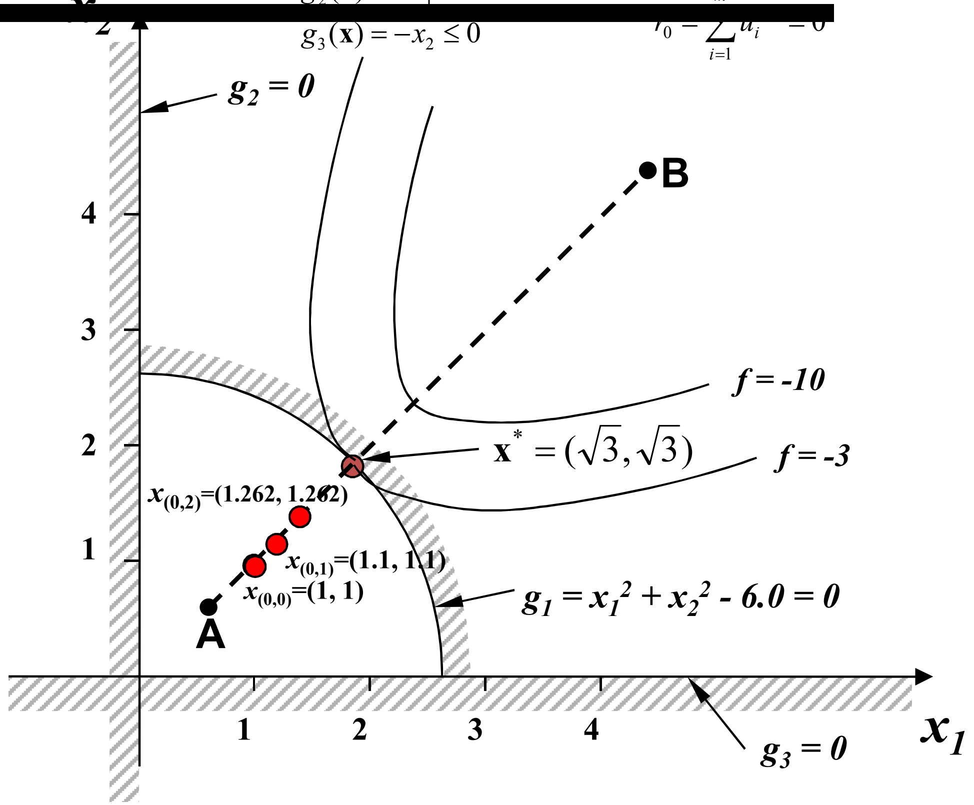
* The initial value of $\alpha_{(k,j)}$ (0.1) is given by the user. It can be given as another value, e.g., 0.5.

Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (3/6)

$$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$
$$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_2 \le 0$$

$$R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$$
$$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$$
$$r_0 = \sum_{i=1}^{m} u_i = 0$$

(vi) Step 6:



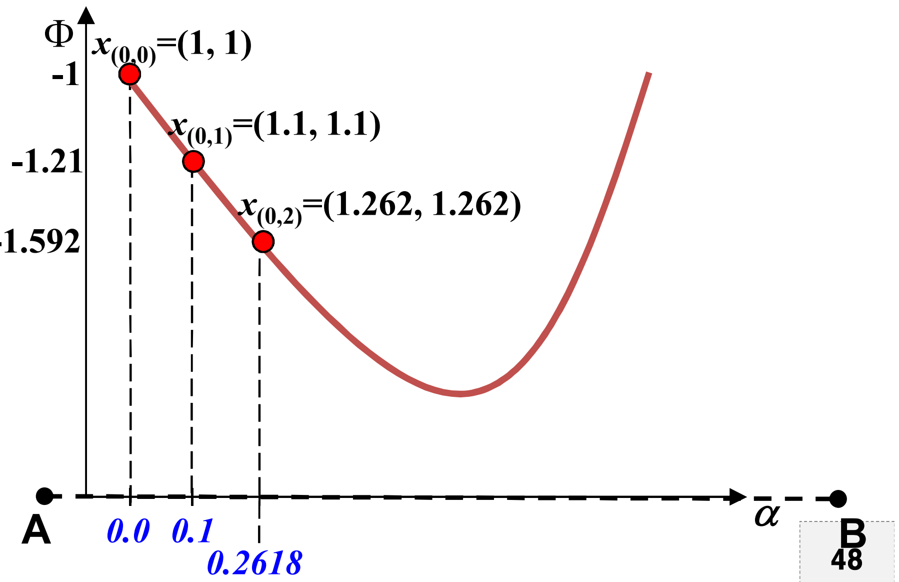
Search direction: $\mathbf{d}_0 = (1,1), \quad k = 0, \ j = 2$

When $\alpha_{(0,j)} = 0.1 + 1.618(0.1) = 0.2618$

$$\mathbf{x}^{(0,j)} = \mathbf{x}^{(0)} + \alpha_{(0,j)} \cdot d^{(0)} = (1,1) + 0.262 \cdot (1,1) = (1.262, 1.262)$$

$$\Phi\left(\mathbf{x}^{(0,j)}\right) = f\left(\mathbf{x}^{(0,j)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,j)}\right) = -1.592 + 10 \times 0 = -1.592$$

$$where,\ V(\mathbf{x}^{(0,2)}) = \max\{0, g_1(\mathbf{x}^{(0,2)}), g_2(\mathbf{x}^{(0,2)}), g_3(\mathbf{x}^{(0,2)})\}$$
$$= \max\{0, -0.469, -1.262, -1.262\} = 0$$



Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (4/6)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

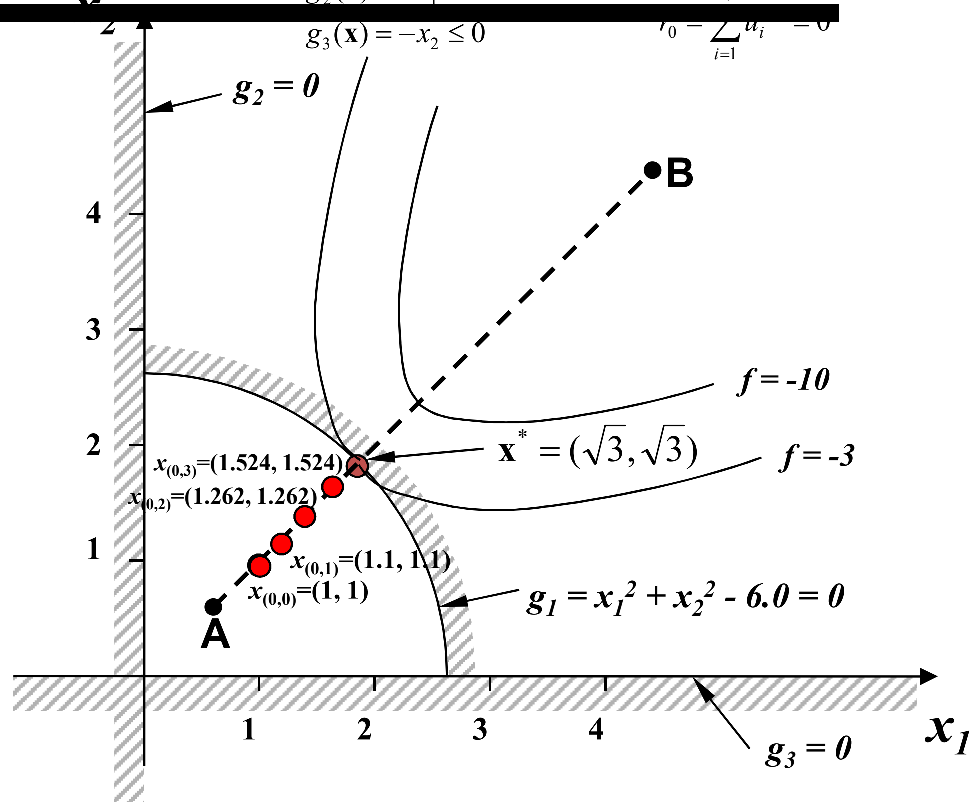
$g_3(\mathbf{x}) = -x_2 \le 0$

$R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$

$r_0 = \sum_{i=1}^{m} u_i = 0$

(vi) Step 6:



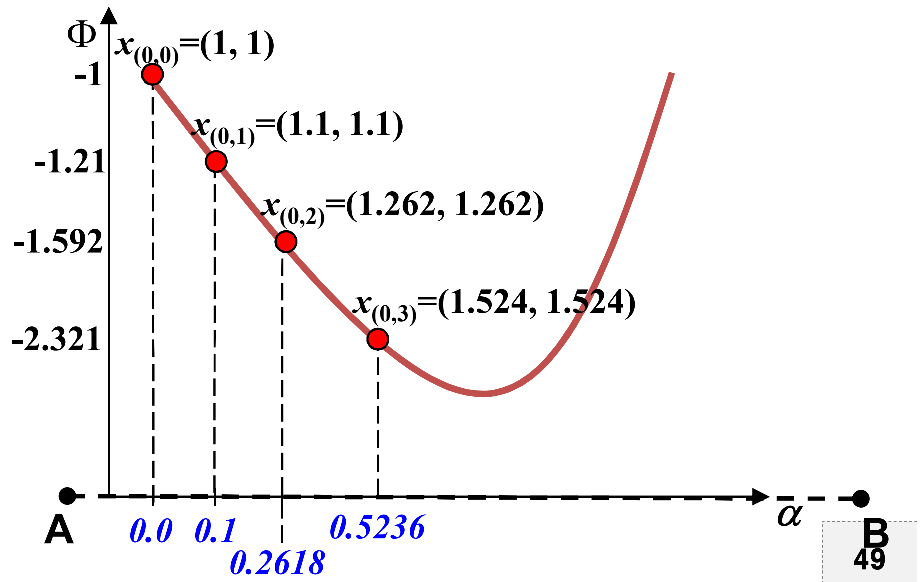
Search direction: $\mathbf{d}_0 = (1,1), \quad k = 0, \ j = 3$

When $\alpha_{(0,j)} = 0.1 + 1.618(0.1) + 1.618^2(0.1) = 0.5236$

$$\mathbf{x}^{(0,j)} = \mathbf{x}^{(0)} + \alpha_{(0,j)} \cdot d^{(0)} = (1,1) + 0.524 \cdot (1,1) = (1.524, 1.524)$$

$$\Phi\left(\mathbf{x}^{(0,j)}\right) = f\left(\mathbf{x}^{(0,j)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,j)}\right) = -2.321 + 10 \times 0 = -2.321$$

$$where,\ V(\mathbf{x}^{(0,j)}) = \max\{0, g_1(\mathbf{x}^{(0,j)}), g_2(\mathbf{x}^{(0,j)}), g_3(\mathbf{x}^{(0,j)})\}$$
$$= \max\{0, -0.226, -1.524, -1.524\} = 0$$



Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (5/6)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

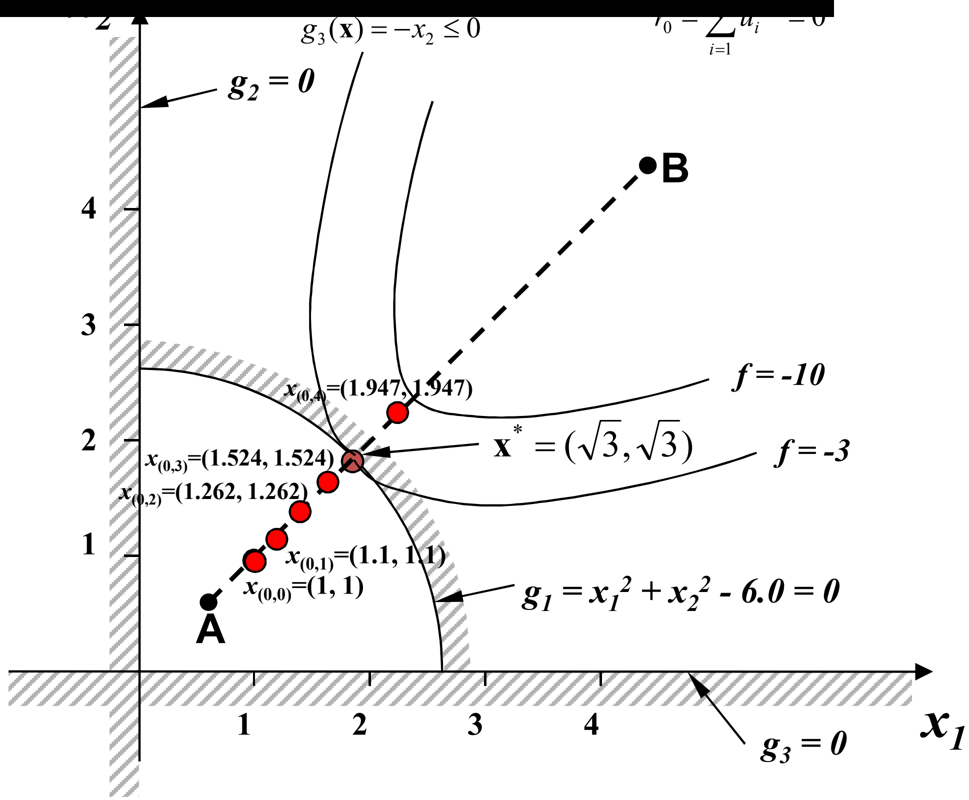
$g_3(\mathbf{x}) = -x_2 \le 0$

$R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$

$r_0 = \sum_{i=1}^{m} u_i = 0$

(vi) Step 6:



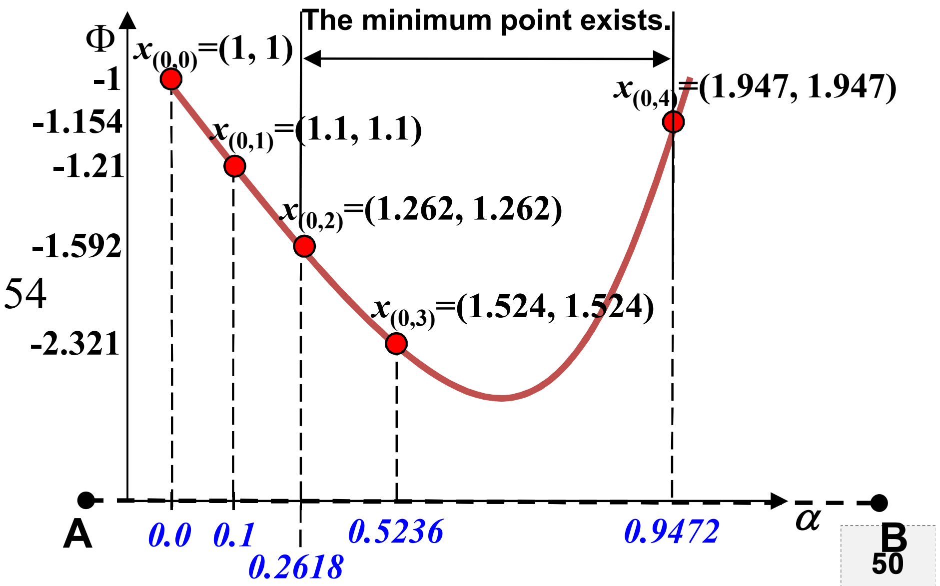
Search direction: $\mathbf{d}_0 = (1,1), \quad k = 0, \; j = 4$

When $\alpha_{(0,j)} = 0.1 + 1.618(0.1) + 1.618^2(0.1) + 1.618^3(0.1) = 0.9472$

$$\mathbf{x}^{(0,j)} = \mathbf{x}^{(0)} + \alpha_{(0,j)} \cdot d^{(0)} = (1,1) + 0.947 \cdot (1,1) = (1.947, 1.947)$$

$$\Phi\left(\mathbf{x}^{(0,j)}\right) = f\left(\mathbf{x}^{(0,j)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,j)}\right) = -3.792 + 10 \times 0.2638 = -1.154$$

$$where,\ V(\mathbf{x}^{(0,4)}) = \max\{0, g_1(\mathbf{x}^{(0,4)}), g_2(\mathbf{x}^{(0,4)}), g_3(\mathbf{x}^{(0,4)})\} = \max\{0, 0.2638, -1.947, -1.947\} = 0.2638$$



Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (6/6)

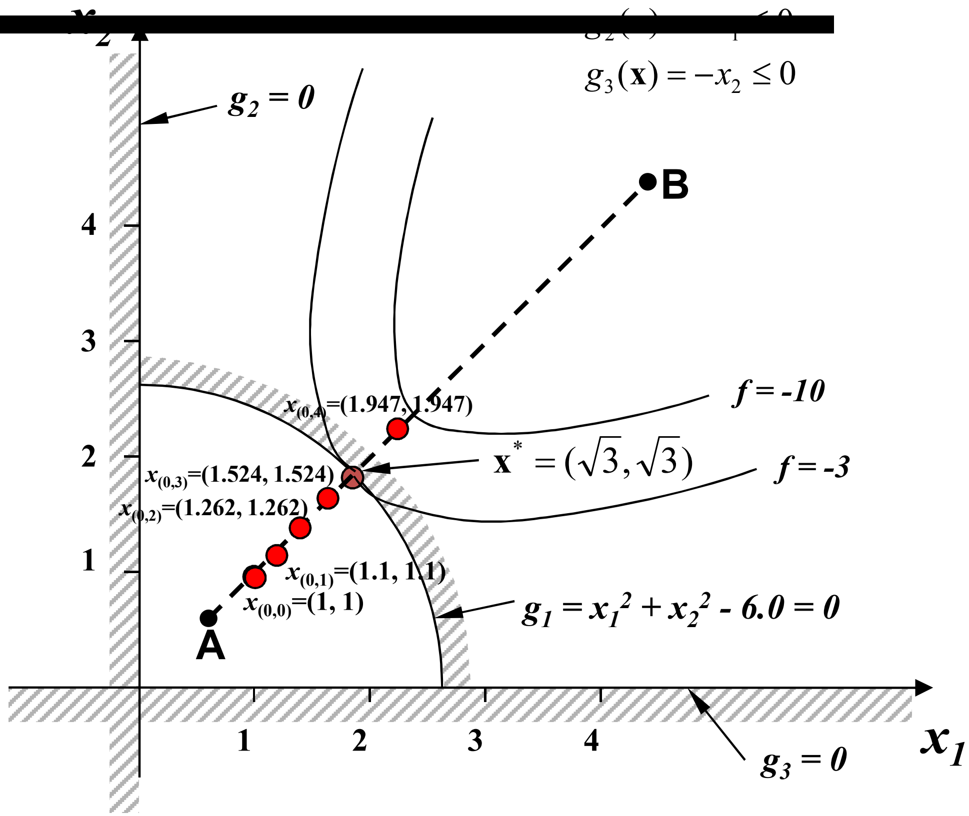
$$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$
$$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_2 \le 0$$

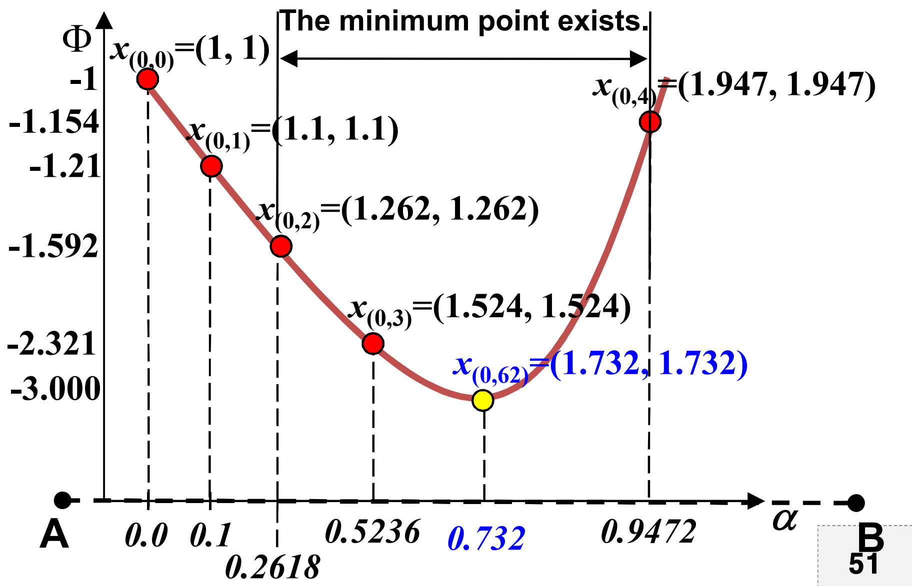
(vi) Step 6:

The value of the $\alpha_0 = 0.732$ is found at which the penalty function is minimized in the interval between $\mathbf{x}^{(0,2)}$ and $\mathbf{x}^{(0,4)}$.

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha_0 \cdot \mathbf{d}^{(0)} = (1,1) + 0.732 \cdot (1,1) = (1.732, 1.732)$$

$$f\left(\mathbf{x}^{(1)}\right) = f(1.732, 1.732) = -3$$





Solution Procedure of SQP Using the Example - Determination of the Search Direction (1/3) [Iteration 2]

$$\text{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$
$$\text{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_2 \le 0$$

(2) Iteration 2(k = 1)

(i) Step 1: Calculate maximum constraint violation of all the constraints.

From the previous stage,

$$\mathbf{x}^{(1)} = (1.732,1.732)$$

$$f\left(\mathbf{x}^{(1)}\right) = f\left(1.732,1.732\right) = -2.999824$$

$$g_1\left(\mathbf{x}^{(1)}\right) = g_1\left(1.732,1.732\right) = -5.866\times10^{-5}$$ ➡ **Constraint is satisfied.**

$$g_2\left(\mathbf{x}^{(1)}\right) = -1.732$$ ➡ **Constraint is satisfied.**

$$g_3\left(\mathbf{x}^{(1)}\right) = -1.732$$ ➡ **Constraint is satisfied.**

$$V_1 = V\left(\mathbf{x}^{(1)}\right) = \max\{0; -5.866\times10^{-5}, -1.732, -1.732\} = 0$$

And,

$$\nabla f(\mathbf{x}^{(1)}) = (2x_1 - 3x_2, 2x_2 - 3x_1) = (-1.732,-1.732)$$

$$\nabla g_1(\mathbf{x}^{(1)}) = (\tfrac{1}{3}x_1, \tfrac{1}{3}x_2) = (0.577,0.577), \nabla g_2 = (-1,0), \nabla g_3 = (0,-1)$$

Solution Procedure of SQP Using the Example - Determination of the Search Direction (2/3)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

(ii&iii) Step 2&3: Define and solve the QP problem to determine the search direction($\mathbf{d}^{(1)}$).

