Ship Stability

September 2013

Myung-Il Roh

Department of Naval Architecture and Ocean Engineering
Seoul National University

Ship Stability

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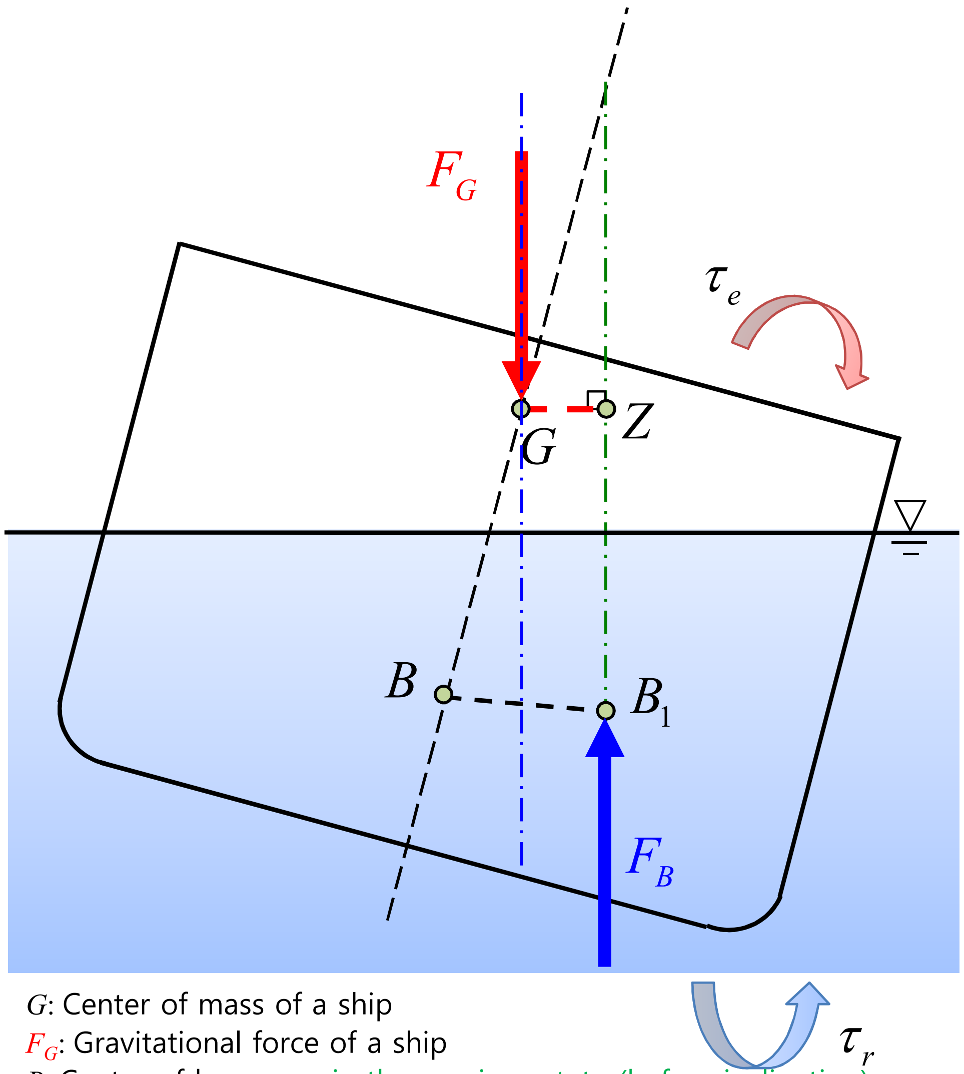
Ch. 4 Initial Transverse Stability

Transverse Metacentric Height (GM)

Righting Arm (GZ, Restoring Arm)

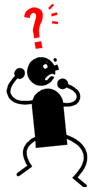


G: Center of mass of a ship
F_G: Gravitational force of a ship
B: Center of buoyancy in the previous state (before inclination)
F_B: Buoyant force acting on a ship
B_1: New position of center of buoyancy after the ship has been inclined
Z : The intersection point of a vertical line through the new position of the center of buoyancy(B_1) with the transversely parallel line to a waterline through the center of mass(G)

- **Transverse Righting Moment**

$$\tau_{righting} = F_B \cdot \underline{GZ}$$

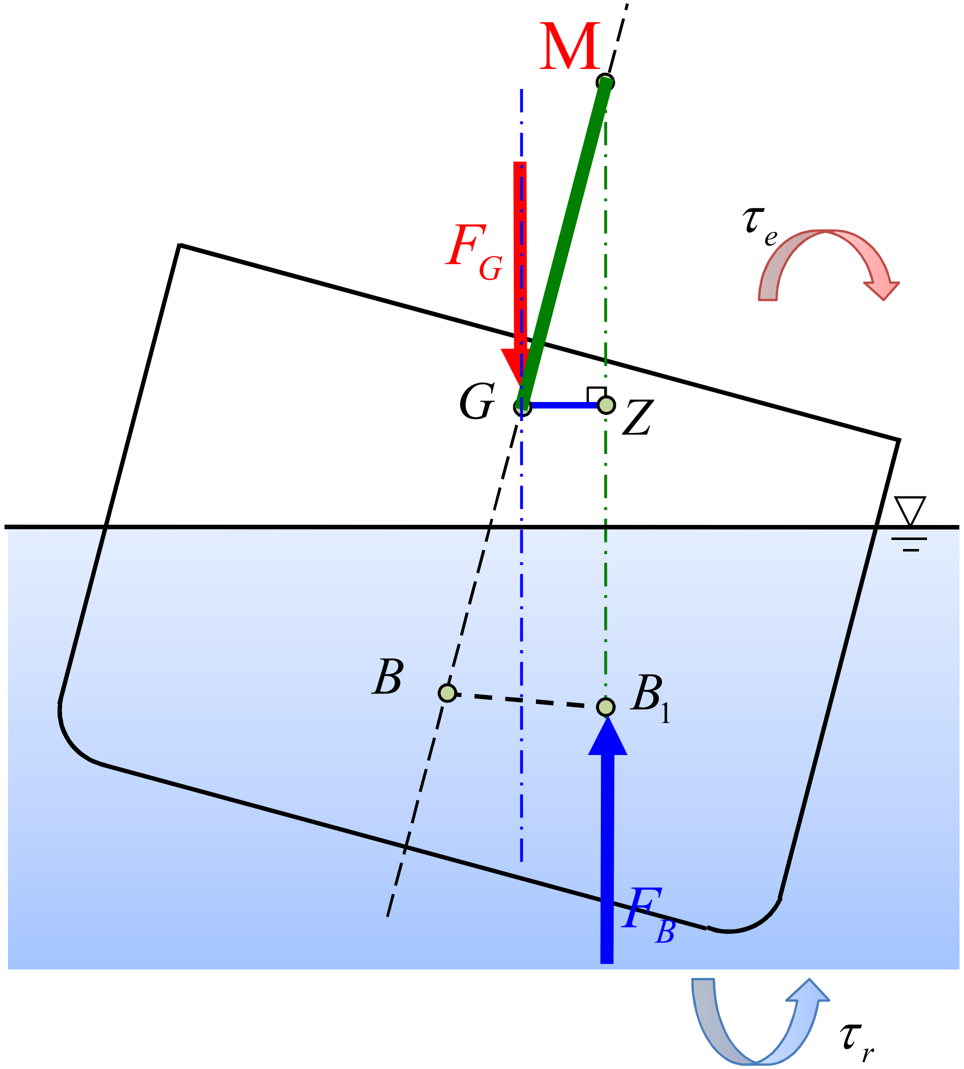
How can we find GZ **in small angle of inclination**?



Metacenter (M)

- **Righting Moment**

$$\tau_{righting} = F_B \cdot GZ$$



Z: The intersection point of the line of buoyant force through B_1 with the transverse line through G

Definition of M (Metacenter)

- The intersection point of the vertical line through the center of buoyancy at **previous position** (*B*) with the vertical line through the center of buoyancy at **new position** (*B_1*) **after inclination**

- The term **meta** was selected as a prefix for center because its Greek meaning implies **movement**. The **metacenter** therefore is a **moving center**.

- GM ➡ **Metacentric height**

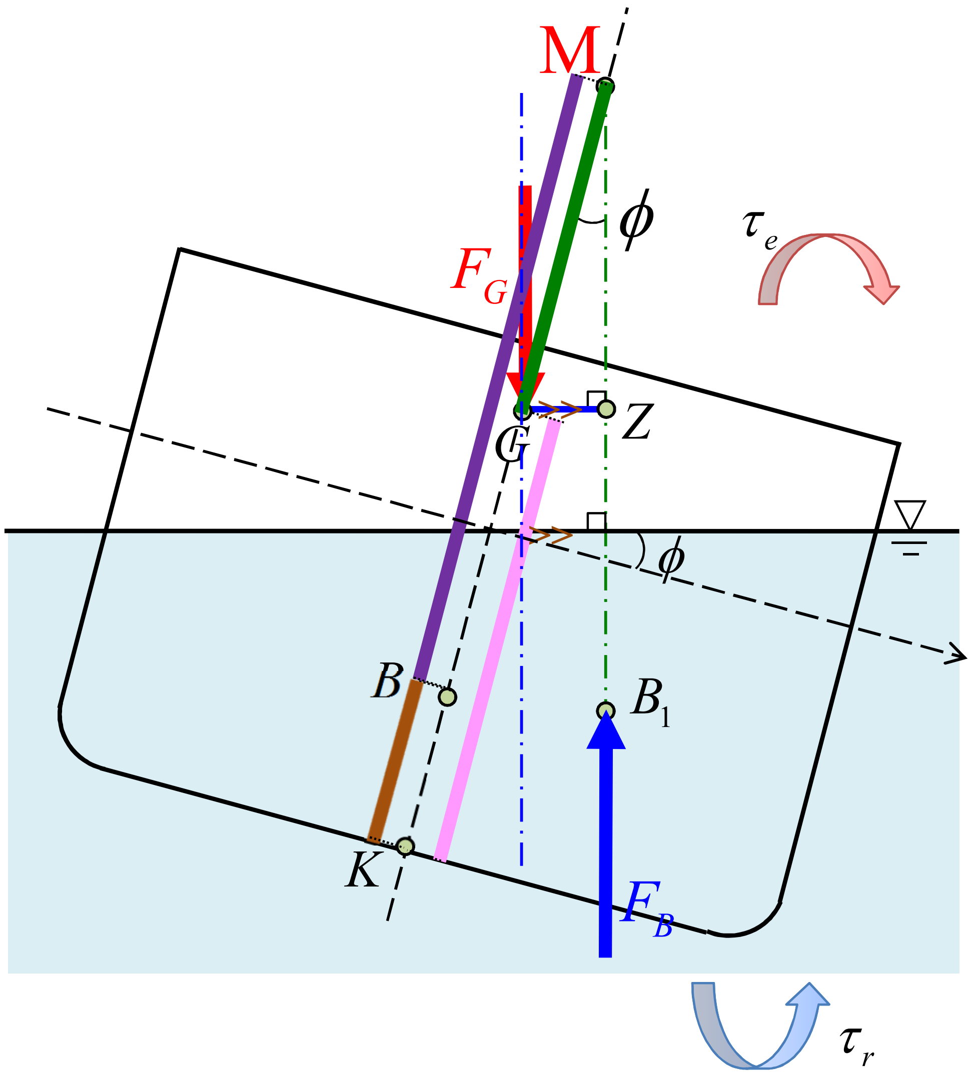
- From the figure, GZ can be obtained with assumption that *M* does not change within a **small angle of inclination** (about 7° to 10°), as below.

$$GZ \approx GM \cdot \sin\phi$$

Metacentric Height (GM) of a Ship for Small Angle of Inclination

- **Righting Moment**

$$\tau_{righting} = F_B \cdot GZ$$



Z: The intersection point of the line of buoyant force through B_1 with the transverse line through G

M: Metacenter

GM: Metacentric height

θ: Angle of heel

K: Keel, the lower most point on the ship vertical centre line

Righting Arm

$$GZ \approx GM \cdot \sin\phi$$

- From the geometrical configuration of the ship, GM is made up as follows:

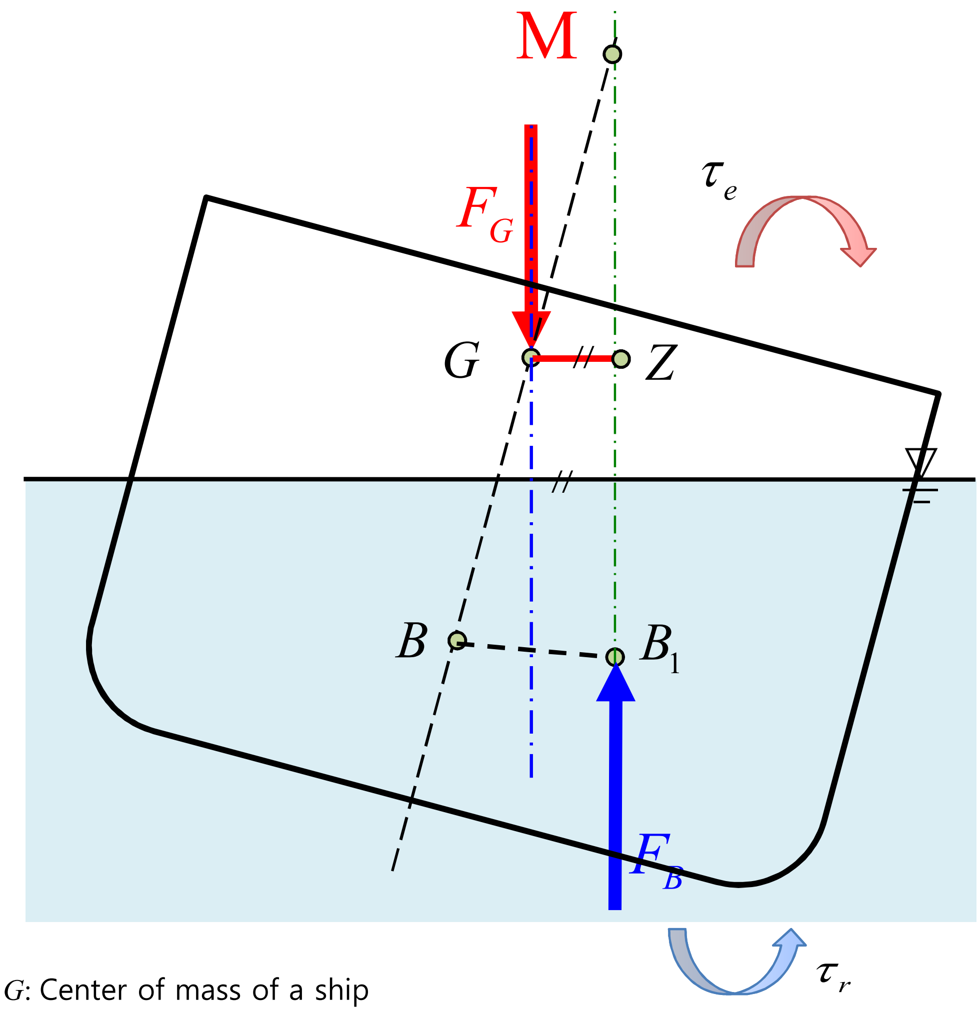
$$GM = KB + BM - KG$$

KB ≒ 51~52% draft

Center of gravity of the ship

How can you get the value of the BM(metacentric radius)?

Righting Moment at Large Angle of Inclination



G: Center of mass of a ship
F_G: Gravitational force of a ship
B: Center of buoyancy in the previous state (before inclination)
F_B: Buoyant force acting on a ship
B_1: New position of center of buoyancy after the ship has been inclined
Z : The intersection point of a vertical line through the new position of the center of buoyancy(B_1) with the transversely parallel line to a waterline through the center of mass(G)

① Apply a large rotational angle to the ship by an external moment

- **Transverse Righting Moment**

$$\tau_{righting} = F_B \cdot \boxed{GZ}$$

Righting Arm

- The use of metacentric height(GM) as the righting arm is **not valid for a ship at a large angle of inclination**.

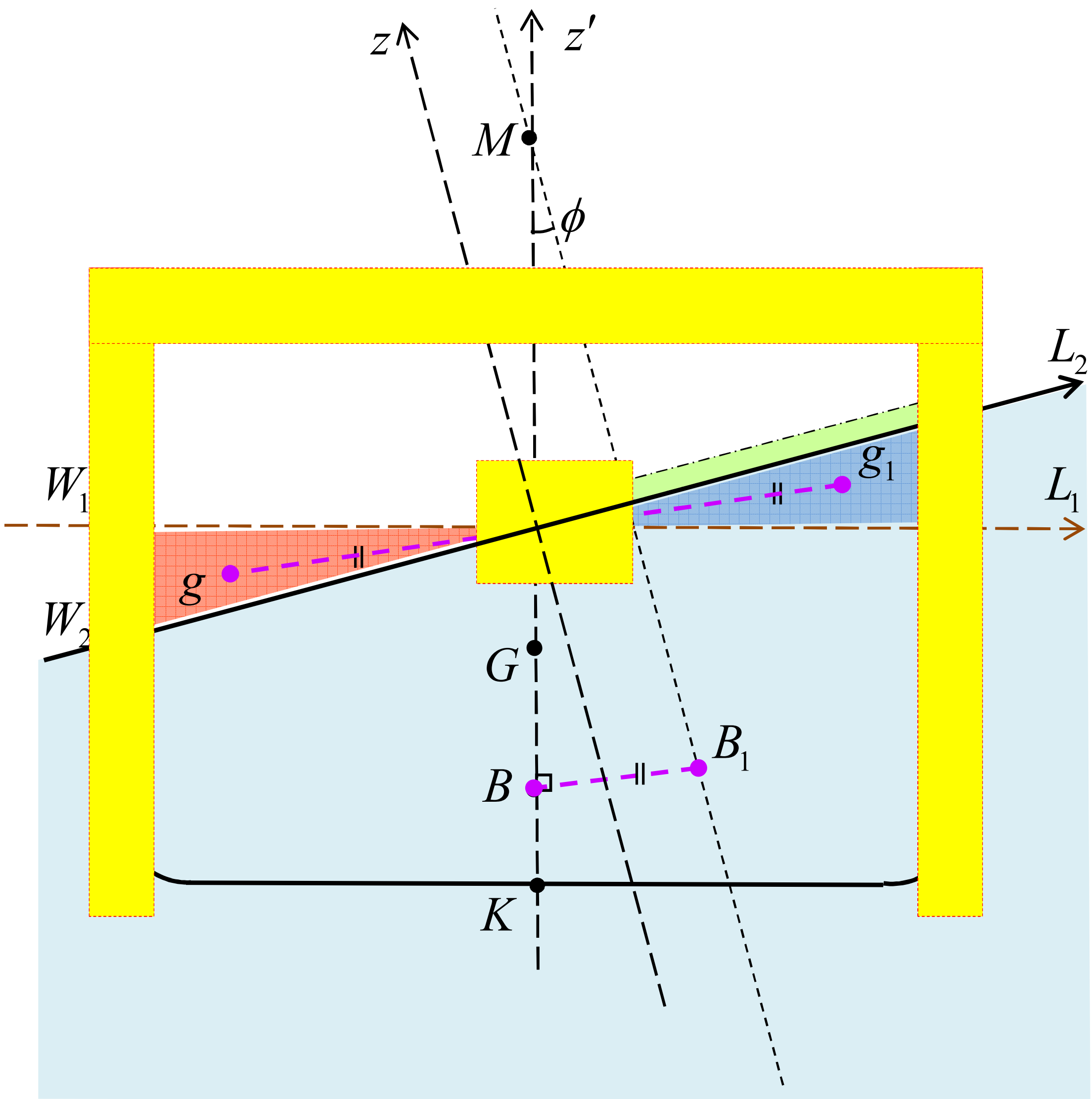
To determine the righting arm "GZ" of the ship at a large angle of inclination, it is necessary to know the accurate GZ which corresponds to the distance from the center of mass(G) to the vertical line through the new position of the center of buoyancy(B_1).

$$GZ \neq GM \cdot \sin\phi$$

for a large angle of inclination

Derivation of Transverse Metacentric Radius (BM_T)

Derivation of BM_T (1/12) (BM_T : Transverse Metacentric Radius)



B : Center of buoyancy in the previous state (before inclination)
B_1: Center of buoyancy in the present state (after inclination)
M : Intersection of the line of the buoyant force through B_1 in the present state with the line of the buoyant force B in the previous state

Let us derive the transverse metacentric radius "BM" in case of a **wall-sided ship** with a simple section shape.

