Chapter 4. Engineering Wave Properties

4.1 Introduction

$$\left.\begin{array}{l}\text{Velocity potential}\\ \text{Surface elevation}\end{array}\right\}\rightarrow \text{Wave kinematics, pressure field, wave energy, etc.}$$

Wave transformation (Shoaling, Refraction, Diffraction)

Wave breaking

4.2 Wave Kinematics for Progressive Waves

<u>Particle velocity</u>

$$\phi = -\frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx-\sigma t)$$

$$u = -\frac{\partial\phi}{\partial x} = \frac{H}{2}\frac{gk}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx-\sigma t)$$
$$= \frac{H}{2}\sigma\frac{\cosh k(h+z)}{\sinh kh}\cos(kx-\sigma t)$$

$$w = -\frac{\partial\phi}{\partial z} = \frac{H}{2}\frac{gk}{\sigma}\frac{\sinh k(h+z)}{\cosh kh}\sin(kx-\sigma t)$$
$$= \frac{H}{2}\sigma\frac{\sinh k(h+z)}{\sinh kh}\sin(kx-\sigma t)$$

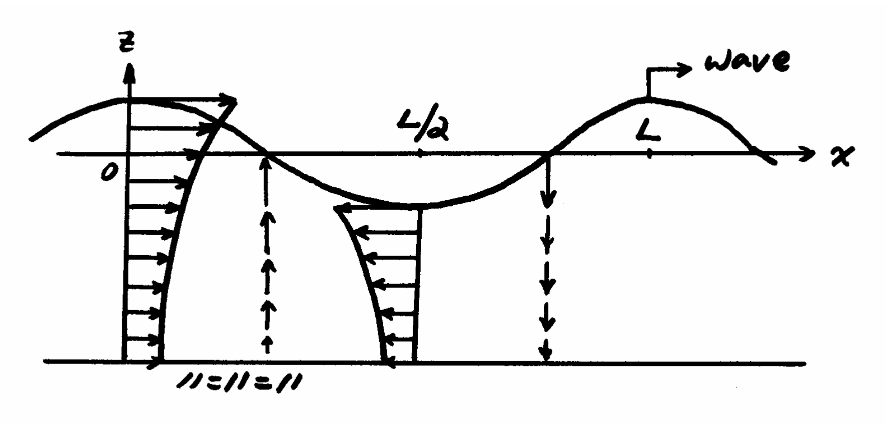
where we used the dispersion relationship, $\sigma^2 = gk\tanh kh$. Note that u and w are 90° out of phase.

<u>Acceleration</u>

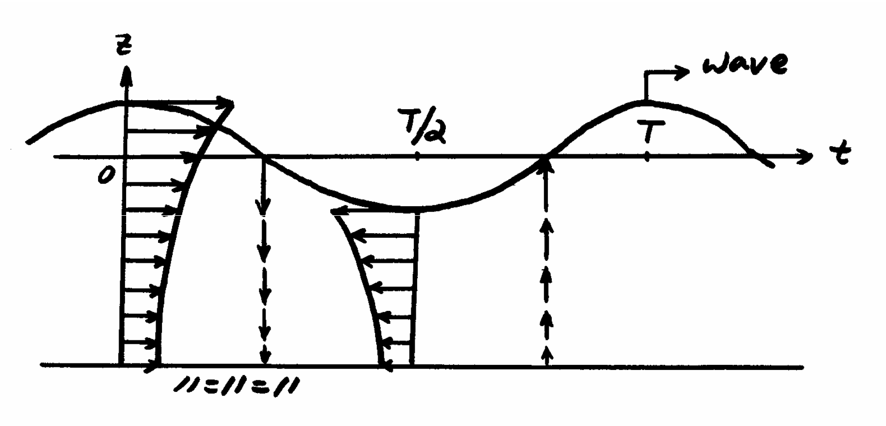
$$\frac{du}{dt} \cong \frac{\partial u}{\partial t} = \frac{H}{2}\sigma^2\frac{\cosh k(h+z)}{\sinh kh}\sin(kx-\sigma t)$$

$$\frac{dw}{dt} \cong \frac{\partial w}{\partial t} = -\frac{H}{2}\sigma^2 \frac{\sinh k(h+z)}{\sinh kh}\cos(kx - \sigma t)$$