Constrained Optimal Design Problem (Original problem)

$$\textit{Minimize}\quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to}\quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_1 \le 0$$

① $\Rightarrow$

$$f(1.732, 1.732) = -3, \nabla f = (-1.732, -1.732)$$
$$g_1(1.732, 1.732) = -5.866\times10^{-5}, \nabla g_1 = (0.577, 0.577)$$
$$g_2(1.732, 1.732) = -1.732, \nabla g_2 = (-1, 0)$$
$$g_3(1.732, 1.732) = -1.732, \nabla g_3 = (0, -1)$$

Quadratic Programming Problem

$$\textit{Minimize}\quad \bar{f} = (-1.732d_1 - 1.732d_2) + 0.5(d_1^2 + d_2^2)$$

$$\textit{Subject to}\quad 0.577d_1 + 0.577d_2 \le 5.866\times10^{-5}$$

$$-d_1 \le 1.732$$

$$-d_2 \le 1.732$$

where,

$$d_1 = x_1 - 1.732,$$
$$d_2 = x_2 - 1.732$$

②

Lagrange function

$$L = (-1.732d_1 - 1.732d_2) + 0.5(d_1^2 + d_2^2)$$
$$+ u_1[0.577(d_1 + d_2) - 5.866\times10^{-5} + s_1^2]$$
$$+ u_2(-d_1 - 1.732 + s_2^2)$$
$$+ u_3(-d_2 - 1.732 + s_3^2)$$

③ $\Rightarrow$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{d}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial d_1} = -1.732 + d_1 + 0.577u_1 - u_2 = 0$$

$$\frac{\partial L}{\partial d_2} = -1.732 + d_2 + 0.577u_1 - u_3 = 0$$

$$\frac{\partial L}{\partial u_1} = 0.577\left(d_1 + d_2\right) - 5.866\times10^{-6} + s_1^2 = 0$$

$$\frac{\partial L}{\partial u_2} = -d_1 - 1.732 + s_2^2 = 0$$

$$\frac{\partial L}{\partial u_3} = -d_2 - 1.732 + s_3^2 = 0$$

$$\frac{\partial L}{\partial s_i} = u_i s_i = 0,\ u \ge 0,\ i = 1, 2, 3$$

The search direction is

$$\mathbf{d}^{(1)} = (d_1, d_2) = (5.081\times10^{-5}, 5.081\times10^{-5})$$

$$\mathbf{u}^{(1)} = (u_1, u_2, u_3) = (3, 0, 0)$$

$$\mathbf{s}^{(1)} = (s_1, s_2, s_3) = (0,\ 1.316,\ 1.316)$$

* The search direction also can be determined using the Simplex method.

Solution Procedure of SQP Using the Example - Determination of the Search Direction (3/3)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

(iv) Step 4: Check for the following stopping criteria.

$$\mathbf{d}^{(1)} = (d_1, d_2) = (5.081\times10^{-5}, \quad 5.081\times10^{-5})$$

$$\left\|\mathbf{d}^{(1)}\right\| = \sqrt{\left(5.081\times10^{-5}\right)^2 + \left(5.081\times10^{-5}\right)^2} = 7.186\times10^{-5} < \varepsilon_2 (= 0.001)$$

The stopping criteria is satisfied.

(v) Step 5: Stop the iteration.

The candidate minimum solution: $\mathbf{x}^* = (\sqrt{3}, \sqrt{3}), f(\mathbf{x}^*) = -3$

where the Lagrange multiplier are:

$$\mathbf{u}^* = (3, 0, 0), \mathbf{s}^* = (0, \ 1.316, \ 1.316)$$

Summary of Sequential Quadratic Programming(SQP)

Optimization Problem

Minimize $f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$

Subject to $h_i(\mathbf{x}) = 0, \quad i = 1,\ldots,p$ **Equality constraints**

$g_i(\mathbf{x}) = 0, \quad i = 1,\ldots,m$ **Inequality constraints**

Pshenichny's descent function:

the penalty function is constructed by adding a penalty for possible constraint violations to the current value of the objective function

$$\Phi(\mathbf{x}^{(k)}) = f(\mathbf{x}^{(k)}) + R_k \cdot V(\mathbf{x}^{(k)})$$

(k is the iteration number how many times the QP problem is defined.)

$V(\mathbf{x}^{(k)})$ is either the maximum constraint violation of all the constraints or zero.

$V(\mathbf{x}^{(k)})$ is nonnegative. If all the constraints are satisfied, the value of the $V(\mathbf{x}^{(k)})$ is zero.

$$V(\mathbf{x}^{(k)}) = \max\{0; |h_1|, |h_2|, \cdots, |h_p|; g_1, g_2, \cdots, g_m\}$$

R_k is a positive number called the penalty parameter(initially specified by the user).

$$R_k = \max\left\{R_0,\ r_k \left(= \sum_{i=1}^{p} \left|v_i^{(k)}\right| + \sum_{i=1}^{m} u_i^{(k)}\right)\right\}$$

r_k: Summation of all the Lagrange multipliers

The improved design point is determined as follows:

$$\underline{\mathbf{x}^{(k+1)}} = \underline{\mathbf{x}^{(k)}} + \underline{\alpha_k} \cdot \underline{\mathbf{d}^{(k)}}$$

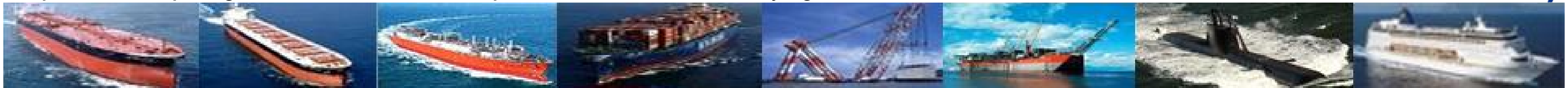
Improved design point — $\mathbf{x}^{(k+1)}$

Current design point — $\mathbf{x}^{(k)}$

Step size calculated by one dimensional search method(ex. Golden section search method) — α_k

Search direction obtained from the QP problem — $\mathbf{d}^{(k)}$

7.4 Determine the Search Direction of the Quadratic Programming Problem by Using the Simplex Method



Formulation of the Quadratic Programming Problem (1/5)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Solve the QP problem to determine the search direction($d^{(0)}$)

Constrained Optimal Design Problem (Original problem)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

①

$f(1,1) = -1, g_1(1,1) = -\frac{2}{3},$
$g_2(1,1) = -1, g_3(1,1) = -1$
$\nabla f = (-1,-1), \nabla g_1 = (\frac{1}{3}, \frac{1}{3}),$
$\nabla g_2 = (-1,0), \nabla g_3 = (0,-1)$

Quadratic Programming Problem

Minimize $\bar{f} = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$

Subject to $\frac{1}{3}d_1 + \frac{1}{3}d_2 \le \frac{2}{3}$

$-d_1 \le 1$

$-d_2 \le 1$

where

$d_1 = x_1 - 1, \; d_2 = x_2 - 1$

②

Lagrange function

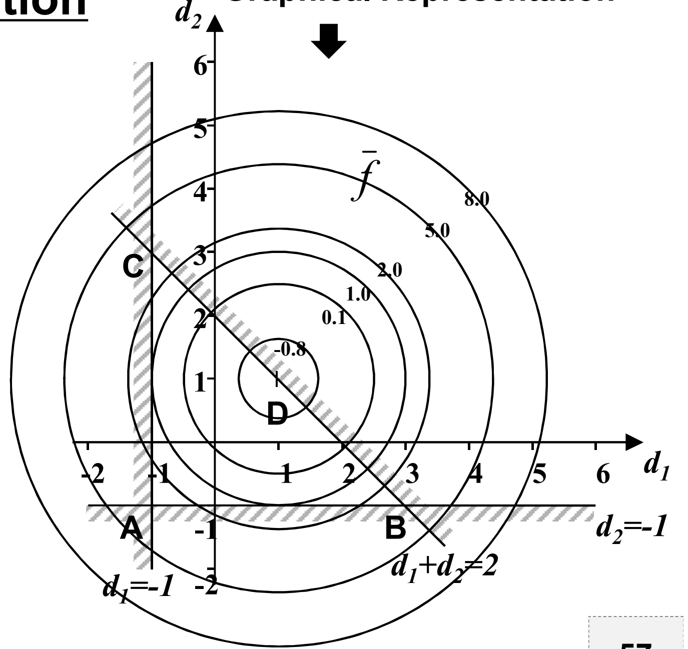
$$L = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$$
$$+ u_1[\tfrac{1}{3}(d_1 + d_2 - 2) + s_1^2]$$
$$+ u_2(-d_1 - 1 + s_2^2)$$
$$+ u_3(-d_2 - 1 + s_3^2)$$

③

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$
$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$
$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}\left(d_1 + d_2 - 2\right) + s_1^2 = 0$$
$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2^2 = 0$$
$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3^2 = 0$$
$$\frac{\partial L}{\partial s_i} = u_i s_i = 0, \; u_i \ge 0, \; i = 1, 2, 3$$

Graphical Representation



Formulation of the Quadratic Programming Problem (2/5)

Quadratic Programming Problem

Minimize $\bar{f} = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$

Subject to $\frac{1}{3}d_1 + \frac{1}{3}d_2 \le \frac{2}{3}$

$-d_1 \le 1$

$-d_2 \le 1$

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$

$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$

$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}\left(d_1 + d_2 - 2\right) + s_1^2 = 0$$

$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2^2 = 0$$

$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3^2 = 0$$

$$\frac{\partial L}{\partial s_i} = \underline{u_i s_i} = 0,\ u_i \ge 0,\ i = 1,2,3 \Rightarrow u_i s_i^2 = 0,\ u_i \ge 0,\ i = 1,2,3$$

Multiply the both side of equations by s_i

Replace s_i^2 with s_i'

$s_i^2 = s_i' \ge 0$

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$

$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$

$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}\left(d_1 + d_2 - 2\right) + s_1' = 0$$

$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2' = 0$$

$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3' = 0$$

$$\frac{\partial L}{\partial s_i} = u_i s_i' = 0$$

$$u_i, s_i' \ge 0,\ i = 1,2,3$$

Represent s_i' to s_i for the convenience

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$

$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$

$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}\left(d_1 + d_2 - 2\right) + s_1 = 0$$

$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2 = 0$$

$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3 = 0$$

$$\frac{\partial L}{\partial s_i} = u_i s_i = 0$$

$$u_i, s_i \ge 0,\ i = 1,2,3$$

Formulation of the Quadratic Programming Problem (3/5)

Quadratic Programming Problem

$$\text{Minimize} \quad \bar{f} = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$$

$$\text{Subject to} \quad \tfrac{1}{3}d_1 + \tfrac{1}{3}d_2 \le \tfrac{2}{3}$$

$$-d_1 \le 1$$

$$-d_2 \le 1$$

Matrix form

$$\text{Minimize} \quad \bar{f} = \mathbf{c}^T{}_{(1\times2)}\mathbf{d}_{(2\times1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times2)}\mathbf{H}_{(2\times2)}\mathbf{d}_{(2\times1)}$$

$$\text{Subject to} \quad \mathbf{A}^T{}_{(3\times2)}\mathbf{d}_{(2\times1)} \le \mathbf{b}_{(3\times1)}$$

Assume that $\mathbf{H}_{(2\times2)}$ is equal to $\mathbf{I}_{(2\times2)}$.

where,

$$\mathbf{d}_{(2\times1)} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \mathbf{c}_{(2\times1)} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \mathbf{H}_{(2\times2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_{(2\times3)} = \begin{bmatrix} \frac{1}{3} & -1 & 0 \\ \frac{1}{3} & 0 & -1 \end{bmatrix}, \mathbf{b}_{(3\times1)} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$$

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$

$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$

$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}\left(d_1 + d_2 - 2\right) + s_1 = 0$$

$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2 = 0$$

$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3 = 0$$

$$\frac{\partial L}{\partial s_i} = u_i s_i = 0$$

$$u_i, s_i \ge 0, \; i = 1, 2, 3$$

How can we express the Kuhn-Tucker necessary condition in matrix form($\mathbf{d}$, $\mathbf{c}$, $\mathbf{H}$, $\mathbf{A}$, $\mathbf{b}$)?

Formulation of the Quadratic Programming Problem (4/5)

$$\mathbf{d}_{(2\times1)} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \mathbf{c}_{(2\times1)} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \mathbf{H}_{(2\times2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_{(2\times3)} = \begin{bmatrix} \frac{1}{3} & -1 & 0 \\ \frac{1}{3} & 0 & -1 \end{bmatrix}, \mathbf{b}_{(3\times1)} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$$

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$

$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$

$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}\left(d_1 + d_2 - 2\right) + s_1 = 0$$

$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2 = 0$$

$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3 = 0$$

$$\frac{\partial L}{\partial s_i} = u_i s_i = 0$$

$$u_i, s_i \ge 0, \; i = 1, 2, 3$$

$$\begin{bmatrix} -1 + d_1 + \frac{1}{3}u_1 - u_2 \\ -1 + d_2 + \frac{1}{3}u_1 - u_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & -1 & 0 \\ \frac{1}{3} & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \mathbf{c}_{(2\times1)} + \mathbf{H}_{(2\times2)}\mathbf{d}_{(2\times1)} + \mathbf{A}_{(2\times3)}\mathbf{u}_{(3\times1)} = \mathbf{0}$$

$$\begin{bmatrix} \frac{1}{3}(d_1 + d_2 - 2) + s_1 \\ -d_1 - 1 + s_2 \\ -d_2 - 1 + s_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$$

$$= \mathbf{A}^T{}_{(3\times2)}\mathbf{d}_{(2\times1)} + \mathbf{s}_{(3\times1)} - \mathbf{b}_{(3\times1)} = \mathbf{0}$$

Matrix form

Formulation of the Quadratic Programming Problem (5/5)

In order to use the Simplex method, we decompose into two variables, because the design variables $\mathbf{d}_{(n\times 1)}$ are free in sign:

$$\mathbf{d}_{(2\times1)} = \mathbf{d}^{+}_{(2\times1)} - \mathbf{d}^{-}_{(2\times1)}$$

<u>**Kuhn-Tucker necessary condition in Matrix form:**</u> $\nabla L(\mathbf{d}^{+}, \mathbf{d}^{-}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\underbrace{\left[\begin{array}{c:c:c:c} \mathbf{H}_{(2\times2)} & -\mathbf{H}_{(2\times2)} & \mathbf{A}_{(2\times3)} & \mathbf{0}_{(2\times3)} \\ \hdashline \mathbf{A}^{T}{}_{(3\times2)} & -\mathbf{A}^{T}{}_{(3\times2)} & \mathbf{0}_{(3\times3)} & \mathbf{I}_{(3\times3)} \end{array}\right]}_{=\mathbf{B}_{(5\times10)}} \underbrace{\begin{bmatrix} \mathbf{d}^{+}_{(2\times1)} \\ \mathbf{d}^{-}_{(2\times1)} \\ \mathbf{u}_{(3\times1)} \\ \mathbf{s}_{(3\times1)} \end{bmatrix}}_{=\mathbf{X}_{(10\times1)}} = \underbrace{\begin{bmatrix} -\mathbf{c}_{(2\times1)} \\ \mathbf{b}_{(3\times1)} \end{bmatrix}}_{=\mathbf{D}_{(5\times1)}}$$

where $\mathbf{d}^{+}_{(2\times1)} = \begin{bmatrix} d_1^{+} \\ d_2^{+} \end{bmatrix}, \mathbf{d}^{-}_{(2\times1)} = \begin{bmatrix} d_1^{-} \\ d_2^{-} \end{bmatrix}, \mathbf{c}_{(2\times1)} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \mathbf{H}_{(2\times2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{A}_{(2\times3)} = \begin{bmatrix} \frac{1}{3} & -1 & 0 \\ \frac{1}{3} & 0 & -1 \end{bmatrix}, \mathbf{b}_{(3\times1)} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$

$$\mathbf{B}_{(5\times10)} = \left[\begin{array}{cc:cc:ccc:ccc} 1 & 0 & -1 & 0 & \frac{1}{3} & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{3} & 0 & -1 & 0 & 0 & 0 \\ \hdashline \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{X}^{T}{}_{(1\times10)} = \begin{bmatrix} d_1^{+} & d_2^{+} & d_1^{-} & d_2^{-} & u_1 & u_2 & u_3 & s_1 & s_2 & s_3 \end{bmatrix}, \mathbf{D}^{T}{}_{(1\times5)} = \begin{bmatrix} 1 & 1 & \frac{2}{3} & 1 & 1 \end{bmatrix}$$

Formulation of the Quadratic Programming Problem to Find the Search Direction by Using the Simplex Method

Kuhn-Tucker necessary condition(matrix form)

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)} = \mathbf{D}_{(5\times1)}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & \frac{1}{3} & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{3} & 0 & -1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1^+ \\ d_2^+ \\ d_1^- \\ d_2^- \\ u_1 \\ u_2 \\ u_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$$

We want to find.

- **This problem is to find X in the linear programming problem only having the equality constraints.**
- $u_i s_i = 0; i = 1\ to\ 3$ **: Check whether the solution obtained from the linear indeterminate equation satisfies the nonlinear indeterminate equation and determine the solution.**

Determine the Search Direction by Using the Simplex Method - Iteration 1 (1/6)

Simplex method to solve the quadratic programming problem

1. The problem to solve the Kuhn-Tucker necessary condition is the same with the problem having only the equality constraints (Linear Programming problem).
2. To solve the linear indeterminate equation, we introduce the artificial variables, define the artificial objective function , and then determine the initial basic feasible solution by using the Simplex method.

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)}+\underbrace{\mathbf{Y}_{(5\times1)}}_{\text{Artificial variables}}=\mathbf{D}_{(5\times1)}$$

3. The artificial objective function is defined as follows.

$$w=\sum_{i=1}^{5}Y_i=\sum_{i=1}^{5}D_i-\sum_{j=1}^{10}\sum_{i=1}^{5}B_{ij}X_j=w_0+\sum_{j=1}^{10}C_jX_j$$

where $C_j=-\sum_{i=1}^{5}B_{ij}$: Add the elements of the j th column of the matrix B and change the its sign.(Initial relative objective coefficient).

$$w_0=\sum_{i=1}^{5}D_i=1+1+\tfrac{2}{3}+1+1=\tfrac{14}{3}$$

: Initial value of the artificial objective function (summation of the all elements of the matrix D)

4. Solve the linear programming problem by using the Simplex and check whether the solution satisfies the following nonlinear equation.

$u_is_i=0; i=1\ to\ 3$: Check whether the solution obtained from the linear indeterminate equation satisfies the nonlinear indeterminate equation and determine the solution.

Determine the Search Direction by Using the Simplex Method - Iteration 1 (2/6)

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)} + \underline{\mathbf{Y}_{(5\times1)}} = \mathbf{D}_{(5\times1)} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & \frac{1}{3} & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{3} & 0 & -1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1^+(=X_1) \\ d_2^+(=X_2) \\ d_1^-(=X_3) \\ d_2^-(=X_4) \\ u_1(=X_5) \\ u_2(=X_6) \\ u_3(=X_7) \\ s_1(=X_8) \\ s_2(=X_9) \\ s_3(=X_{10}) \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$$

Artificial variables

Define the artificial objective function for using the Simplex method:

Sum all the rows(1~5): $\frac{1}{3}X_1 + \frac{1}{3}X_2 - \frac{1}{3}X_3 - \frac{1}{3}X_4 + \frac{2}{3}X_5 - X_6 - X_7 + X_8 + X_9 + X_{10} + \underbrace{Y_1 + Y_2 + Y_3 + Y_4 + Y_5}_{w} = \frac{14}{3}$

Express the artificial function as w and rearrange:

$$-\frac{1}{3}X_1 - \frac{1}{3}X_2 + \frac{1}{3}X_3 + \frac{1}{3}X_4 - \frac{2}{3}X_5 + X_6 + X_7 - X_8 - X_9 - X_{10} = w - \frac{14}{3}$$

1

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
Y1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
Y2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
Y3	1/3	1/3	-1/3	-1/3	0	0	0	**1**	0	0	0	0	1	0	0	2/3	**2/3**
Y4	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1	-
Y5	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1	-
A. Obj.	-1/3	-1/3	1/3	1/3	-2/3	1	1	**-1**	-1	-1	0	0	0	0	0	w-14/3	-

Artificial objective function

Sum all the elements of the row and change the its sign (ex. Row 1: -(1+0+1/3-1+0)=-1/3)

Determine the Search Direction by Using the Simplex Method - Iteration 1 (3/6)

2

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
Y1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
Y2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
X8	1/3	1/3	-1/3	-1/3	0	0	0	1	0	0	0	0	1	0	0	2/3	-
Y4	-1	0	1	0	0	0	0	0	**1**	0	0	0	0	1	0	1	**1**
Y5	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1	-
A. Obj.	0	0	0	0	-2/3	1	1	0	**-1**	-1	0	0	1	0	0	w-4	-

3

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
Y1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
Y2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
X8	1/3	1/3	-1/3	-1/3	0	0	0	1	0	0	0	0	1	0	0	2/3	-
X9	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1	-
Y5	0	-1	0	1	0	0	0	0	0	**1**	0	0	0	0	1	1	**1**
A. Obj.	-1	0	1	0	-2/3	1	1	0	0	**-1**	0	0	1	1	0	w-3	-

4

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
Y1	**1**	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	**1**
Y2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
X8	1/3	1/3	-1/3	-1/3	0	0	0	1	0	0	0	0	1	0	0	2/3	2
X9	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1	-
X10	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1	-
A. Obj.	**-1**	-1	1	1	-2/3	1	1	0	0	0	0	0	1	1	1	w-2	-

Determine the Search Direction by Using the Simplex Method - Iteration 1 (4/6)

5

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
Y2	0	**1**	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	**1**
X8	0	1/3	0	-1/3	-1/9	1/3	0	1	0	0	-1/3	0	1	0	0	1/3	1
X9	0	0	0	0	1/3	-1	0	0	1	0	1	0	0	1	0	2	-
X10	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1	-
A. Obj.	0	**-1**	0	1	-1/3	0	1	0	0	0	1	0	1	1	1	w-1	-

6

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
X2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
X8	0	0	0	0	-2/9	1/3	1/3	1	0	0	-1/3	-1/3	1	0	0	0	-
X9	0	0	0	0	1/3	-1	0	0	1	0	1	0	0	1	0	2	-
X10	0	0	0	0	1/3	0	-1	0	0	1	0	1	0	0	1	2	-
A. Obj.	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	w-0	-

Since the value of the objective function becomes zero, the initial basic feasible solution is obtained.

Determine the Search Direction by Using the Simplex Method - Iteration 1 (5/6)

6

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
X2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
X8	0	0	0	0	-2/9	1/3	1/3	1	0	0	-1/3	-1/3	1	0	0	0	-
X9	0	0	0	0	1/3	-1	0	0	1	0	1	0	0	1	0	2	-
X10	0	0	0	0	1/3	0	-1	0	0	1	0	1	0	0	1	2	-
A. Obj.	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	w-0	-

$$\mathbf{X}^T_{(1\times10)} = \begin{bmatrix} d_1^+ & d_2^+ & d_1^- & d_2^- & u_1 & u_2 & u_3 & s_1 & s_2 & s_3 \end{bmatrix}$$

Since the value of the objective function becomes zero, the initial basic feasible solution is obtained.

Basic solution:

$$X_1 = 1, \quad X_2 = 1, \quad X_8 = 0, \quad X_9 = 2, \quad X_{10} = 2$$

Nonbasic solution:

$$X_3 = X_4 = X_5 = X_6 = X_7 = 0$$

This solution satisfies the nonlinear indeterminate equation$(X_i X_{3+i} = 0; i = 5 \; to \; 7, X_i \geq 0; i = 1 \; to \; 10)$

So, the optimal solution is $d_1 = d_2 = 1, u_1 = u_2 = u_3 = 0, s_1 = 0, s_2 = s_3 = 2.$

➡ **Caution: In the Pivot step, if the smallest(i.e., the most negative) coefficient of the artificial objective function or the smallest positive ratio"bi/ai" appears more than one time, the initial basic feasible solution can be changed depending on the selection of the pivot element in the pivot procedure.**

➡ **We have to check the solution until the nonlinear indeterminate equation(u_i*s_i=0) are satisfied.**

Determine the Search Direction by Using the Simplex Method - Iteration 1 (6/6)

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_2 \le 0$$

The optimal solution for this problem is $d_1 = d_2 = 1, u_1 = u_2 = u_3 = 0, s_1 = 0, s_2 = s_3 = 2$.