- **Wall sided ship**

: When a ship has a perpendicular side shell to water plane, the ship is called as "wall-sided ship".

Assumption

1. Wall sided ship

- When the ship is inclined, the submerged volume is the same as the emerged volume without any change in the displacement volume.

2. Main deck is not submerged.

3. Axis of inclination do not change.

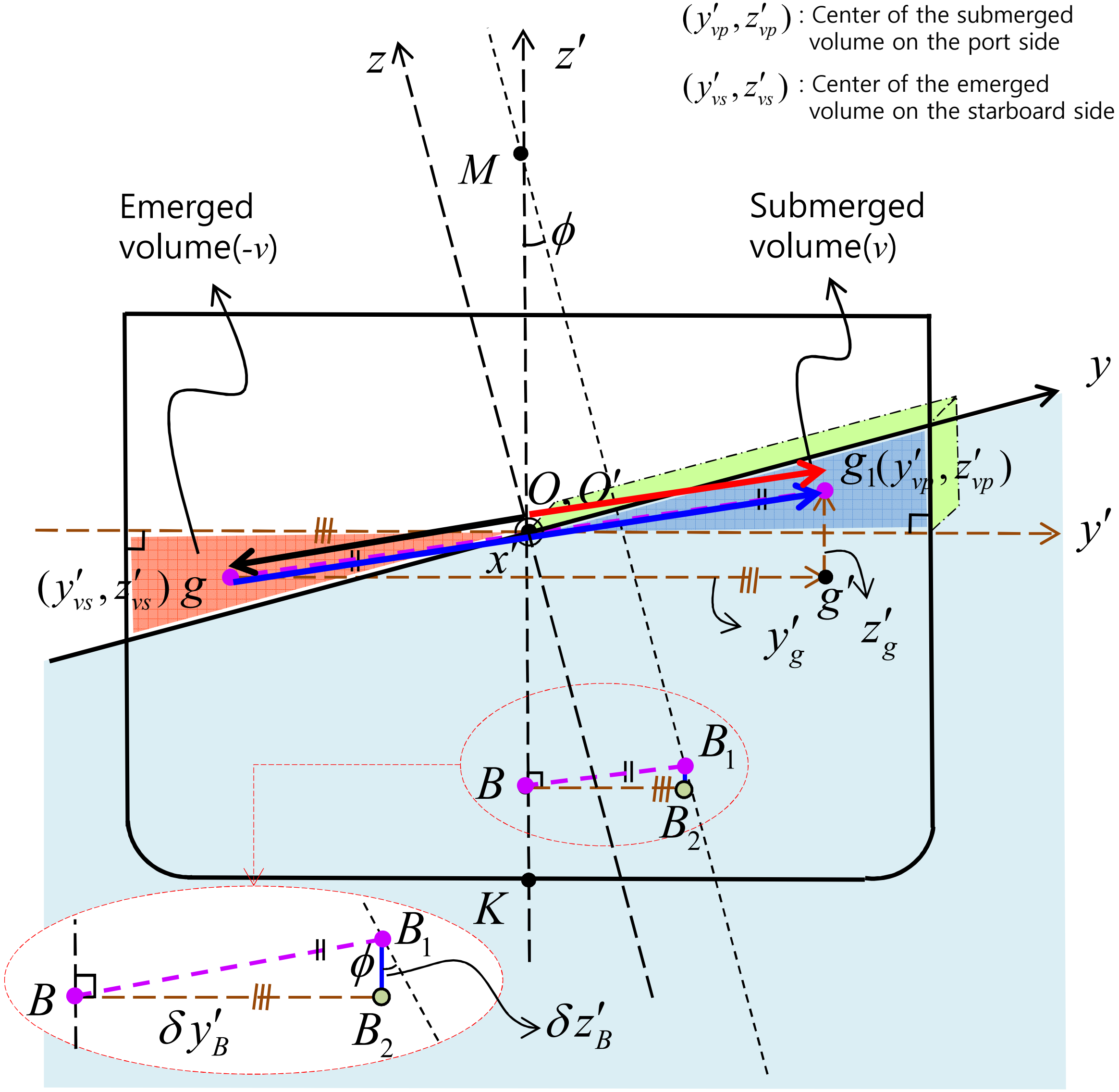
- In this case, the axis of inclination passes through the origin "O" of the inertial frame.

Derivation of BM_T (2/12)

∇: Displacement volume
v: Changed displacement volume (wedge)
BB_1 $(\delta y'_B, \delta z'_B)$: Distance between the initial center of buoyancy and the changed B_1
gg_1 (y'_g, z'_g): Distance between the center of wedges

(y'_{vp}, z'_{vp}) : Center of the submerged volume on the port side

(y'_{vs}, z'_{vs}) : Center of the emerged volume on the starboard side



Translation of the center of buoyancy caused by the movement of the small volume v

$$\begin{cases} \delta y'_B \cdot \nabla = y'_g \cdot v & \cdots\cdots (1) \\ \delta z'_B \cdot \nabla = z'_g \cdot v & \cdots\cdots (2) \end{cases}$$

, where v is the each volume of the submerged and emerged volume.

∇ is total volume of the ship.

$$\overrightarrow{O'g} + \overrightarrow{gg_1} = \overrightarrow{O'g_1}$$

$$\overrightarrow{gg_1} = \overrightarrow{O'g_1} - \overrightarrow{O'g}$$

$$(y'_g, z'_g) = (y'_{vp}, z'_{vp}) - (y'_{vs}, z'_{vs})$$

$$y'_g = y'_{vp} - y'_{vs} \quad \cdots\cdots (3)$$

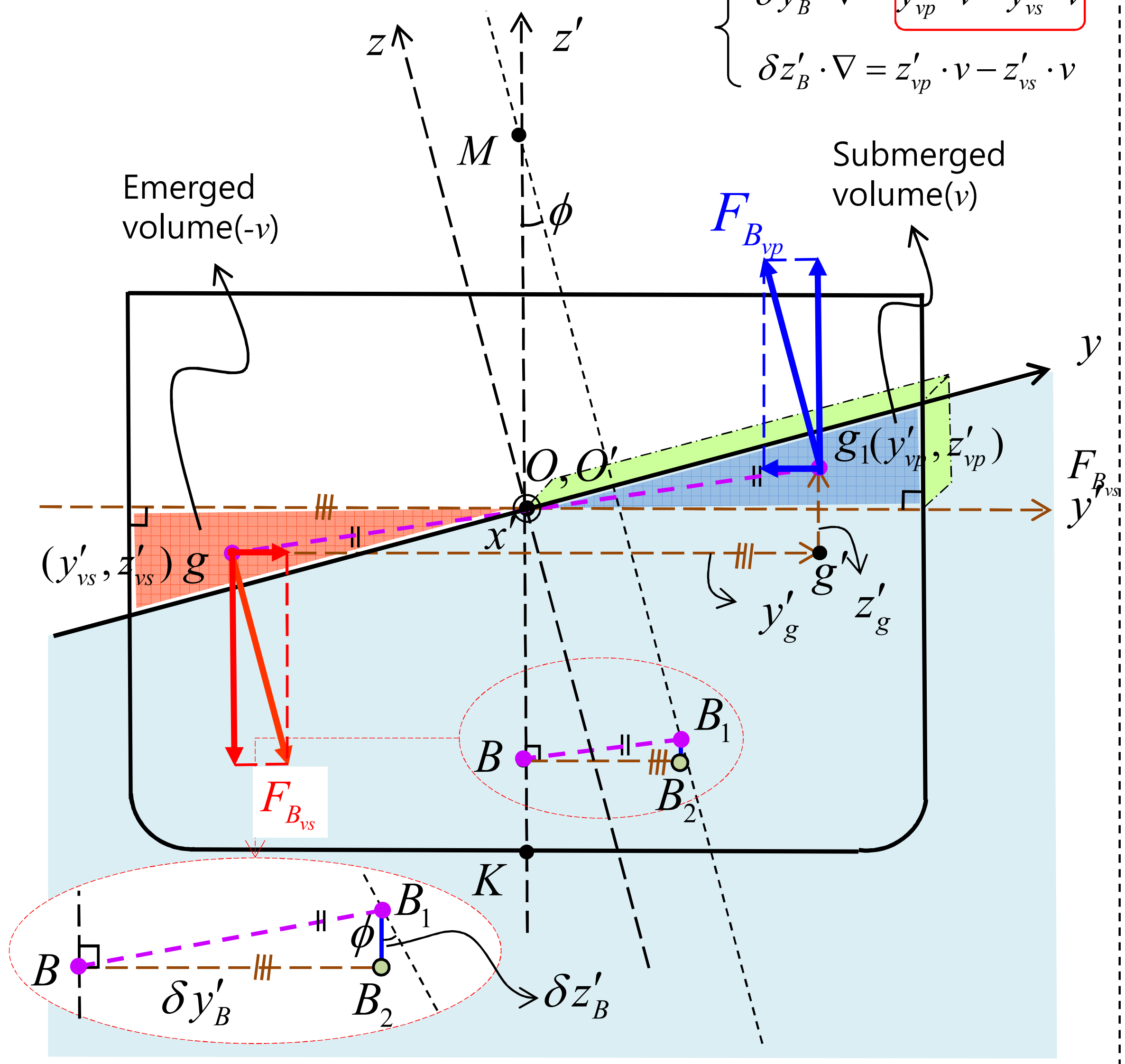
$$z'_g = z'_{vp} - z'_{vs} \quad \cdots\cdots (4)$$

Substituting Eq. (3) and (4) into the Eq. (1) and (2), respectively.

$$\begin{cases} \delta y'_B \cdot \nabla = y'_{vp} \cdot v - y'_{vs} \cdot v \\ \delta z'_B \cdot \nabla = z'_{vp} \cdot v - z'_{vs} \cdot v \end{cases}$$

Derivation of BM_T (3/12)

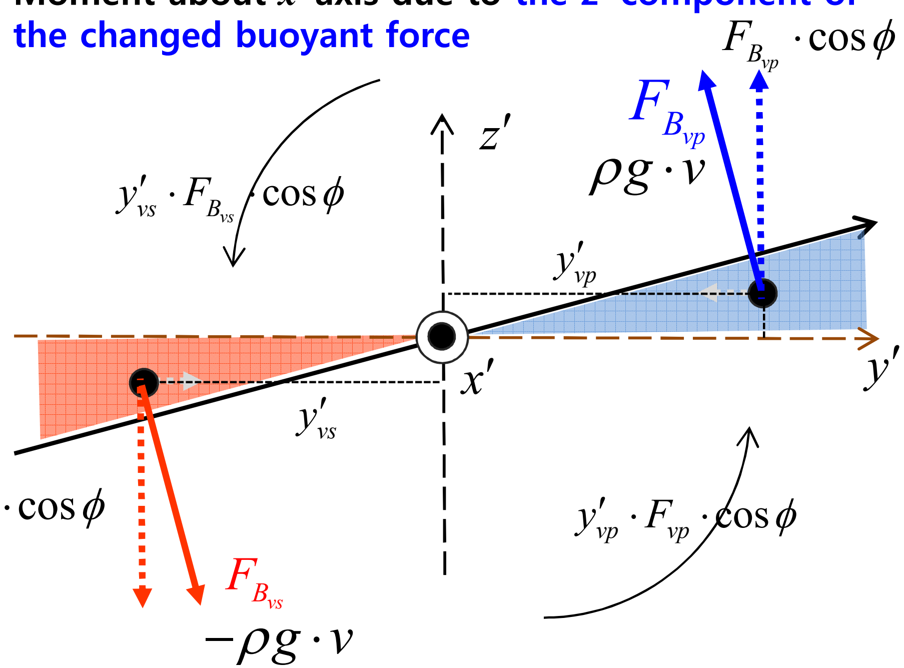
$$\begin{cases} \delta y'_B \cdot \nabla = \boxed{y'_{vp} \cdot v - y'_{vs} \cdot v} \\ \delta z'_B \cdot \nabla = z'_{vp} \cdot v - z'_{vs} \cdot v \end{cases}$$



(y'_{vp}, z'_{vp}) : The center of the changed volume on the port side

(y'_{vs}, z'_{vs}) : The center of the changed volume on the starboard side

Moment about x' axis due to the z′ component of the changed buoyant force



$$\delta y'_B \cdot \rho g \cdot \nabla \cdot \cos\phi = \boxed{y'_{vp} \cdot F_{B_{vp}} \cdot \cos\phi + y'_{vs} \cdot F_{B_{vs}} \cdot \cos\phi}$$

$$\delta y'_B \cdot \rho g \cdot \nabla = y'_{vp} \cdot F_{B_{vp}} + y'_{vs} \cdot F_{B_{vs}}$$

$$\downarrow \; F_{B_{vs}} = -\rho g \cdot v, \; F_{B_{vp}} = \rho g \cdot v$$

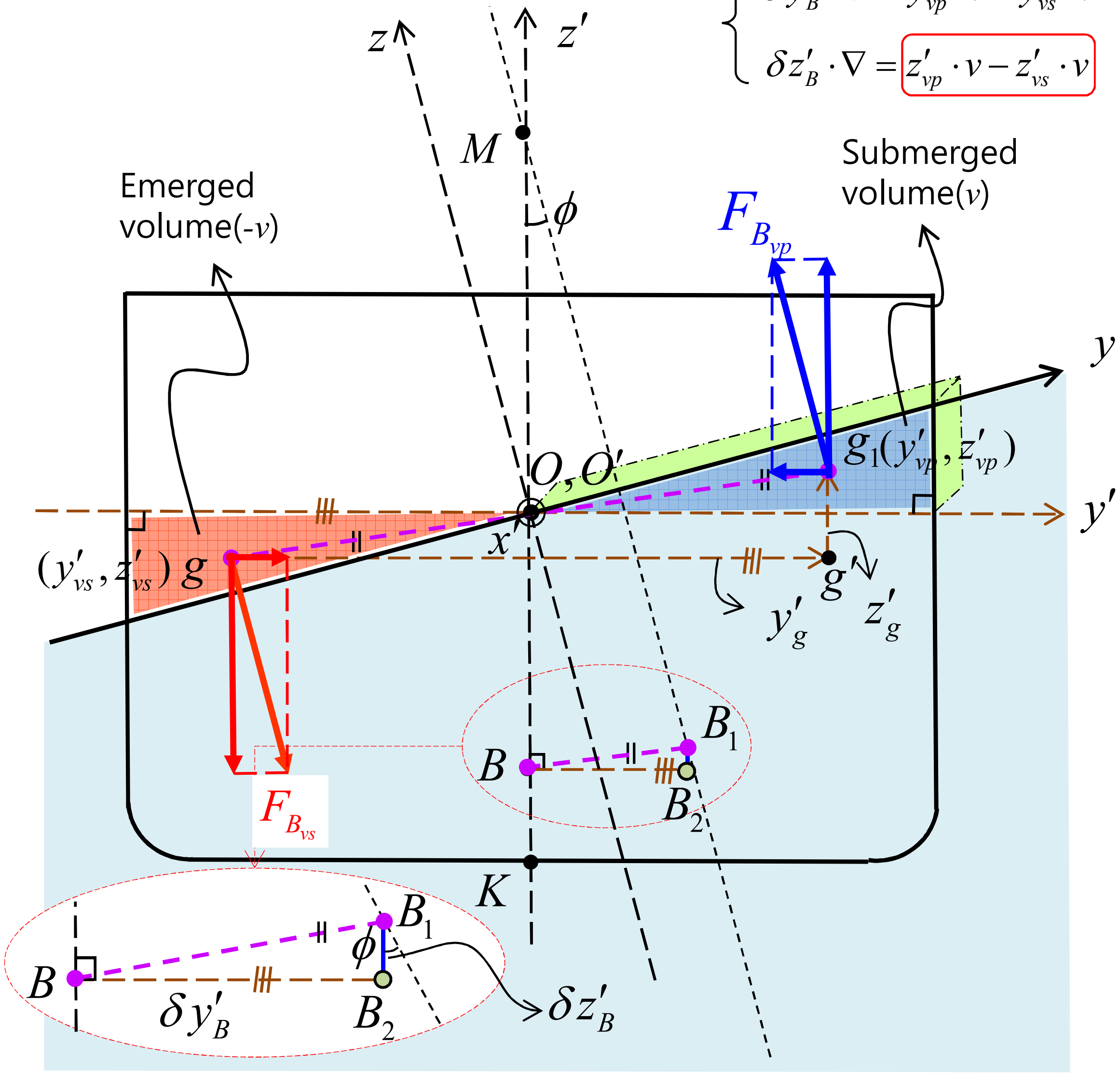
$$\rho g \cdot \delta y'_B \cdot \nabla = y'_{vp} \cdot \rho g \cdot v - y'_{vs} \cdot \rho g \cdot v$$

$$\delta y'_B \cdot \nabla = \boxed{y'_{vp} \cdot v - y'_{vs} \cdot v}$$

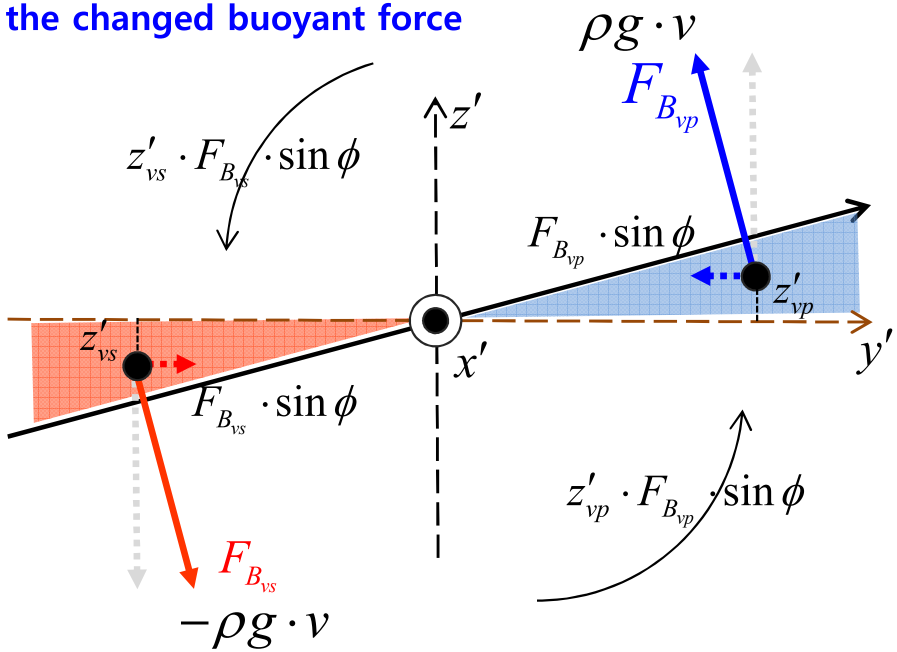
: Moment about x' axis due to the z′ component of the changed volume

Derivation of BM$_T$ (4/12)

$$\begin{cases} \delta y'_B \cdot \nabla = y'_{vp} \cdot v - y'_{vs} \cdot v \\ \delta z'_B \cdot \nabla = \boxed{z'_{vp} \cdot v - z'_{vs} \cdot v} \end{cases}$$



Moment about x' axis due to the y′ component of the changed buoyant force



$$\delta z'_B \cdot \rho g \cdot \nabla \cdot \sin\phi = \boxed{z'_{vp} \cdot F_{B_{vp}} \cdot \sin\phi + z'_{vs} \cdot F_{B_{vs}} \cdot \sin\phi}$$

$$\delta z'_B \cdot \rho g \cdot \nabla = z'_{vp} \cdot F_{B_{vp}} + z'_{vs} \cdot F_{B_{vs}}$$