At $t = 0$



At $x = 0$



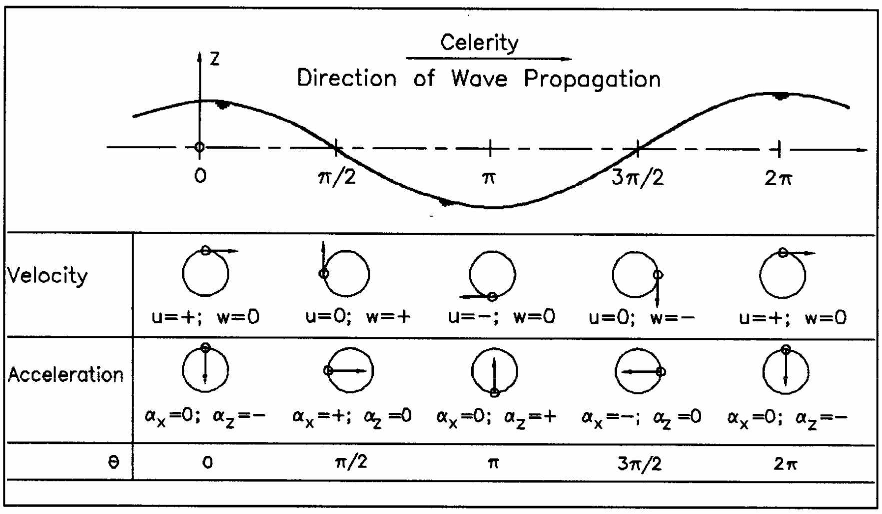


Figure II-1-2. Local fluid velocities and accelerations

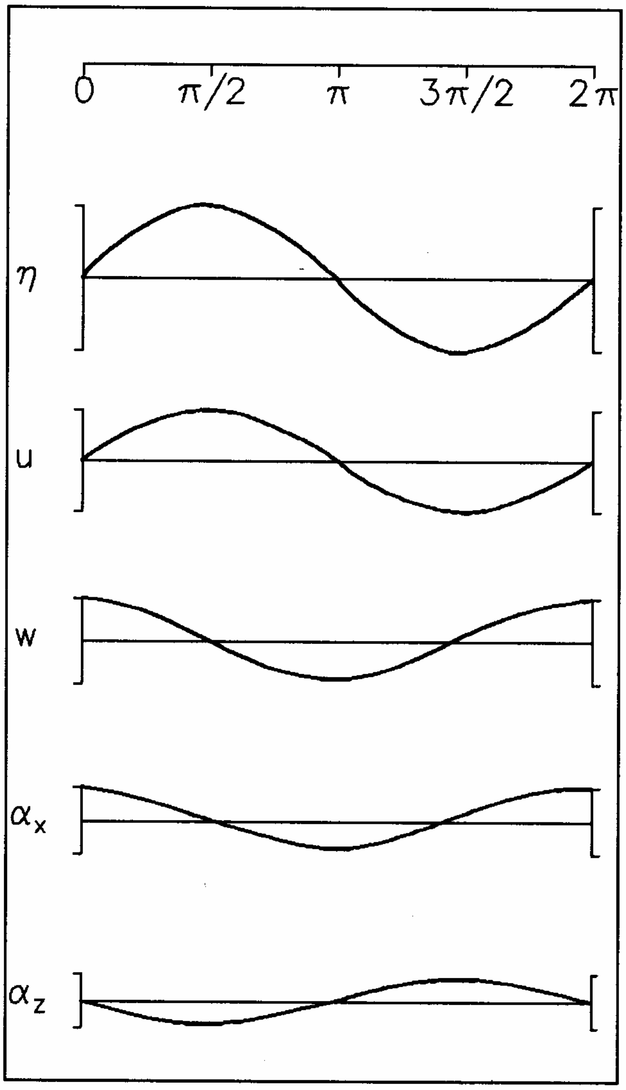
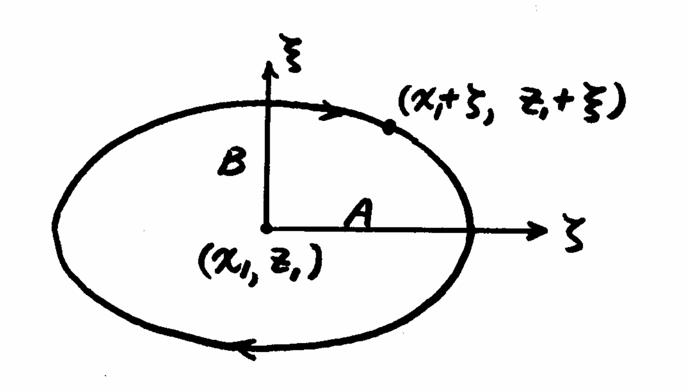


Figure II-1-3. Profiles of particle velocity and acceleration by Airy theory in relation to the surface elevation

Particle displacements

Consider a water particle moving around the mean position (x_1, z_1):



$$\varsigma(x_1, z_1, t) = \int u(x_1 + \varsigma, z_1 + \xi) dt$$
$$= \int \left[u(x_1, z_1) + \varsigma \frac{\partial u}{\partial x}\bigg|_{(x_1, z_1)} + \xi \frac{\partial u}{\partial z}\bigg|_{(x_1, z_1)} + \cdots \right] dt$$
$$\left(\frac{H}{T}\right) \quad \left(H \frac{H/T}{L}\right)$$
$$\cong \int u(x_1, z_1) dt$$
$$= -\frac{H}{2} \frac{\cosh k(h + z_1)}{\sinh kh} \sin(kx_1 - \sigma t)$$

Similarly,

$$\xi(x_1, z_1, t) = \int w(x_1 + \varsigma, z_1 + \xi) dt$$
$$= \frac{H}{2} \frac{\sinh k(h + z_1)}{\sinh kh} \cos(kx_1 - \sigma t)$$

Let

$$\varsigma(x_1, z_1, t) = -A(z_1) \sin(kx_1 - \sigma t)$$
$$\xi(x_1, z_1, t) = B(z_1) \cos(kx_1 - \sigma t)$$

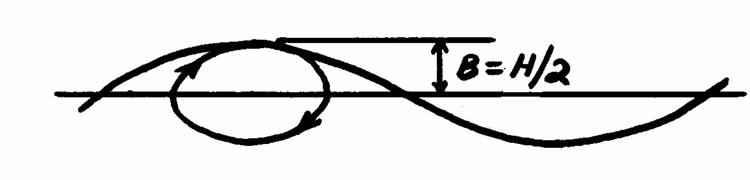
in which

$$A(z_1) = \frac{H}{2}\frac{\cosh k(h+z_1)}{\sinh kh}$$
$$B(z_1) = \frac{H}{2}\frac{\sinh k(h+z_1)}{\sinh kh}$$

Then

$$\left(\frac{\varsigma}{A}\right)^2 + \left(\frac{\xi}{B}\right)^2 = \sin^2(kx_1 - \sigma t) + \cos^2(kx_1 - \sigma t) = 1$$

which is the equation of an ellipse. Note that A is always greater than B because $\cosh x > \sinh x$. Also note that $B = H/2$ at $z_1 = 0$, that is, the water particle on the free surface moves vertically between crest and trough.