Why are the values of $\mathbf{u_1}$ and $\mathbf{s_1}$ are zero at the same time?

Quadratic Programming Problem

$$\textit{Minimize} \quad \bar{f} = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$$

$$\textit{Subject to} \quad \tfrac{1}{3}d_1 + \tfrac{1}{3}d_2 \le \tfrac{2}{3}$$

$$-d_1 \le 1$$

$$-d_2 \le 1$$

This example is graphically represented in the right side.

$s_1 = 0 \longrightarrow$

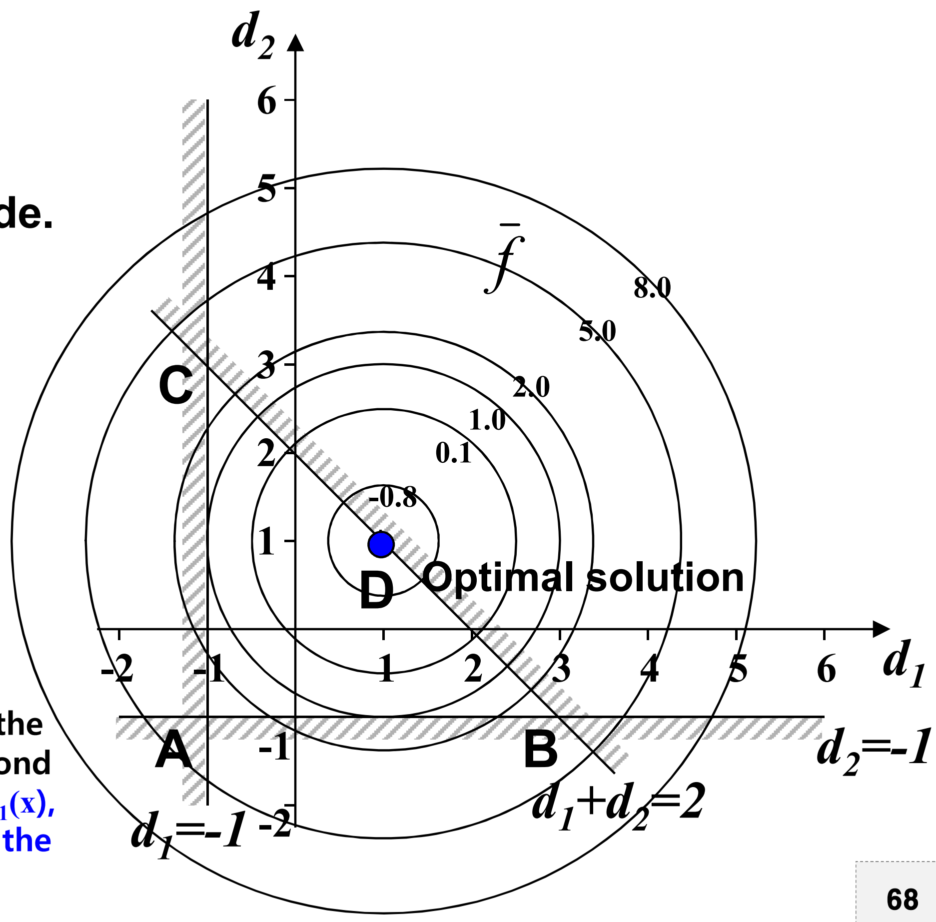
However, fortunately, the optimal solution is on the linearized inequality constraint(g_1(x), $d_1+d_2=2$).

$u_2 = u_3 = 0 \longrightarrow$

The optimal solution is in the region satisfying the inequality constraints of $d_1 = -1 > 0$, $d_2 = -1 > 0$ which are inactive.

$u_1 = 0 \longrightarrow$

The optimal solution is on the inequality constraint(g_1(x)) and is equal to the optimal solution of the objective function to be approximated to the second order. Therefore, although we do not consider the inequality constraint g_1(x), the optimal solution of QP problem is not changed. (g_1(x) does not affect the optimal solution of this problem.)

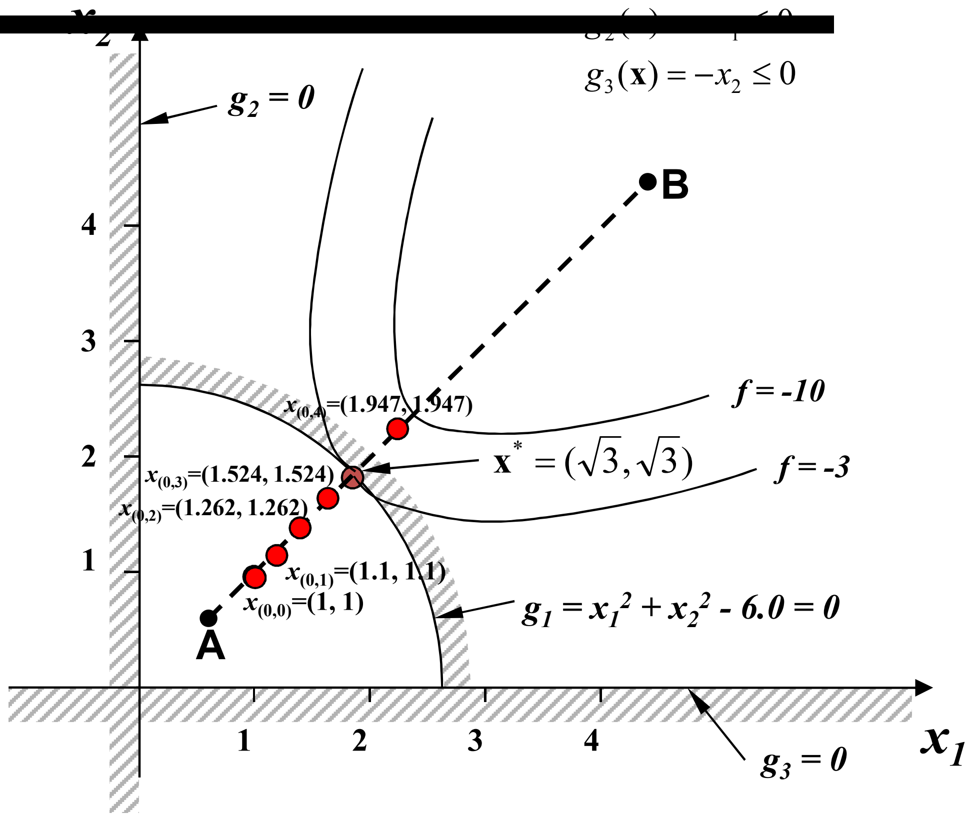


Determination of the Step Size for the Penalty Function of the Pshenichny's Descent Function by Using the Golden Section Search Method

$$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

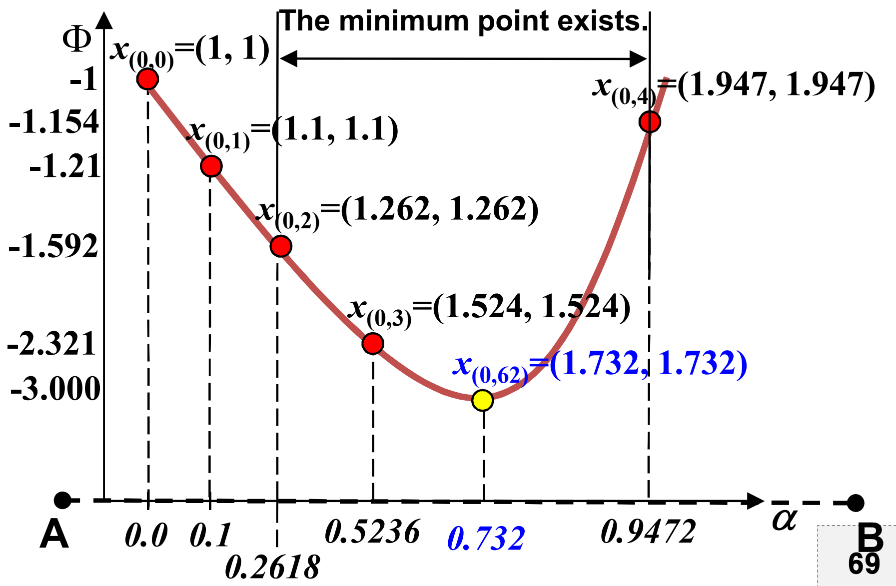
$$g_3(\mathbf{x}) = -x_2 \le 0$$



The value of the $\alpha_0 = 0.732$ is found at which the penalty function is minimized in the interval between $\mathbf{x}^{(0,2)}$ and $\mathbf{x}^{(0,4)}$.

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha_0 \cdot \mathbf{d}^{(0)} = (1,1) + 0.732 \cdot (1,1) = (1.732, 1.732)$$

$$f\left(\mathbf{x}^{(1)}\right) = f\left(1.732, 1.732\right) = -3$$



Determine the Search Direction by Using the Simplex Method - Iteration 2 (1/10)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Solve the QP problem to determine the search direction($d^{(0)}$)

Constrained Optimal Design Problem (Original problem)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_1 \le 0$

$f(1.732, 1.732) = -3, \nabla f = (-1.732, -1.732)$

$g_1(1.732, 1.732) = -5.866\times10^{-5}, \nabla g_1 = (0.577, 0.577)$

$g_2(1.732, 1.732) = -1.732, \nabla g_2 = (-1, 0)$

$g_3(1.732, 1.732) = -1.732, \nabla g_3 = (0, -1)$

①

Quadratic Programming Problem

Minimize $\bar{f} = (-1.732d_1 - 1.732d_2) + 0.5(d_1^2 + d_2^2)$

Subject to $0.577d_1 + 0.577d_2 \le 5.866\times10^{-5}$

$-d_1 \le 1.732$

$-d_2 \le 1.732$

where

$d_1 = x_1 - 1.732,$

$d_2 = x_2 - 1.732$

②

Lagrange function

$$L = (-1.732d_1 - 1.732d_2) + 0.5(d_1^2 + d_2^2)$$
$$+ u_1[0.577(d_1 + d_2) - 5.866\times10^{-5} + s_1^2]$$
$$+ u_2(-d_1 - 1.732 + s_2^2)$$
$$+ u_3(-d_2 - 1.732 + s_3^2)$$

③

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{d}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial d_1} = -1.732 + d_1 + 0.577u_1 - u_2 = 0$$

$$\frac{\partial L}{\partial d_2} = -1.732 + d_2 + 0.577u_1 - u_3 = 0$$

$$\frac{\partial L}{\partial u_1} = 0.577\left(d_1 + d_2\right) - 5.866\times10^{-6} + s_1^2 = 0$$

$$\frac{\partial L}{\partial u_2} = -d_1 - 1.732 + s_2^2 = 0$$

$$\frac{\partial L}{\partial u_3} = -d_2 - 1.732 + s_3^2 = 0$$

$$\frac{\partial L}{\partial s_i} = u_i s_i = 0,\ u \ge 0,\ i = 1, 2, 3$$

1. Multiply the both side by s_i and replace s_i^2 with s_i'.

2. Represent s_i' to s_i for the convenience.

Determine the Search Direction by Using the Simplex Method - Iteration 2 (2/10)

Quadratic Programming Problem

$$\textit{Minimize} \quad \bar{f} = (-1.732d_1 - 1.732d_2) + 0.5(d_1^2 + d_2^2)$$

$$\textit{Subject to} \quad 0.577d_1 + 0.577d_2 \le 0$$

$$-d_1 \le 1.732$$

$$-d_2 \le 1.732$$

Matrix form

$$\textit{Minimize} \quad \bar{f} = \mathbf{c}^T{}_{(1\times2)}\mathbf{d}_{(2\times1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times2)}\mathbf{H}_{(2\times2)}\mathbf{d}_{(2\times1)}$$

$$\textit{Subject to} \quad \mathbf{A}^T{}_{(3\times2)}\mathbf{d}_{(2\times1)} \le \mathbf{b}_{(3\times1)}$$

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1} = -1.732 + d_1 + 0.577u_1 - u_2 = 0$$

$$\frac{\partial L}{\partial d_2} = -1.732 + d_2 + 0.577u_1 - u_3 = 0$$

$$\frac{\partial L}{\partial u_1} = 0.577\left(d_1 + d_2\right) - 5.866\times10^{-6} + s_1 = 0$$

$$\frac{\partial L}{\partial u_2} = -d_1 - 1.732 + s_2 = 0$$

$$\frac{\partial L}{\partial u_3} = -d_2 - 1.732 + s_3 = 0$$

$$\frac{\partial L}{\partial s_i} = u_i s_i = 0,\ u_i, s_i \ge 0,\ i = 1,2,3$$

Assume that $\mathbf{H}_{(2\times2)}$ is equal to $\mathbf{I}_{(2\times2)}$.

where

$$\mathbf{d}_{(2\times1)} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \mathbf{c}_{(2\times1)} = \begin{bmatrix} -1.732 \\ -1.732 \end{bmatrix}, \mathbf{H}_{(2\times2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_{(2\times3)} = \begin{bmatrix} 0.577 & -1 & 0 \\ 0.577 & 0 & -1 \end{bmatrix}, \mathbf{b}_{(3\times1)} = \begin{bmatrix} 0 \\ 1.732 \\ 1.732 \end{bmatrix}$$

Determine the Search Direction by Using the Simplex Method - Iteration 2 (3/10)

$$\mathbf{d}_{(2\times1)}=\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \mathbf{c}_{(2\times1)}=\begin{bmatrix} -1.732 \\ -1.732 \end{bmatrix}, \mathbf{H}_{(2\times2)}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_{(2\times3)}=\begin{bmatrix} 0.577 & 1 & 0 \\ 0.577 & 0 & -1 \end{bmatrix}, \mathbf{b}_{(3\times1)}=\begin{bmatrix} 0 \\ 1.732 \\ 1.732 \end{bmatrix}$$

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1}=-1.732+d_1+0.577u_1-u_2=0$$

$$\frac{\partial L}{\partial d_2}=-1.732+d_2+0.577u_1-u_3=0$$

$$\frac{\partial L}{\partial u_1}=0.577(d_1+d_2)-5.866\times10^{-6}+s_1=0$$

$$\frac{\partial L}{\partial u_2}=-d_1-1.732+s_2=0$$

$$\frac{\partial L}{\partial u_3}=-d_2-1.732+s_3=0$$

$$\frac{\partial L}{\partial s_i}=u_i s_i=0,\ u_i, s_i\ge 0,\ i=1,2,3$$

$$\begin{bmatrix} -1.732+d_1+0.577u_1-u_2 \\ -1.732+d_2+0.577u_1-u_3 \end{bmatrix}=\begin{bmatrix} -1.732 \\ -1.732 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}+\begin{bmatrix} 0.577 & -1 & 0 \\ 0.577 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$=\mathbf{c}_{(2\times1)}+\mathbf{H}_{(2\times2)}\mathbf{d}_{(2\times1)}+\mathbf{A}_{(2\times3)}\mathbf{u}_{(3\times1)}=\mathbf{0}$$

$$\begin{bmatrix} -0.577d_1-0.577d_2+s_1 \\ -d_1-1.732+s_2 \\ -d_2-1.732+s_3 \end{bmatrix}=\begin{bmatrix} 0.577 & 0.577 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}+\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}-\begin{bmatrix} 0 \\ 1.732 \\ 1.732 \end{bmatrix}$$

$$=\mathbf{A}^T{}_{(3\times2)}\mathbf{d}_{(2\times1)}+\mathbf{s}_{(3\times1)}-\mathbf{b}_{(3\times1)}=\mathbf{0}$$

Matrix form

Since the design variables $\mathbf{d}_{(n\times1)}$ are free in sign, we may decompose them as follows for using the Simplex method.

$$\mathbf{d}_{(2\times1)}=\mathbf{d}^+_{(2\times1)}-\mathbf{d}^-_{(2\times1)}$$

$$\underbrace{\begin{bmatrix} \mathbf{H}_{(2\times2)} & -\mathbf{H}_{(2\times2)} & \mathbf{A}_{(2\times3)} & \mathbf{0}_{(2\times3)} \\ \mathbf{A}^T{}_{(3\times2)} & -\mathbf{A}^T{}_{(3\times2)} & \mathbf{0}_{(3\times3)} & \mathbf{I}_{(3\times3)} \end{bmatrix}}_{=\mathbf{B}_{(5\times10)}}\underbrace{\begin{bmatrix} \mathbf{d}^+_{(2\times1)} \\ \mathbf{d}^-_{(2\times1)} \\ \mathbf{u}_{(3\times1)} \\ \mathbf{s}_{(3\times1)} \end{bmatrix}}_{=\mathbf{X}_{(10\times1)}}=\underbrace{\begin{bmatrix} -\mathbf{c}_{(2\times1)} \\ \mathbf{b}_{(3\times1)} \end{bmatrix}}_{=\mathbf{D}_{(5\times1)}}$$

Determine the Search Direction by Using the Simplex Method - Iteration 2 (4/10)

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{d}^+, \mathbf{d}^-, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\underbrace{\left[\begin{array}{c:c:c:c} \mathbf{H}_{(2\times2)} & -\mathbf{H}_{(2\times2)} & \mathbf{A}_{(2\times3)} & \mathbf{0}_{(2\times3)} \\ \hdashline \mathbf{A}^T{}_{(3\times2)} & -\mathbf{A}^T{}_{(3\times2)} & \mathbf{0}_{(3\times3)} & \mathbf{I}_{(3\times3)} \end{array}\right]}_{=\mathbf{B}_{(5\times10)}} \underbrace{\begin{bmatrix} \mathbf{d}^+_{(2\times1)} \\ \mathbf{d}^-_{(2\times1)} \\ \mathbf{u}_{(3\times1)} \\ \mathbf{s}_{(3\times1)} \end{bmatrix}}_{=\mathbf{X}_{(10\times1)}} = \underbrace{\begin{bmatrix} -\mathbf{c}_{(2\times1)} \\ \mathbf{b}_{(3\times1)} \end{bmatrix}}_{=\mathbf{D}_{(5\times1)}}$$

where, $\mathbf{d}^+_{(2\times1)} = \begin{bmatrix} d_1^+ \\ d_2^+ \end{bmatrix}, \mathbf{d}^-_{(2\times1)} = \begin{bmatrix} d_1^- \\ d_2^- \end{bmatrix}, \mathbf{c}_{(2\times1)} = \begin{bmatrix} -1.732 \\ -1.732 \end{bmatrix}, \mathbf{H}_{(2\times2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{A}_{(2\times3)} = \begin{bmatrix} 0.577 & -1 & 0 \\ 0.577 & 0 & -1 \end{bmatrix}, \mathbf{b}_{(3\times1)} = \begin{bmatrix} 0 \\ 1.732 \\ 1.732 \end{bmatrix}$

$$\mathbf{B}_{(5\times10)} = \left[\begin{array}{cc:cc:ccc:ccc} 1 & 0 & -1 & 0 & 0.577 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0.577 & 0 & -1 & 0 & 0 & 0 \\ \hdashline 0.577 & 0.577 & -0.577 & -0.577 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{X}^T{}_{(1\times10)} = \begin{bmatrix} d_1^+ & d_2^+ & d_1^- & d_2^- & u_1 & u_2 & u_3 & s_1 & s_2 & s_3 \end{bmatrix}, \mathbf{D}^T{}_{(1\times5)} = \begin{bmatrix} 1.732 & 1.732 & 0 & 1.732 & 1.732 \end{bmatrix}$$

Determine the Search Direction by Using the Simplex Method - Iteration 2 (5/10)

Kuhn-Tucker necessary condition(matrix form)

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)} = \mathbf{D}_{(5\times1)}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0.577 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0.577 & 0 & -1 & 0 & 0 & 0 \\ 0.577 & 0.577 & -0.577 & -0.577 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1^+ \\ d_2^+ \\ d_1^- \\ d_2^- \\ u_1 \\ u_2 \\ u_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1.732 \\ 1.732 \\ 0 \\ 1.732 \\ 1.732 \end{bmatrix}$$

We want to find.

➡ **This problem is to find X in the linear programming problem only having the equality constraints.**

➡ $u_i s_i = 0; i = 1\ to\ 3$ **: Check whether the solution obtained from the linear indeterminate equation satisfies the nonlinear indeterminate equation and determine the solution.**

Determine the Search Direction by Using the Simplex Method - Iteration 2 (6/10)

Simplex method to solve the quadratic programming problem

1. The problem to solve the Kuhn-Tucker necessary condition is the same with the problem having only the equality constraints(linear programming problem).
2. To solve the linear indeterminate equation, we introduce the artificial variables, define the artificial objective function and determine the initial basic feasible solution by using the Simplex method.

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)} + \underbrace{\mathbf{Y}_{(5\times1)}}_{\text{Artificial variables}} = \mathbf{D}_{(5\times1)}$$

3. The artificial objective function is defined as follows.

$$w = \sum_{i=1}^{5} Y_i = \sum_{i=1}^{5} D_i - \sum_{j=1}^{10}\sum_{i=1}^{5} B_{ij} X_j = w_0 + \sum_{j=1}^{10} C_j X_j$$

where $C_j = -\sum_{i=1}^{5} B_{ij}$: Add the elements of the j th column of the matrix B and change the its sign.(Initial relative objective coefficient).

$$w_0 = \sum_{i=1}^{5} D_i = 1 + 1 + \tfrac{2}{3} + 1 + 1 = \tfrac{14}{3}$$

: Initial value of the artificial objective function (summation of the all elements of the matrix D)

4. Solve the linear programming problem by using the Simplex and check whether the solution satisfies the following equation.

$u_i s_i = 0; i = 1\ to\ 3$: Check whether the solution obtained from the linear indeterminate equation satisfies the nonlinear indeterminate equation and determine the solution.