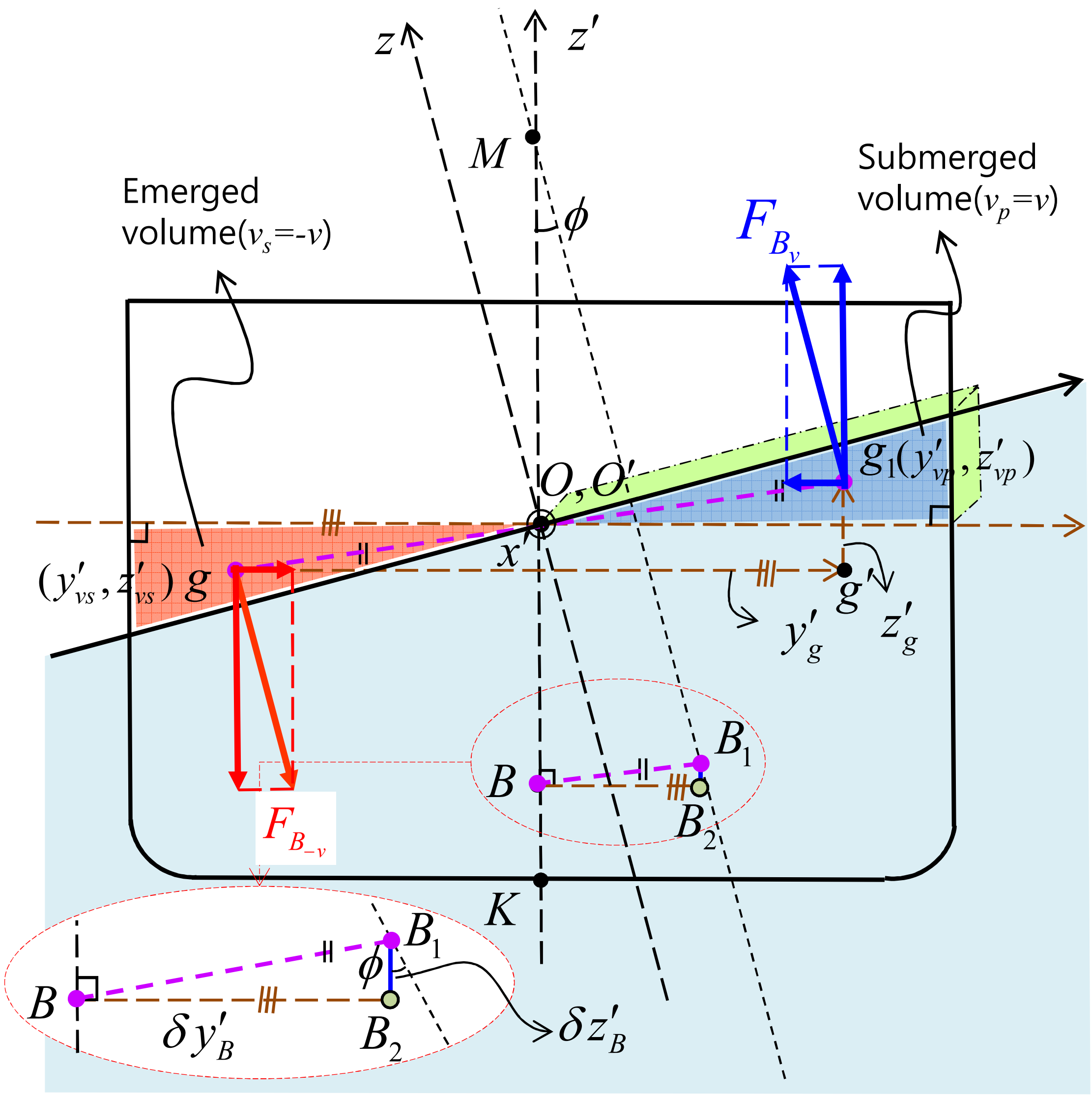
$$\Big\downarrow \; F_{B_{vs}} = -\rho g \cdot v,\ F_{B_{vp}} = \rho g \cdot v$$

$$\delta z'_B \cdot \rho g \cdot \nabla = \rho g \cdot z'_{vp} \cdot v - \rho g \cdot z'_{vs} \cdot v$$

$$\delta z'_B \cdot \nabla = \boxed{z'_{vp} \cdot v - z'_{vs} \cdot v}$$

∴ Moment about x' axis due to the y′ component of the changed volume

Derivation of BM$_T$ (5/12)



$$\delta y'_B \cdot \nabla = \boxed{y'_{vp} \cdot v - y'_{vs} \cdot v}$$

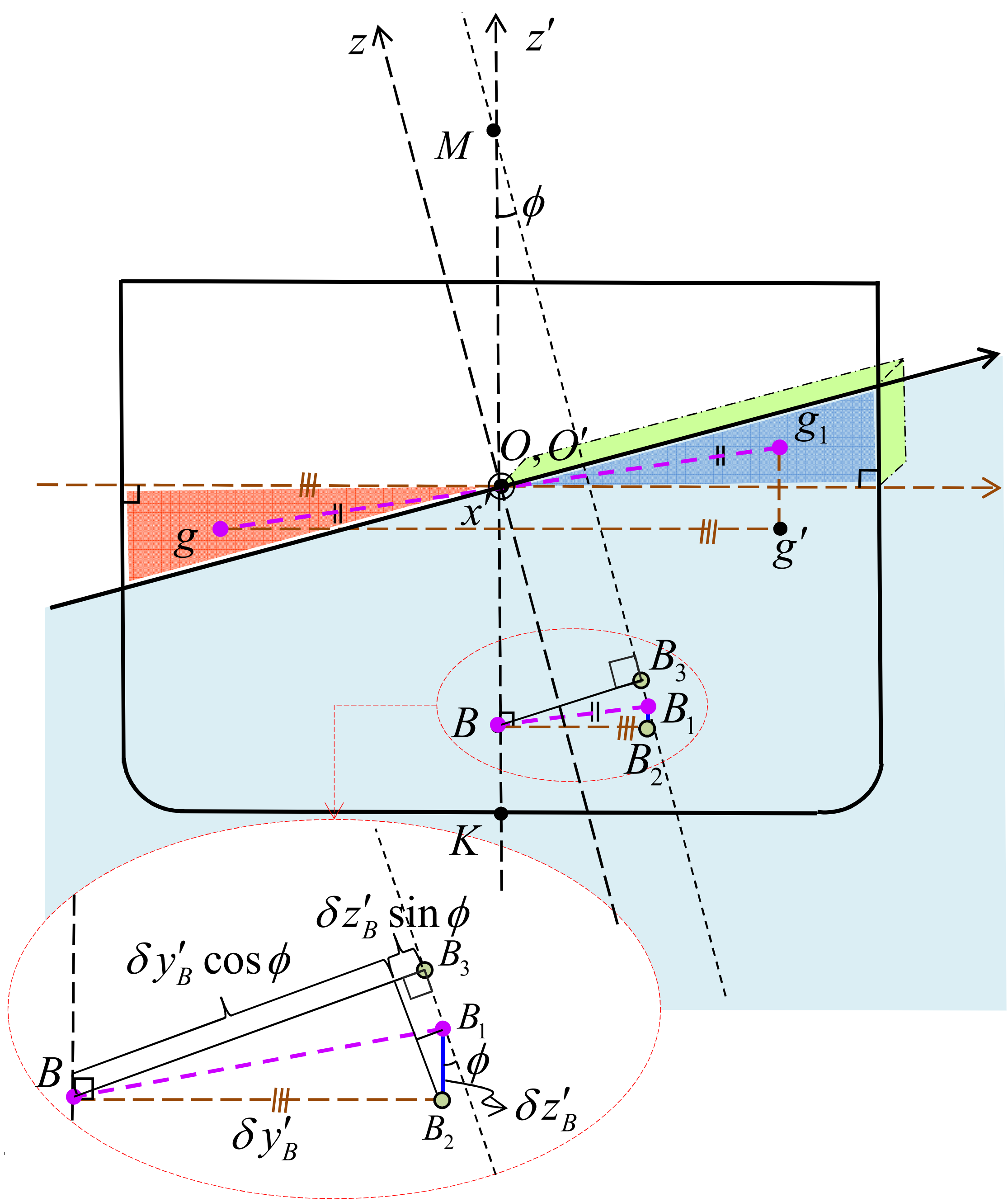
: Moment about x' axis due to the z′ component of the changed volume

$$\delta z'_B \cdot \nabla = \boxed{z'_{vp} \cdot v - z'_{vs} \cdot v}$$

: Moment about x' axis due to the y′ component of the changed volume

$$\Downarrow \longleftarrow \quad v = v_p,\ -v = v_s$$

$$\begin{cases} \delta y'_B \cdot \nabla = y'_{vp} \cdot v_p + y'_{vs} \cdot v_s \\ \delta z'_B \cdot \nabla = z'_{vp} \cdot v_p + z'_{vs} \cdot v_s \end{cases}$$



$$\begin{cases} \delta y'_B \cdot \nabla = y'_{vp} \cdot v_p + y'_{vs} \cdot v_s \\ \delta z'_B \cdot \nabla = z'_{vp} \cdot v_p + z'_{vs} \cdot v_s \end{cases}$$

: Moment about x' axis due to the z′ component of the changed volume

: Moment about x' axis due to the y′ component of the changed volume

What we want to find is BM.

$$BM \sin\phi = BB_3$$

$$BM = \frac{BB_3}{\sin\phi}$$

$$= \frac{1}{\sin\phi}\left(\delta y'_B \cos\phi + \delta z'_B \sin\phi\right)$$

$$= \frac{\cos\phi}{\sin\phi}\left(\delta y'_B + \delta z'_B \frac{\sin\phi}{\cos\phi}\right)$$

$$= \frac{1}{\tan\phi}\left(\delta y'_B + \delta z'_B \tan\phi\right)$$

$$\delta y'_B = \left(y'_{vp} \cdot v_p + y'_{vs} \cdot v_s\right)/\nabla$$

$$\delta z'_B = \left(z'_{vp} \cdot v_p + z'_{vs} \cdot v_s\right)/\nabla$$

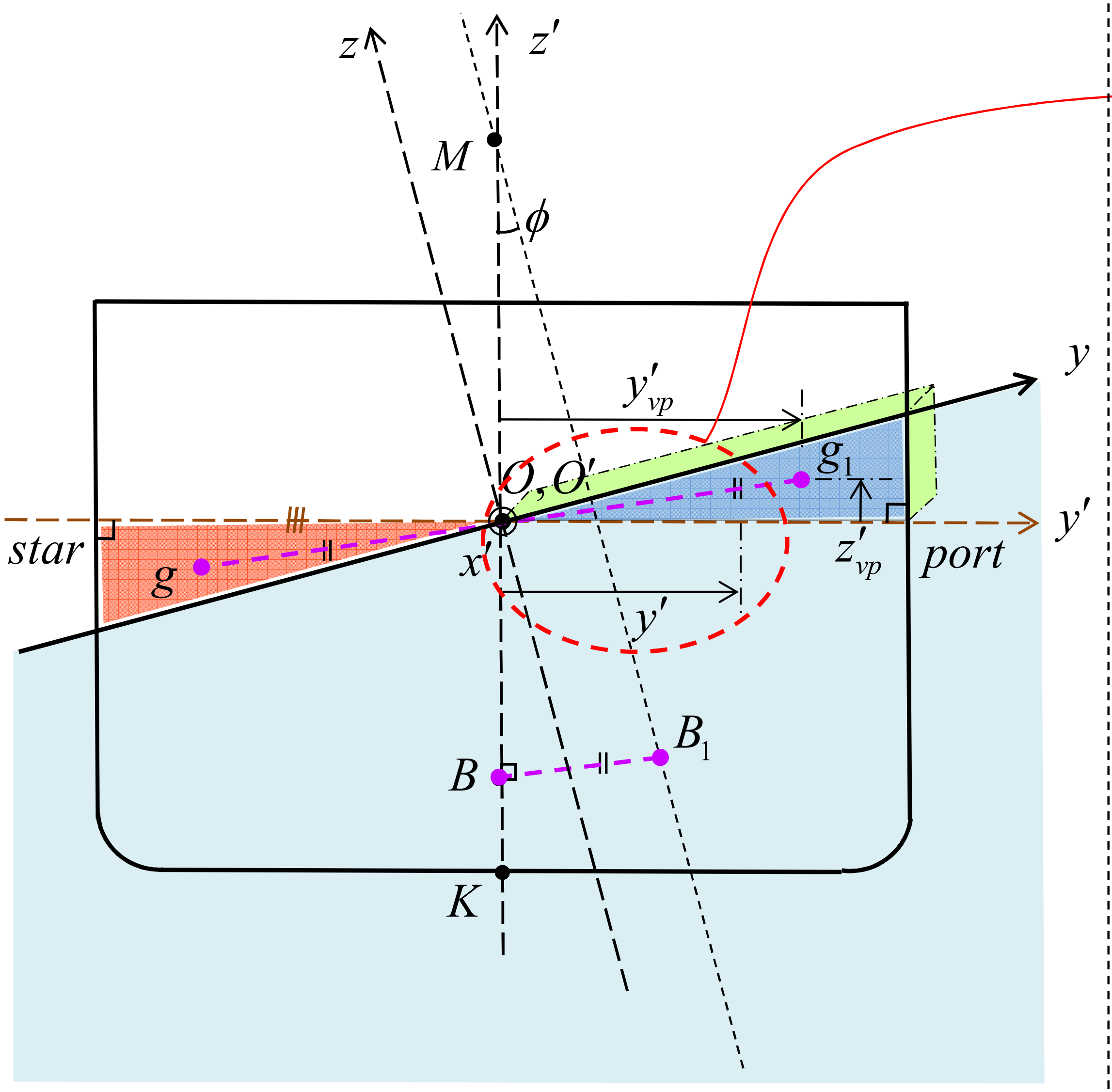
$$= \frac{1}{\tan\phi}\left(\frac{y'_{vp} \cdot v_p + y'_{vs} \cdot v_s}{\nabla} + \frac{z'_{vp} \cdot v_p + z'_{vs} \cdot v_s}{\nabla} \cdot \tan\phi\right)$$

$$BM = \frac{1}{\nabla \cdot \tan\phi}\left(\boxed{y'_{vp} \cdot v_p} + \boxed{y'_{vs} \cdot v_s} + \left(\boxed{z'_{vp} \cdot v_p} + \boxed{z'_{vs} \cdot v_s}\right) \cdot \tan\phi\right)$$

Find!

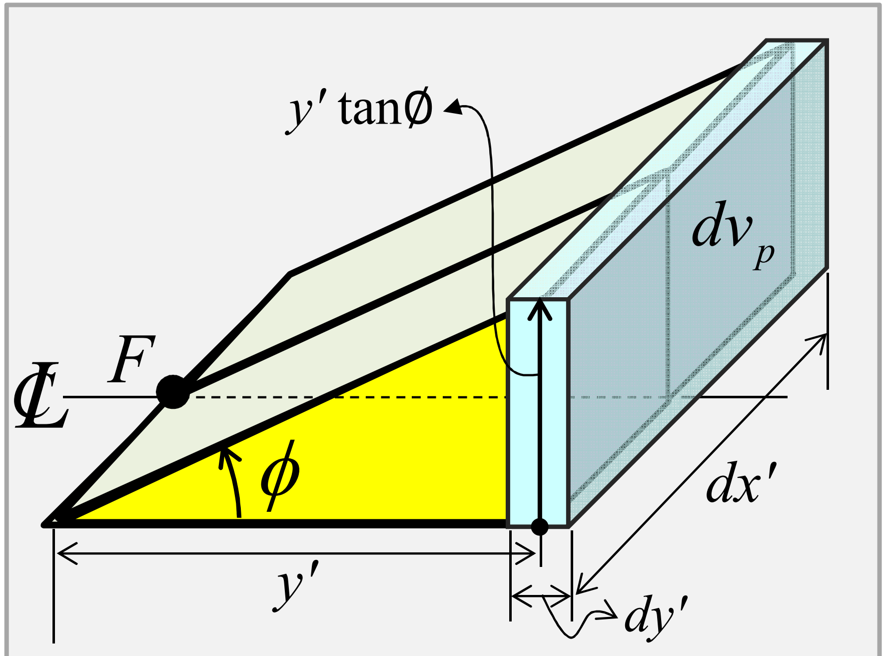
Derivation of BM$_T$ (7/12)

$$BM = \frac{1}{\nabla \cdot \tan\phi}\left(\underbrace{y'_{vp} \cdot v_p}_{(A)} + \underbrace{y'_{vs} \cdot v_s}_{(B)} + \left(\underbrace{z'_{vp} \cdot v_p}_{(C)} + \underbrace{z'_{vs} \cdot v_s}_{(D)} \right) \cdot \tan\phi \right)$$



(y'_{vp}, z'_{vp}) : The center of the changed volume on the port side

(y'_{vs}, z'_{vs}) : The center of the changed volume on the starboard side



$dv_p = y' \cdot \tan\phi \cdot dx' \cdot dy'$

$$v_p = \int_0^{port}\int_{aft}^{fore} dv_p = \int_0^{port}\int_{aft}^{fore} y' \cdot \tan\phi \cdot dx' \cdot dy'$$

(A) $v_p \cdot y'_{vp}$: The moment about x'-axis due to the z' component of the changed volume of **port side**

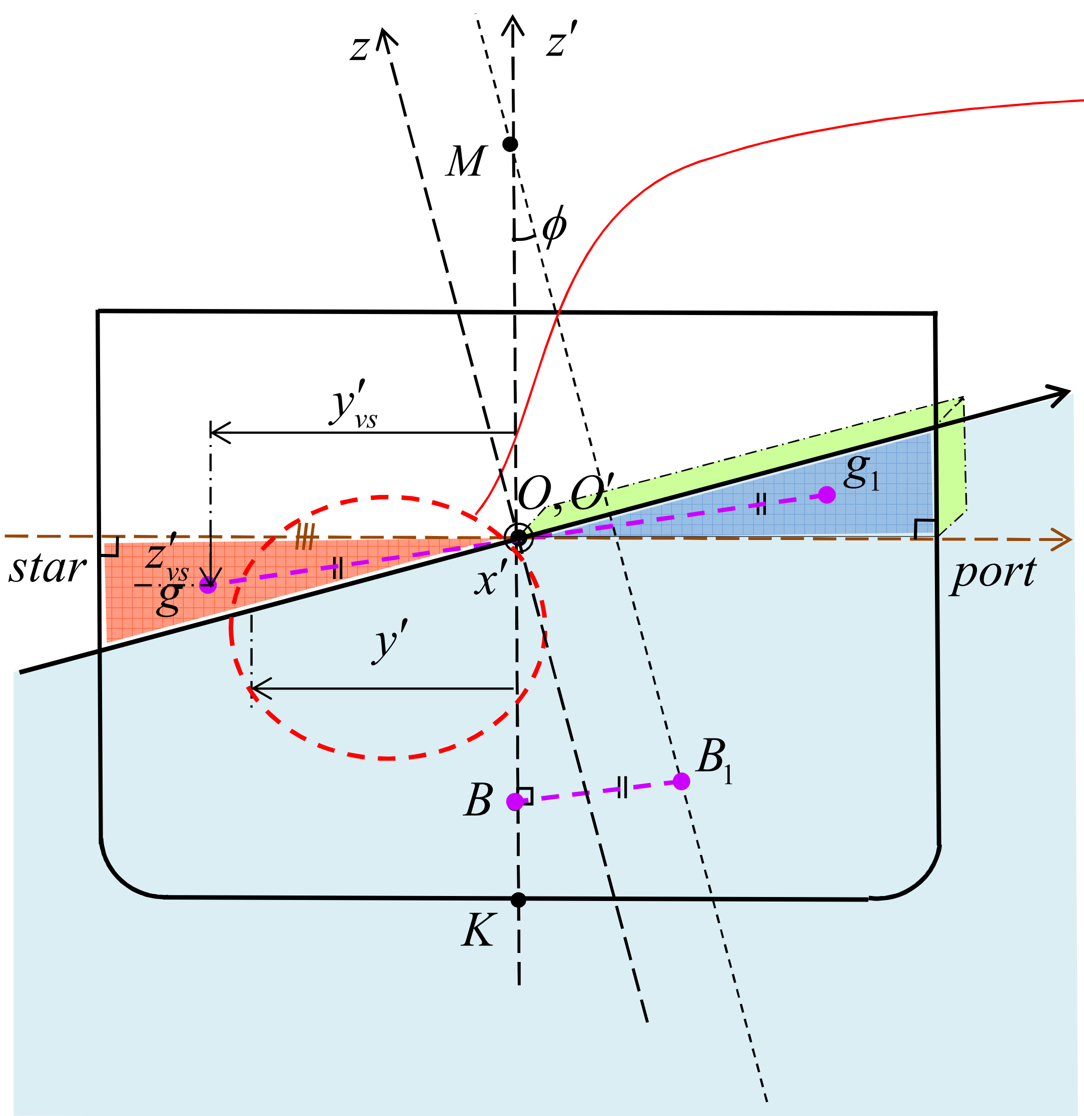
$$= \int_0^{port}\int_{aft}^{fore} y' \cdot dv_p$$

$$= \int_0^{port}\int_{aft}^{fore} y' \cdot y' \cdot \tan\phi \cdot dx' \cdot dy'$$

$$= \tan\phi \int_0^{port}\int_{aft}^{fore} y' \cdot y' \cdot dx' dy'$$

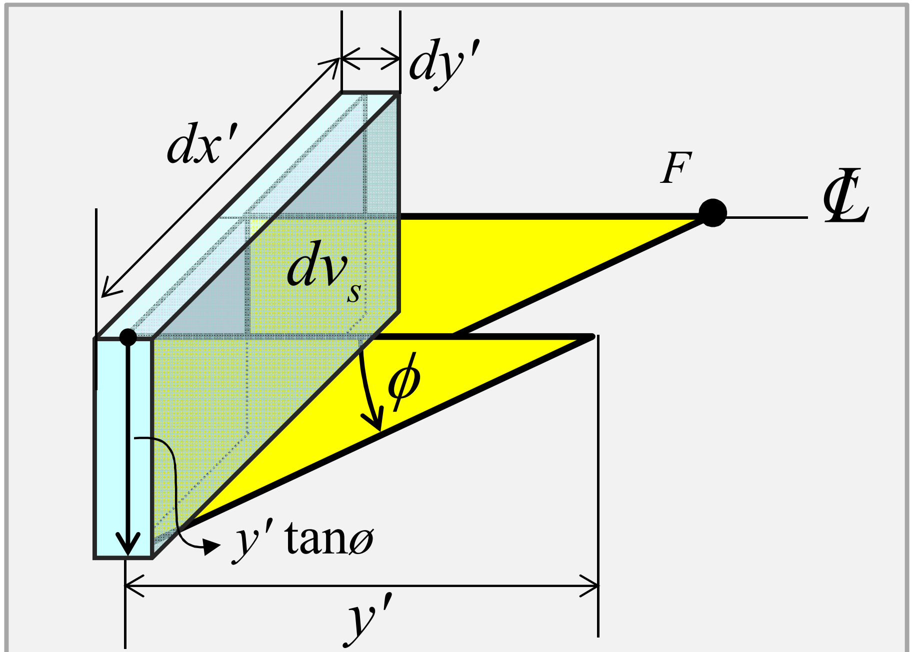
Derivation of BM$_T$ (8/12)

$$BM = \frac{1}{\nabla \cdot \tan\phi}\left(\underset{\text{(A)}}{y'_{vp} \cdot v_p} + \underset{\text{(B)}}{y'_{vs} \cdot v_s} + \left(\underset{\text{(C)}}{z'_{vp} \cdot v_p} + \underset{\text{(D)}}{z'_{vs} \cdot v_s} \right) \cdot \tan\phi \right)$$