In shallow water,

$$A = \frac{H}{2}\frac{1}{kh} \neq f(z_1) \text{ but constant over the depth}$$

$$B = \frac{H}{2}\frac{k(h+z_1)}{kh} = \frac{H}{2}\left(1+\frac{z_1}{h}\right) \text{ varies from } H/2 \text{ at surface to } 0 \text{ at bottom}$$

In deep water,

$$A = \frac{H}{2}\frac{e^{kh}e^{kz_1}}{e^{kh}} = \frac{H}{2}e^{kz_1}$$

$B = \frac{H}{2}\frac{e^{kh}e^{kz_1}}{e^{kh}} = \frac{H}{2}e^{kz_1} = A$: Circle with diameter decreasing exponentially with depth

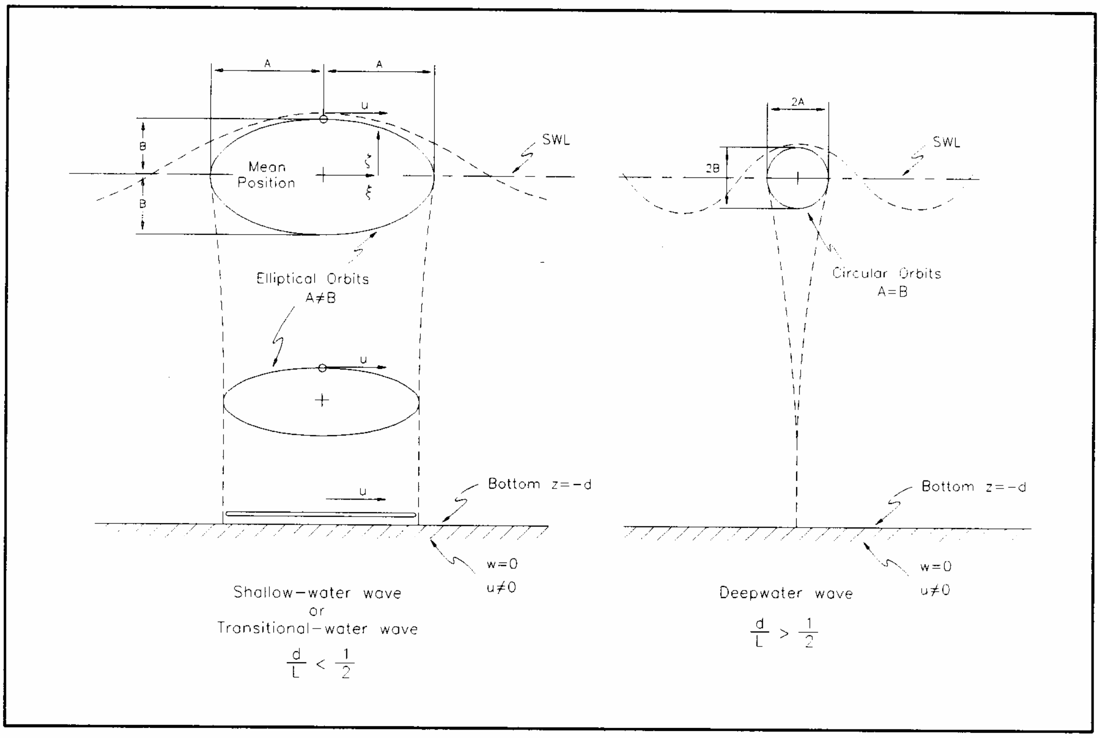


Figure II-1-4. Water particle displacements from mean position for shallow-water and deepwater waves

4.3 Pressure Field under a Progressive Wave

Consider the Bernoulli equation:

$$\frac{p}{\rho} + gz + \frac{1}{2}\left(u^2 + w^2\right) - \frac{\partial \phi}{\partial t} = C(t)$$

Neglecting small nonlinear terms,

$$\left(\frac{p}{\rho} + gz - \frac{\partial \phi}{\partial t}\right)_z = \left(\frac{p}{\rho} + gz - \frac{\partial \phi}{\partial t}\right)_{z=\eta} \cong g\eta - \left.\frac{\partial \phi}{\partial t}\right|_{z=0} = 0$$

$$\frac{p}{\rho} = -gz + \frac{\partial \phi}{\partial t} = -gz + \frac{H}{2} g \frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \sigma t)$$

$$\therefore \quad p = -\rho g z + \rho g \frac{H}{2}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \sigma t)$$