Determine the Search Direction by Using the Simplex Method - Iteration 2 (7/10)

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)}+\underbrace{\mathbf{Y}_{(5\times1)}}_{\text{Artificial variables}}=\mathbf{D}_{(5\times1)} \;\Rightarrow\; \begin{bmatrix} 1 & 0 & -1 & 0 & 0.577 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0.577 & 0 & -1 & 0 & 0 & 0 \\ 0.577 & 0.577 & -0.577 & -0.577 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1^+ \\ d_2^+ \\ d_1^- \\ d_2^- \\ u_1 \\ u_2 \\ u_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1.732 \\ 1.732 \\ 0 \\ 1.732 \\ 1.732 \end{bmatrix}$$

Define the artificial objective function for using the Simplex method

Sum the all rows(1~5): $0.577X_1+0.577X_2-0.577X_3-0.577X_4+1.154X_5-X_6-X_7+X_8+X_9+X_{10}+\underbrace{Y_1+Y_2+Y_3+Y_4+Y_5}_{w}=6.928$

Replace the summation of the all artificial to w and rearrange: $-0.577X_1-0.577X_2+0.577X_3+0.577X_4-1.154X_5+X_6+X_7-X_8-X_9-X_{10}=w-6.928$

1	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
Y1	1	0	-1	0	0.577	-1	0	0	0	0	1	0	0	0	0	1.732	3
Y2	0	1	0	-1	0.577	0	-1	0	0	0	0	1	0	0	0	1.732	3
Y3	0.577	0.577	-0.577	-0.577	0	0	0	1	0	0	0	0	1	0	0	0	-
Y4	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1.732	-
Y5	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1.732	-
A. Obj.	-0.577	-0.577	0.577	0.577	-1.154	1	1	-1	-1	-1	0	0	0	0	0	w-6.928	-

Artificial objective function

Sum all the elements of the row and change the its sign (ex. 1 row: -(1+0+1/3-1+0)=-1/3)

Determine the Search Direction by Using the Simplex Method - Iteration 2 (8/10)

2	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	1.732	0.000	-1.732	0.000	1.000	-1.732	0.000	0.000	0.000	0.000	1.732	0.000	0.000	0.000	0.000	3.000	-1.732
Y2	-1.000	1.000	1.000	-1.000	0.000	1.000	-1.000	0.000	0.000	0.000	-1.000	1.000	0.000	0.000	0.000	0.000	0.000
Y3	0.577	0.577	-0.577	-0.577	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
Y4	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	0.000	1.732	1.732
Y5	0.000	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	1.732	-
A. Obj.	1.423	-0.577	-1.423	0.577	0.000	-1.000	1.000	-1.000	-1.000	-1.000	2.000	0.000	0.000	0.000	0.000	w-3.464	

3	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	0.000	1.732	0.000	-1.732	1.000	0.000	-1.732	0.000	0.000	0.000	0.000	1.732	0.000	0.000	0.000	3.000	-
X3	-1.000	1.000	1.000	-1.000	0.000	1.000	-1.000	0.000	0.000	0.000	-1.000	1.000	0.000	0.000	0.000	0.000	-
Y3	0.000	1.155	0.000	-1.155	0.000	0.577	-0.577	1.000	0.000	0.000	-0.577	0.577	1.000	0.000	0.000	0.000	-
Y4	0.000	-1.000	0.000	1.000	0.000	-1.000	1.000	0.000	1.000	0.000	1.000	-1.000	0.000	1.000	0.000	1.732	-
Y5	0.000	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	1.732	1.732
A. Obj.	0.000	0.845	0.000	-0.845	0.000	0.423	-0.423	-1.000	-1.000	-1.000	0.577	1.423	0.000	0.000	0.000	w-3.464	

4	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	0.000	1.732	0.000	-1.732	1.000	0.000	-1.732	0.000	0.000	0.000	0.000	1.732	0.000	0.000	0.000	3.000	-
X3	-1.000	1.000	1.000	-1.000	0.000	1.000	-1.000	0.000	0.000	0.000	-1.000	1.000	0.000	0.000	0.000	0.000	-
Y3	0.000	1.155	0.000	-1.155	0.000	0.577	-0.577	1.000	0.000	0.000	-0.577	0.577	1.000	0.000	0.000	0.000	-
Y4	0.000	-1.000	0.000	1.000	0.000	-1.000	1.000	0.000	1.000	0.000	1.000	-1.000	0.000	1.000	0.000	1.732	1.732
X10	0.000	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	1.732	-
A. Obj.	0.000	-0.155	0.000	0.155	0.000	0.423	-0.423	-1.000	-1.000	0.000	0.577	1.423	0.000	0.000	1.000	w-1.732	

Determine the Search Direction by Using the Simplex Method - Iteration 2 (9/10)

5

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	0.000	1.732	0.000	-1.732	1.000	0.000	-1.732	0.000	0.000	0.000	0.000	1.732	0.000	0.000	0.000	3.000	1.732
X3	-1.000	1.000	1.000	-1.000	0.000	1.000	-1.000	0.000	0.000	0.000	-1.000	1.000	0.000	0.000	0.000	0.000	0.000
Y3	0.000	1.155	0.000	-1.155	0.000	0.577	-0.577	1.000	0.000	0.000	-0.577	0.577	1.000	0.000	0.000	0.000	0.000
X9	0.000	-1.000	0.000	1.000	0.000	-1.000	1.000	0.000	1.000	0.000	1.000	-1.000	0.000	1.000	0.000	1.732	-1.732
X10	0.000	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	1.732	-1.732
A. Obj.	0.000	-1.155	0.000	1.155	0.000	-0.577	0.577	-1.000	0.000	0.000	1.577	0.423	0.000	1.000	1.000	w-0.000	

6

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	1.732	0.000	-1.732	0.000	1.000	-1.732	0.000	0.000	0.000	0.000	1.732	0.000	0.000	0.000	0.000	3.000	1.732
X2	-1.000	1.000	1.000	-1.000	0.000	1.000	-1.000	0.000	0.000	0.000	-1.000	1.000	0.000	0.000	0.000	0.000	0.000
Y3	1.155	0.000	-1.155	0.000	0.000	-0.577	0.577	1.000	0.000	0.000	0.577	-0.577	1.000	0.000	0.000	0.000	0.000
X9	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	0.000	1.732	-1.732
X10	-1.000	0.000	1.000	0.000	0.000	1.000	-1.000	0.000	0.000	1.000	-1.000	1.000	0.000	0.000	1.000	1.732	-1.732
A. Obj.	-1.155	0.000	1.155	0.000	0.000	0.577	-0.577	-1.000	0.000	0.000	0.423	1.577	0.000	1.000	1.000	w-0.000	

7

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	0.000	0.000	0.000	0.000	1.000	-0.866	-0.866	-1.500	0.000	0.000	0.866	0.866	-1.500	0.000	0.000	3.000	
X2	0.000	1.000	0.000	-1.000	0.000	0.500	-0.500	0.866	0.000	0.000	-0.500	0.500	0.866	0.000	0.000	0.000	
X1	1.000	0.000	-1.000	0.000	0.000	-0.500	0.500	0.866	0.000	0.000	0.500	-0.500	0.866	0.000	0.000	0.000	
X9	0.000	0.000	0.000	0.000	0.000	-0.500	0.500	0.866	1.000	0.000	0.500	-0.500	0.866	1.000	0.000	1.732	
X10	0.000	0.000	0.000	0.000	0.000	0.500	-0.500	0.866	0.000	1.000	-0.500	0.500	0.866	0.000	1.000	1.732	
A. Obj.	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	1.000	1.000	1.000	w-0.000	

Determine the Search Direction by Using the Simplex Method - Iteration 2 (10/10)

7	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	0.000	0.000	0.000	0.000	1.000	-0.866	-0.866	-1.500	0.000	0.000	0.866	0.866	-1.500	0.000	0.000	3.000	
X2	0.000	1.000	0.000	-1.000	0.000	0.500	-0.500	0.866	0.000	0.000	-0.500	0.500	0.866	0.000	0.000	0.000	
X1	1.000	0.000	-1.000	0.000	0.000	-0.500	0.500	0.866	0.000	0.000	0.500	-0.500	0.866	0.000	0.000	0.000	
X9	0.000	0.000	0.000	0.000	0.000	-0.500	0.500	0.866	1.000	0.000	0.500	-0.500	0.866	1.000	0.000	1.732	
X10	0.000	0.000	0.000	0.000	0.000	0.500	-0.500	0.866	0.000	1.000	-0.500	0.500	0.866	0.000	1.000	1.732	
A. Obj.	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	1.000	1.000	1.000	w-0.000	

$$\mathbf{X}^T{}_{(1\times10)} = \begin{bmatrix} d_1^+ & d_2^+ & d_1^- & d_2^- & u_1 & u_2 & u_3 & s_1 & s_2 & s_3 \end{bmatrix}$$

Basic solution:

$$X_5 = 3, \quad X_2 = 0, \quad X_1 = 0, \quad X_9 = 1.732, \quad X_{10} = 1.732$$

Nonbasic solution:

$$X_3 = X_4 = X_6 = X_7 = X_8 = 0$$

This solution satisfy the nonlinear indeterminate equation($X_i X_{3+i} = 0; i = 5\ to\ 7, X_i \geq 0; i = 1\ to\ 10$)**.**

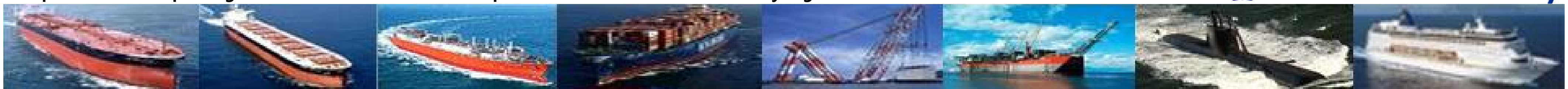
So, the optimal solution is $d_1 = d_2 = 0,\ u_1 = 3,\ u_2 = u_3 = 0,\ s_1 = 0,\ s_2 = s_3 = 1.732$.

➡ **In the Pivot step, if the smallest(i.e., the most negative) coefficient of the artificial objective function or the smallest positive ratio "bi/ai" appears more than one time, the initial basic feasible solution can be changed by depending on the selection of the pivot element in the pivot procedure.**

➡ **We have to find and check the solution until the nonlinear indeterminate equation(u_i*s_i=0) is satisfied.**

7.5 Summary of the Sequential Quadratic Programming(SQP)

SYstem Design Laboratory



Formulation of the Quadratic Programming Problem to Determine the Search Direction

Minimize $f(\mathbf{x}+\Delta\mathbf{x}) \cong f(\mathbf{x}) + \nabla f^T(\mathbf{x})\Delta\mathbf{x} + 0.5\Delta\mathbf{x}^T\mathbf{H}\Delta\mathbf{x}$

The second-order Taylor series expansion of the objective function

Subject to $h_j(\mathbf{x}+\Delta\mathbf{x}) \cong h_j(\mathbf{x}) + \nabla h_j^T(\mathbf{x})\Delta\mathbf{x} = 0;\ j = 1\ to\ p$

The first-order(linear) Taylor series expansion of the equality constraints

$g_j(\mathbf{x}+\Delta\mathbf{x}) \cong g_j(\mathbf{x}) + \nabla g_j^T(\mathbf{x})\Delta\mathbf{x} \le 0;\ j = 1\ to\ m$

The first-order(linear) Taylor series expansion of the inequality constraints

Define: $\bar{f} = f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x}),\ e_j = -h_j(\mathbf{x}),\ b_j = -g_j(\mathbf{x}),$

$c_i = \partial f(\mathbf{x})/\partial x_i,\ n_{ij} = \partial h_j(\mathbf{x})/\partial x_i,\ a_{ij} = \partial g_j(\mathbf{x})/\partial x_i,$

$d_i = \Delta x_i$

Matrix form

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$: Quadratic objective function

Subject to $\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$: Linear equality constraints

$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$: Linear inequality constraints

Determination of the Step Size by Using the Golden Section Search Method

Trial design point for which the descent condition is checked

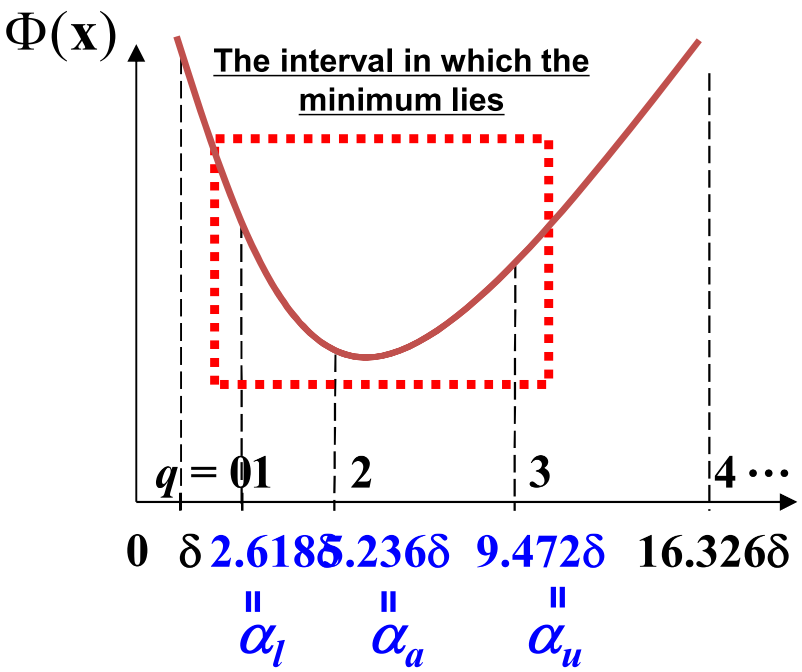
$$\mathbf{x}^{(k,j)} = \mathbf{x}^{(k)} + \alpha_{(k,j)}\mathbf{d}^{(k)}$$

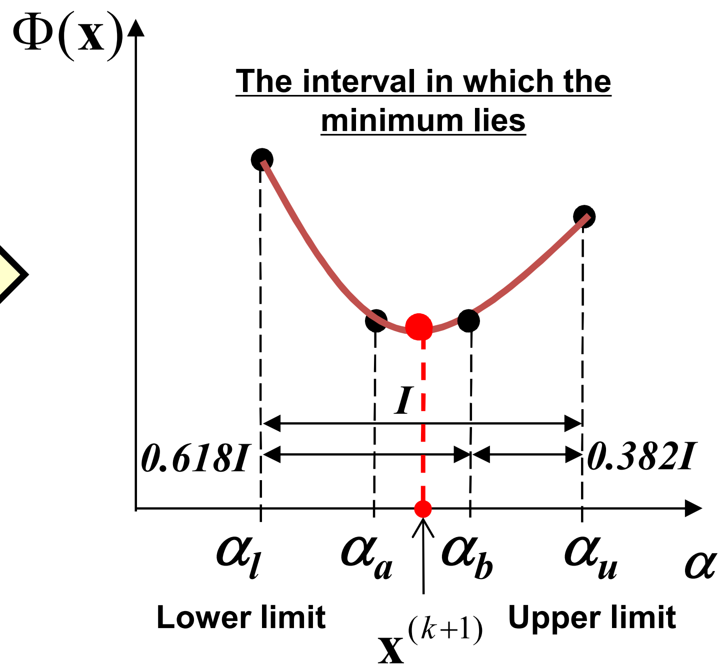
➡ How can we determine the value of the $\alpha_{(k,j)}$ to find the improved design point?

Find the improved design point which minimizes the descent function more than the current point by changing $\alpha_{(k,j)}$. (One dimensional search method, such as the Golden section search method, can be used.)

Determination of the improved design point $\mathbf{x}^{(k+1)}$ by using the one dimensional search method such as the Golden section search method($\mathbf{x}^{(k,j)}$ is changed to $\mathbf{x}^{(k+1)}$.)

After finding the interval in which the minimum lies, find the minimum point, x, by reducing the interval(Golden section search method).





Formulation of the Quadratic Programming Problem

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$

Subject to $\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$

$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$

Assumption: $\mathbf{H}_{(n\times n)} = \mathbf{I}_{(n\times n)}$

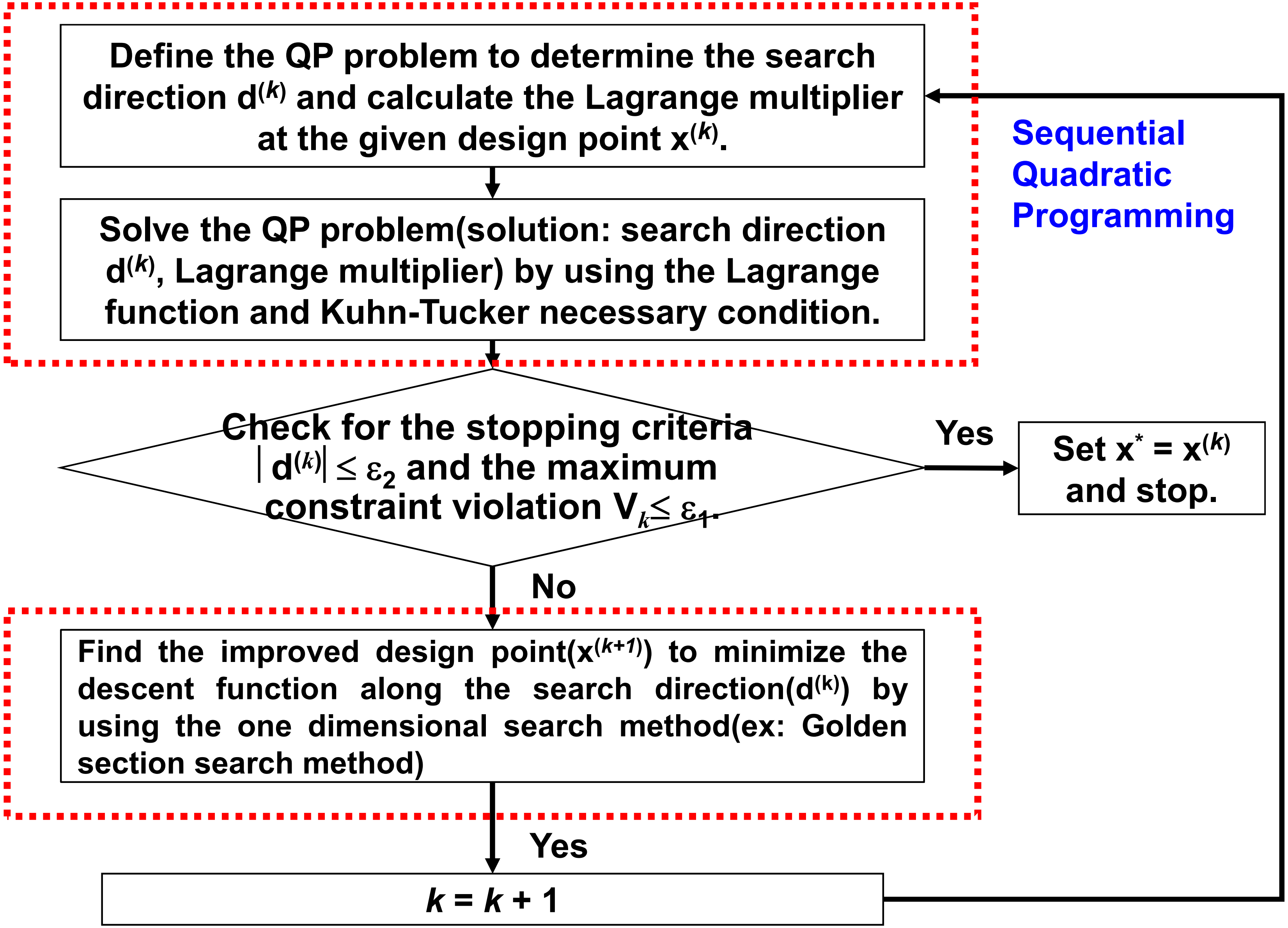
Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{I}_{(n\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)}$

Subject to $\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$

$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$

- Since $\mathbf{H}_{(n\times n)} = \mathbf{I}_{(n\times n)}$, the objective function is a quadratic form.
- All constraints are linear.
- This problem is called the convex programming problem and any local optimum solution is also a global optimum solution.

Flow Diagram of the SQP Algorithm



Summary of the SQP Algorithm (1/2)

☑ Step 1: Set $k = 0$. Estimate the initial value for the design variables as $\mathbf{x}^{(0)}$. Select an appropriate initial value for the penalty parameter R_0, and two small number ε_1, ε_2 that define the permissible constraint violations and convergence parameter values, respectively.

☑ Step 2: At $\mathbf{x}^{(k)}$, compute the objective and constraint functions and their gradient. Calculate the maximum constraint violation V_k.

☑ Step 3: Using the objective and constraints function values and their gradients, define the QP problem. Solve the QP problem to obtain the search direction $\mathbf{d}^{(k)}(= \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)})$ and Lagrange multiplier $\mathbf{v}^{(k)}$ and $\mathbf{u}^{(k)}$.

Summary of the SQP Algorithm (2/2)

☑ Step 4: Check for the stopping criteria $|\mathbf{d}^{(k)}| \le \varepsilon_2$ and the maximum constraint violation $V_k \le \varepsilon_1$. If these criteria are satisfied then stop. Otherwise continue.

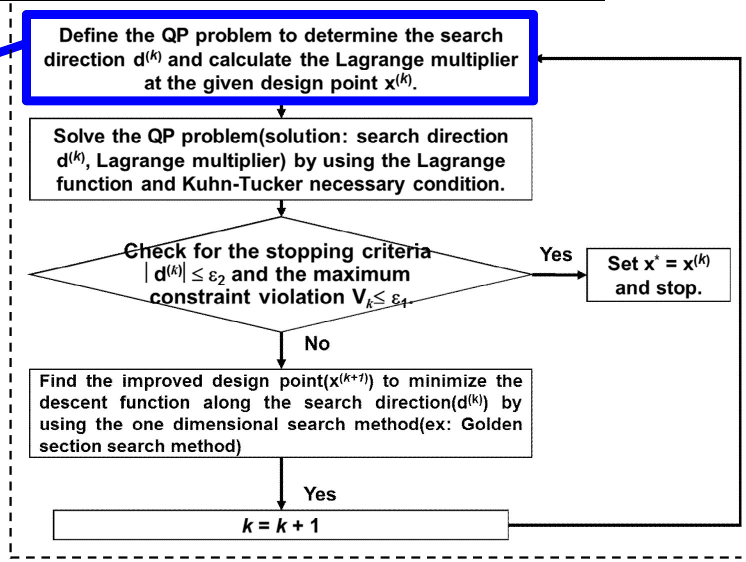
☑ Step 5: Calculate the sum r_k of the Lagrange multiplier. Set $R = \max\{R_k, r_k\}$.

☑ Step 6: Set $\mathbf{x}^{(k,j)} = \mathbf{x}^{(k)} + \alpha_{(k,j)}\mathbf{d}^{(k)}$, where $\alpha = \alpha_{(k,j)}$ is a proper step size. As for the unconstrained problems, the step size can be obtained by minimizing the descent function along the search direction $\mathbf{d}^{(k)}$. The one dimensional search method, such as the Golden section search method, can be used to determine the optimum step size.
(If the one dimensional search method is completed, the current design point $\mathbf{x}^{(k,j)}$ is changed to $\mathbf{x}^{(k+1)}$.)

☑ Step 7: Save the current penalty parameter as R_k = R. Update the iteration counter as $k = k+1$ and go to Step 2.

Effect of the Starting Point in the SQP Algorithm

The starting point can affect performance of the algorithm.



For example, at some points, the Quadratic Programming problem defined to determine the search direction may not have any solution.
This does not mean that the original problem is infeasible.
The original problem may be highly nonlinear, so that the linearized constraints may be inconsistent giving infeasible Quadratic Programming problem.

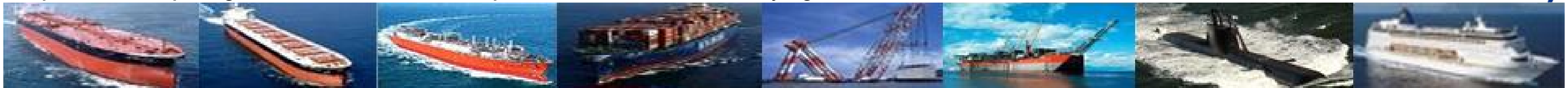
This situation can be handled by either temporarily deleting the inconsistent constraints or starting from another point.

Naval Architecture & Ocean Engineering

[Appendix] Another Method for Solving Nonlinear Indeterminate Equations

Applying the Simplex Method

SYstem Design Laboratory



Applying the Simplex Method

- Eliminate variables using relevant equations and introduce new 'virtual' linear variables x'.