(y'_{vp}, z'_{vp}) : The center of the changed volume on the port side

(y'_{vs}, z'_{vs}) : The center of the changed volume on the starboard side



$$dv_s = y' \cdot \tan\phi \cdot dx' \cdot dy'$$

$$v_s = \int_{star}^{0}\int_{aft}^{fore} dv_s = \int_{star}^{0}\int_{aft}^{fore} y' \cdot \tan\phi \cdot dx' \cdot dy'$$

(B) $v_s \cdot y'_{vs}$: The moment about x'-axis due to the z' component of the changed volume of **starboard side**

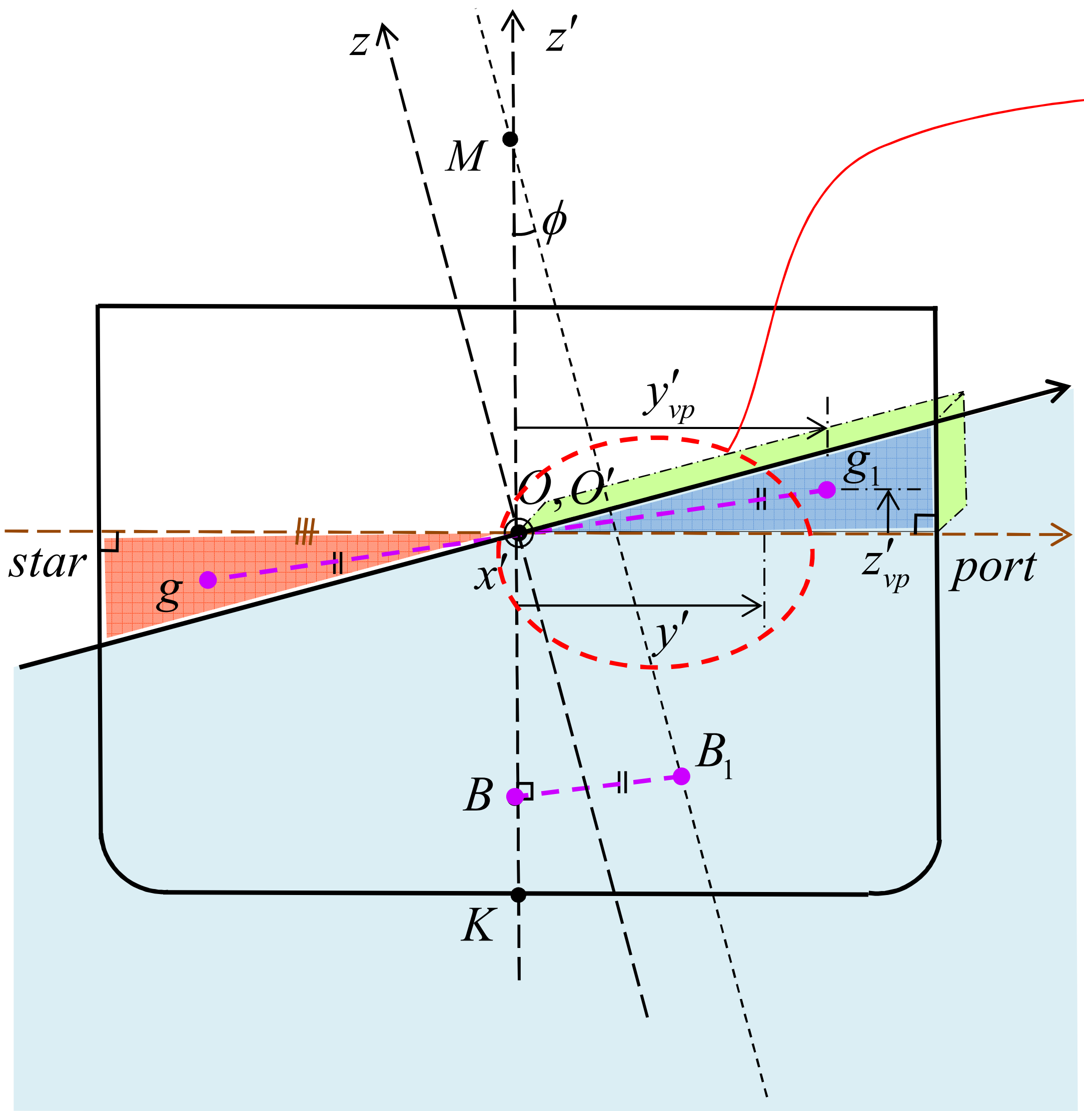
$$= \int_{star}^{0}\int_{aft}^{fore} y' \cdot dv_s$$

$$= \int_{star}^{0}\int_{aft}^{fore} y' \cdot y' \cdot \tan\phi \cdot dx' \cdot dy'$$

$$= \tan\phi \int_{star}^{0}\int_{aft}^{fore} y' \cdot y' \cdot dx' dy'$$

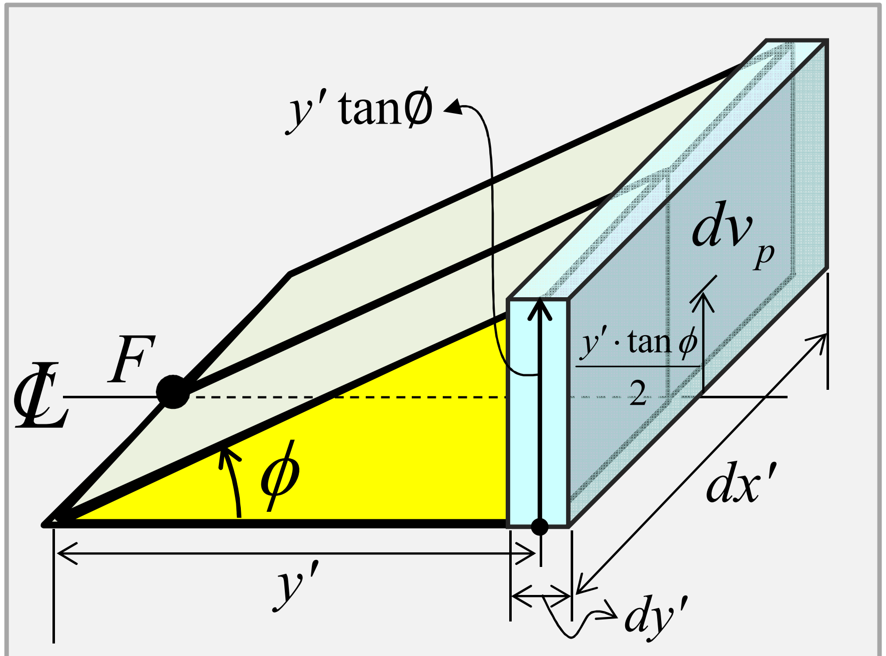
Derivation of BM_T (9/12)

$$BM = \frac{1}{\nabla \cdot \tan\phi}\left(\underbrace{y'_{vp} \cdot v_p}_{(A)} + \underbrace{y'_{vs} \cdot v_s}_{(B)} + \left(\underbrace{z'_{vp} \cdot v_p}_{(C)} + \underbrace{z'_{vs} \cdot v_s}_{(D)} \right) \cdot \tan\phi \right)$$



(y'_{vp}, z'_{vp}) : The center of the changed volume on the port side

(y'_{vs}, z'_{vs}) : The center of the changed volume on the starboard side



$$dv_p = y' \cdot \tan\phi \cdot dx' \cdot dy'$$

$$v_p = \int_0^{port}\int_{aft}^{fore} dv_p = \int_0^{port}\int_{aft}^{fore} y' \cdot \tan\phi \cdot dx' \cdot dy'$$

(C) $v_p \cdot z'_{vp}$: The moment about x'-axis due to the y' component of the changed volume of **port side**

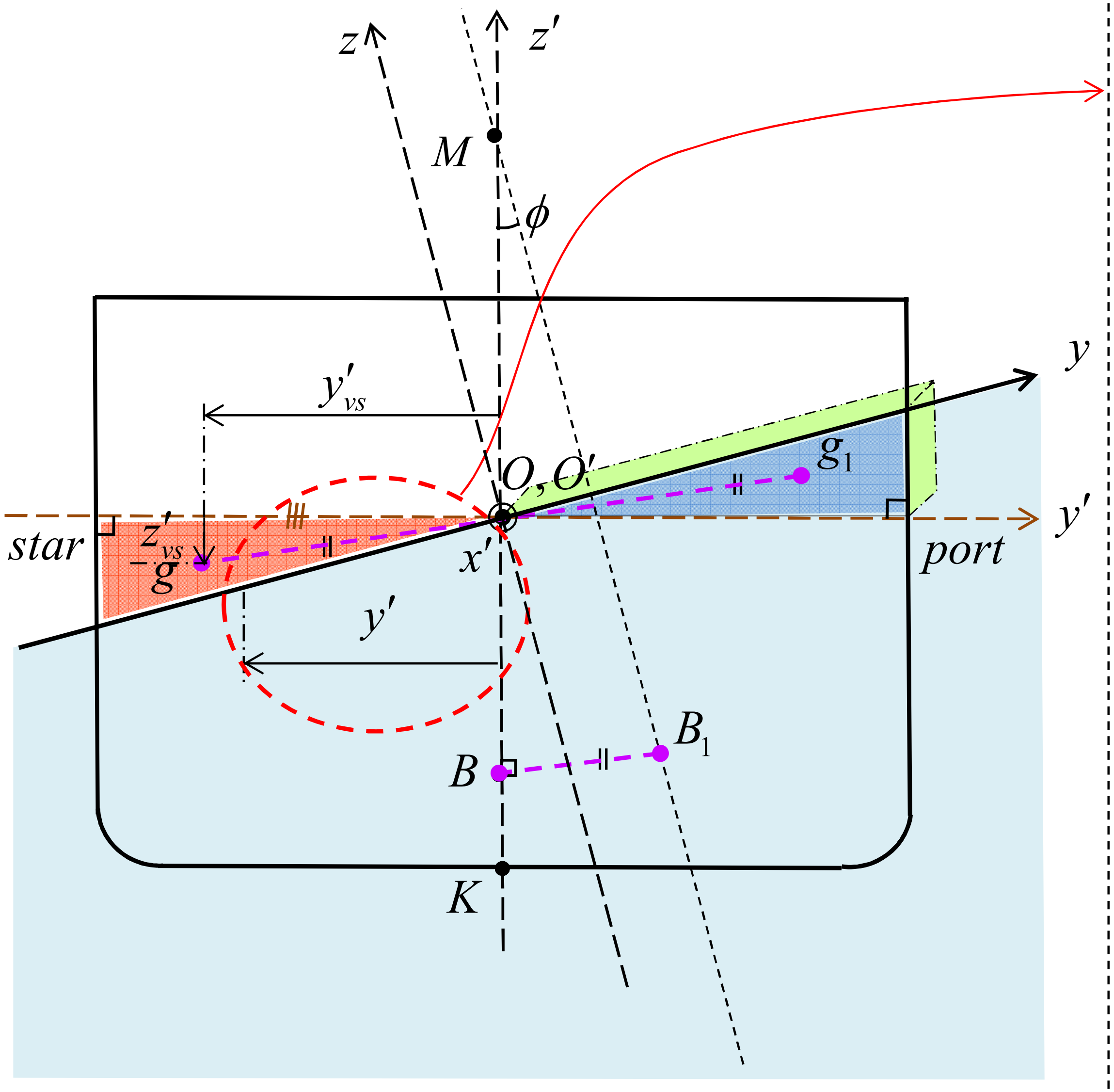
$$= \int_0^{port}\int_{aft}^{fore} \frac{y' \cdot \tan\phi}{2} \cdot dv_p$$

$$= \int_0^{port}\int_{aft}^{fore} \frac{y' \cdot \tan\phi}{2} \cdot y' \cdot \tan\phi \cdot dx' \cdot dy'$$

$$= \frac{\tan^2\phi}{2}\int_0^{port}\int_{aft}^{fore} y' \cdot y' \cdot dx' \cdot dy'$$

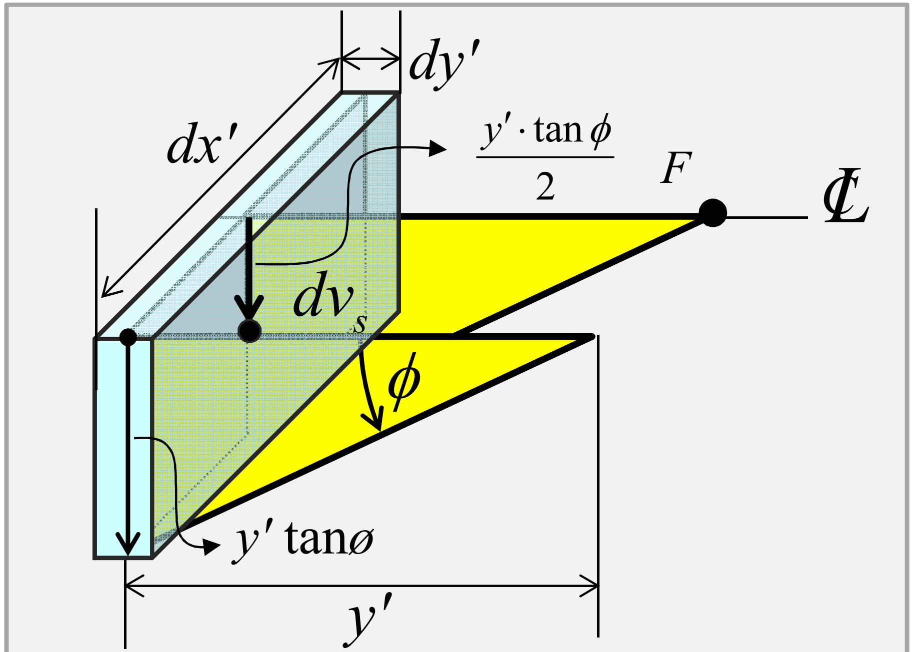
Derivation of BM_T (10/12)

$$BM = \frac{1}{\nabla \cdot \tan\phi}\left(\underset{\text{(A)}}{\boxed{y'_{vp} \cdot v_p}} + \underset{\text{(B)}}{\boxed{y'_{vs} \cdot v_s}} + \left(\underset{\text{(C)}}{\boxed{z'_{vp} \cdot v_p}} + \underset{\text{(D)}}{\boxed{z'_{vs} \cdot v_s}} \right) \cdot \tan\phi \right)$$



(y'_{vp}, z'_{vp}) : The center of the changed volume on the port side

(y'_{vs}, z'_{vs}) : The center of the changed volume on the starboard side



$$dv_s = y' \cdot \tan\phi \cdot dx' \cdot dy'$$

$$v_s = \int_{star}^{0}\int_{aft}^{fore} dv_s = \int_{star}^{0}\int_{aft}^{fore} y' \cdot \tan\phi \cdot dx' \cdot dy'$$

(D) $v_s \cdot z'_{vs}$: The moment about x'-axis due to the y' component of the changed volume of **starboard side**

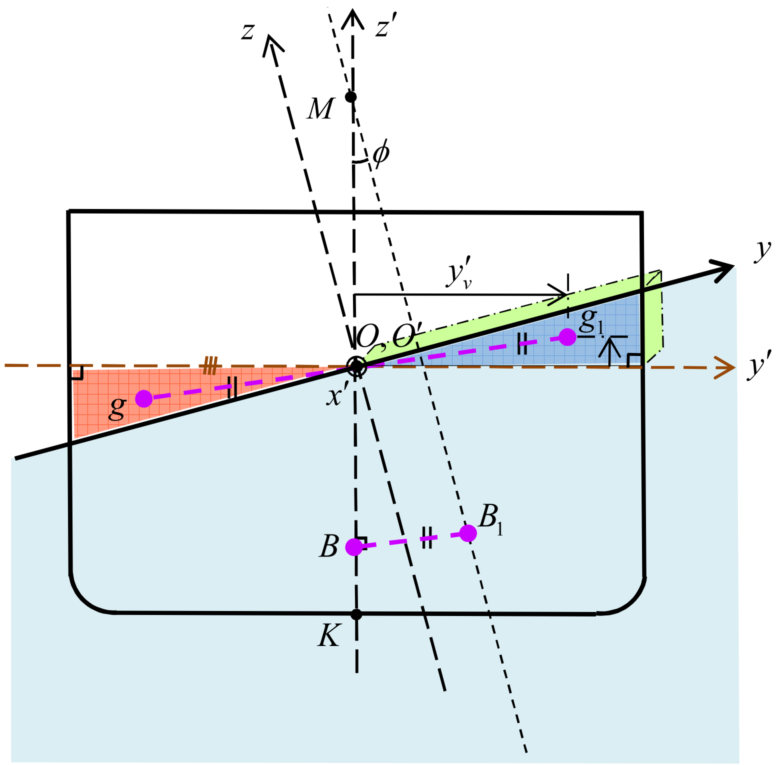
$$= \int_{star}^{0}\int_{aft}^{fore} \frac{y' \cdot \tan\phi}{2} \cdot dv_s$$

$$= \int_{star}^{0}\int_{aft}^{fore} \frac{y' \cdot \tan\phi}{2} \cdot y' \cdot \tan\phi \cdot dx' \cdot dy'$$

$$= \frac{\tan^2\phi}{2}\int_{star}^{0}\int_{aft}^{fore} y' \cdot y' \cdot dx' \cdot dy'$$

Derivation of BM$_T$ (11/12)

$$BM = \frac{1}{\nabla \cdot \tan\phi}\left(\underbrace{y'_{vp} \cdot v_p}_{(A)} + \underbrace{y'_{vs} \cdot v_s}_{(B)} + \left(\underbrace{z'_{vp} \cdot v_p}_{(C)} + \underbrace{z'_{vs} \cdot v_s}_{(D)} \right) \cdot \tan\phi \right)$$