$$= -\rho g z + \rho g \eta K_p(z)$$

in which

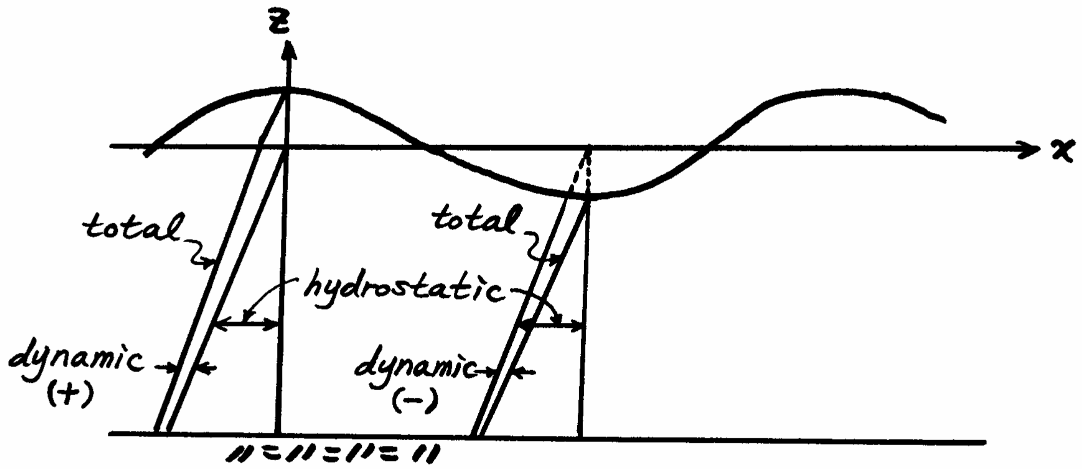
$$K_p(z) = \frac{\cosh k(h+z)}{\cosh kh}$$

is the pressure response factor, which varies from $1/\cosh kh$ at $z = -h$ to 1 at $z = 0$. Note that $K_p(z) \le 1$. In the equation for pressure, the first term represents the hydrostatic pressure, and the second term dynamic pressure due to waves. The hydrostatic pressure increases linearly with depth, but the dynamic pressure decreases exponentially with depth.

Above the mean water level, i.e., in the range of $0 < z_1 < \eta$,

$$
\begin{aligned}
p(z_1) &= p(0) + z_1 \left.\frac{\partial p}{\partial z}\right|_{z=0} + \cdots \\
&= \left(-\rho g z + \rho g \eta K_p\right)_{z=0} + z_1 \frac{\partial}{\partial z}\left(-\rho g z + \rho g \eta K_p\right)_{z=0} + \cdots \\
&= \rho g \eta - \rho g z_1 \\
&= \rho g (\eta - z_1)
\end{aligned}
$$

That is, the pressure is hydrostatic for $0 < z < \eta$.



Wave measurement using a pressure transducer

$$p_d = \rho g \eta K_p(z) \quad \Rightarrow \quad \eta = \frac{p_d}{\rho g K_p(z)}$$

If the pressure transducer is located at the bottom ($z = -h$),

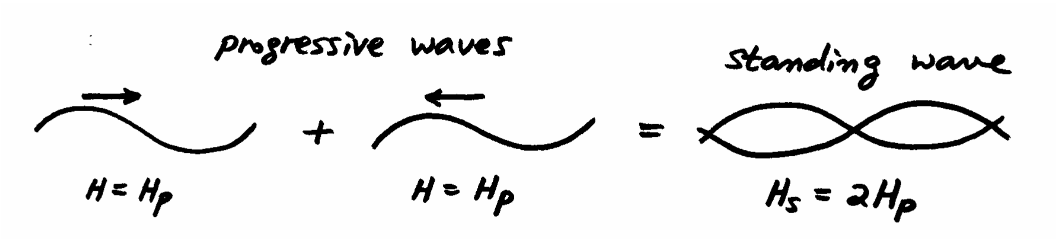
$$\eta = \frac{p_d}{\rho g K_p(-h)}$$

where

$$K_p(-h) = \frac{1}{\cosh kh} \cong \begin{cases} 1 & \text{for shallow water (long - period wave)} \\ 0 & \text{for deep water (short - period wave)} \end{cases}$$

Dynamic pressure due to short-period wave is hardly recorded by a pressure transducer located at the bottom. It is difficult to distinguish real signal and noise of the gauge. Therefore, pressure should be measured near the water surface when water depth is large.

4.4 Wave Kinematics for Standing Waves



$$\phi = -\frac{H_p}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx-\sigma t) + \frac{H_p}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx+\sigma t)$$

$$= \frac{H_p}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}(-\sin kx\cos\sigma t + \cos kx\sin\sigma t + \sin kx\cos\sigma t + \cos kx\sin\sigma t)$$

$$= H_p\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\cos kx\sin\sigma t$$

$$= \frac{H_s}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\cos kx\sin\sigma t$$

$$u = -\frac{\partial\phi}{\partial x} = \frac{H_s}{2}\frac{gk}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\sin kx\sin\sigma t$$

$$= \frac{H_s}{2}\sigma\frac{\cosh k(h+z)}{\sinh kh}\sin kx\sin\sigma t$$

$$w = -\frac{\partial\phi}{\partial z} = -\frac{H_s}{2}\sigma\frac{\sinh k(h+z)}{\sinh kh}\cos kx\sin\sigma t$$

$$\eta = \frac{H_s}{2}\cos kx\cos\sigma t$$

At node, $\cos kx = 0$, $\sin kx = 1$, $w = 0$, $kx = \left(n+\frac{1}{2}\right)\pi$