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x}, \mathbf{u}, \mathbf{s}, \boldsymbol{\zeta}, \boldsymbol{\delta}) = 0$

$$\frac{\partial L}{\partial x_1} = -2 + 2x_1 - 2u_1 - u_2 - \zeta_1 = 0, \qquad \frac{\partial L}{\partial x_2} = -2 + 2x_2 - u_1 - 2u_2 - \zeta_2 = 0$$

$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1^2 = 0, \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_2^2 = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1^2 = 0, \frac{\partial L}{\partial s_2} = 2u_2 s_2^2 = 0 \qquad \frac{\partial L}{\partial \delta_1} = 2\zeta_1 \delta_1^2 = 0, \frac{\partial L}{\partial \delta_2} = 2\zeta_2 \delta_2^2 = 0$$

Substitute

$$\boxed{\frac{\partial L}{\partial \zeta_1} = -x_1 + \delta_1^2 = 0 \qquad \frac{\partial L}{\partial \zeta_2} = -x_2 + \delta_2^2 = 0}$$

$$\Rightarrow \text{Redefine } \mathbf{s}_i^2 \textit{ as } \mathbf{s}_i'$$

where $u_i, \zeta_i, s_i' \geq 0$

Reformulated Kuhn-Tucker necessary condition:

$$\frac{\partial L}{\partial x_1} = -2 + 2x_1 - 2u_1 - u_2 - \zeta_1 = 0, \qquad \frac{\partial L}{\partial x_2} = -2 + 2x_2 - u_1 - 2u_2 - \zeta_2 = 0$$

$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1' = 0, \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_2' = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1' = 0, \frac{\partial L}{\partial s_2} = 2u_2 s_2' = 0 \qquad \frac{\partial L}{\partial \delta_1} = 2\zeta_1 x_1 = 0, \frac{\partial L}{\partial \delta_2} = 2\zeta_2 x_2 = 0$$

where $u_i, \zeta_i, s_i', x_i \geq 0$

We eliminate two variables using relevant two equations and also introduce new variable s' instead of s^2.

Applying the Simplex Method of Phase 1 for Solving Linear Indeterminate Equations (1/9)

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x}, \mathbf{u}, \mathbf{s}, \boldsymbol{\zeta}, \boldsymbol{\delta}) = 0$

Linear indeterminate equations

$$\frac{\partial L}{\partial x_1} = -2 + 2x_1 - 2u_1 - u_2 - \zeta_1 = 0, \quad \frac{\partial L}{\partial x_2} = -2 + 2x_2 - u_1 - 2u_2 - \zeta_2 = 0$$

$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1' = 0, \quad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_2' = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1' = 0, \frac{\partial L}{\partial s_2} = 2u_2 s_2' = 0 \quad \frac{\partial L}{\partial \delta_1} = 2\zeta_1 x_1 = 0, \frac{\partial L}{\partial \delta_2} = 2\zeta_2 x_2 = 0$$

where $u_i, \zeta_i, s_i', x_i \ge 0$

Solve the linear indeterminate equations by using the Simplex method of Phase 1.

Define the standard LP problem

$$\begin{bmatrix} 2 & 0 & -2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 & -1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ \zeta_1 \\ \zeta_2 \\ s_1' \\ s_2' \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -4 \\ -4 \end{bmatrix}$$

Multiply both side of the constraints by -1

$$\Rightarrow \begin{bmatrix} 2 & 0 & -2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ \zeta_1 \\ \zeta_2 \\ s_1' \\ s_2' \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

Applying the Simplex Method of Phase 1 for Solving Linear Indeterminate Equations (2/9)

Introduce the artificial variables to treat the linear equality constraints.

$$\begin{bmatrix} 2 & 0 & -2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(=X_1) \\ x_2(=X_2) \\ u_1(=X_3) \\ u_2(=X_4) \\ \zeta_1(=X_5) \\ \zeta_2(=X_6) \\ s_1'(=X_7) \\ s_2'(=X_8) \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

Define the artificial objective function as sum of all the artificial variables (Y_1+Y_2+Y_3+Y_4).

$$5x_1 + 5x_2 - 3u_1 - 3u_2 - \zeta_1 - \zeta_2 - s_1' - s_2' + \underbrace{Y_1 + Y_2 + Y_3 + Y_4}_{w} = 12$$

$$-5x_1 - 5x_2 + 3u_1 + 3u_2 + \zeta_1 + \zeta_2 + s_1' + s_2' = w - 12$$ **: Artificial objective function**

Applying the Simplex Method of Phase 1 for Solving Linear Indeterminate Equations (3/9)

$$\begin{bmatrix} 2 & 0 & -2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(=X_1) \\ x_2(=X_2) \\ u_1(=X_3) \\ u_2(=X_4) \\ \zeta_1(=X_5) \\ \zeta_2(=X_6) \\ s_1'(=X_7) \\ s_2'(=X_8) \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

$-5x_1 - 5x_2 + 3u_1 + 3u_2 + \zeta_1 + \zeta_2 + s_1' + s_2' = w - 12$ **: Artificial objective function**

1

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
Y1	2	0	-2	-1	-1	0	0	0	1	0	0	0	2	1
Y2	0	2	-1	-2	0	-1	0	0	0	1	0	0	2	-
Y3	2	1	0	0	0	0	-1	0	0	0	1	0	4	2
Y4	1	2	0	0	0	0	0	-1	0	0	0	1	4	4
A. Obj.	-5	-5	3	3	1	1	1	1	0	0	0	0	w-12	-

Applying the Simplex Method of Phase 1 for Solving Linear Indeterminate Equations (4/9)

2

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	-1	-1/2	-1/2	0	0	0	1/2	0	0	0	1	-
Y2	0	**2**	-1	-2	0	-1	0	0	0	1	0	0	2	**1**
Y3	0	1	2	1	1	0	-1	0	-1	0	1	0	2	2
Y4	0	2	1	1/2	1/2	0	0	-1	-1/2	0	0	1	3	3/2
A. Obj.	0	**-5**	-2	1/2	-3/2	1	1	1	5/2	0	0	0	w-7	-

3

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	-1	-1/2	-1/2	0	0	0	1/2	0	0	0	1	-
X2	0	1	-1/2	-1	0	-1/2	0	0	0	1/2	0	0	1	-
Y3	0	0	**5/2**	2	1	1/2	-1	0	-1	-1/2	1	0	1	**2/5**
Y4	0	0	2	5/2	1/2	1	0	-1	-1/2	-1	0	1	1	1/2
A. Obj.	0	0	**-9/2**	-9/2	-3/2	-3/2	1	1	5/2	5/2	0	0	w-2	-

4

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	0	3/10	-1/10	1/5	-2/5	0	1/10	-1/5	2/5	0	7/5	14/3
X2	0	1	0	-3/5	1/5	-2/5	-1/5	0	-1/5	2/5	1/5	0	6/5	-
X3	0	0	1	4/5	2/5	1/5	-2/5	0	-2/5	-1/5	2/5	0	2/5	1/2
Y4	0	0	0	**9/10**	-3/10	3/5	4/5	-1	3/10	-3/5	-4/5	1	1/5	**2/9**
A. Obj.	0	0	0	**-9/10**	3/10	-3/5	-4/5	1	7/10	8/5	9/5	0	w-1/5	-

Applying the Simplex Method of Phase 1 for Solving Linear Indeterminate Equations (5/9)

5

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	0	0	0	0	-2/3	1/3	0	0	2/3	-1/3	4/3	-
X2	0	1	0	0	0	0	7/15	-2/3	2/5	0	-7/15	2/15	4/3	-
X3	0	0	1	0	2/3	-1/3	-10/9	8/9	-2/3	7/15	10/9	-8/45	2/9	-
X4	0	0	0	1	-1/3	2/3	8/9	-10/9	1/3	-2/3	-8/9	2/9	2/9	-
A. Obj.	0	0	0	0	0	0	0	0	1	1	1	1	w-0	-

Since the value of the objective function becomes zero, the initial basic feasible solution is obtained.

$$\mathbf{X}^T_{(1\times 8)} = \begin{bmatrix} x_1 & x_2 & u_1 & u_2 & \zeta_1 & \zeta_2 & s_1{}' & s_2{}' \end{bmatrix}$$

One of the initial basic feasible solutions is $X_1=X_2=4/3, X_3=X_4=2/9, X_5=X_6=X_7=X_8=0$.

$$x_1 = x_2 = \tfrac{4}{3}, u_1 = u_2 = \tfrac{2}{9}, \zeta_1 = \zeta_2 = s_1{}' = s_2{}' = 0$$

And this solution satisfies the following nonlinear equations(constraints).

$$u_2 s_2' = 0, \quad u_1 s_1' = 0, \quad \zeta_1 x_1 = 0, \quad \zeta_2 x_2 = 0$$

Therefore, the optimal solution of this problem is $x_1 = x_2 = \tfrac{4}{3}, u_1 = u_2 = \tfrac{2}{9}, \zeta_1 = \zeta_2 = s_1' = s_2' = 0$.

This result is same as that obtained by the first method.

Applying the Simplex Method of Phase 1 for Solving Linear Indeterminate Equations (6/9)

Instead of choosing the first column as the pivot column, we can choose the second column, because the coefficient of the objective function is same.

1

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
Y1	2	0	-2	-1	-1	0	0	0	1	0	0	0	2	-
Y2	0	2	-1	-2	0	-1	0	0	0	1	0	0	2	1
Y3	2	1	0	0	0	0	-1	0	0	0	1	0	4	4
Y4	1	2	0	0	0	0	0	-1	0	0	0	1	4	2
A. Obj.	-5	-5	3	3	1	1	1	1	0	0	0	0	w-12	-

2

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
Y1	2	0	-2	-1	-1	0	0	0	1	0	0	0	2	1
X2	0	1	-1/2	-1	0	-1/2	0	0	0	1/2	0	0	1	-
Y3	2	0	1/2	1	0	1/2	-1	0	0	-1/2	1	0	3	3/2
Y4	1	0	1	2	0	1	0	-1	0	-1	0	1	2	2
A. Obj.	-5	0	1/2	-2	1	-3/2	1	1	0	5/2	0	0	w-7	-

3

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	-1	-1/2	-1/2	0	0	0	1/2	0	0	0	1	-
X2	0	1	-1/2	-1	0	-1/2	0	0	0	1/2	0	0	1	-
Y3	0	0	5/2	2	1	1/2	-1	0	-1	-1/2	1	0	1	2/5
Y4	0	0	2	5/2	1/2	1	0	-1	-1/2	-1	0	1	1	1/2
A. Obj.	0	0	-9/2	-9/2	-3/2	-3/2	1	1	5/2	5/2	0	0	w-2	-

Applying the Simplex Method of Phase 1 for Solving Linear Indeterminate Equations (7/9)

4

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	0	3/10	-1/10	1/5	-2/5	0	1/10	-1/5	2/5	0	7/5	-
X2	0	1	0	-6/10	1/5	-2/5	-1/5	0	-1/5	2/5	1/5	0	6/5	-
X3	0	0	1	4/5	2/5	1/5	-2/5	0	-2/5	-1/5	2/5	0	2/5	-
Y4	0	0	0	9/10	-3/10	3/5	4/5	-1	3/10	-3/5	-4/5	1	1/5	1/4
A. Obj.	0	0	0	-9/10	3/10	-3/5	-4/5	1	7/10	8/5	9/5	0	w-1/5	-

5

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	0	3/4	-1/4	1/2	0	-1/2	-1/4	-1/2	0	1/2	3/2	-
X2	0	1	0	-3/8	1/8	-1/4	0	-1/4	-1/8	1/4	0	1/4	5/4	-
X3	0	0	1	5/4	1/4	1/2	0	-1/2	-1/4	-1/2	0	1/2	1/2	-
X7	0	0	0	9/8	-3/8	3/4	1	-5/4	3/8	-3/4	-1	5/4	1/4	-
A. Obj.	0	0	0	0	0	0	0	0	1	1	1	1	w-0	-

$$\mathbf{X}^T_{(1\times 8)} = \begin{bmatrix} x_1 & x_2 & u_1 & u_2 & \zeta_1 & \zeta_2 & s_1' & s_2' \end{bmatrix}$$

Since the value of the objective function becomes zero, the initial basic feasible solution is obtained.

The another initial basic feasible solution is $X_1=3/2, X_2=5/4, X_3=1/2, X_4=X_5=X_6=0, X_7=1/4, X_8=0$.

$$x_1 = \tfrac{2}{3}, x_2 = \tfrac{5}{4}, u_1 = \tfrac{1}{2}, u_2 = \zeta_1 = \zeta_2 = 0, s_1' = \tfrac{1}{4}, s_2' = 0$$

But this solution does not satisfy the constraint of $u_1 s_1' = 0$.

Therefore, this solution cannot be the optimal solution.

- **When the smallest(i.e., the most negative) coefficient of the artificial objective function or the smallest positive ratio "b_i/a_i" appears more than one entry, the initial basic feasible solution can be changed depending on the selection of the pivot element in the pivot operation.**
- **We have to check whether the solution obtained by the Simplex method satisfies the nonlinear equation. (constraint, $u_i * s_i' = 0$).**

Applying the Simplex Method of Phase 1 for Solving Linear Indeterminate Equations (8/9)

In the tableau 3, if we choose the column 4 as the pivot column which has the same coefficient of the artificial objective function as of the column 3, what will happen?

3

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	-1	-1/2	-1/2	0	0	0	1/2	0	0	0	1	-
X2	0	1	-1/2	-1	0	-1/2	0	0	0	1/2	0	0	1	-
Y3	0	0	5/2	2	1	1/2	-1	0	-1	-1/2	1	0	1	1/2
Y4	0	0	2	5/2	1/2	1	0	-1	-1/2	-1	0	1	1	2/5
A. Obj.	0	0	-9/2	-9/2	-3/2	-3/2	1	1	5/2	5/2	0	0	w-2	-

4

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	-6/10	0	-2/5	1/5	0	-1/5	2/5	-1/5	0	1/5	6/5	-
X2	0	1	3/10	0	1/5	-1/10	0	-2/5	-1/5	1/10	0	2/5	7/5	-
Y3	0	0	9/10	0	3/5	-3/10	-1	4/5	-3/5	3/10	1	-4/5	1/5	1/4
X4	0	0	4/5	1	1/5	2/5	0	-2/5	-1/5	-2/5	0	2/5	2/5	-
A. Obj.	0	0	-9/10	0	-3/5	3/10	1	-4/5	8/5	7/10	0	9/5	w-1/5	-

5

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	-3/8	0	-1/4	1/8	-1/4	0	1/4	-1/8	1/4	0	5/4	-
X2	0	1	3/4	0	1/2	-1/4	-1/2	0	-1/2	-1/4	1/2	0	3/2	-
X8	0	0	9/8	0	3/4	-3/8	-5/4	1	3/4	3/8	5/4	-1	1/4	-
X4	0	0	5/4	1	1/2	1/4	-1/2	0	-1/2	-1/4	1/2	0	1/2	-
A. Obj.	0	0	0	0	0	0	0	0	1	1	1	1	w-0	-

Applying the Simplex Method of Phase 1 for Solving Linear Indeterminate Equations (9/9)

5

	X1	X2	X3	X4	X5	X6	X7	X8	Y1	Y2	Y3	Y4	bi	bi/ai
X1	1	0	-3/8	0	-1/4	1/8	-1/4	0	1/4	-1/8	1/4	0	5/4	-
X2	0	1	3/4	0	1/2	-1/4	-1/2	0	-1/2	-1/4	1/2	0	3/2	-
X8	0	0	9/8	0	3/4	-3/8	-5/4	1	3/4	3/8	5/4	-1	1/4	-
X4	0	0	5/4	1	1/2	1/4	-1/2	0	-1/2	-1/4	1/2	0	1/2	-
A. Obj.	0	0	0	0	0	0	0	0	1	1	1	1	w-0	-

Since the value of the objective function becomes zero, the initial basic feasible solution is obtained.

$$\mathbf{X}^T{}_{(1\times 8)} = \begin{bmatrix} x_1 & x_2 & u_1 & u_2 & \zeta_1 & \zeta_2 & s_1 & s_2 \end{bmatrix}$$

The another initial basic feasible solution is X_1=5/4, X_2=3/2, X_3=0, X_4=1/2, X_5=X_6=0=X_7=0, X_8=1/4.

$$x_1 = \tfrac{4}{3}, x_2 = \tfrac{5}{4}, u_1 = 0, u_2 = \tfrac{1}{2}, \zeta_1 = \zeta_2 = s_1' = 0, s_2' = \tfrac{1}{4}$$

But this solution does not satisfy the constraint of $u_2 s_2' = 0$.

Therefore, this solution cannot be the optimal solution.

[Reference] Reason to Decompose the Unrestricted Variable into the Difference of Two Nonnegative Design Variables for Using the Simplex Method (1/2)

To use the Simplex method, the variables must be nonnegative in the LP problem.

We can use the Simplex method only for the case that all the variables are nonnegative at the optimal point.

The variables unrestricted in sign at the optimal point should be decomposed into the difference of two nonnegative variables in the LP problem.

$$
\begin{aligned}
&\textit{Minimize} \quad z = -y_1 - 2y_2 \\
&\textit{Subject to} \quad 3y_1 + 2y_2 \le 12 \\
&\qquad 2y_1 + 3y_2 \ge 6 \\
&\qquad y_1 \ge 0 \\
&\qquad y_2 \text{ is unrestricted in sign.}
\end{aligned}
$$

$$\boxed{y_2 = y_2^+ - y_2^-}$$

$$y_2^+, y_2^- \ge 0$$

$$
\begin{aligned}
&\textit{Minimize} \quad f = -y_1 \boxed{-2y_2^+ + 2y_2^-} \\
&\textit{Subject to} \quad 3y_1 + \boxed{2y_2^+ - 2y_2^-} \le 12 \\
&\qquad 2y_1 + \boxed{3y_2^+ - 3y_2^-} \ge 6 \\
&\qquad y_1, y_2^+, y_2^- \ge 0
\end{aligned}
$$

Since y_2 is free in sign, it should be decomposed into the difference of two nonnegative variables.

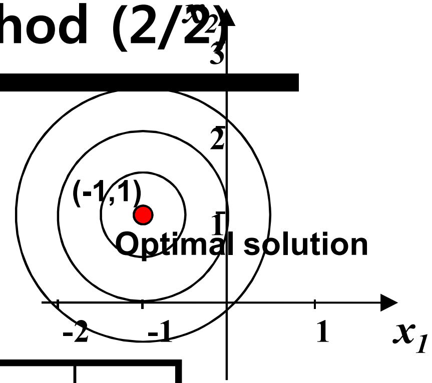
For "≤" type inequality constraint, we Introduce the slack variable.

For "≥" type inequality constraint, we Introduce the surplus variable and the artificial variable.

$$
\begin{aligned}
&\textit{Minimize} \quad f = -y_1 \boxed{-2y_2^+ + 2y_2^-} \\
&\textit{Subject to} \quad 3y_1 + \boxed{2y_2^+ - 2y_2^-} + \boxed{x_1} = 12 \\
&\qquad 2y_1 + \boxed{3y_2^+ - 3y_2^-} - \boxed{x_2} + \boxed{x_3} = 6 \\
&\qquad y_1, y_2^+, y_2^- \ge 0,\ x_i \ge 0; i = 1 \text{ to } 3
\end{aligned}
$$

Solve the problem using the Simplex method.

[Reference] Reason to Decompose the Unrestricted Variable into the Difference of Two Nonnegative Design Variables for Using the Simplex Method (2/2)



Minimize $f(x) = x_1^2 + x_2^2 + 2x_1 - 2x_2$: Quadratic objective function

where, x_1, x_2 are free in sign.

Minimize $f(x) = x_1^2 + x_2^2 + 2x_1 - 2x_2$

Definition of the Lagrange function

$$L(x_1, x_2) = x_1^2 + x_2^2 + 2x_1 - 2x_2$$

$$\frac{\partial L}{\partial x_1} = 2x_1 + 2 = 0 \quad \cdots\cdots\cdots ①'$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 2 = 0 \quad \cdots\cdots\cdots ②'$$

We try to solve this problem by using the Simplex method

$$-2x_1 = 2$$
$$2x_2 = 2$$

Since these constraints are the equality constraints, we must introduce artificial variables for the equality constraints and define an artificial objective function, and solve it .

$$-2x_1 + y_1 = 2 \quad ---③$$
$$2x_2 + y_2 = 2 \quad ---④$$

Since the artificial objective function is the sum of all the artificial variables, its minimum value must be zero.

Eq. ③+④ $\longrightarrow -2x_1 + 2x_2 + y_1 + y_2 = 4$

$$2x_1 - 2x_2 = \underbrace{y_1 + y_2}_{w} - 4$$

Redefine the variables as $x_1 = X_1, x_2 = X_2, y_1 = Y_1, y_2 = Y_2$ **and express in Matrix from.**

Basic variable	X1	X2	Y1	Y2	bi	bi/ai
Y1	-2	0	1	0	2	-
Y2	0	2	0	1	2	1
A. Obj.	2	-2	0	0	w-4	-

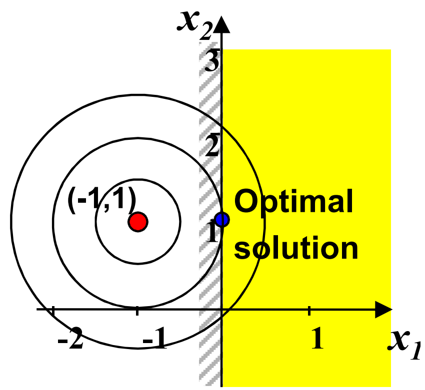
Basic variable	X1	X2	Y1	Y2	bi	bi/ai
Y1	-2	0	1	0	2	2
X2	0	1	0	1/2	1	-
A. Obj.	2	0	0	1	w-2	-

- **All coefficients of the artificial objective function become nonnegative.**
- **However, the sum of all the artificial variables(w) does not become zero.** ⇨ **Stop the simplex**

$$x_1 = 0, x_2 = 1, y_1 = 2, y_2 = 0$$

- **The simplex method does not give the optimum solution of x_1=-1, x_2=1, rather x_1=0, x_2=1.**
- **The reason is that the simplex method assume that all the variables are nonnegative, whereas the variables x_1, x_2 of this example are free in sign. From this, we can see that to use the simplex method, the unrestricted variables must be decomposed into the two nonnegative variables.**

[Review]



Minimize $f(x) = x_1^2 + x_2^2 + 2x_1 - 2x_2$ **: Quadratic objective function**

Subject to $x_1 \geq 0$ **: Linearized inequality constraint**

Minimize $f(x) = x_1^2 + x_2^2 + 2x_1 - 2x_2$

Subject to $x_1 \geq 0 \rightarrow -x_1 \leq 0 \rightarrow -x_1 + \delta^2 = 0$

Definition of the Lagrange function

$$L(x_1, x_2, \zeta, \delta) = x_1^2 + x_2^2 + 2x_1 - 2x_2 + \zeta(-x_1 + \delta^2)$$

Kuhn-Tucker necessary condition: $\nabla L(x_1, x_2, \zeta, \delta) = 0$

$$\frac{\partial L}{\partial x_1} = 2x_1 + 2 - \zeta = 0 \quad \cdots\cdots ①$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 2 = 0 \quad \cdots\cdots ②$$

$$\frac{\partial L}{\partial \zeta} = -x_1 + \delta^2 = 0 \quad \cdots\cdots ③$$

$$\frac{\partial L}{\partial \delta} = 2\zeta\delta = 0 \quad \cdots\cdots ④$$

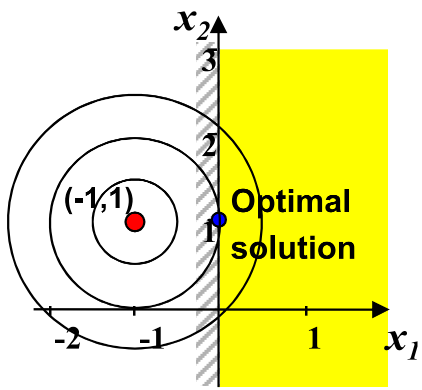
If we assume $\zeta = 0$, $x_1 = -1$ ➡ **The equation ③ is not satisfied.**

If we assume $\delta = 0$, $x_1 = 0, x_2 = 1, \zeta = 2$

Solving the Problem by Using the Simplex Method (1/2)

$$\textit{Minimize} \quad f(x) = x_1^2 + x_2^2 + 2x_1 - 2x_2$$: Quadratic objective function

$$\textit{Subject to} \quad x_1 \ge 0$$: Linearized inequality constraint



$$\textit{Minimize} \quad f(x) = x_1^2 + x_2^2 + 2x_1 - 2x_2$$

$$\textit{Subject to} \quad x_1 \ge 0 \rightarrow -x_1 \le 0 \rightarrow -x_1 + \delta^2 = 0$$

Definition of the Lagrange function

$$L(x_1, x_2, \zeta, \delta) = x_1^2 + x_2^2 + 2x_1 - 2x_2 + \zeta(-x_1 + \delta^2)$$

Kuhn-Tucker necessary condition: $\nabla L(x_1, x_2, \zeta, \delta) = 0$

We try to solve this problem by using the Simplex method.

$$\frac{\partial L}{\partial x_1} = 2x_1 + 2 - \zeta = 0$$
$$\frac{\partial L}{\partial x_2} = 2x_2 - 2 = 0$$
$$\frac{\partial L}{\partial \zeta} = -x_1 + \delta^2 = 0$$
$$\frac{\partial L}{\partial \delta} = 2\zeta\delta = 0$$

$$\boxed{x_2 = x_2^+ - x_2^-}$$
$$x_2^+, x_2^- \ge 0$$

Since x_2 is free in sign, we may decompose it as $x_2 = x_2^+ - x_2^-$

$$\frac{\partial L}{\partial x_1} = 2x_1 + 2 - \zeta = 0 \quad \cdots\cdots ①$$
$$\frac{\partial L}{\partial x_2} = \boxed{2x_2^+ - 2x_2^-} - 2 = 0 \quad \cdots ②$$
$$\frac{\partial L}{\partial \zeta} = -x_1 + \delta^2 = 0 \quad \cdots\cdots ③$$
$$\frac{\partial L}{\partial \delta} = 2\zeta\delta = 0 \quad \cdots\cdots ④$$

① $\rightarrow$ $2x_1 - \zeta = -2$

② $\rightarrow$ $2x_2^+ - 2x_2^- = 2$

③ $\rightarrow$ $x_1 = \delta^2$

④ $\rightarrow$ $2\zeta\delta^2 = 0$

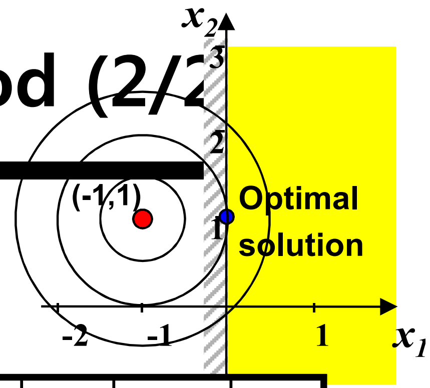
$\zeta x_1 = 0 \cdots\cdots$ ⑤

Multiply the both side of equation ④ by δ and substitute the equation ③ into that.
We eliminate one variable δ and one equation.