(A)+(B) Moment about x' axis due to the z' component of the changed volume

$$y'_{vp} \cdot v_p + y'_{vs} \cdot v_s = \tan\phi \int_0^{port}\int_{aft}^{fore} y' \cdot y' \cdot dx'dy' + \tan\phi \int_{star}^{0}\int_{aft}^{fore} y' \cdot y' \cdot dx'dy'$$

$$= \tan\phi \int_{star}^{port}\int_{aft}^{fore} y' \cdot y' \cdot dx'dy'$$

$$= \tan\phi \int_{star}^{port}\int_{aft}^{fore} y'^2 \cdot dx'dy'$$

$$= \tan\phi \cdot I_T$$

(C)+(D) Moment about x' axis due to the y' component of the changed volume

$$z'_{vp} \cdot v_p + z'_{vs} \cdot v_s = \frac{\tan^2\phi}{2} \int_0^{port}\int_{aft}^{fore} y' \cdot y' \cdot dx' \cdot dy' + \frac{\tan^2\phi}{2} \int_{star}^{0}\int_{aft}^{fore} y' \cdot y' \cdot dx' \cdot dy'$$

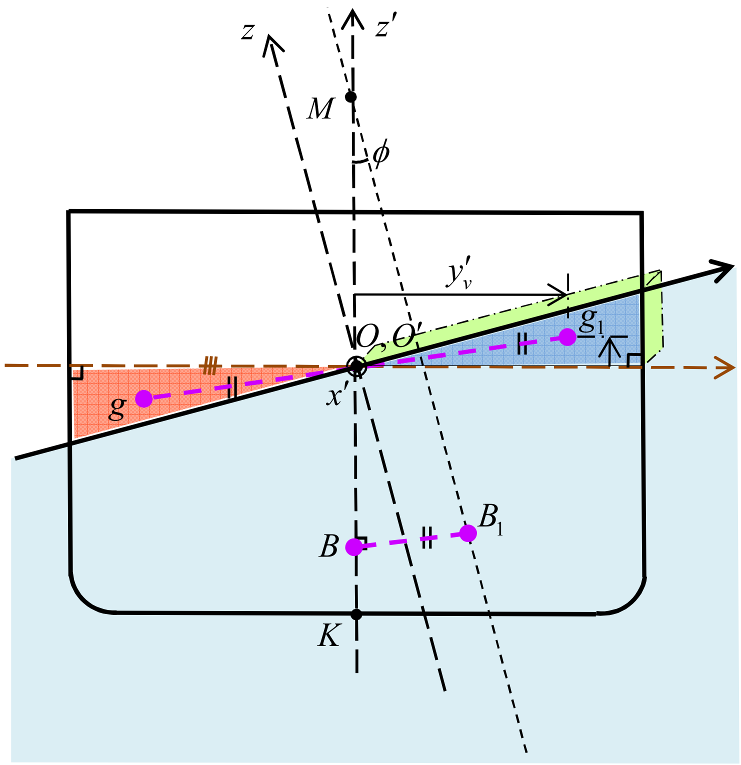
$$= \frac{\tan^2\phi}{2} \int_{star}^{port}\int_{aft}^{fore} y' \cdot y' \cdot dx' \cdot dy'$$

$$= \frac{\tan^2\phi}{2} \int_{star}^{port}\int_{aft}^{fore} y'^2 \cdot dx' \cdot dy'$$

$$= \frac{\tan^2\phi}{2} \cdot I_T$$

Derivation of BM_T (12/12)

$$BM = \frac{1}{\nabla \cdot \tan\phi}\left(\underbrace{y'_{vp}\cdot v_p}_{(A)} + \underbrace{y'_{vs}\cdot v_s}_{(B)} + \left(\underbrace{z'_{vp}\cdot v_p}_{(C)} + \underbrace{z'_{vs}\cdot v_s}_{(D)}\right)\cdot\tan\phi\right)$$



(A)+(B) Moment about x' axis due to the z' component of the changed volume

$$y'_{vp}\cdot v_p + y'_{vs}\cdot v_s = \tan\phi\cdot I_T$$

(C)+(D) Moment about x' axis due to the y' component of the changed volume

$$z'_{vp}\cdot v_p + z'_{vs}\cdot v_s = \frac{\tan^2\phi}{2}\cdot I_T$$

$$BM = \frac{1}{\nabla\cdot\tan\phi}\left(\tan\phi\cdot I_T + \frac{1}{2}.\tan^3\phi\cdot I_T\right)$$

$$\boxed{BM = \frac{I_T}{\nabla}\left(1+\frac{1}{2}\tan^2\phi\right)}$$

$$BM = \frac{I_T}{\nabla}\left(1+\cancel{\frac{1}{2}\tan^2\phi}\right)$$

if ϕ is small
$\tan^2\phi \approx \phi^2 = 0$

$$BM = \frac{I_T}{\nabla}$$

which is generally known as BM.

This BM does not consider the change of the center of buoyancy **in vertical direction**.

In order to distinguish those, we will indicate two BM as follows.

$$\boxed{BM_0 = \frac{I_T}{\nabla}(1+\frac{1}{2}\tan^2\phi)}$$

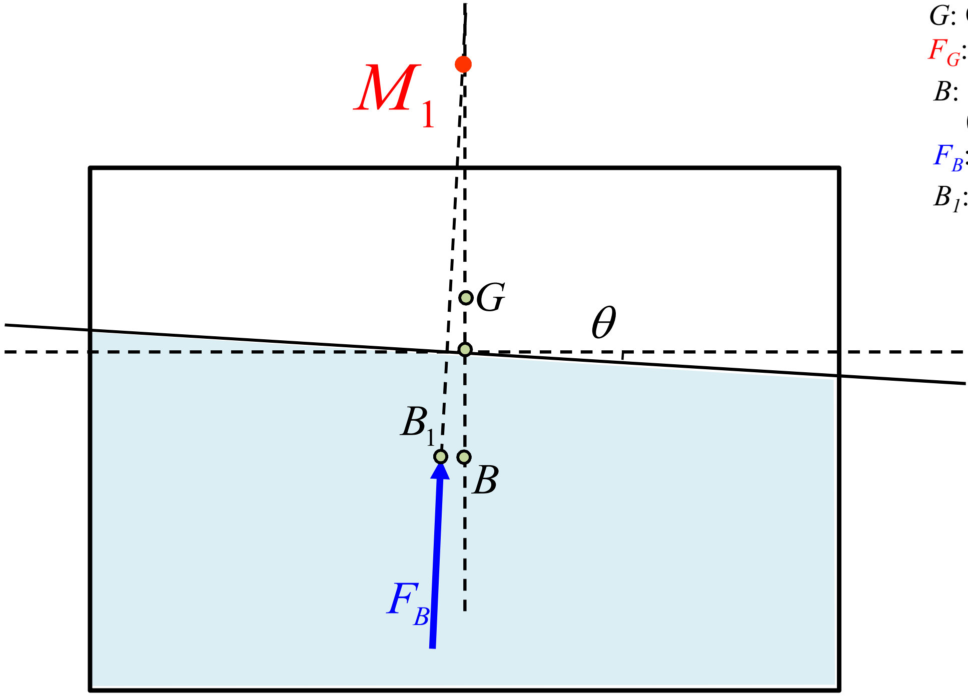
(**Considering** the change of the center of buoyancy in vertical direction)

$$\boxed{BM = \frac{I_T}{\nabla}}$$

(**Without considering** the change of the center of buoyancy in vertical direction)

Change of the Metacenter for Large Angle of Inclination

Metacenter (M)



G: Center of mass of a ship
F_G: Gravitational force of a ship
B: Center of buoyancy in the previous state (before inclination)
F_B: Buoyant force acting on a ship
B_1: New position of center of buoyancy after the ship has been inclined

Definition

Which vertical line is used to define the metacenter "M"?

Metacenter (*M*)

※ Metacenter "M" is valid for small angle of inclination.

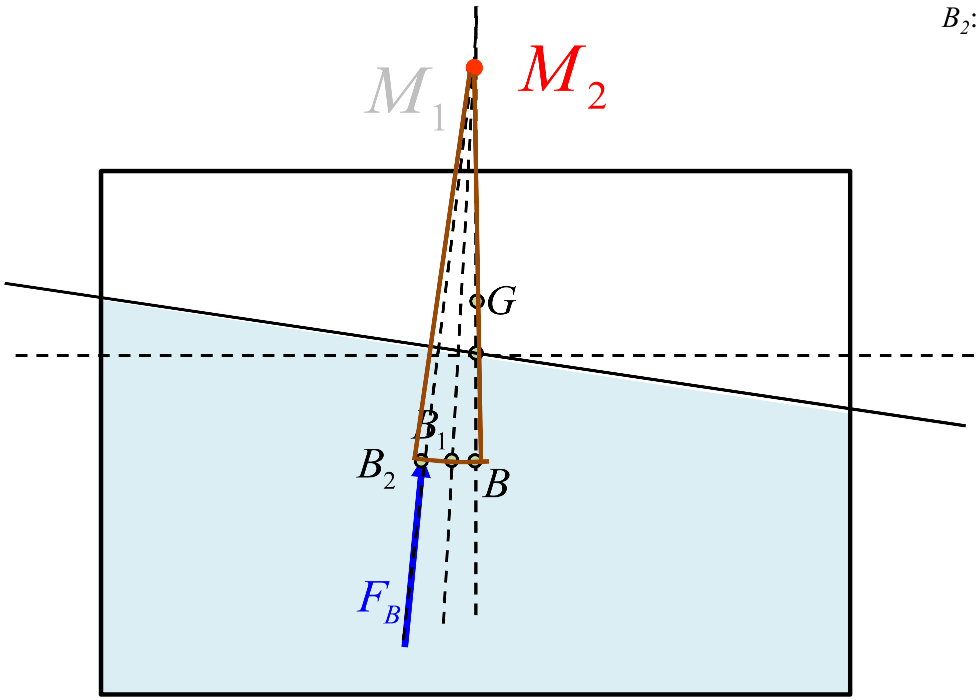
The **intersection point** of

a vertical line through the center of buoyancy **at a previous position** (***B***)

with a vertical line through the center of buoyancy **at the present position** (***B_1***)

* Dage, J.L. et al., Stability and Trim for the Ship's Officer, D. Van Nostrand Company, p.43, 1946

Metacenter (M) at Small Angles



B_2: New position of center of buoyancy after the ship has been inclined with small angle

M **remains at the same position** for small angles of inclination, up to **about 7~10 degrees**.

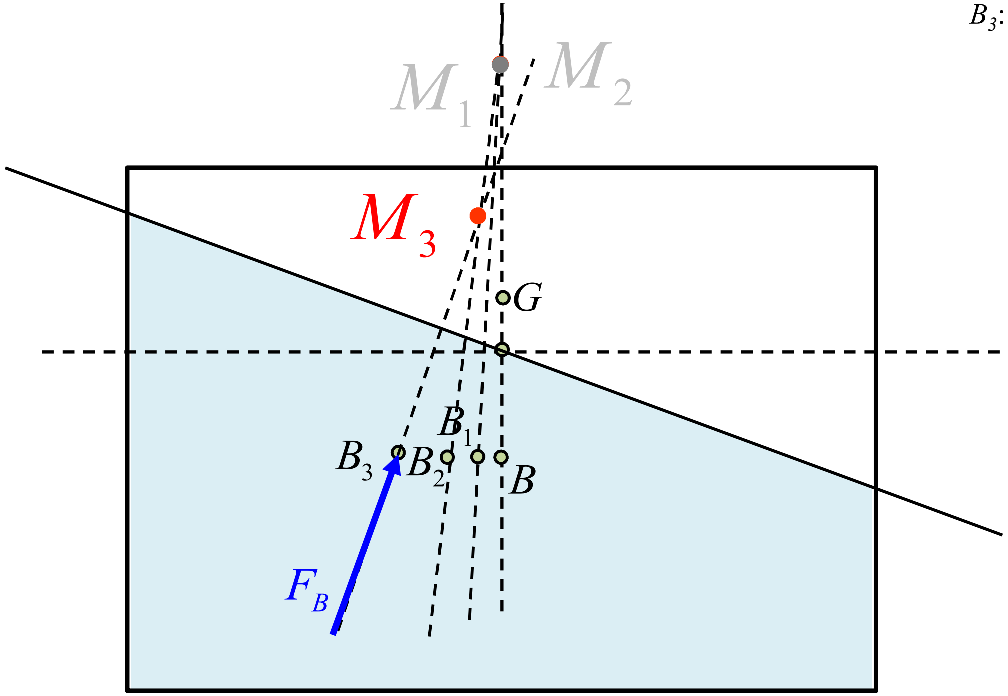
As the ship is inclined with a small angle, **B** moves **along the arc of a circle** whose center is at **M**.

The BM is **metacentric radius**.

The GM is **metacentric height**.

※ The term **meta** was selected as a prefix for center because its Greek meaning implies **movement**. Therefore, the **metacenter** is a **"moving center"**.

Metacenter (M) at Large Angles



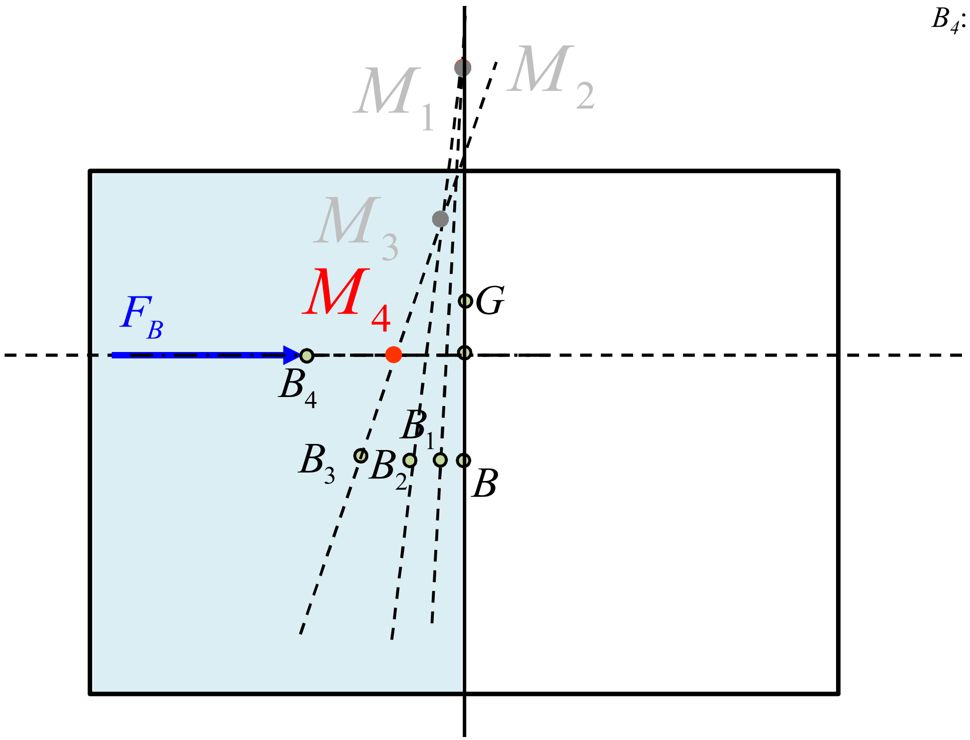
B_3: New position of center of buoyancy after the ship has been inclined with large angle

M **does not remain in the same position** for large angles of inclination **over 10 degrees**.

Thus, the metacenter, M, is only **valid for a small angle of inclination**.

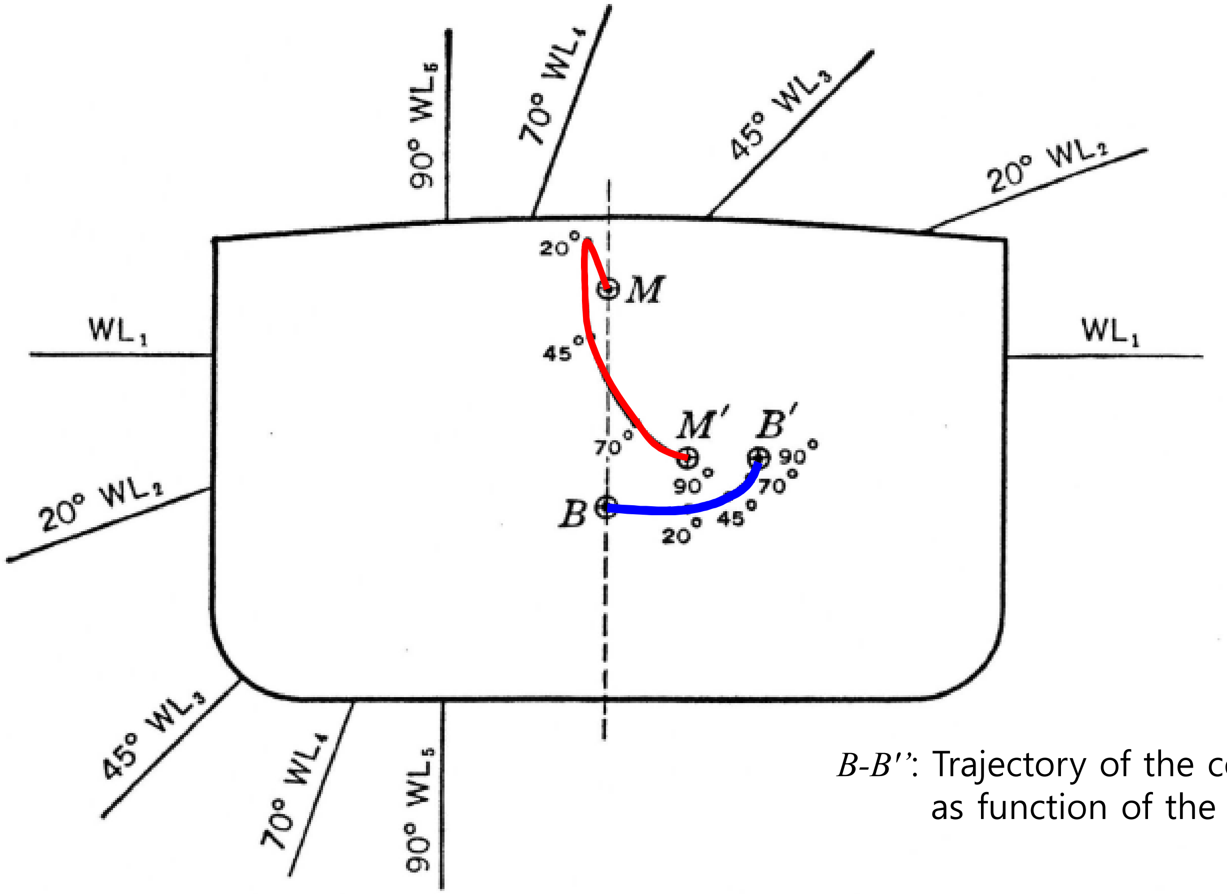
Meaning of Metacenter (M)

B_4: New position of center of buoyancy after the ship has been inclined with large angle



The term **meta** was selected as a prefix for center because its Greek meaning implies **movement**. The **metacenter** therefore is a **"moving center"**.