At antinode, $\cos kx = 1$, $\sin kx = 0$, $u = 0$, $kx = n\pi$

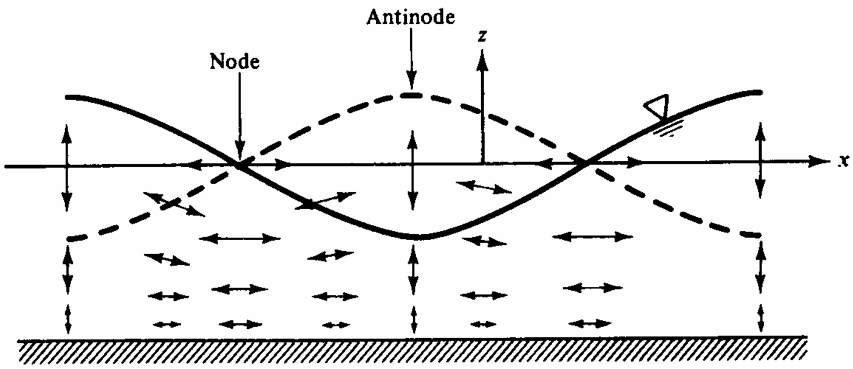
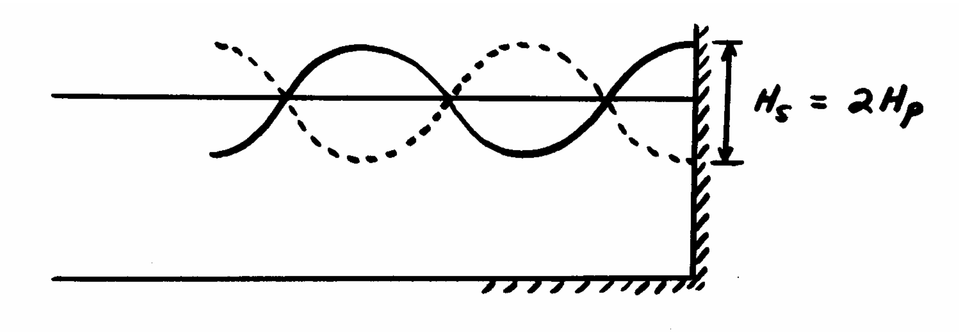


Figure 4.6 Distribution of water particle velocities in a standing water wave.

At vertical wall,

$$u = -\frac{\partial\phi}{\partial x} = 0 \quad \Rightarrow \quad \text{Antinode at the wall}$$



Local accelerations:

$$\frac{\partial u}{\partial t} = \frac{H_s}{2}\sigma^2 \frac{\cosh k(h+z)}{\sinh kh} \sin kx \cos \sigma t \; ; \quad \frac{\partial u}{\partial t} \quad \text{is max at node and zero at antinode.}$$

$$\frac{\partial w}{\partial t} = -\frac{H_s}{2}\sigma^2 \frac{\sinh k(h+z)}{\sinh kh} \cos kx \cos \sigma t \; ; \quad \frac{\partial w}{\partial t} \quad \text{is max at antinode and zero at node.}$$

Particle displacements:

$$\begin{aligned}
\varsigma(x_1, z_1) &= \int u(x_1 + \varsigma, z_1 + \xi)dt \cong \int u(x_1, z_1)dt \\
&= -\frac{H_s}{2} \frac{\cosh k(h+z_1)}{\sinh kh} \sin kx_1 \cos \sigma t \\
&= -A(x_1, z_1) \cos \sigma t
\end{aligned}$$

$$\begin{aligned}
\xi(x_1, z_1) &= \frac{H_s}{2} \frac{\sinh k(h+z_1)}{\sinh kh} \cos kx_1 \cos \sigma t \\
&= B(x_1, z_1) \cos \sigma t
\end{aligned}$$

$$\frac{\xi}{\varsigma} = -\frac{B}{A} \quad \Rightarrow \quad \xi = -\left(\frac{B}{A}\right)\varsigma \quad \text{: Straight line with slope } -\left(\frac{B}{A}\right)$$

4.5 Pressure Field under a Standing Wave

Consider the Bernoulli equation:

$$\frac{p}{\rho} + gz + \frac{1}{2}\left(u^2 + w^2\right) - \frac{\partial \phi}{\partial t} = C(t)$$

By the same way as for progressive waves,

$$\begin{aligned} p &= -\rho g z + \rho \frac{\partial \phi}{\partial t} \\ &= -\rho g z + \rho g \frac{H_s}{2} \frac{\cosh k(h+z)}{\cosh kh} \cos kx \cos \sigma t \\ &= -\rho g z + \rho g \eta K_p(z) \end{aligned}$$

Again

$$p(z_1) = \rho g(\eta - z_1) \quad \text{for} \quad 0 < z_1 < \eta$$

<u>Wave force acting on a vertical wall</u>

$$F = \int_{-h}^{\eta_w} p(z)dz \qquad \eta_w = \eta \text{ at the wall}$$

$$= \int_{-h}^{0} \left[-\rho g z + \rho g \eta_w \frac{\cosh k(h+z)}{\cosh kh} \right] dz + \int_{0}^{\eta_w} \rho g(\eta_w - z)dz$$

The first integral is a first-order term, and the second integral is a second-order term. However, note that this F is neither first-order nor second-order. For a complete second order solution, the nonlinear terms (i.e., u^2 terms) should be included from the beginning.

To the first order,

$$
\begin{aligned}
F &= \int_{-h}^{0}\left[-\rho g z + \rho g \eta_w \frac{\cosh k(h+z)}{\cosh kh}\right] dz \\
&= \left[-\frac{1}{2}\rho g z^2 + \frac{\rho g}{k}\eta_w \frac{\sinh k(h+z)}{\cosh kh}\right]_{-h}^{0} \\
&= \frac{\rho g}{k}\eta_w \tanh kh + \frac{1}{2}\rho g h^2
\end{aligned}
$$

The first term is the wave force due to dynamic pressure, while the second term is the hydrostatic pressure force.