1st stage: Find the solution satisfying the equation ①, ②, and ⑤.

2nd stage: Check whether the solution obtained in the 1st stage satisfies the nonlinear equation ⑤ or not.

Solving the Problem by Using the Simplex Method (2/2)



Minimize $f(x) = x_1^2 + x_2^2 + 2x_1 - 2x_2$: Quadratic objective function

Subject to $x_1 \geq 0$: Linearized inequality constraint

$$2x_1 - \zeta = -2$$
$$2x_2^+ - 2x_2^- = 2$$

The right hand sides of the equations have to be nonnegative.

$$-2x_1 + \zeta = 2$$
$$2x_2^+ - 2x_2^- = 2$$

$$-2x_1 + \zeta = 2$$
$$2x_2^+ - 2x_2^- = 2$$

Since the constraints are the equality constraints, introduce the artificial variables.

$$-2x_1 + \zeta + y_1 = 2$$
$$2x_2^+ - 2x_2^- + y_2 = 2$$

The artificial variables have to be zero.
Since the artificial objective function is the sum of all the artificial variables, its minimum value is clearly zero.

$$-2x_1 + 2x_2^+ - 2x_2^- + \zeta + y_1 + y_2 = 4$$
$$2x_1 - 2x_2^+ + 2x_2^- - \zeta = \underbrace{\boxed{y_1 + y_2}}_{w} - 4$$

Change the variables as $x_1 = X_1, x_2^+ = X_2, x_2^- = X_3, \zeta = X_4, y_1 = Y_1, y_2 = Y_2$ and express these as the Matrix from.

Basic variable	X1	X2	X3	X4	Y1	Y2	bi	bi/ai
Y1	**-2**	0	0	1	1	0	2	-
Y2	0	2	-2	0	0	1	2	1
A. Obj.	**2**	-2	2	-1	0	0	w-4	-

Basic variable	X1	X2	X3	X4	Y1	Y2	bi	bi/ai
Y1	**-2**	0	0	1	1	0	2	-
Y2	0	2	-2	0	0	1	2	1
A. Obj.	**2**	-2	2	-1	0	0	w-4	-

Basic variable	X1	X2	X3	X4	Y1	Y2	bi	bi/ai
Y1	**-2**	0	0	1	1	0	2	**2**
X2	0	1	-1	0	0	1/2	1	-
A. Obj.	**2**	0	0	-1	0	1	w-2	-

Basic variable	X1	X2	X3	X4	Y1	Y2	bi	bi/ai
X4	**-2**	0	0	1	1	0	2	-
X2	0	1	-1	0	0	1/2	1	-
A. Obj.	**0**	0	0	0	1	1	w-0	-

$$X_1 = 0, X_2 = 1, X_3 = 0, X_4 = 2, Y_1 = 0, Y_2 = 0$$
$$x_1 = 0, x_2^+ = 1, x_2^- = 0, \zeta = 2, y_1 = 0, y_2 = 0$$
$$x_2 = x_2^+ - x_2^- = 1 - 0 = 1$$
$$\zeta x_1 = 2 \cdot 0 = 0 \cdots\cdots ⑤$$

Since the equation ⑤ is satisfied, this is the solution of this problem.

[Reference] Solution of the Problem Having the Design Variables whose Sign is Unrestricted (1/2)

Matrix Form

Number of design variable $2n+2m+2p$

Number of equation $n+2m+p$

$$\underbrace{\begin{bmatrix} \mathbf{H}_{(n\times n)} & -\mathbf{H}_{(n\times n)} & \mathbf{A}_{(n\times m)} & \mathbf{0}_{(n\times m)} & \mathbf{N}_{(n\times p)} & -\mathbf{N}_{(n\times p)} \\ \mathbf{A}^T_{(m\times n)} & -\mathbf{A}^T_{(m\times n)} & \mathbf{0}_{(m\times m)} & \mathbf{I}_{(m\times m)} & \mathbf{0}_{(m\times p)} & \mathbf{0}_{(m\times p)} \\ \mathbf{N}^T_{(p\times n)} & -\mathbf{N}^T_{(p\times n)} & \mathbf{0}_{(p\times m)} & \mathbf{0}_{(p\times m)} & \mathbf{0}_{(p\times p)} & \mathbf{0}_{(p\times p)} \end{bmatrix}}_{=\mathbf{B}_{((n+m+p)\times(2n+2m+2p))}} \underbrace{\begin{bmatrix} \mathbf{d}^+_{(n\times 1)} \\ \mathbf{d}^-_{(n\times 1)} \\ \mathbf{u}_{(m\times 1)} \\ \mathbf{s}'_{(m\times 1)} \\ \mathbf{y}_{(p\times 1)} \\ \mathbf{z}_{(p\times 1)} \end{bmatrix}}_{=\mathbf{X}_{((2n+2m+2p)\times 1)}} = \underbrace{\begin{bmatrix} -\mathbf{c}_{(n\times 1)} \\ \mathbf{b}_{(m\times 1)} \\ \mathbf{e}_{(p\times 1)} \end{bmatrix}}_{=\mathbf{D}_{((n+m+p)\times 1)}}$$

$$u_i s'_i = 0; i = 1\ to\ m$$

$$\mathbf{B}_{((n+m+p)\times(2n+2m+2p))}\mathbf{X}_{((2n+2m+2p)\times 1)} = \mathbf{D}_{((n+m+p)\times 1)}$$

The number of the design variables is the same with that of the equations as $n+2m+p$ in the original problem.

Since the equations $\mathbf{v}_{(p\times1)} = \mathbf{y}_{(p\times1)} - \mathbf{z}_{(p\times1)}$ and $\mathbf{d}_{(n\times1)} = \mathbf{d}^+_{(n\times1)} - \mathbf{d}^-_{(n\times1)}$ are introduced, the number of the design variables is also increased by $n+p$.

The interesting variables v_i and d_i are determined by the equation $\mathbf{v}_{(p\times1)} = \mathbf{y}_{(p\times1)} - \mathbf{z}_{(p\times1)}$.

Example

Equation

$$x+y+z=2$$
$$2x+2y+z=6$$
$$2x+y=5$$

Replace z as $z_1 - z_2$ $\longrightarrow$ $(z_1, z_2 \geq 0)$

$$x+y+z_1-z_2=2$$
$$2x+2y+z_1-z_2=6$$
$$2x+y=5$$

Solution $x=1, y=3, z=-2$ $\qquad$ $x=1, y=3, z_1=0, z_2=2$

After replacing the variable, this problem becomes the indeterminate equation. The value of z_1-z_2 is always -2.

[Reference] Solution of the Problem Having the Design Variables whose Sign is Unrestricted (2/2)

Example Equation

$$x + y + z = 5$$
$$2x + 3y + z = 11$$
$$xz = 0$$

Replace z as $z_1 - z_2$ $\longrightarrow$ $(z_1, z_2 \ge 0)$

$$x + y + z_1 - z_2 = 5$$
$$2x + 3y + z_1 - z_2 = 11$$
$$xz = 0$$

Case #1

Introduce the artificial variables for using the Simplex method

$$x + y + z + Y_1 = 5$$
$$2x + 3y + z + Y_2 = 11$$

$\longleftarrow$

- **Number of the design variables: 5**
- **Number of the linear independent equation: 2**

Solve this problem by assuming the three design variables as zero.
Stop the Simplex method, if the sum of all the artificial variables(Y1+Y2) zero.

	(x,	y,	z,	Y1,	Y2)
①	(4,	1,	0,	0,	0)
②	(6,	0,	-1,	0,	0)
③	(0,	3,	2,	0,	0)

Between the solution ① and ③ obtained by using the Simplex method, the final solution has to satisfy the equation $xz = 0$.

If the solution whose value of $z(=z_1-z_2)$ is negative is excluded, the solution of the Case #1 is the same with that of the Case #2.

Case #2

Introduce the artificial variables for using the Simplex method

$$x + y + z_1 - z_2 + Y_1 = 5$$
$$2x + 3y + z_1 - z_2 + Y_2 = 11$$

$\longleftarrow$

- **Number of the design variables: 6**
- **Number of the linear independent equation: 2**

Solve this problem by assuming the four design variables as zero.
Stop the Simplex method, if the sum of all the artificial variables(Y1+Y2) zero.

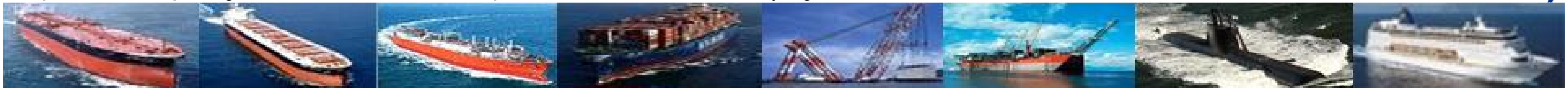
			$z = z_1 - z_2$			
	(x,	y,	z_1,	z_2,	Y1,	Y2)
①	(4,	1,	0,	0,	0,	0)
②	(6,	0,	0,	1,	0,	0)
③	(6,	0,	-1,	0,	0,	0)
④	(0,	0,	-,	-,	0,	0)
⑤	(0,	3,	0,	-2,	0,	0)
⑥	(0,	3,	2,	0,	0,	0)

Among the solution ①, ②, and ⑥ obtained by using the Simplex method, the final solution has to satisfy the equation $xz = 0$.

[Appendix] Another Method for Sequential Quadratic Programming(SQP)

Use of the Descent Condition Method for SQP Instead of the Golden Section Search Method

SYstem Design Laboratory



Use of the Descent Condition Method for SQP Instead of the Golden Section Search Method (1/4)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

(vi) Step 6: By using the one dimensional search method(ex. Descent condition method), calculate the step size to minimize the descent function along the search direction($\mathbf{d}^{(0)}$) and determine the improved design point.

$$\Phi(\mathbf{x}^{(k,j)}) = f(\mathbf{x}^{(k,j)}) + R_k \cdot V(\mathbf{x}^{(k,j)})$$

$$= x_1^2 + x_2^2 - 3x_1x_2 + 10 \cdot V(\mathbf{x}^{(k,j)})$$

$$V(\mathbf{x}^{(k,j)}) = \max\{0, g_1(\mathbf{x}^{(k,j)}), g_2(\mathbf{x}^{(k,j)}), g_3(\mathbf{x}^{(k,j)})\}, \ (k=0)$$

$$\mathbf{x}^{(k,j)} = \mathbf{x}^{(k)} + t_{(k,j)}\mathbf{d}^{(k)}$$

$\mathbf{x}^{(k,j)}$ ⇒ **k iteration of CSD algorithm**
j iteration of one dimensional search method

$$\mathbf{d}_0 = (1,1) \qquad \mathbf{x}^{(0,0)} = (1,1),$$

$$\Phi\left(\mathbf{x}^{(1,j)}\right) = \Phi\left(\mathbf{x}^{(0)} + t_{(0,j)}\mathbf{d}^{(0)}\right) = \boxed{\Phi\left(t_{(0,j)}\right)}$$

$$\boxed{\Phi\left(\mathbf{x}^{(0,0)}\right) - t_{(0,j)}\beta_k} \quad \text{where, } \beta_k = \gamma\left\|\mathbf{d}^{(k)}\right\|^2, \ (\gamma = 0.5, \text{Defined by user})$$

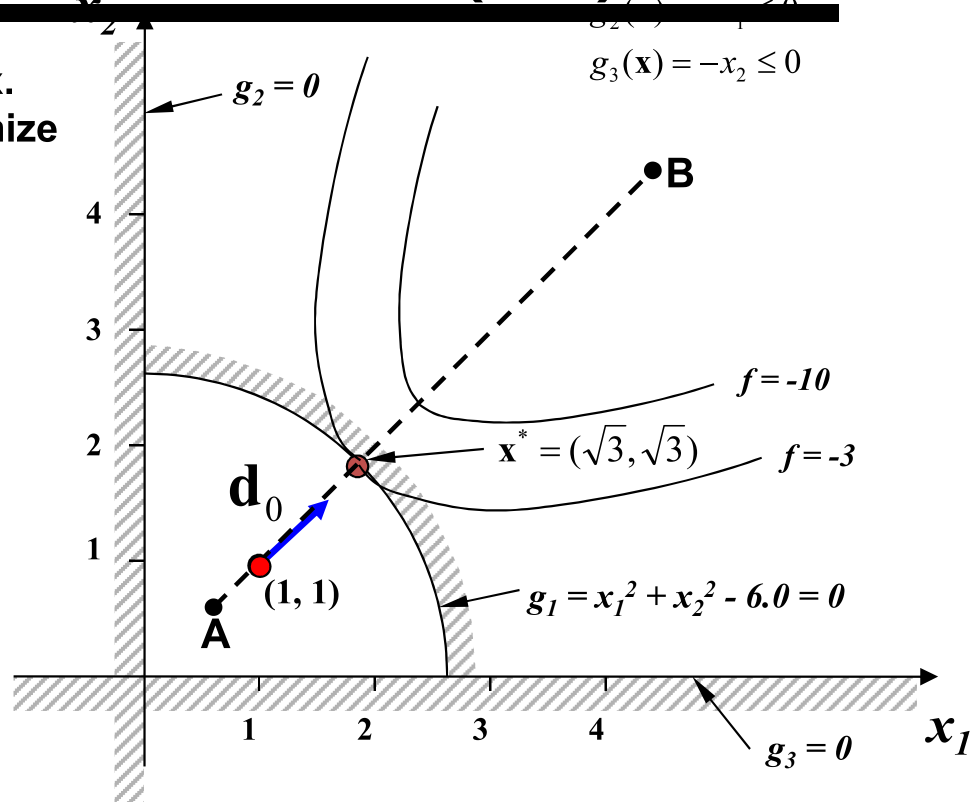
By reducing the value of *t* from 1 to a half, find the point to satisfy the following equation.

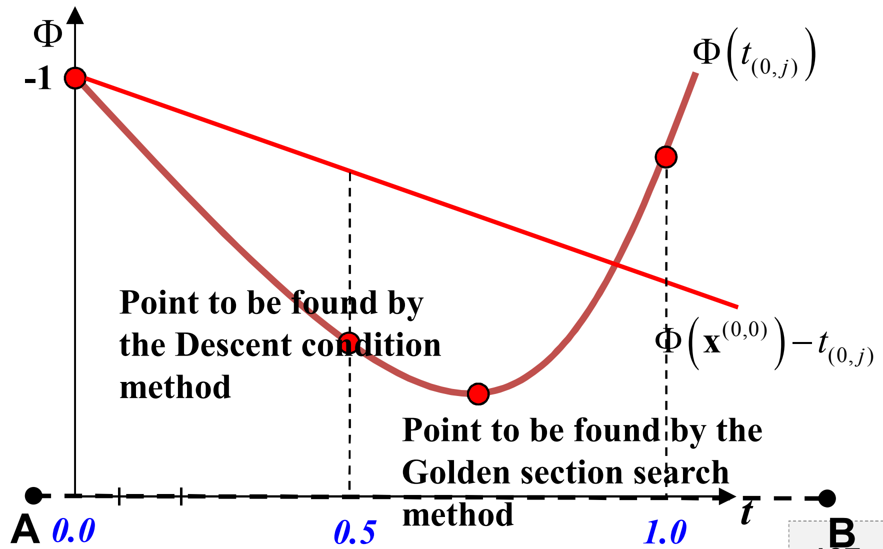
$$\boxed{\Phi\left(t_{(0,j)}\right)} \le \boxed{\Phi\left(\mathbf{x}^{(0,0)}\right) - t_{(0,j)}\beta_k}$$

$$\Phi\left(t_{(0,j)}\right) \le -1 - t_{(0,j)} \qquad \text{where, } \beta_k = \gamma\left\|\mathbf{d}^{(k)}\right\|^2 = 0.5(1^2 + 1^2) = 1$$

$$\Phi\left(\mathbf{x}^{(0,0)}\right) = f\left(\mathbf{x}^{(0,0)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,0)}\right) = -1 + 10 \times 0 = -1$$

$$V(\mathbf{x}^{(0,0)}) = \max\{0, -\tfrac{2}{3}, -1, -1\} = 0$$





Use of the Descent Condition Method for SQP Instead of the Golden Section Search Method (2/4)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

(vi) Step 6: By using the one dimensional search method(ex. Descent condition method), calculate the step size to minimize the descent function along the search direction($d^{(0)}$) and determine the improved design point.

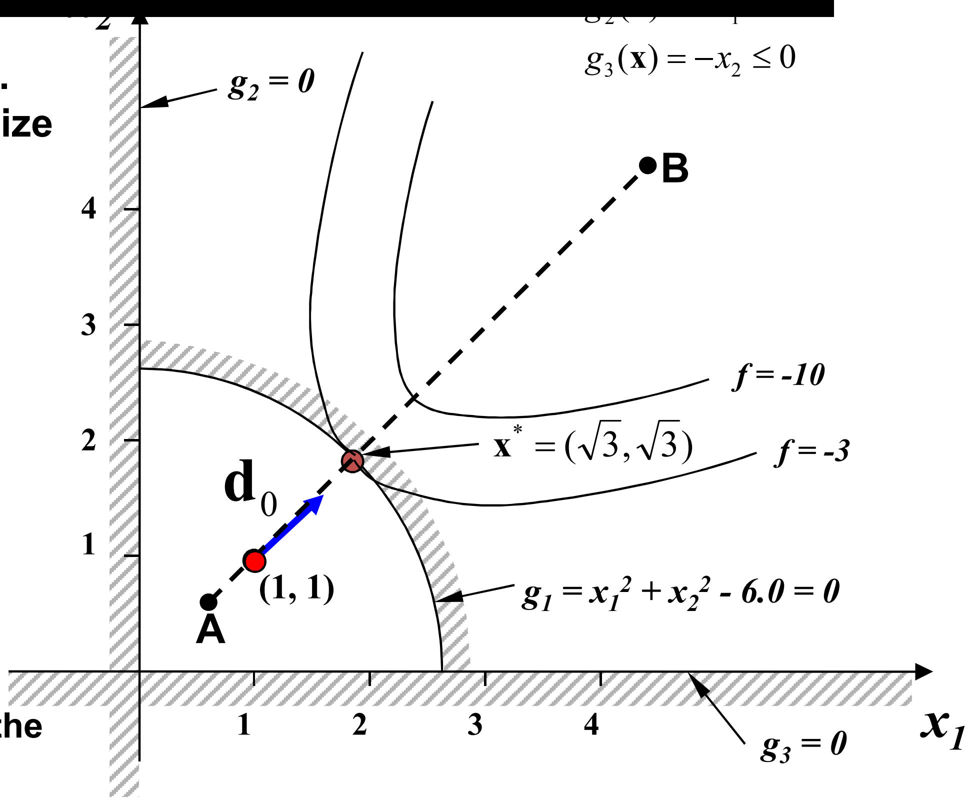
$$\Phi(\mathbf{x}^{(k,j)}) = f(\mathbf{x}^{(k,j)}) + R_k \cdot V(\mathbf{x}^{(k,j)})$$

$$= x_1^2 + x_2^2 - 3x_1x_2 + 10 \cdot V(\mathbf{x}^{(k,j)})$$

$$V(\mathbf{x}^{(k,j)}) = \max\{0, g_1(\mathbf{x}^{(k,j)}), g_2(\mathbf{x}^{(k,j)}), g_3(\mathbf{x}^{(k,j)})\}, \ (k=0)$$

$$\mathbf{x}^{(k,j)} = \mathbf{x}^{(k)} + t_{(k,j)}\mathbf{d}^{(k)}$$

$\mathbf{x}^{(k,j)}$ ⇨ k iteration of CSD algorithm
j iteration of one dimensional search method



By reducing the value of t from 1 to a half, find the point to satisfy the following equation.

$$\Phi\left(t_{(0,j)}\right) \le -1 - t_{(0,j)} \qquad k = 0,\ j = 0$$

When $t_{(0,j)} = 1$

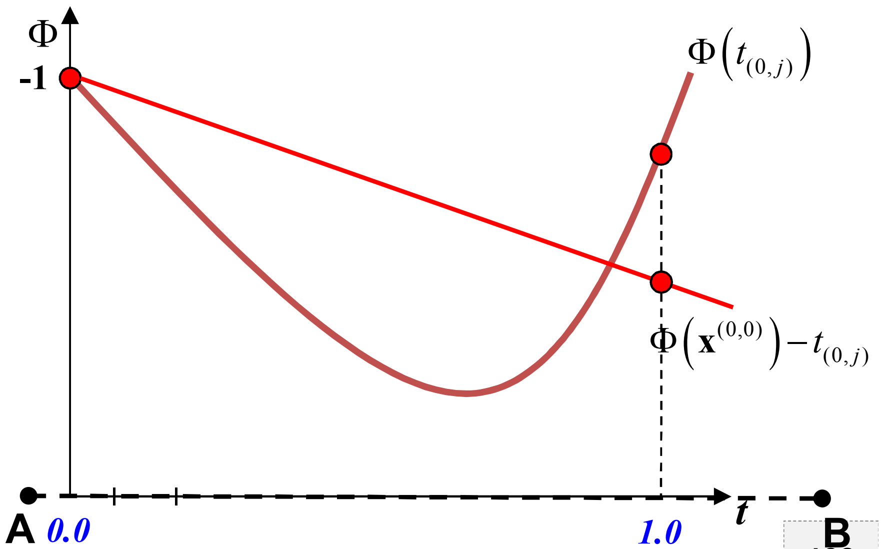
$$\mathbf{x}^{(0,j)} = \mathbf{x}^{(0)} + t_{(0,j)} \cdot d^{(0)} = (1,1) + 1 \cdot (1,1) = (2,2)$$

$$\Phi\left(t_{(0,j)}\right) = f(2,2) + R_0 \cdot V\left(\mathbf{x}^{(0,j)}\right) = -4 + 10 \times 0.333 = -0.667$$

$$where,\ V(\mathbf{x}^{(0,j)}) = \max\{0, \tfrac{1}{3}, -2, -2\} = 0.333$$

$$-1 - t_{(0,j)} = -1 - 1 = -2$$

If $\Phi\left(t_{(0,j)}\right) \le -1 - t_{(0,j)}$ is not satisfied, t is reduced to 0.5.