Example of Metacenter (M)

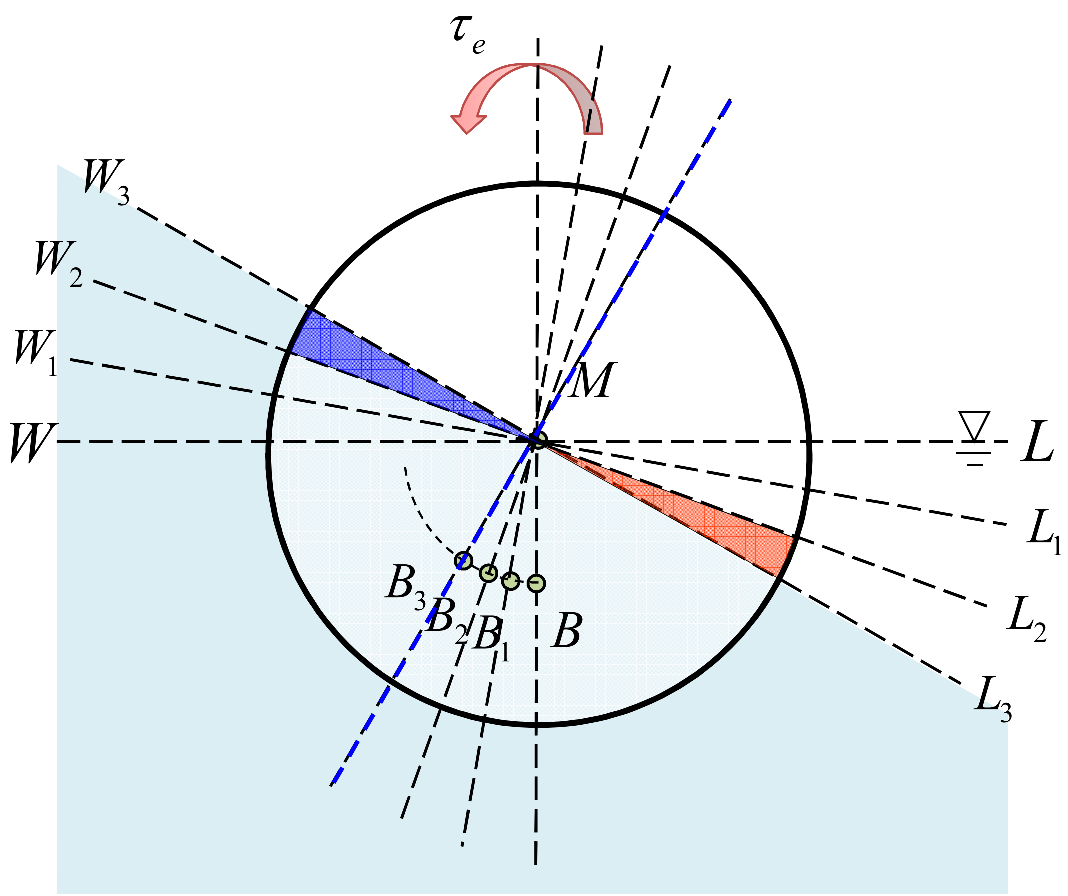


B-B'': Trajectory of the center of buoyancy as function of the inclination angle

Typical locus of metacenter and centers of buoyancy for an average form merchant ship

Metacenter (M) of Circular Section

Metacenter (M) : The **intersection point** of a vertical line through the center of buoyancy in the **previous state** with a vertical line through the center of buoyancy in the **present state**.

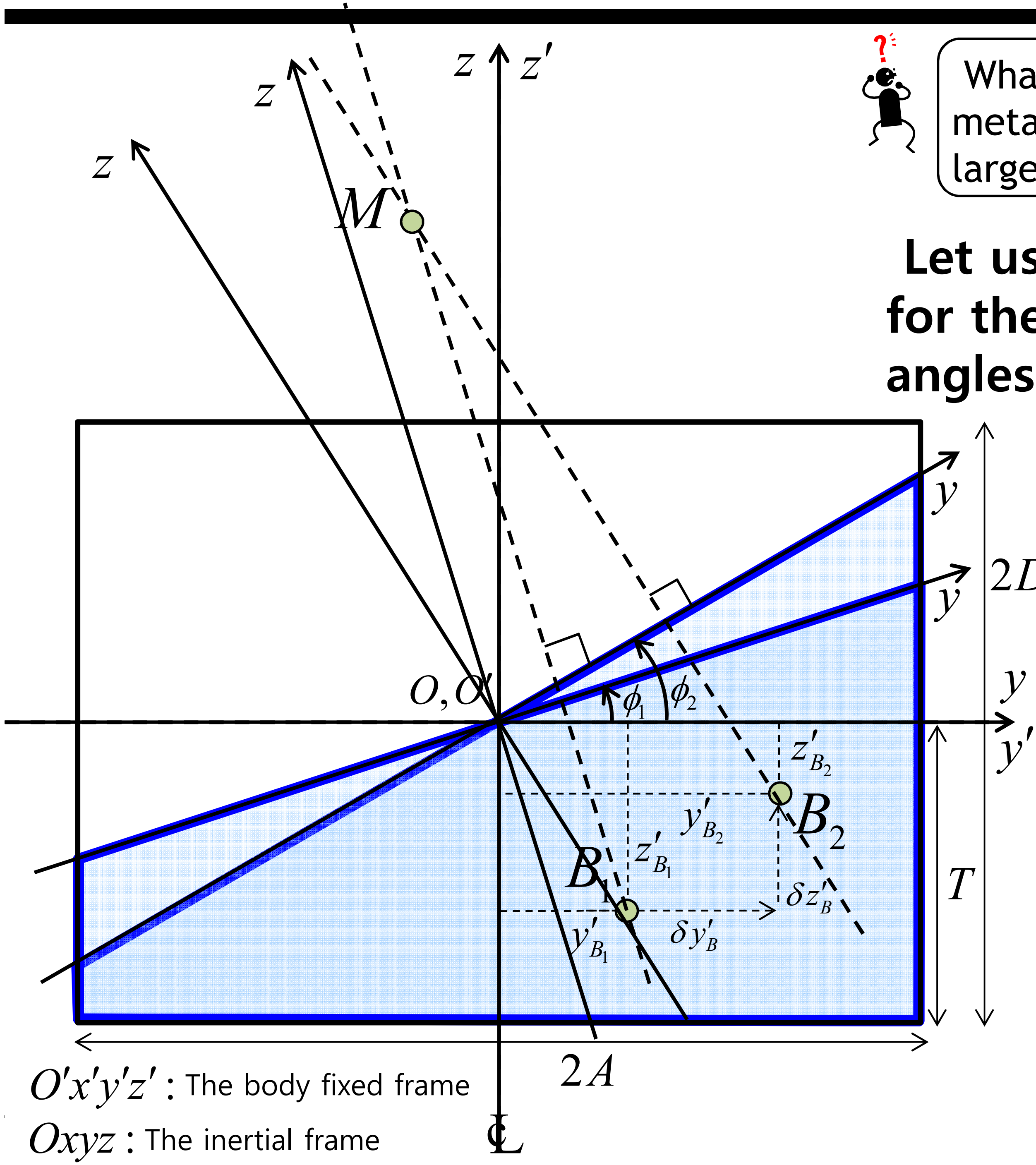


This figure shows a ship with a circular section.

The **M does not move either vertically or off the center line**.

Since the **shape of the immersed section (under water plane)** remains the **same**, the position of the metacenter **M** has the same value regardless of the inclination.

Metacenter (M) of a Box-shaped Ship (1/4)



What will be the location of the metacenter(M) for the box-shaped ship at large angles of heel?

Let us calculate the metacenter(M) for the barge-shaped ship at various angles of heel.

Given

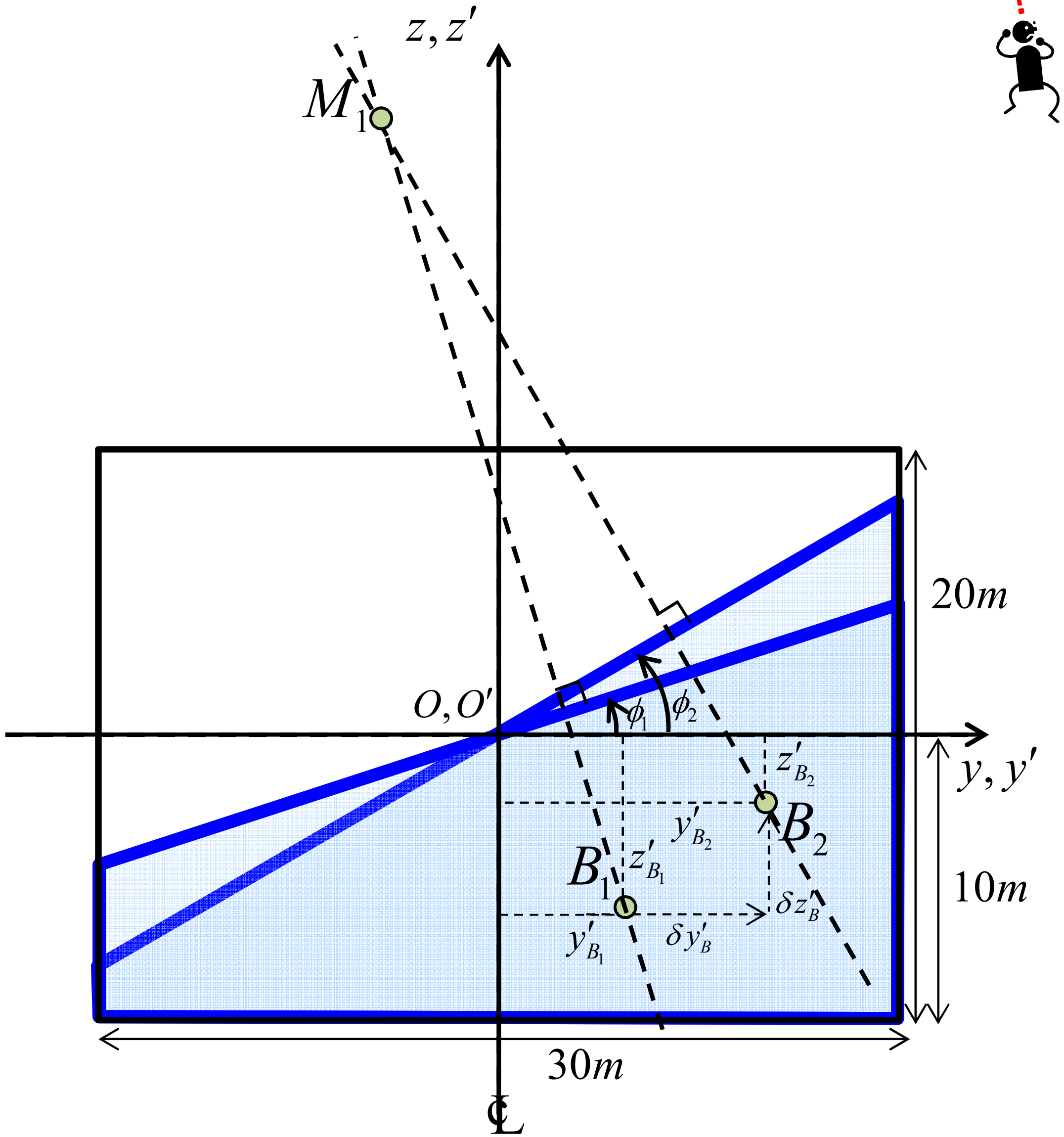
Geometry of the box-shaped ship
: A, D, T
Angles of heel
: angle of heel in the previous state, ϕ_1
angle of heel in the present state, ϕ_2

Find

Metacenter 'M' at given angles of heel

$O'x'y'z'$: The body fixed frame
$Oxyz$: The inertial frame

Metacenter (M) of a Box-shaped Ship (2/4)



What will be the location of the metacenter(M) for the box-shaped ship at large angles of heel?

$O'x'y'z'$: The body fixed frame

$Oxyz$: The inertial frame

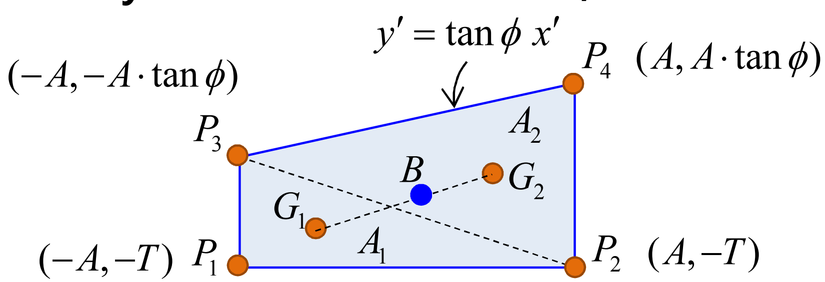
Metacenter (M) of a Box-shaped Ship (3/4)
- Center of Buoyancy at a Given Angle of Heel

What will be the location of the metacenter(M) for the box-shaped ship at large angles of heel?

(1) Center of Buoyancy(B) at a given angle of heel ϕ

※ Assumption: Deck will not be immersed and the bottom will not emerge.

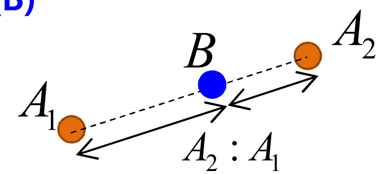
In the body fixed reference frame,

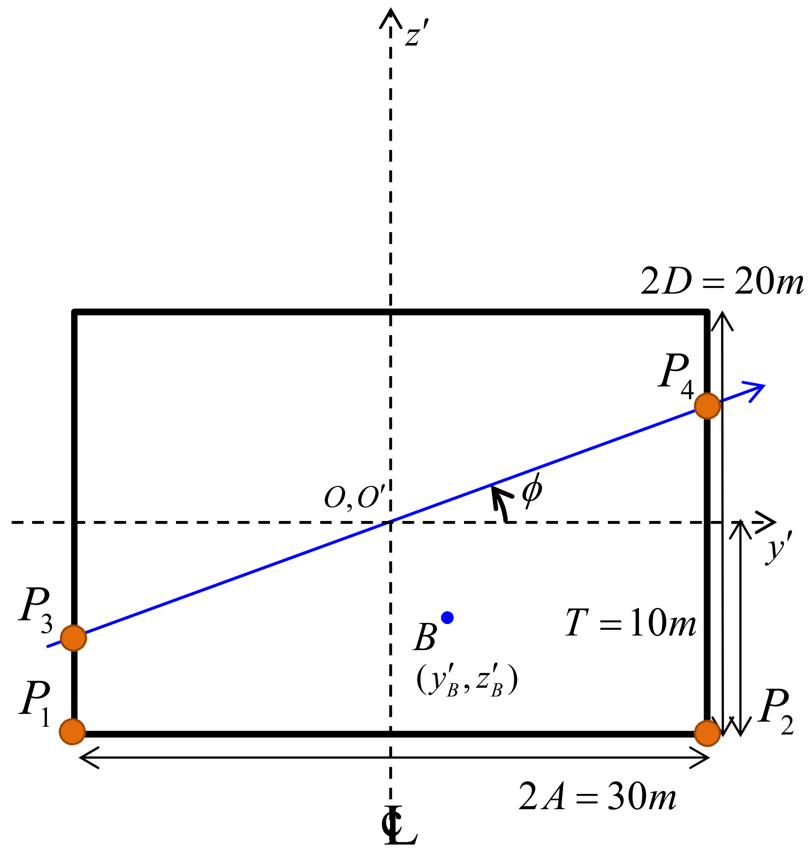


- **Area of A_1** $= (T - A\cdot\tan\phi)\cdot 2A\cdot\dfrac{1}{2}$
- **Area of A_2** $= (T + A\cdot\tan\phi)\cdot 2A\cdot\dfrac{1}{2}$
- **Centroid of $A_1(G_1)$** $= \left(\dfrac{-A+A-A}{3}, \dfrac{-T-T-A\cdot\tan\phi}{3}\right)$
- **Centroid of $A_2(G_2)$** $= \left(\dfrac{A-A+A}{3}, \dfrac{-T+A\cdot\tan\phi-A\cdot\tan\phi}{3}\right)$
- **Centroid of $P_1P_2P_3P_4$ = Center of Buoyancy (B)**

$$= \frac{A_1}{A_1+A_2}G_1 + \frac{A_2}{A_1+A_2}G_2$$

$$= \left(\frac{A^2\tan\phi}{3T}, \frac{-3T^2 + A^2\tan\phi}{6T}\right)$$





- Geometry of the box-shaped ship
: 2A=30m, 2D=20m, T=10m

$O'x'y'z'$: The body fixed frame
$Oxyz$: The inertial frame

Metacenter (M) of a Box-shaped Ship (3/4)
- Vertical Line through the Center of Buoyancy at a Given Angle of Heel

(2) Vertical line through the center of buoyancy(B) at a given angle of heel ϕ

※ Assumption: Deck will not be immersed and the bottom will not emerge.

- **Slope of the vertical line** $= -\dfrac{1}{\tan\phi}$

- **Vertical line through the center of buoyancy(B)**

$$z' = -\frac{1}{\tan\phi}(y' - y'_B) + z'_B$$

$$z' = -\frac{1}{\tan\phi}y' + \frac{1}{\tan\phi}y'_B + z'_B$$

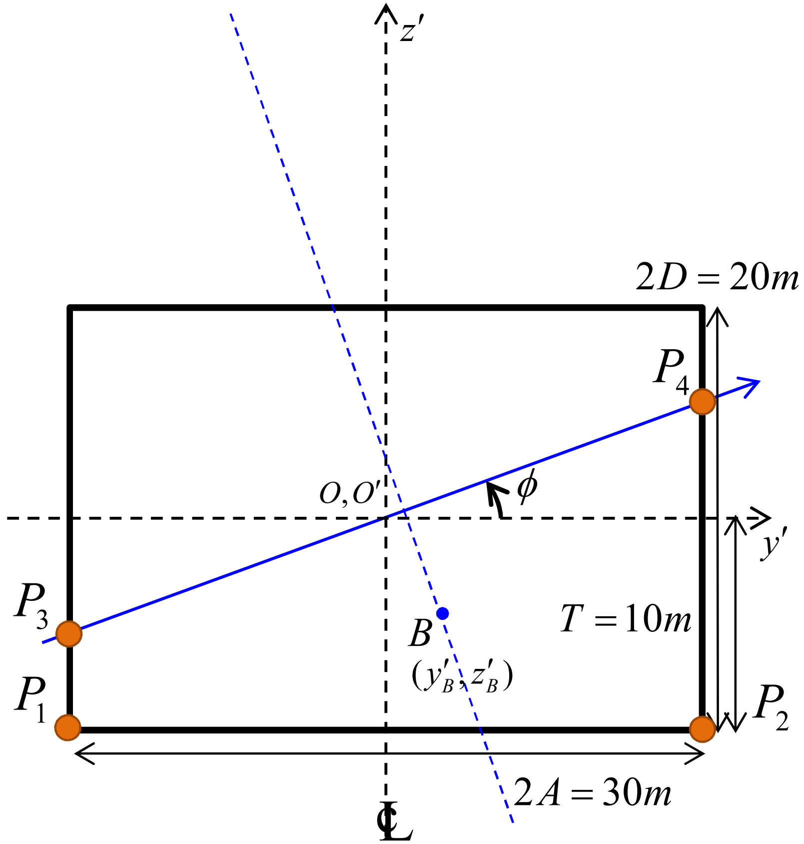
$$z' = -\frac{1}{\tan\phi}y' + \frac{1}{\tan\phi}\frac{A^2\tan\phi}{3T} + \frac{-3T^2 + A^2\tan\phi}{6T}$$

$$z' = -\frac{1}{\tan\phi}y' + \frac{A^2}{3T} + \frac{-3T^2 + A^2\tan\phi}{6T}$$

$$z' = -\frac{1}{\tan\phi}y' + \frac{2A^2 - 3T^2 + A^2\tan\phi}{6T}$$

- **If the angle of heel is zero degree, the vertical line is given as follows.**

$$z' = 0$$



- Geometry of the box-shaped ship
: 2A=30m, 2D=20m, T=10m

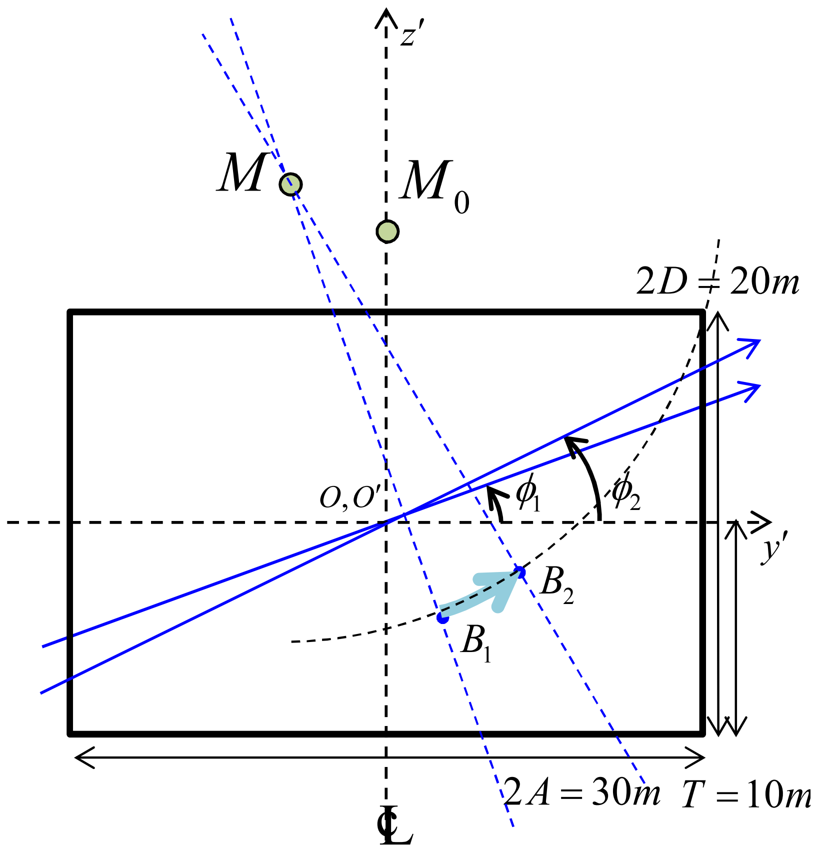
- Center of buoyancy(B) at a angle of heel ϕ

$$= \left(\frac{A^2\tan\phi}{3T}, \frac{-3T^2 + A^2\tan\phi}{6T}\right) \cdots\cdots(1)$$

$O'x'y'z'$: The body fixed frame
$Oxyz$: The inertial frame

Metacenter (M) of a Box-shaped Ship (3/4)
- Metacenter at Given Angles of Heel (1/4)

M_0: metacenter at upright condition



(3) Metacenter(M) at given angles of heel ϕ_1, ϕ_2

where, ϕ_1 is the angle of heel in the previous state
ϕ_2 is the angle of heel in the present state

※ Assumption: Deck will not be immersed and the bottom will not emerge.