Since

$$
\eta_w = \frac{H_s}{2}\cos \sigma t
$$

we have

$$
\begin{aligned}
F_{\max} &= \frac{\rho g}{2k} H_s \tanh kh + \frac{1}{2}\rho g h^2 \\
&= \frac{\rho}{2} H_s C^2 + \frac{1}{2}\rho g h^2
\end{aligned}
$$

4.6 Partial Standing Waves

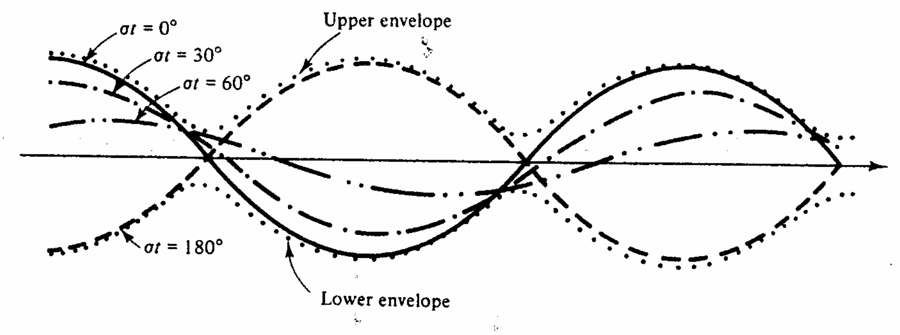
Partial reflection from beaches or breakwaters: $H_i > H_r$ where H_i = incident wave height, and H_r = reflected wave height. The wave potential is

$$\phi = -\frac{H_i}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx-\sigma t) + \frac{H_r}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx+\sigma t+\varepsilon)$$

where ε = phase lag.

$$\eta = \frac{1}{g}\frac{\partial \phi}{\partial t}\bigg|_{z=0} = \frac{H_i}{2}\cos(kx-\sigma t) + \frac{H_r}{2}\cos(kx+\sigma t+\varepsilon)$$

Since $H_i \neq H_r$, there is no true node. Instead, we have wave envelopes within which the surface elevation varies (see Fig. 4.7 of textbook). In other words, the upper and lower envelopes are the upper and lower boundaries of surface elevation, i.e. $\eta_{\max}$ and $\eta_{\min}$. Therefore, $\partial \eta / \partial t = 0$ on envelopes.



Separation of incident and reflected waves

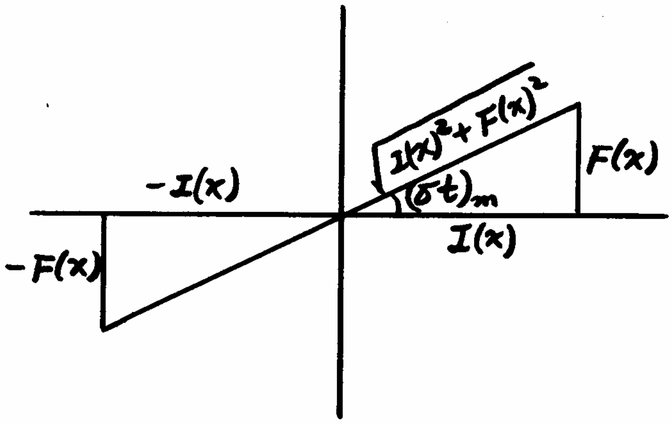
$$\begin{aligned}\eta &= \frac{H_i}{2}(\cos kx \cos\sigma t + \sin kx \sin\sigma t) + \frac{H_r}{2}(\cos(kx+\varepsilon)\cos\sigma t - \sin(kx+\varepsilon)\sin\sigma t)\\ &= \left\{\frac{H_i}{2}\cos kx + \frac{H_r}{2}\cos(kx+\varepsilon)\right\}\cos\sigma t + \left\{\frac{H_i}{2}\sin kx - \frac{H_r}{2}\sin(kx+\varepsilon)\right\}\sin\sigma t\\ &= I(x)\cos\sigma t + F(x)\sin\sigma t\end{aligned}$$

To find the envelopes,

$$\frac{\partial \eta}{\partial t} = 0 = -\sigma I(x)\sin\sigma t + \sigma F(x)\cos\sigma t$$

$$I(x)\sin\sigma t = F(x)\cos\sigma t$$

$$\tan(\sigma t)_m = \frac{F(x)}{I(x)}$$



$$\eta_{\max} = I(x)\frac{I(x)}{\sqrt{I(x)^2 + F(x)^2}} + F(x)\frac{F(x)}{\sqrt{I(x)^2 + F(x)^2}}$$

$$= \frac{I(x)^2 + F(x)^2}{\sqrt{I(x)^2 + F(x)^2}}$$

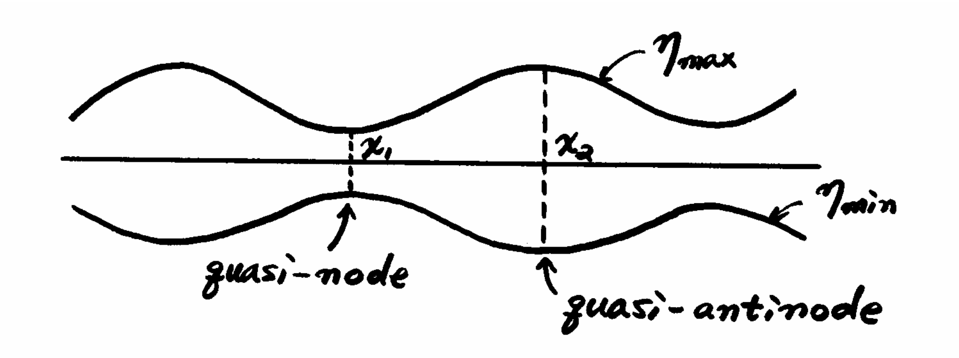
$$= \sqrt{I(x)^2 + F(x)^2}$$

$$\eta_{\min} = I(x)\frac{-I(x)}{\sqrt{I(x)^2 + F(x)^2}} + F(x)\frac{-F(x)}{\sqrt{I(x)^2 + F(x)^2}}$$