Use of the Descent Condition Method for SQP Instead of the Golden Section Search Method (3/4)

$$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$
$$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_2 \le 0$$

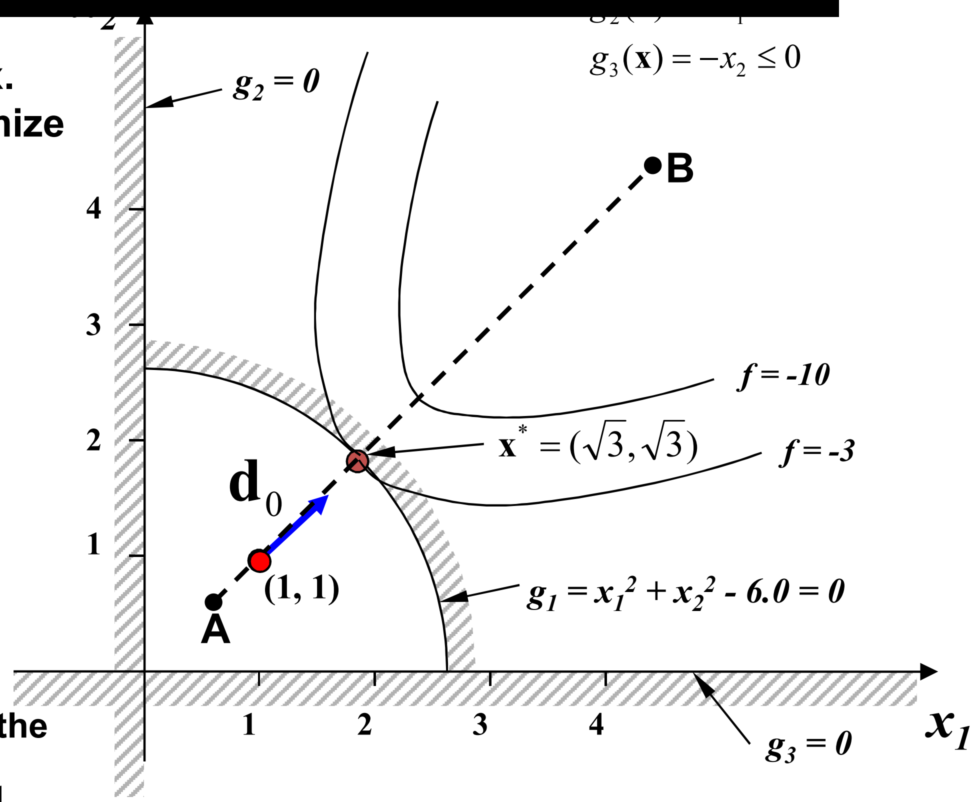
(vi) Step 6: By using the one dimensional search method(ex. Descent Condition method), calculate the step size to minimize the descent function along the search direction($\mathbf{d}^{(0)}$) and determine the improved design point.

$$\Phi(\mathbf{x}^{(k,j)}) = f(\mathbf{x}^{(k,j)}) + R_k \cdot V(\mathbf{x}^{(k,j)})$$
$$= x_1^2 + x_2^2 - 3x_1x_2 + 10 \cdot V(\mathbf{x}^{(k,j)})$$

$$V(\mathbf{x}^{(k,j)}) = \max\{0, g_1(\mathbf{x}^{(k,j)}), g_2(\mathbf{x}^{(k,j)}), g_3(\mathbf{x}^{(k,j)})\}, \ (k=0)$$

$$\mathbf{x}^{(k,j)} = \mathbf{x}^{(k)} + t_{(k,j)}\mathbf{d}^{(k)}$$

$\mathbf{x}^{(k,j)}$ ➡ **k iteration of CSD algorithm**
j iteration of one dimensional search method



By reducing the value of t from 1 to a half, find the point to satisfy the following equation.

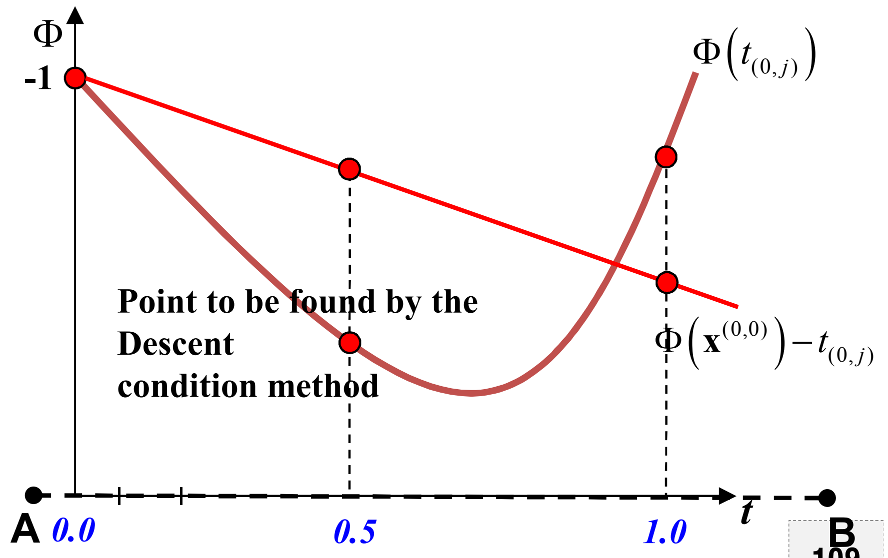
$$-2.25 = \Phi(t_{(0,j)}) \le -1 - t_{(0,j)} = -1.5 \qquad k = 0, \ j = 1$$

When $t_{(0,j)} = 0.5$

$$\mathbf{x}^{(0,j)} = \mathbf{x}^{(0)} + t_{(0,j)} \cdot d^{(0)} = (1,1) + 0.5 \cdot (1,1) = (1.5, 1.5)$$

$$\Phi(t_{(0,j)}) = f(1.5, 1.5) + R_0 \cdot V(\mathbf{x}^{(0,j)}) = -2.25 + 10 \times 0 = -2.25$$
$$where, \ V(\mathbf{x}^{(0,j)}) = \max\{0, -\tfrac{2}{8}, -1.5, -1.5\} = 0$$

$$-1 - t_{(0,j)} = -1 - 0.5 = -1.5$$



Since $\Phi(t_{(0,j)}) \le -1 - t_{(0,j)}$ is satisfied, (1.5, 1.5) is the next design point.

Use of the Descent Condition Method for SQP Instead of the Golden Section Search Method (4/4)

$$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$
$$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_2 \le 0$$

The step size obtained by Descent condition is different from the step size obtained by Golden section search method.

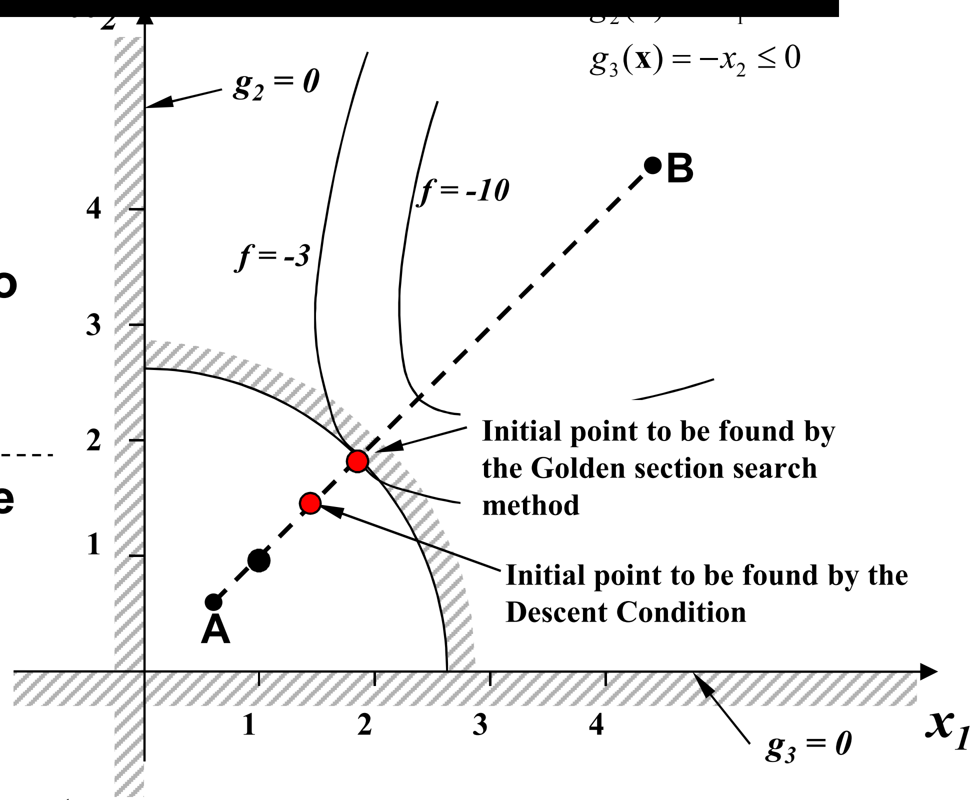
Since the improved design points obtained by two method are different, the number of iteration of defining the QP problem is changed.

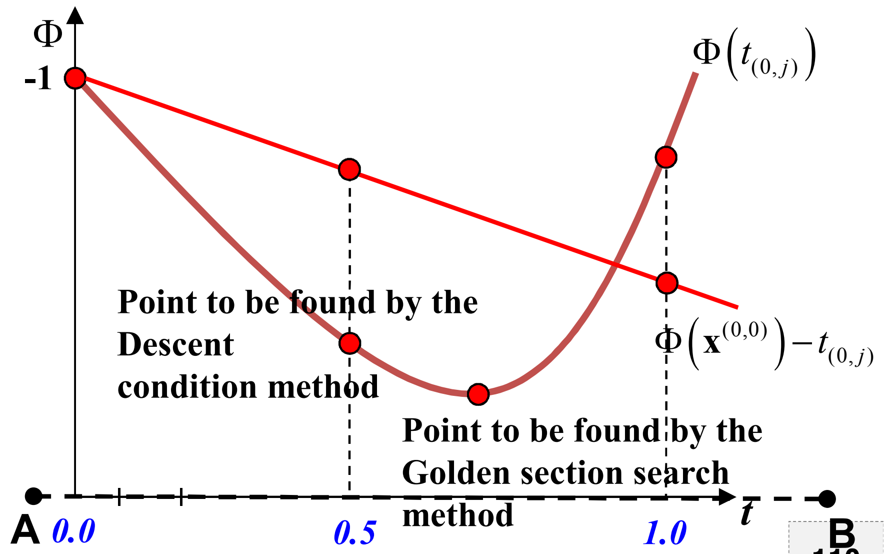
If we use the Golden section search method in the right example,

- The number of iteration of the one dimensional search in the first iteration of CSD is 62.
- By defining the QP problem two times, we can find the optimal design point.
  + The step size obtained by one dimensional search direction is exact size.

If we use the Descent condition method in the right example,

- The number of iteration of the one dimensional search in the first iteration of CSD is 1.
- Since the step size obtained by one dimensional search direction is not exact size, the QP problem should be defined in 20 times to find the optimal solution.



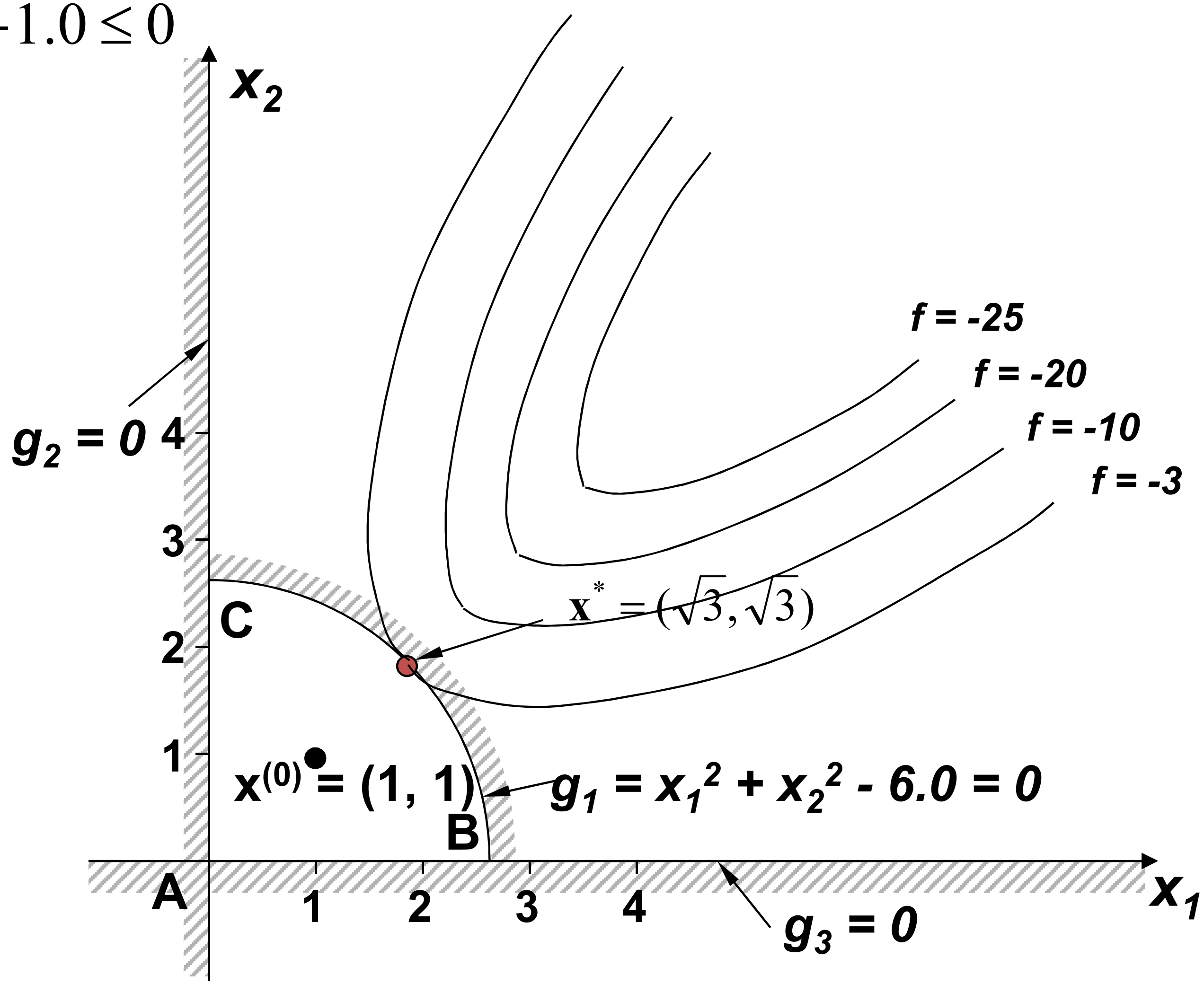


Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_1 \le 0$



Optimal Solution:

$$\mathbf{x}^* = (\sqrt{3}, \sqrt{3}), f(\mathbf{x}^*) = -3$$

Comparison between the Golden Section Search Method and Descent Condition Method - Example #1 (2/2)

Minimize: $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to:

$$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_1 \le 0$$

Solution: $\mathbf{x} = (\sqrt{3},\ \sqrt{3}),\ f(\mathbf{x}) = -3$

Initial Value	Method		Iteration of defining the QP problem	Iteration of one dimensional search method	Local Optimum Point	Optimum Value
(1, 1)	Descent condition method	r = 0.0	19	19	(1.732, 1.732)	-3.0
		r = 0.1	19	19	(1.732, 1.732)	-3.0
		r = 0.5	19	19	(1.732, 1.732)	-3.0
		r = 0.9	19	19	(1.732, 1.732)	-3.0
	Golden section search method		1	62	(1.732, 1.732)	-3.0
(0.1, 0.1)	Descent condition method	r = 0.0	35	85	(1.732, 1.732)	-3.0
		r = 0.1	36	52	(1.732, 1.732)	-3.0
		r = 0.5	29	44	(1.732, 1.732)	-3.0
		r = 0.9	44	124	(1.732, 1.732)	-3.0
	Golden section search method		1	38	(1.732, 1.732)	-3.0
(1.5, 1.5)	Descent condition method	r = 0.0	18	18	(1.732, 1.732)	-3.0
		r = 0.1	18	18	(1.732, 1.732)	-3.0
		r = 0.5	18	18	(1.732, 1.732)	-3.0
		r = 0.9	18	18	(1.732, 1.732)	-3.0
	Golden section search method		2	68	(1.732, 1.732)	-3.0

Comparison between the Golden Section Search Method and Descent Condition Method - Example #2

Minimize: $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$

Solution: $\mathbf{x} = \left(-1.0,\ 1.5\right),\ f(\mathbf{x}) = -1.25$

Initial Value	Method		Iteration of defining the QP problem	Iteration of one dimensional search method	Local Optimum Point	Optimum Value
(0, 0)	Descent condition method	r = 0.0	39	59	(-1.0, 1.5)	-1.25
		r = 0.1	38	58	(-1.0, 1.5)	-1.25
		r = 0.5	41	67	(-1.0, 1.5)	-1.25
		r = 0.9	60	127	(-1.0, 1.5)	-1.25
	Golden section search method		17	329	(-1.0, 1.5)	-1.25
(1, 1)	Descent condition method	r = 0.0	40	63	(-1.0, 1.5)	-1.25
		r = 0.1	40	63	(-1.0, 1.5)	-1.25
		r = 0.5	40	66	(-1.0, 1.5)	-1.25
		r = 0.9	72	194	(-1.0, 1.5)	-1.25
	Golden section search method		17	282	(-1.0, 1.5)	-1.25
(-1, 2)	Descent condition method	r = 0.0	35	55	(-1.0, 1.5)	-1.25
		r = 0.1	35	55	(-1.0, 1.5)	-1.25
		r = 0.5	37	61	(-1.0, 1.5)	-1.25
		r = 0.9	66	177	(-1.0, 1.5)	-1.25
	Golden section search method		18	299	(-1.0, 1.5)	-1.25

Minimize

$$f(x_1, x_2) = -\left[25 - (x_1 - 5)^2 - (x_2 - 5)^2\right]$$

Subject to

$$g_1(x_1, x_2) = -32 + 4x_1 + x_2{}^2 \le 0$$

$$g_2(x_1, x_2) = -x_1 \le 0$$

$$g_3(x_1, x_2) = x_1 \le 10$$

$$g_4(x_1, x_2) = -x_2 \le 0$$

$$g_5(x_1, x_2) = x_2 \le 10$$

Solution

$$x_1{}^* = 4.374, x_2{}^* = 3.808, f(x_1{}^*, x_2{}^*) = -4.815$$

Comparison between the Golden Section Search Method and Descent Condition Method - Example #3 (2/2)

Minimize: $f(x_1, x_2) = -\left[25 - (x_1 - 5)^2 - (x_2 - 5)^2\right]$

Subject to:
$$g_1(x_1, x_2) = -32 + 4x_1 + x_2^{\ 2} \le 0$$
$$g_2(x_1, x_2) = -x_1 \le 0$$
$$g_3(x_1, x_2) = x_1 \le 10$$
$$g_4(x_1, x_2) = -x_2 \le 0$$
$$g_5(x_1, x_2) = x_2 \le 10$$

Solution: $\mathbf{x} = (4.374,\ 3.808),\ f(\mathbf{x}) = -4.815$

Initial Value	Method		Iteration of defining the QP problem	Iteration of one dimensional search method	Local Optimum Point	Optimum Value
(0, 0)	Descent condition method	r = 0.0	22	23	(4.374, 3.808)	-23.188
		r = 0.1	22	23	(4.374, 3.808)	-23.188
		r = 0.5	22	23	(4.374, 3.808)	-23.188
		r = 0.9	22	24	(4.374, 3.808)	-23.188
	Golden section search method		590	13,509	(4.374, 3.808)	-23.188
(7, 1)	Descent condition method	r = 0.0	15	22	(4.374, 3.808)	-23.188
		r = 0.1	15	22	(4.374, 3.808)	-23.188
		r = 0.5	15	22	(4.374, 3.808)	-23.188
		r = 0.9	24	45	(4.374, 3.808)	-23.188
	Golden section search method		1143	26,804	(4.374, 3.808)	-23.188
(-3, -10)	Descent condition method	r = 0.0	19	35	(4.374, 3.808)	-23.188
		r = 0.1	19	35	(4.374, 3.808)	-23.188
		r = 0.5	19	35	(4.374, 3.808)	-23.188
		r = 0.9	28	61	(4.374, 3.808)	-23.188
	Golden section search method		884	20,005	(4.374, 3.808)	-23.188