From the equation (2), the vertical lines at each position are obtained as follows.

- Vertical line through the center of buoyancy(B_1) at a given angle of heel ϕ_1 :

$$z' = -\frac{1}{\tan\phi_1} y' + \frac{2A^2 - 3T^2 + A^2 \tan\phi_1}{6T}$$

- Vertical line through the center of buoyancy(B_2) at a given angle of heel ϕ_2 :

$$z' = -\frac{1}{\tan\phi_2} y' + \frac{2A^2 - 3T^2 + A^2 \tan\phi_2}{6T}$$

If the two lines are described as $z' = ay' + b$ and $z' = cy' + d$, the intersection point is obtained as follows.

- Intersection point: $\left(\frac{d-b}{a-c}, \frac{ad-bc}{a-c} \right)$

- Geometry of the box-shaped ship
: 2A=30m, 2D=20m, T=10m
- Center of buoyancy(B) at a angle of heel ϕ

$$= \left(\frac{A^2 \tan\phi}{3T}, \frac{-3T^2 + A^2 \tan\phi}{6T} \right) \cdots\cdots(1)$$

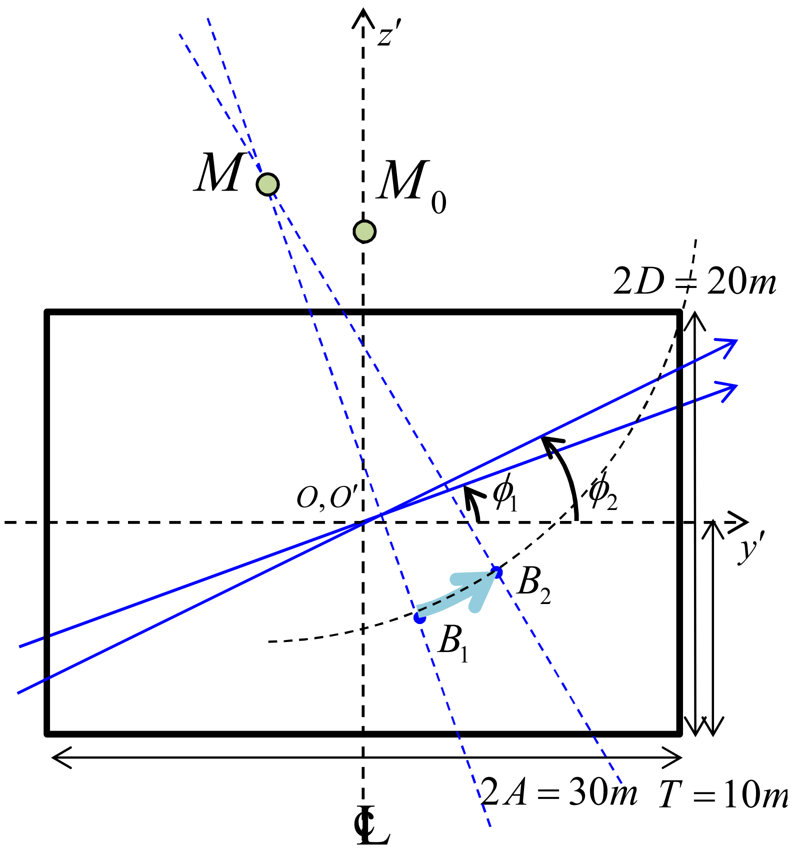
- Vertical line through the center of buoyancy (B) at a angle of heel ϕ

$$z' = -\frac{1}{\tan\phi} y' + \frac{2A^2 - 3T^2 + A^2 \tan\phi}{6T} \cdots\cdots(2)$$

$O'x'y'z'$: The body fixed frame
$Oxyz$: The inertial frame

Metacenter (M) of a Box-shaped Ship (3/4)
- Metacenter at Given Angles of Heel (2/4)

M_0: metacenter at upright condition



(3) Metacenter(M) at given angles of heel ϕ_1, ϕ_2 ,

where, ϕ_1 is the angle of heel in the previous state
ϕ_2 is the angle of heel in the present state

$$\rightarrow \frac{d-b}{a-c} = \frac{\dfrac{2A^2 - 3T^2 + A^2 \tan\phi_2}{6T} - \dfrac{2A^2 - 3T^2 + A^2 \tan\phi_1}{6T}}{-\dfrac{1}{\tan\phi_1} + \dfrac{1}{\tan\phi_2}}$$

$$= \frac{(2A^2 - 3T^2 + A^2 \tan\phi_2) - (2A^2 - 3T^2 + A^2 \tan\phi_1)}{6T(-\tan\phi_2 + \tan\phi_1)} (\tan\phi_1 \cdot \tan\phi_2)$$

$$= \frac{(A^2 \tan\phi_2) - (A^2 \tan\phi_1)}{6T(-\tan\phi_2 + \tan\phi_1)} (\tan\phi_1 \cdot \tan\phi_2)$$

$$= \frac{A^2 (\tan\phi_2 - \tan\phi_1)}{6T(-\tan\phi_2 + \tan\phi_1)} (\tan\phi_1 \cdot \tan\phi_2)$$

$$= \frac{A^2}{6T} (\tan\phi_1 \cdot \tan\phi_2)$$

- Geometry of the box-shaped ship
: 2A=30m, 2D=20m, T=10m

- Center of buoyancy(B) at a angle of heel ϕ

$$= \left(\frac{A^2 \tan\phi}{3T}, \frac{-3T^2 + A^2 \tan\phi}{6T} \right) \cdots\cdots(1)$$

- Vertical line through the center of buoyancy (B) at a angle of heel ϕ

$$z' = -\frac{1}{\tan\phi} y' + \frac{2A^2 - 3T^2 + A^2 \tan\phi}{6T} \cdots\cdots(2)$$

$O'x'y'z'$: The body fixed frame

$Oxyz$: The inertial frame

Metacenter (M) of a Box-shaped Ship (3/4)
- Metacenter at Given Angles of Heel (3/4)

M_0: metacenter at upright condition

(3) Metacenter(M) at given angles of heel ϕ_1, ϕ_2 ,

where, ϕ_1 is the angle of heel **in the previous state**
ϕ_2 is the angle of heel **in the present state**

$$\left(\frac{d-b}{a-c}, \frac{ad-bc}{a-c}\right) \quad , \frac{d-b}{a-c} = \frac{A^2}{6T}(\tan\phi_1 \cdot \tan\phi_2)$$

$$\text{where}\ , a = -\frac{1}{\tan\phi_1},\ b = \frac{2A^2-3T^2+A^2\tan\phi_1}{6T},\ c = -\frac{1}{\tan\phi_2},\ d = \frac{2A^2-3T^2+A^2\tan\phi_2}{6T}$$

$$\rightarrow \frac{ad-bc}{a-c} = \frac{\left(-\frac{1}{\tan\phi_1}\right)\frac{2A^2-3T^2+A^2\tan\phi_2}{6T} - \frac{2A^2-3T^2+A^2\tan\phi_1}{6T}\left(-\frac{1}{\tan\phi_2}\right)}{-\frac{1}{\tan\phi_1}+\frac{1}{\tan\phi_2}}$$

$$= \frac{(-\tan\phi_2)\frac{2A^2-3T^2+A^2\tan\phi_2}{6T} - \frac{2A^2-3T^2+A^2\tan\phi_1}{6T}(-\tan\phi_1)}{(-\tan\phi_2+\tan\phi_1)}$$

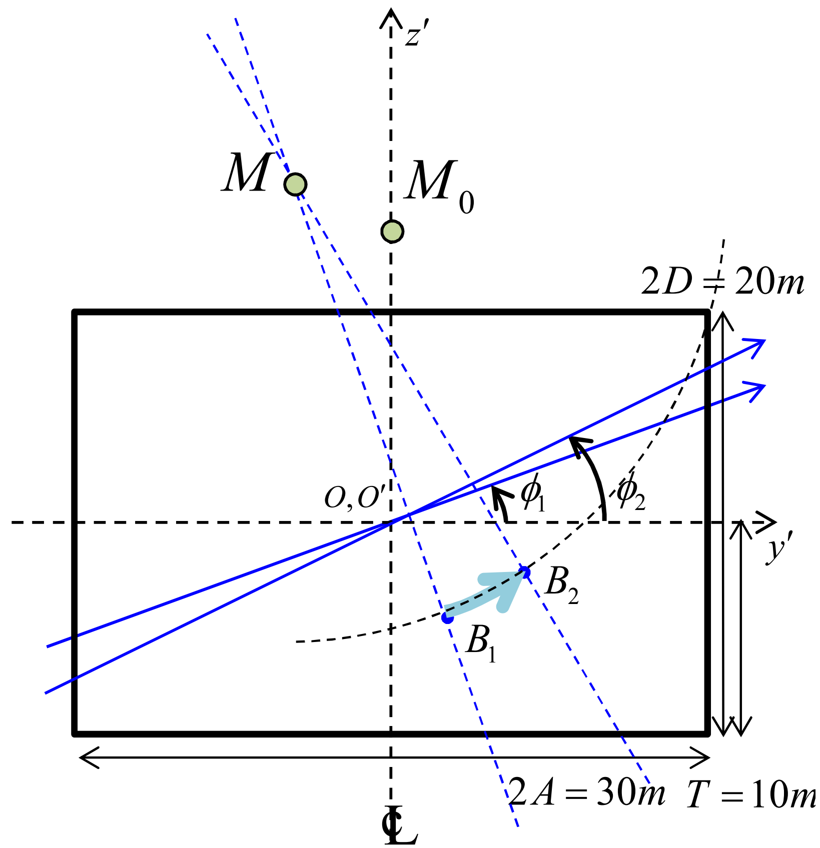
$$= \frac{-\tan\phi_2(2A^2-3T^2+A^2\tan\phi_2)+\tan\phi_1(2A^2-3T^2+A^2\tan\phi_1)}{6T(-\tan\phi_2+\tan\phi_1)}$$

$$= \frac{-\tan\phi_2(2A^2-3T^2+A^2\tan\phi_2)+\tan\phi_1(2A^2-3T^2+A^2\tan\phi_1)}{6T(-\tan\phi_2+\tan\phi_1)}$$

$$= \frac{(-\tan\phi_2+\tan\phi_1)(2A^2-3T^2)+A^2(-\tan^2\phi_2+\tan^2\phi_1)}{6T(-\tan\phi_2+\tan\phi_1)}$$

$$= \frac{(-\tan\phi_2+\tan\phi_1)(2A^2-3T^2)+A^2(-\tan\phi_2+\tan\phi_1)(\tan\phi_2+\tan\phi_1)}{6T(-\tan\phi_2+\tan\phi_1)}$$

$$= \frac{(2A^2-3T^2)+A^2(\tan\phi_2+\tan\phi_1)}{6T}$$



- **Geometry of the box-shaped ship**
: 2A=30m, 2D=20m, T=10m
- **Center of buoyancy(B) at a angle of heel ϕ**

$$= \left(\frac{A^2\tan\phi}{3T}, \frac{-3T^2+A^2\tan\phi}{6T}\right) \cdots\cdots(1)$$

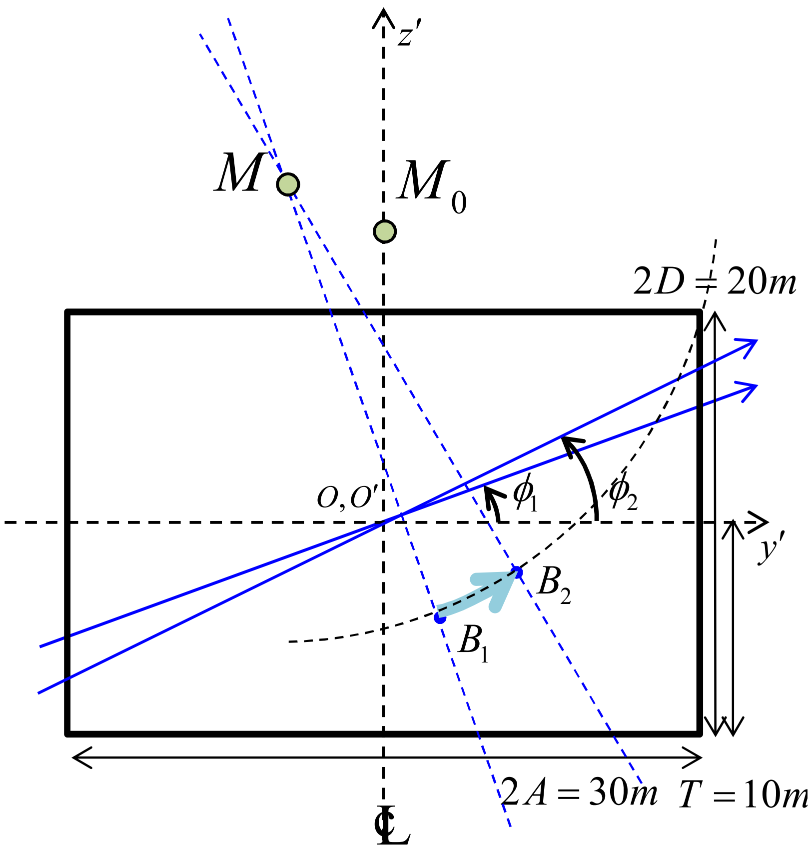
- **Vertical line through the center of buoyancy (B) at a angle of heel ϕ**

$$z' = -\frac{1}{\tan\phi}y' + \frac{2A^2-3T^2+A^2\tan\phi}{6T} \cdots\cdots(2)$$

$O'x'y'z'$: The body fixed frame
$Oxyz$: The inertial frame

Metacenter (M) of a Box-shaped Ship (3/4)
- Metacenter at Given Angles of Heel (4/4)

M_0: metacenter at upright condition



(3) Metacenter(M) at given angles of heel ϕ_1, ϕ_2 ,

where, ϕ_1 is the angle of heel in the previous state
ϕ_2 is the angle of heel in the present state

- Intersection point:

$$\left(\frac{d-b}{a-c}, \frac{ad-bc}{a-c}\right) = \left(\frac{A^2}{6T}(\tan\phi_1 \cdot \tan\phi_2), \frac{(2A^2-3T^2)+A^2(\tan\phi_2+\tan\phi_1)}{6T}\right)$$

where , $a = -\dfrac{1}{\tan\phi_1}$, $b = \dfrac{2A^2-3T^2+A^2\tan\phi_1}{6T}$, $c = -\dfrac{1}{\tan\phi_2}$, $d = \dfrac{2A^2-3T^2+A^2\tan\phi_2}{6T}$

By definition of the metacenter(M), the metacenter is the intersection point.

$$\therefore \text{Metacenter } (M) = \left(\frac{A^2}{6T}(\tan\phi_1 \cdot \tan\phi_2), \frac{(2A^2-3T^2)+A^2(\tan\phi_2+\tan\phi_1)}{6T}\right)$$

If $\phi_1, \phi_2 \ll 1, \quad \tan\phi_1 \simeq 0, \quad \tan\phi_2 \simeq 0$

Thus, y' component of metacenter is approximately zero.

$$\frac{A^2}{6T}(\tan\phi_1 \cdot \tan\phi_2) \approx 0$$

- Geometry of the box-shaped ship
: 2A=30m, 2D=20m, T=10m
- Center of buoyancy(B) at a angle of heel ϕ

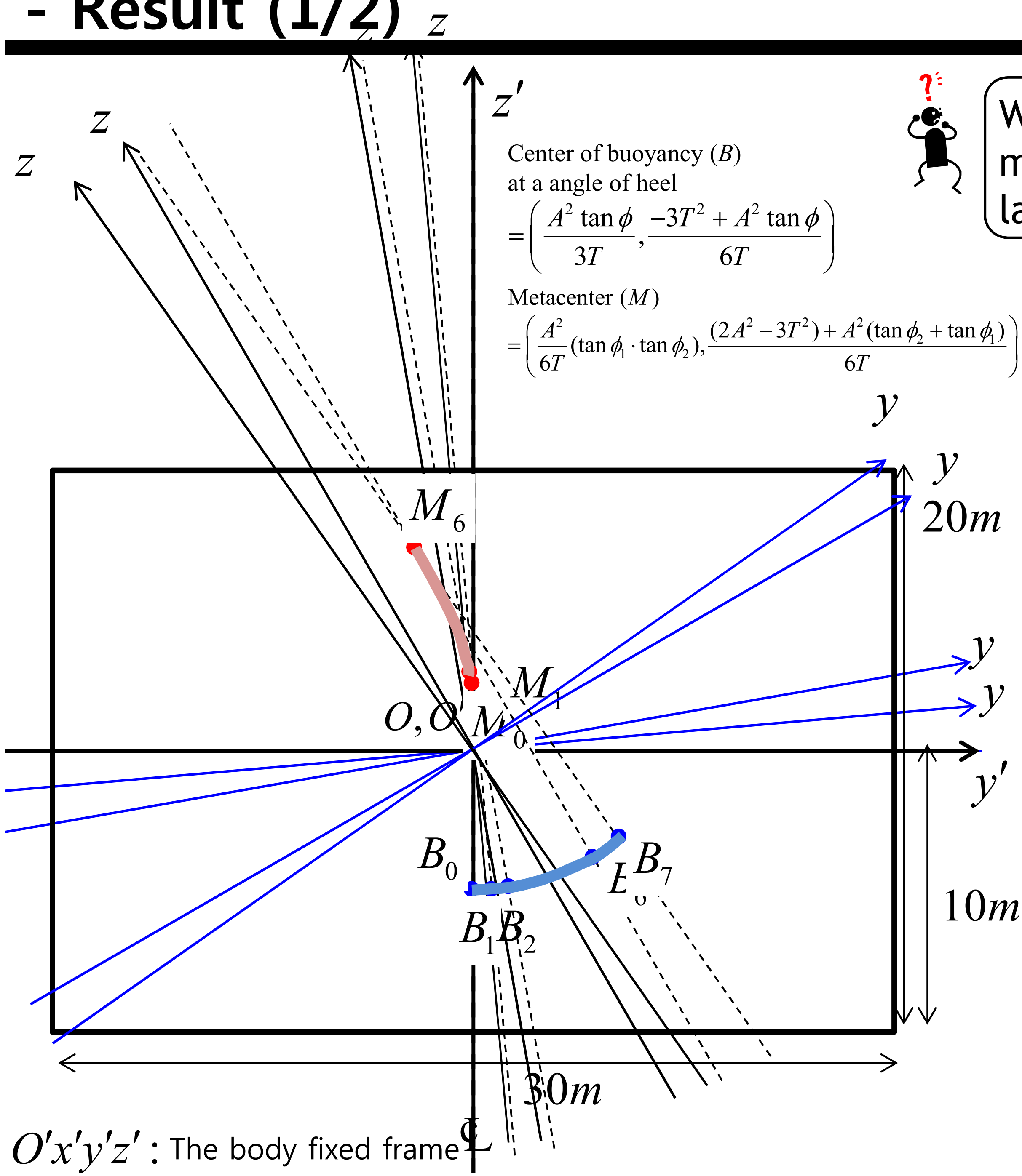
$$= \left(\frac{A^2\tan\phi}{3T}, \frac{-3T^2+A^2\tan\phi}{6T}\right) \quad \cdots\cdots(1)$$

- Vertical line through the center of buoyancy (B) at a angle of heel ϕ

$$z' = -\frac{1}{\tan\phi}y' + \frac{2A^2-3T^2+A^2\tan\phi}{6T} \quad \cdots\cdots(2)$$

$O'x'y'z'$: The body fixed frame
$Oxyz$: The inertial frame

Metacenter (M) of a Box-shaped Ship (4/4) - Result (1/2)



Center of buoyancy (B) at a angle of heel

$$=\left(\frac{A^2 \tan\phi}{3T}, \frac{-3T^2 + A^2 \tan\phi}{6T}\right)$$

Metacenter (M)

$$=\left(\frac{A^2}{6T}(\tan\phi_1 \cdot \tan\phi_2), \frac{(2A^2 - 3T^2) + A^2(\tan\phi_2 + \tan\phi_1)}{6T}\right)$$

$O'x'y'z'$: The body fixed frame

$Oxyz$: The inertial frame

What will be the location of the metacenter(M) for the box-shaped ship at large angles of heel?