$$= -\sqrt{I(x)^2 + F(x)^2}$$

$$\eta_{\substack{\max\\ \min}} = \pm\sqrt{\left\{\frac{H_i}{2}\cos kx + \frac{H_r}{2}\cos(kx+\varepsilon)\right\}^2 + \left\{\frac{H_i}{2}\sin kx - \frac{H_r}{2}\sin(kx+\varepsilon)\right\}^2}$$

$$= \pm\sqrt{\left(\frac{H_i}{2}\right)^2 + \left(\frac{H_r}{2}\right)^2 + \frac{H_i H_r}{2}\cos(2kx+\varepsilon)}$$



$$\cos(2kx+\varepsilon) = -1 \qquad\qquad \cos(2kx+\varepsilon) = 1$$

$$2kx+\varepsilon = (2n-1)\pi \qquad\qquad 2kx+\varepsilon = 2n\pi$$

$$\eta_{\max} = \sqrt{\left(\frac{H_i}{2}\right)^2 + \left(\frac{H_r}{2}\right)^2 - \frac{H_i H_r}{2}} \qquad \eta_{\max} = \sqrt{\left(\frac{H_i}{2}\right)^2 + \left(\frac{H_r}{2}\right)^2 + \frac{H_i H_r}{2}}$$

$$= \frac{1}{2}\sqrt{H_i^2 + H_r^2 - 2H_i H_r} \qquad = \frac{1}{2}\sqrt{H_i^2 + H_r^2 + 2H_i H_r}$$

$$= \frac{1}{2}(H_i - H_r) \qquad = \frac{1}{2}(H_i + H_r)$$

Distance between x_1 and x_2 $= x_2 - x_1$

$$= \frac{2n\pi - \varepsilon}{2k} - \frac{(2n-1)\pi - \varepsilon}{2k}$$

$$= \frac{\pi}{2k}$$

$$= \frac{L}{4}$$ same as perfect reflection

$$(\eta_{\max})_{\text{node}} = \frac{1}{2}(H_i - H_r)$$

$$(\eta_{\max})_{\text{antinode}} = \frac{1}{2}(H_i + H_r)$$

$$\therefore\ H_i = (\eta_{\max})_{\text{antinode}} + (\eta_{\max})_{\text{node}}$$
$$H_r = (\eta_{\max})_{\text{antinode}} - (\eta_{\max})_{\text{node}}$$

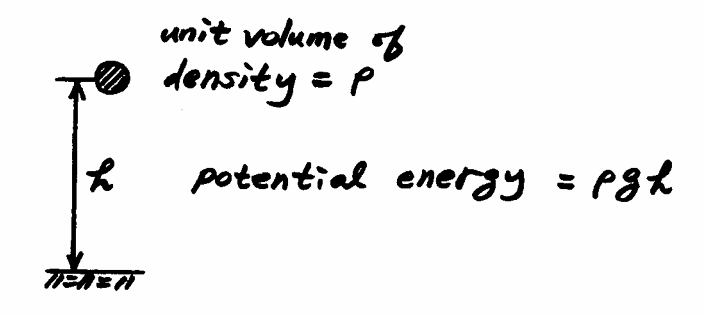
To measure the wave envelopes, we have to measure waves as slowly moving a wave gauge along the wave tank (see Fig. 4.9 of textbook). This technique for separation of incident and reflected waves is applicable only to regular waves. For irregular waves, use multiple fixed gauge method (Suh et al. 2001, Coastal Engineering, 43, 149-159).