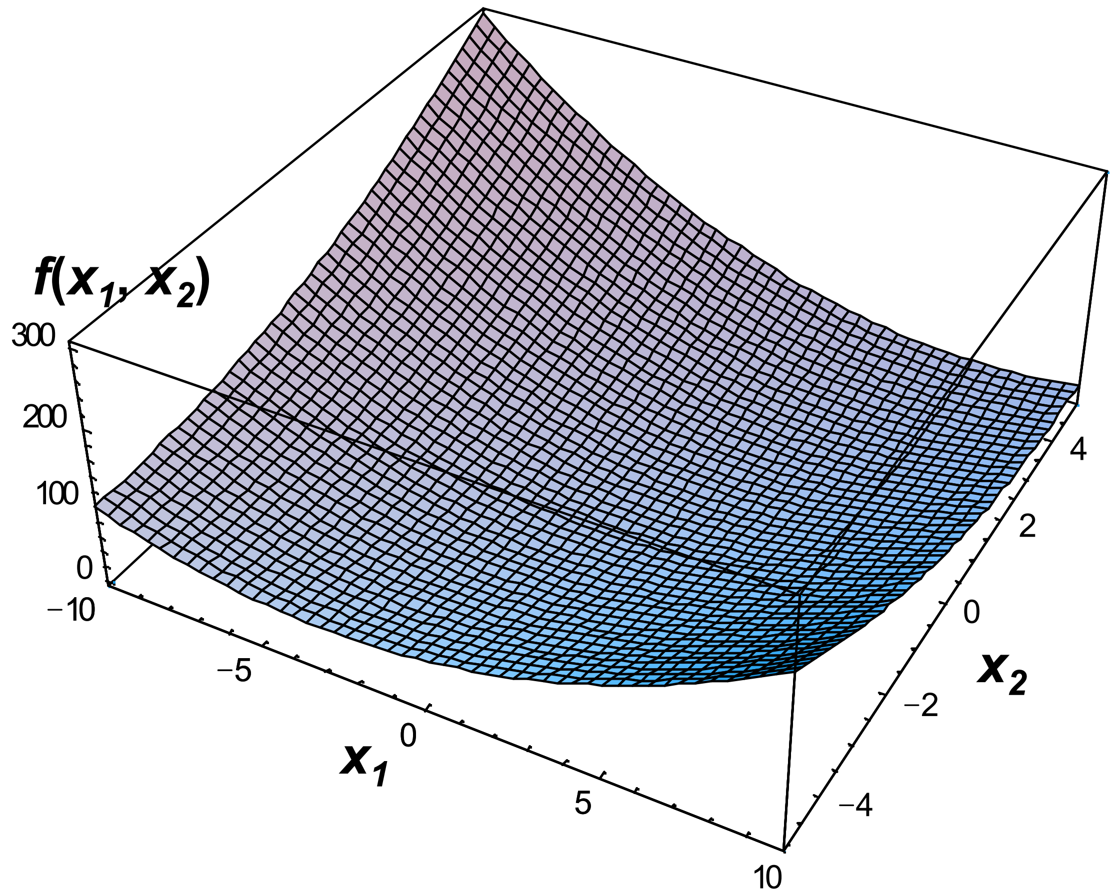
Comparison between the Golden Section Search Method and Descent Condition Method - Example #4 (1/2)

Find $x_1(=B/T), x_2(=1/C_B)$

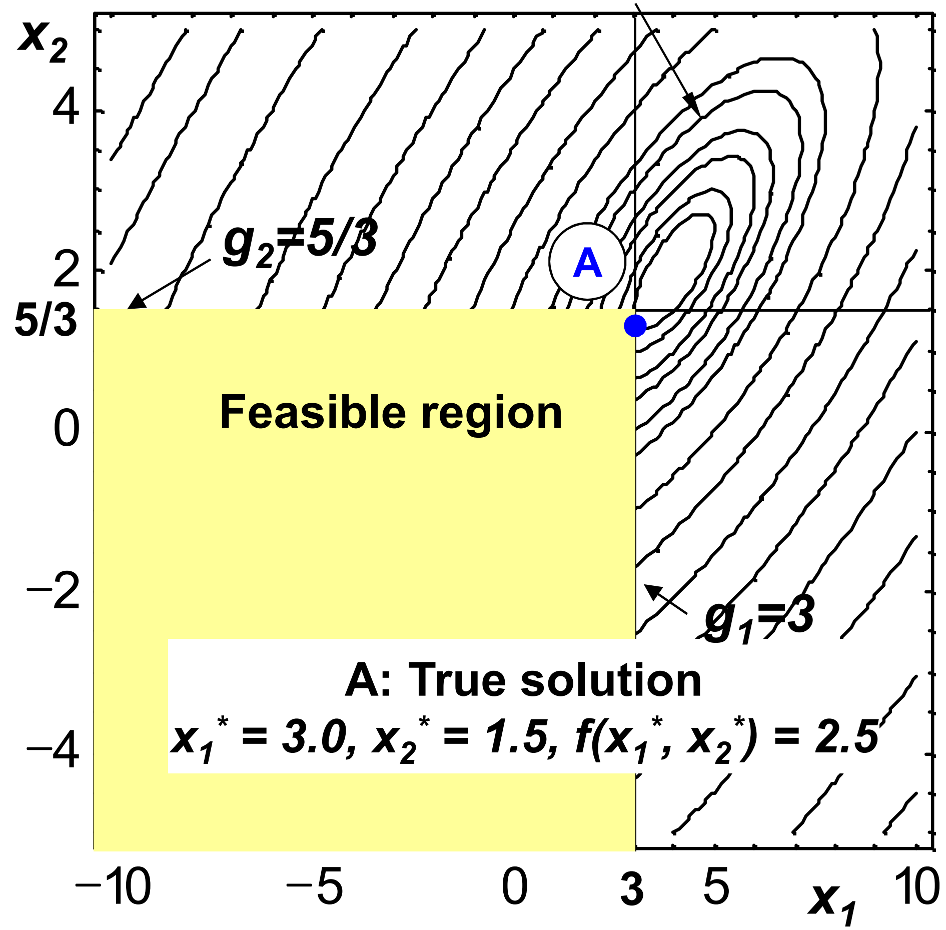
Minimize $f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_1 - 2x_1x_2 + 10$

Subject to $g_1(x_1, x_2) = x_1 - 3 \le 0$ ➡ **Optimization problem having two unknown variables and two inequality constraints**

$g_2(x_1, x_2) = x_2 - 5/3 \le 0$



Contour line(f = const.) of objective function



Comparison between the Golden Section Search Method and Descent Condition Method - Example #4 (2/2)

Minimize: $f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_1 - 2x_1x_2 + 10$

Subject to: $g_1(x_1, x_2) = x_1 - 3 \le 0$

$g_2(x_1, x_2) = x_2 - 5/3 \le 0$

Solution: $\mathbf{x} = (3.0,\ 1.5),\ f(\mathbf{x}) = 2.5$

Initial Value	Method		Iteration of defining the QP problem	Iteration of one dimensional search method	Local Optimum Point	Optimum Value
(0, 0)	Descent condition method	r = 0.0	22	24	(3.0, 1.5)	2.5
		r = 0.1	22	24	(3.0, 1.5)	2.5
		r = 0.5	22	26	(3.0, 1.5)	2.5
		r = 0.9	24	33	(3.0, 1.5)	2.5
	Golden section search method		13	203	(3.0, 1.5)	2.5
(2, 1)	Descent condition method	r = 0.0	19	20	(3.0, 1.5)	2.5
		r = 0.1	19	20	(3.0, 1.5)	2.5
		r = 0.5	19	20	(3.0, 1.5)	2.5
		r = 0.9	19	20	(3.0, 1.5)	2.5
	Golden section search method		4	89	(3.0, 1.5)	2.5
(-3, -5)	Descent condition method	r = 0.0	26	52	(3.0, 1.5)	2.5
		r = 0.1	25	28	(3.0, 1.5)	2.5
		r = 0.5	25	28	(3.0, 1.5)	2.5
		r = 0.9	25	30	(3.0, 1.5)	2.5
	Golden section search method		9	255	(3.0, 1.5)	2.5

Comparison between the Golden Section Search Method and Descent Condition Method - Example #5 (1/2)

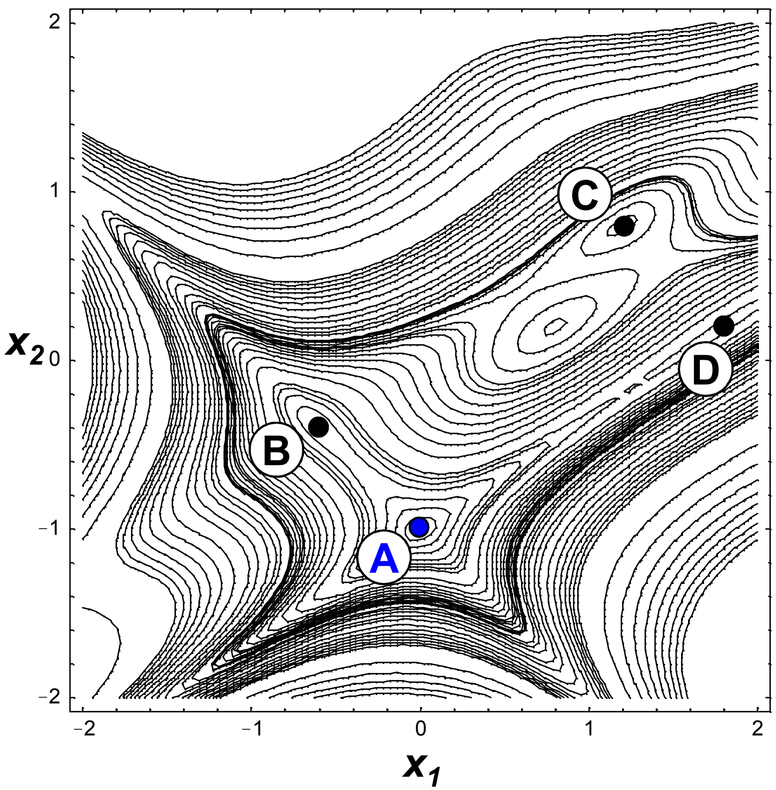
Goldstein-Price Function

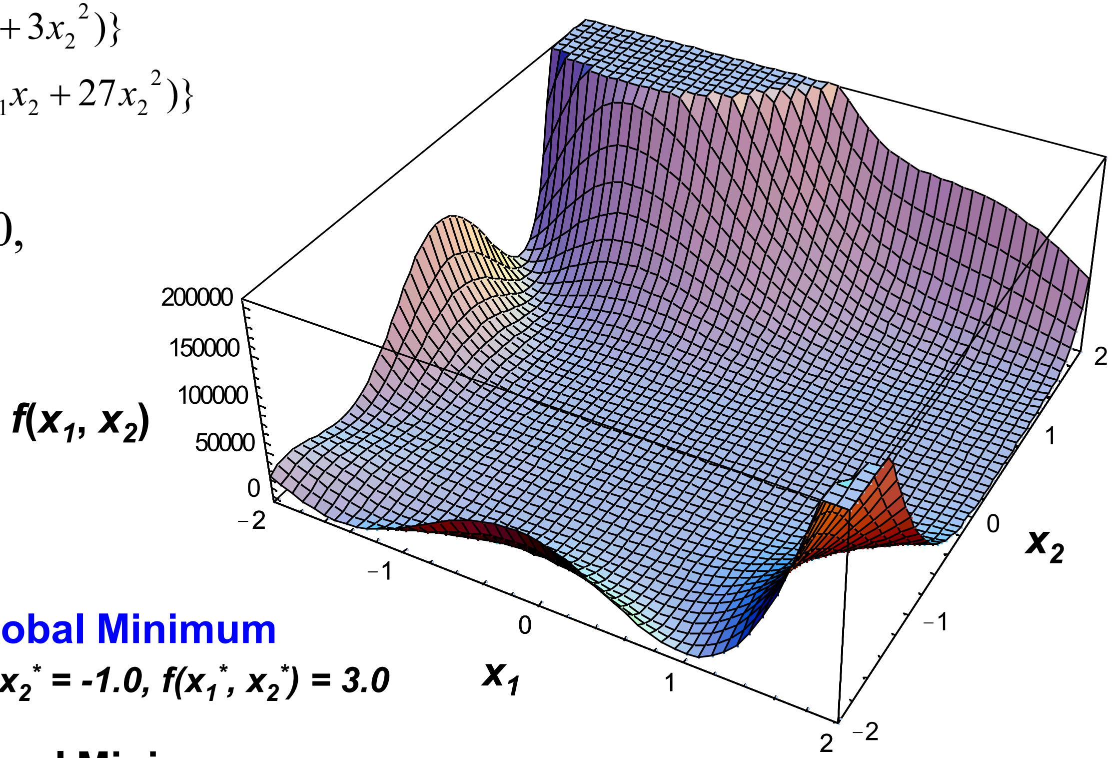
Minimize

$$f(x_1,x_2)=\{1+(x_1+x_2+1)^2\cdot(19-14x_1+3x_1^{\ 2}-14x_2+6x_1x_2+3x_2^{\ 2})\}$$
$$\cdot\{30+(2x_1-3x_2)^2\cdot(18-32x_1+12x_1^{\ 2}+48x_2-36x_1x_2+27x_2^{\ 2})\}$$

Subject to

$$g_1(x_1,x_2)=-2-x_1\le 0,\ g_2(x_1,x_2)=-2-x_2\le 0,$$
$$g_3(x_1,x_2)=x_1-2\le 0,\ g_4(x_1,x_2)=x_2-2\le 0$$





A : Global Minimum

$x_1^* = 0.0,\ x_2^* = -1.0,\ f(x_1^*, x_2^*) = 3.0$

B : Local Minimum

$x_1^* = -0.6,\ x_2^* = -0.4,\ f(x_1^*, x_2^*) = 30.0$

C : Local Minimum

$x_1^* = 1.2,\ x_2^* = 0.8,\ f(x_1^*, x_2^*) = 840.0$

D : Local Minimum

$x_1^* = 1.8,\ x_2^* = 0.2,\ f(x_1^*, x_2^*) = 84.0$

Comparison between the Golden Section Search Method and Descent Condition Method - Example #5 (2/2)

Minimize:

$$f(x_1,x_2)=\{1+(x_1+x_2+1)^2$$
$$\times(19-14x_1+3x_1^2-14x_2+6x_1x_2+3x_2^2)\}$$
$$\times\{30+(2x_1-3x_2)^2$$
$$\times(18-32x_1+12x_1^2+48x_2-36x_1x_2+27x_2^2)\}$$

Subject to:

$$g_1(x_1,x_2)=-2-x_1\le 0,\ g_2(x_1,x_2)=-2-x_2\le 0,$$
$$g_3(x_1,x_2)=x_1-2\le 0,\ g_4(x_1,x_2)=x_2-2\le 0$$

In this example, since there are some local minimum design points, the optimal solution to be obtained is changed depending on the initial design point. So, the calculating the optimal solutions by assuming the initial design point in many times and comparing the results are needed.

Initial Value	Method		Iteration of defining the QP problem	Iteration of one dimensional search method	Local Optimum Point	Optimum Value
(0, 0)	Descent condition method	r = 0.0	30	302	(-0.6, -0.4)	30.0
		r = 0.1	26	258	(-0.6, -0.4)	30.0
		r = 0.5	21	208	(-0.6, -0.4)	30.0
		r = 0.9	62	739	(-0.6, -0.4)	30.0
	Golden section search method		15	467	(-0.6, -0.4)	30.0
(2, 3)	Descent condition method	r = 0.0	77	605	(0.0, -1.0)	3.0
		r = 0.1	31	194	(0.0, -1.0)	3.0
		r = 0.5	28	172	(0.0, -1.0)	3.0
		r = 0.9	56	523	(0.0, -1.0)	3.0
	Golden section search method		13	417	(0.0, -1.0)	3.0
(-5, -5)	Descent condition method	r = 0.0	70	545	(0.0, -1.0)	3.0
		r = 0.1	24	135	(0.0, -1.0)	3.0
		r = 0.5	24	136	(0.0, -1.0)	3.0
		r = 0.9	51	459	(0.0, -1.0)	3.0
	Golden section search method		17	497	(0.0, -1.0)	3.0

Rastrigin's Function

Minimize

$$f(x_1, x_2) = 20 + x_1^2 - 10\cos(2\pi \cdot x_1) + x_2^2 - 10\cos(2\pi \cdot x_2)$$

Subject to

$$g_1(x_1, x_2) = -5.12 - x_1 \le 0$$

$$g_2(x_1, x_2) = -5.12 - x_2 \le 0$$

$$g_3(x_1, x_2) = x_1 - 5.12 \le 0$$

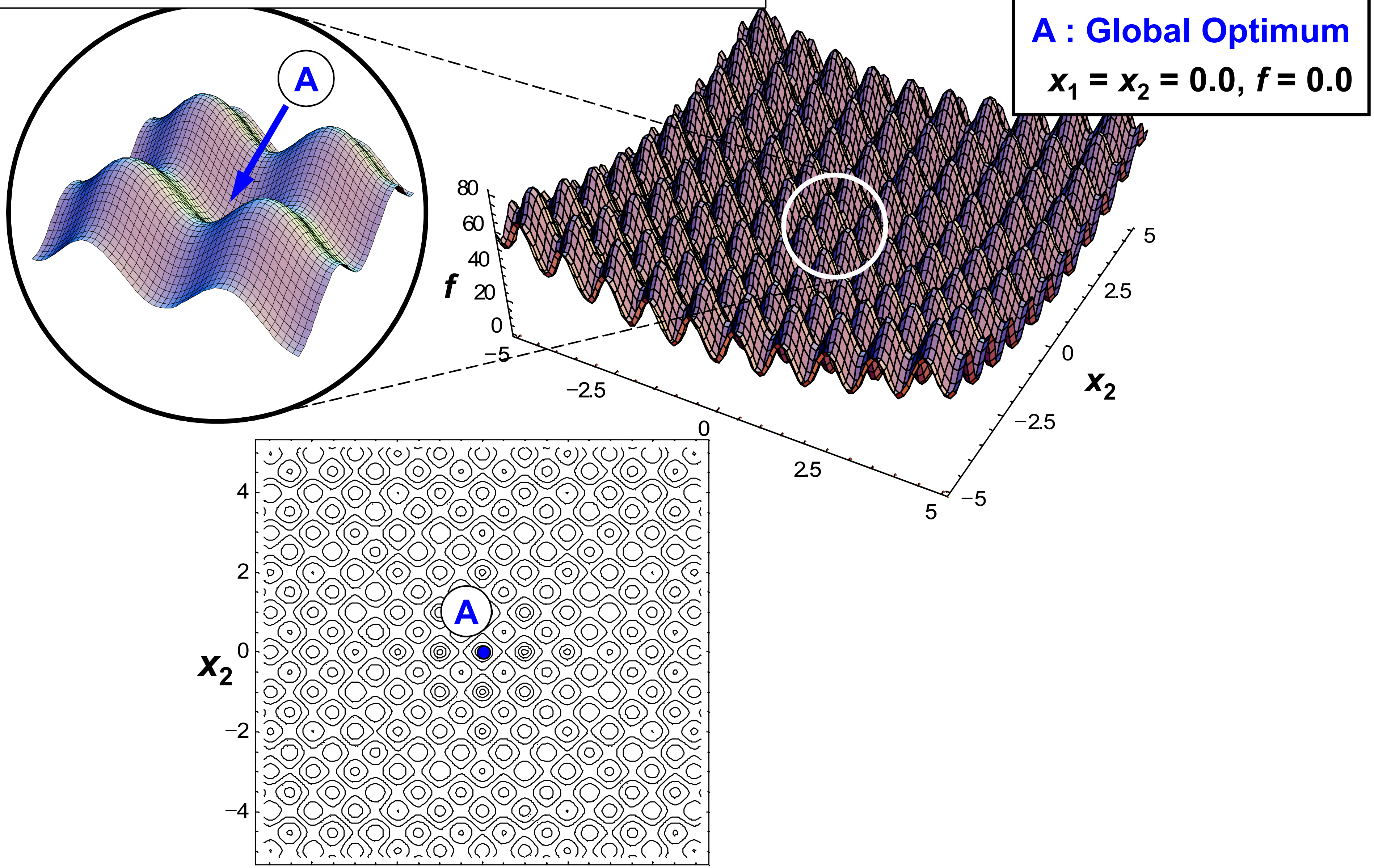
$$g_4(x_1, x_2) = x_2 - 5.12 \le 0$$

Solution

$$x_1^* = 0.0, x_2^* = 0.0, f(x_1^*, x_2^*) = 0.0$$

Comparison between the Golden Section Search Method and Descent Condition Method - Example #6 (2/3)

Global and Local minimum point of the Rastrigins's Function



Comparison between the Golden Section Search Method and Descent Condition Method - Example #6 (3/3)

Minimize:

$$f(x_1,x_2) = 20 + x_1^2 - 10\cos(2\pi \cdot x_1) + x_2^2 - 10\cos(2\pi \cdot x_2)$$

Subject to:

$$g_1(x_1,x_2) = -5.12 - x_1 \le 0$$
$$g_2(x_1,x_2) = -5.12 - x_2 \le 0$$
$$g_3(x_1,x_2) = x_1 - 5.12 \le 0$$
$$g_4(x_1,x_2) = x_2 - 5.12 \le 0$$

In this example, since there are some local minimum design points, the optimal solution to be obtained is changed depending on the initial design point. So, the calculating the optimal solutions by assuming the initial design point in many times and comparing the results are needed.

Initial Value	Method		Iteration of defining the QP problem	Iteration of one dimensional search method	Local Optimum Point	Optimum Value
(0.1, 0.1)	Descent condition method	r = 0.0	18	147	(0.0, 0.0)	0.0
		r = 0.1	18	147	(0.0, 0.0)	0.0
		r = 0.5	9	82	(0.0, 0.0)	0.0
		r = 0.9	39	427	(0.0, 0.0)	0.0
	Golden section search method		1	47	(0.0, 0.0)	0.0
(2.1, 2.1)	Descent condition method	r = 0.0	16	134	(1.990, 1.990)	7.960
		r = 0.1	16	134	(1.990, 1.990)	7.960
		r = 0.5	7	69	(1.990, 1.990)	7.960
		r = 0.9	32	358	(1.990, 1.990)	7.960
	Golden section search method		1	45	(1.990, 1.990)	7.960
(-2.1, -3)	Descent condition method	r = 0.0	18	144	(-1.990, -2.985)	12.934
		r = 0.1	18	144	(-1.990, -2.985)	12.934
		r = 0.5	9	82	(-1.990, -2.985)	12.934
		r = 0.9	36	395	(-1.990, -2.985)	12.934
	Golden section search method		7	229	(-1.990, -2.985)	12.934

Comparison between the Golden Section Search Method and Descent Condition

Step of calculation	Descent Condition method	Golden Section Search method
[redacted]	Many	Little
Iteration number of one dimensional search method	Little	Many

☑ **Comparison between the Golden Section Search Method and Descent Condition Method**

- **When we use the one dimensional search method, we have to calculate the value of the objective function and constraints repetitively.**
- **If it takes much time to calculate the value of the objective function and constraints, the Descent condition method is more useful.**

Reference Slides

[Reference] Taylor Series Expansion for the Function of Two Variables

The second-order Taylor series expansion of $f(x_1, x_2)$ at (x_1^*, x_2^*)

$$f(x_1,x_2) = f(x_1^*,x_2^*) + \frac{\partial f(x_1^*,x_2^*)}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f(x_1^*,x_2^*)}{\partial x_2}(x_2 - x_2^*)$$

$$+\frac{1}{2}\left(\frac{\partial^2 f(x_1^*,x_2^*)}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f(x_1^*,x_2^*)}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f(x_1^*,x_2^*)}{\partial x_2^2}(x_2 - x_2^*)^2 \right) \quad \cdots\cdots ①$$

define: $\mathbf{c} = \nabla f(\mathbf{x}^*),\ \mathbf{d} = (\mathbf{x} - \mathbf{x}^*)$

$$\Rightarrow f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}\left(\mathbf{x} - \mathbf{x}^*\right)^T \mathbf{H}(\mathbf{x}^*)\left(\mathbf{x} - \mathbf{x}^*\right) = f(\mathbf{x}^*) + \mathbf{c}^T\mathbf{d} + \frac{1}{2}\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\,\mathbf{d} \quad \cdots\cdots ②$$