ϕ_1 : angle of heel at present position
ϕ_2 : angle of heel at next position
(y'_B, z'_B) : center of buoyancy at present position(B)
(y'_M, z'_M) : metacenter(M)

The calculation result of the metacenter(M) and center of buoyancy(B) is as follows;

$\phi_1 = 0^\circ, \phi_2 = 5^\circ$: $(y'_{B_0}, z'_{B_0}) = (0.000, \ -5.000)$, $(y'_{B_1}, z'_{B_1}) = (0.656, \ -4.971)$
$(y'_{M_0}, z'_{M_0}) = (\underline{0.000}, \ 2.529)$

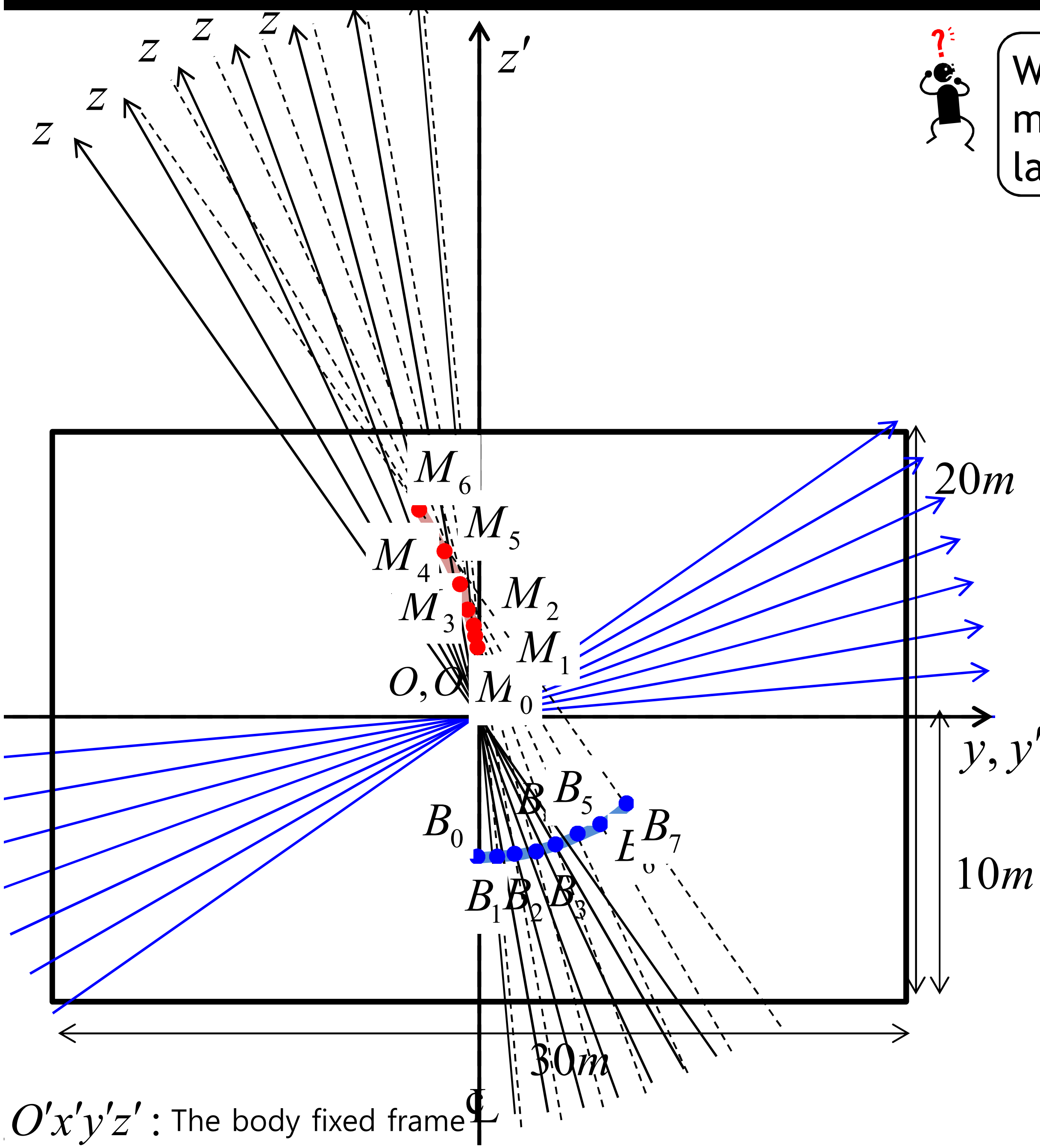
$\phi_1 = 5^\circ, \phi_2 = 10^\circ$: $(y'_{B_1}, z'_{B_1}) = (0.656, \ -4.971)$, $(y'_{B_2}, z'_{B_2}) = (1.322, \ -4.883)$
$(y'_{M_1}, z'_{M_1}) = (\underline{-0.015}, \ 2.703)$

: At small angle of heel, y′ component of metacenter is approximately zero.

$\phi_1 = 30^\circ, \phi_2 = 35^\circ$: $(y'_{B_6}, z'_{B_6}) = (4.330, \ -3.750)$, $(y'_{B_7}, z'_{B_7}) = (5.252, \ -3.161)$
$(y'_{M_6}, z'_{M_6}) = (\underline{-3.392}, \ 9.182)$

: At large angle of heel, y′ component of metacenter is not negligible.

Metacenter (M) of a Box-shaped Ship (4/4) - Result (2/2)



$O'x'y'z'$: The body fixed frame

$Oxyz$: The inertial frame

What will be the location of the metacenter(M) for the box-shaped ship at large angles of heel?

ϕ_1 : angle of heel at present position
ϕ_2 : angle of heel at next position
(y'_B, z'_B) : center of buoyancy at present position(B)
(y'_M, z'_M) : metacenter(M)

The calculation result of the metacenter(M) and center of buoyancy(B) is as follows;

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$(y'_{M_0}, z'_{M_0}) = (0.000, \ 2.529)$

$\phi_1 = 5^\circ, \phi_2 = 10^\circ$: $(y'_{B_1}, z'_{B_1}) = (0.656, \ -4.971)$, $(y'_{B_2}, z'_{B_2}) = (1.322, \ -4.883)$
$(y'_{M_1}, z'_{M_1}) = (-0.015, \ 2.703)$

$\phi_1 = 10^\circ, \phi_2 = 15^\circ$: $(y'_{B_2}, z'_{B_2}) = (1.322, -4.883)$, $(y'_{B_3}, z'_{B_3}) = (2.010, -4.731)$
$(y'_{M_2}, z'_{M_2}) = (-0.079, \ 3.063)$

$\phi_1 = 15^\circ, \phi_2 = 20^\circ$: $(y'_{B_3}, z'_{B_3}) = (2.010, -4.731)$, $(y'_{B_4}, z'_{B_4}) = (2.730, \ -4.503)$
$(y'_{M_3}, z'_{M_3}) = (-0.231, \ 3.632)$

$\phi_1 = 20^\circ, \phi_2 = 25^\circ$: $(y'_{B_4}, z'_{B_4}) = (2.730, \ -4.503)$, $(y'_{B_5}, z'_{B_5}) = (3.497, \ -4.185)$
$(y'_{M_4}, z'_{M_4}) = (-1.054, \ 5.575)$

$\phi_1 = 25^\circ, \phi_2 = 30^\circ$: $(y'_{B_5}, z'_{B_5}) = (3.497, \ -4.185)$, $(y'_{B_6}, z'_{B_6}) = (4.330, \ -3.750)$
$(y'_{M_5}, z'_{M_5}) = (-1.937, \ 7.105)$

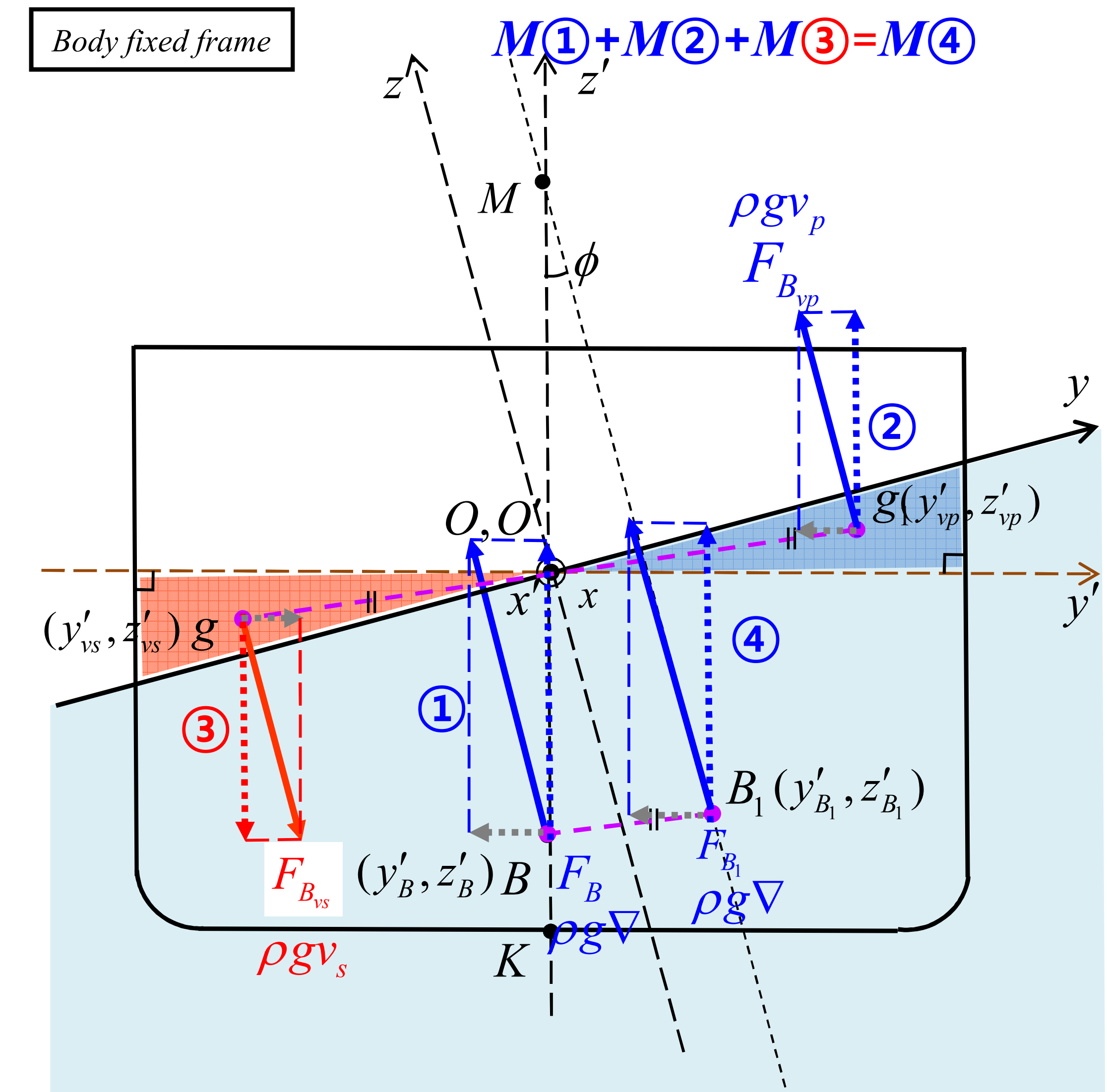
$\phi_1 = 30^\circ, \phi_2 = 35^\circ$: $(y'_{B_6}, z'_{B_6}) = (4.330, \ -3.750)$, $(y'_{B_7}, z'_{B_7}) = (5.252, \ -3.161)$
$(y'_{M_6}, z'_{M_6}) = (-3.392, \ 9.182)$

Another Approach to Derive the Following Formula

$$\begin{cases} \delta y'_B \cdot \nabla = y'_{vp} \cdot v_p + y'_{vs} \cdot v_s \\ \delta z'_B \cdot \nabla = z'_{vp} \cdot v_p + z'_{vs} \cdot v_s \end{cases}$$

Derivation of BM_T (1/2)

$$M=\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ r_x & r_y & r_z\\ F_x & F_y & F_z\end{bmatrix}=\underbrace{\boxed{\mathbf{i}(r_y\cdot F_z-r_z\cdot F_y)}}_{M_{O,x'}}+\mathbf{j}(-r_x\cdot F_z+r_z\cdot F_x)+\mathbf{k}(r_x\cdot F_y-r_y\cdot F_x)$$



B: The center of buoyancy before inclination
B₁: The center of buoyancy after inclination
∇: Displacement volume
v: Changed displacement volume (wedge)
BB₁: Distance of changed center of buoyancy
gg₁: Distance of changed center of wedge

For the convenience of calculation, the forces are decomposed in body fixed frame.

Moment about x' axis through point O

1. Moment about x' axis due to the z′ component of the changed buoyant force

M④=M①+M②+M③

$$y'_{B_1}\cdot F_{B_1,z'}=y'_B\cdot F_{B,z'}+y'_{vp}\cdot F_{B_{vp},z'}+y'_{vs}\cdot F_{B_{vs},z'}$$

$$y'_{B_1}\cdot(\rho g\nabla\cdot\cos\phi)$$
$$=y'_B\cdot(\rho g\nabla\cdot\cos\phi)+y'_{vp}\cdot(\rho g v_p\cdot\cos\phi)+y'_{vs}(\rho g v_s\cdot\cos\phi)$$

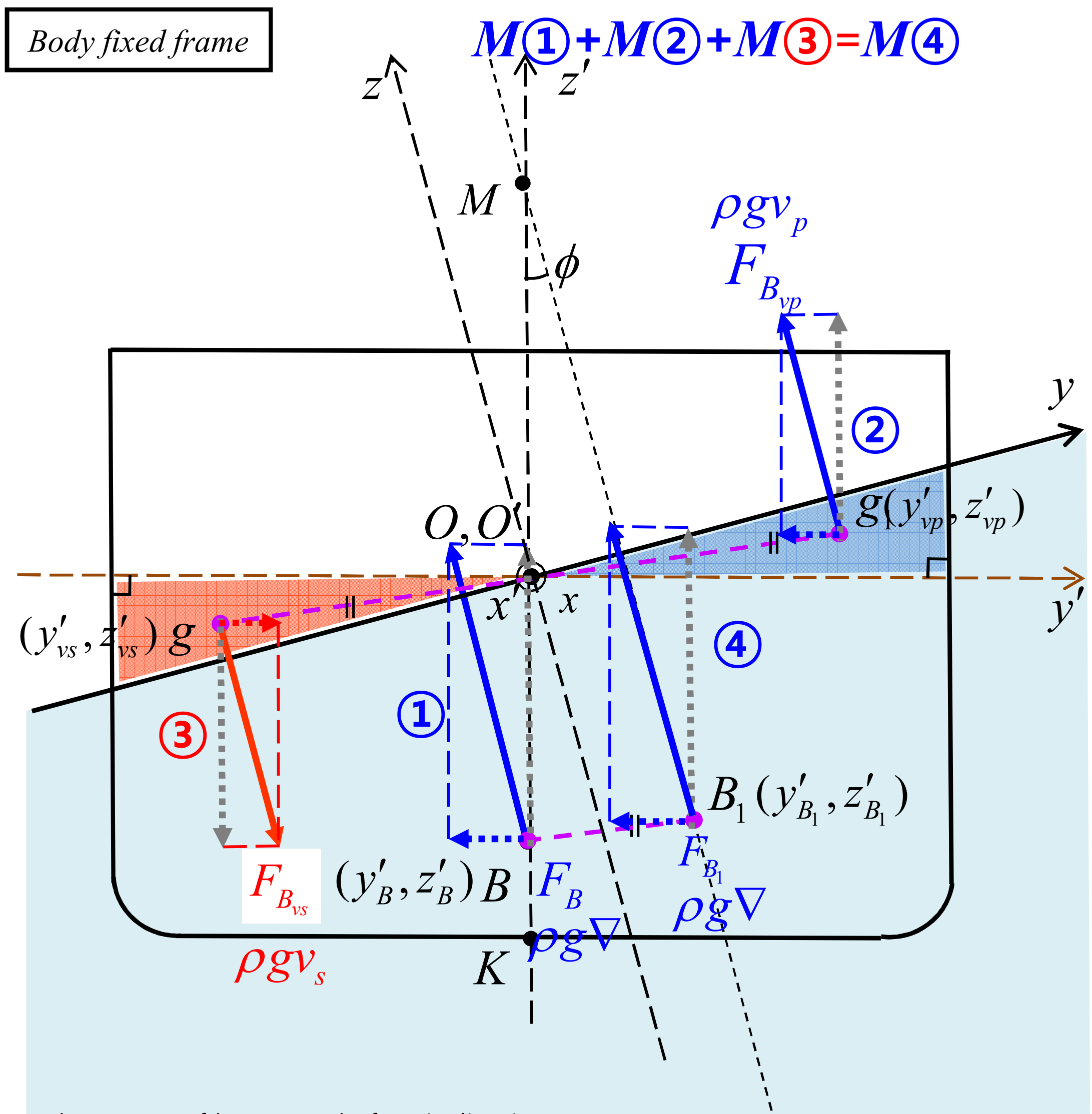
$$\underbrace{\boxed{(y'_{B_1}-y'_B)}}_{=\delta y'_B}\cdot(\rho g\nabla\cdot\cos\phi)=y'_{vp}\cdot(\rho g v_p\cdot\cos\phi)+y'_{vs}(\rho g v_s\cdot\cos\phi)$$

$$(\delta y'_B)\cdot(\cancel{\rho g}\nabla\cdot\cancel{\cos\phi})=y'_{vp}\cdot(\cancel{\rho g}v_p\cdot\cancel{\cos\phi})+y'_{vs}(\cancel{\rho g}v_s\cdot\cancel{\cos\phi})$$

$$\delta y'_B\cdot\nabla=y'_{vp}\cdot v_p+y'_{vs}\cdot v_s$$

Derivation of BM_T (2/2)

$$M = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{bmatrix} = \underbrace{\mathbf{i}(r_y \cdot F_z - r_z \cdot F_y)}_{M_{O,x'}} + \mathbf{j}(-r_x \cdot F_z + r_z \cdot F_x) + \mathbf{k}(r_x \cdot F_y - r_y \cdot F_x)$$



B: The center of buoyancy before inclination

B_1: The center of buoyancy after inclination

∇: Displacement volume

v: Changed displacement volume (wedge)

BB_1: Distance of changed center of buoyancy

gg_1: Distance of changed center of wedge

For the convenience of calculation, the forces are decomposed in body fixed frame.

Moment about x' axis through point O

2. Moment about x' axis due to the y′ component of the changed buoyant force

M④=M①+M②+M③

$$-z'_{B_1} \cdot F_{B_1,y'} = -z'_B \cdot F_{B,y'} - z'_{vp} \cdot F_{B_{vp},y'} - z'_{vs} \cdot F_{B_{vs},y'}$$

$$-z'_{B_1} \cdot (\rho g \nabla \cdot \sin\phi) = -z'_B \cdot (\rho g \nabla \cdot \sin\phi) - z'_{vp} \cdot (\rho g v_p \cdot \sin\phi) - z'_{vs}(\rho g v_s \cdot \sin\phi)$$

$$-\underbrace{(z'_{B_1} - z'_B)}_{=\delta z'_B} \cdot (\rho g \nabla \cdot \sin\phi) = -z'_{vp} \cdot (\rho g v_p \cdot \sin\phi) - z'_{vs}(\rho g v_s \cdot \sin\phi)$$

$$(\delta z'_B) \cdot (\cancel{\rho g} \nabla \cdot \cancel{\sin\phi}) = z'_{vp} \cdot (\cancel{\rho g} v_p \cdot \cancel{\sin\phi}) + z'_{vs}(\cancel{\rho g} v_s \cdot \cancel{\sin\phi})$$

$$\delta z'_B \cdot \nabla = z'_{vp} \cdot v_p + z'_{vs} \cdot v_s$$