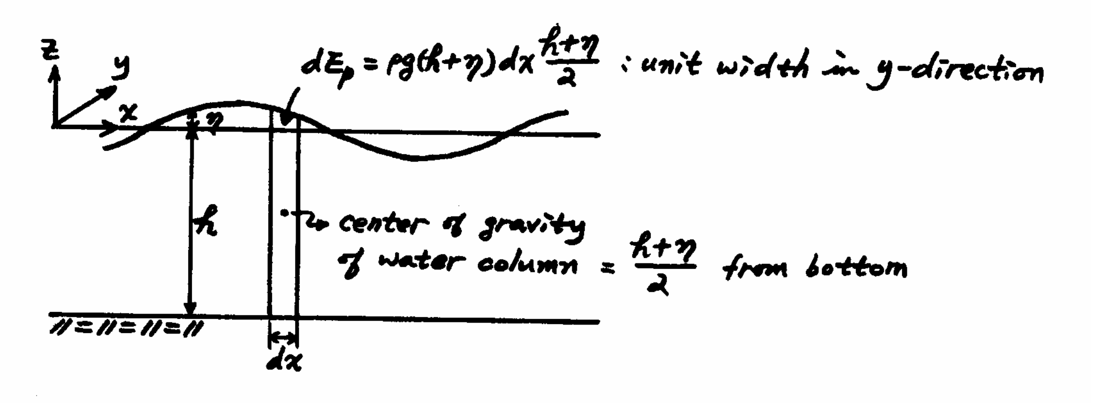
4.7 Wave Energy

Total energy (E) = potential energy (E_p) due to displacement of free surface

+ kinetic energy (E_k) due to water particle movement

Potential energy





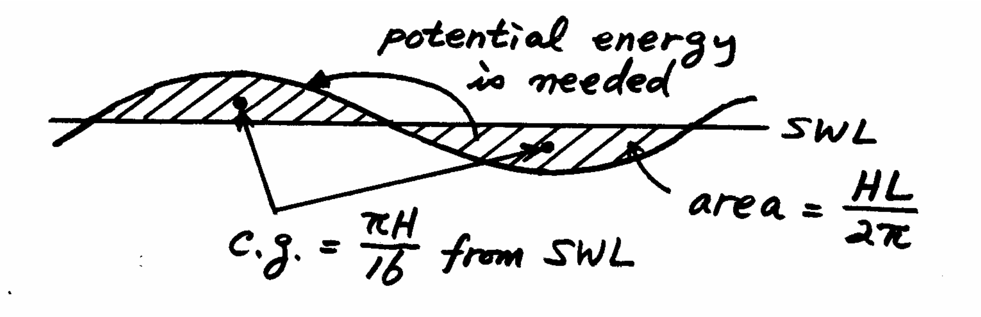
Potential energy per unit surface area:

$$
\begin{aligned}
E_p &= \frac{1}{L}\int_x^{x+L} \frac{\rho g}{2}(h+\eta)^2\,dx \\
&= \frac{\rho g}{L}\int_x^{x+L} \frac{1}{2}\left(h^2 + 2\eta h + \eta^2\right)dx \\
&= \frac{\rho g}{L}\left\{\frac{1}{2}h^2 L + h\int_x^{x+L}\eta dx + \frac{1}{2}\int_x^{x+L}\eta^2 dx\right\} \\
&= \frac{1}{2}\rho g h^2 + \frac{1}{16}\rho g H^2
\end{aligned}
$$

The first term is the potential energy in still water, while the second term is the potential energy due to wave. Therefore, the potential energy of only wave is

$$E_p = \frac{1}{16}\rho g H^2$$

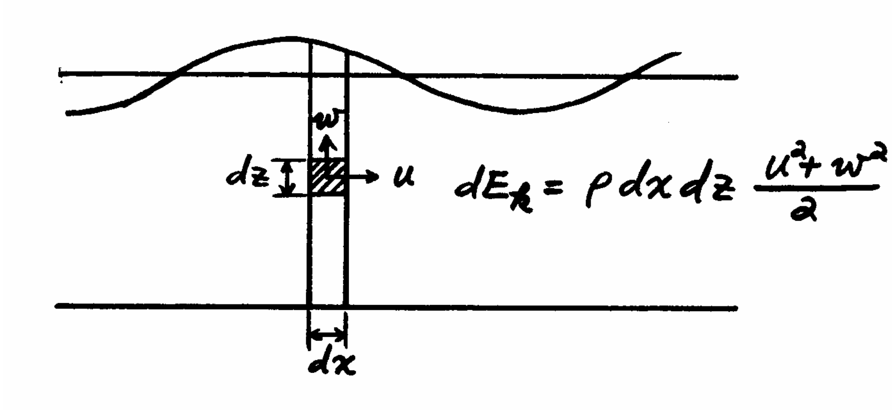
Another view:



Potential energy = $\rho g \dfrac{HL}{2\pi} \times \dfrac{\pi H}{16} \times 2 = \dfrac{\rho g}{16} H^2 L$

$\therefore$ For unit area, $E_p = \dfrac{1}{16}\rho g H^2$

<u>Kinetic energy</u>



$$
\begin{aligned}
E_k &= \frac{1}{L}\int_x^{x+L}\int_{-h}^{\eta}\frac{\rho}{2}\left(u^2+w^2\right)dzdx \\
&= \frac{\rho}{2L}\left(\frac{H}{2}\frac{gk}{\sigma}\frac{1}{\cosh kh}\right)^2\int_x^{x+L}\int_{-h}^{\eta}\{\cosh^2 k(h+z)\cos^2(kx-\sigma t) \\
&\qquad\qquad + \sinh^2 k(h+z)\sin^2(kx-\sigma t)\}dzdx \\
&\cong \frac{\rho}{2L}\left(\frac{H}{2}\frac{gk}{\sigma}\frac{1}{\cosh kh}\right)^2\int_x^{x+L}\int_{-h}^{0}\frac{1}{2}\left[\cosh 2k(h+z)+\cos 2(kx-\sigma t)\right]dzdx \\
&= \frac{1}{16}\rho g H^2
\end{aligned}
$$

Now, the total energy per unit surface area is

$$
E = E_p + E_k = \frac{1}{8}\rho g H^2 = \frac{1}{2}\rho g a^2 \quad \left(\frac{\mathrm{N\cdot m}}{\mathrm{m}^2} \text{ or } \frac{\mathrm{Joule}}{\mathrm{m}^2}\right)
$$

<u>Energy flux</u>

Energy flux, or rate of work done by wave, is

$$
F = \int_{-h}^{\eta} p_d u dz
$$

Average energy flux over one wave period is

$$
\begin{aligned}
\overline{F} &= \frac{1}{T}\int_{t}^{t+T}\int_{-h}^{\eta} p_d u dz dt \\
&= \frac{1}{T}\int_{t}^{t+T}\int_{-h}^{\eta}\left\{\rho g\eta\frac{\cosh k(h+z)}{\cosh kh}\right\}\left\{\frac{H}{2}\frac{gk}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx-\sigma t)\right\}dzdt \\
&= \frac{1}{8}\rho g H^2\frac{\sigma}{k}\left[\frac{1}{2}\left(1+\frac{2kh}{\sinh 2kh}\right)\right] \\
&= ECn \\
&= EC_g
\end{aligned}
$$

where C_g = group velocity at which wave energy is transmitted.

$$C_g = nC = \frac{d\sigma}{dk}$$

where $\sigma^2 = gk \tanh kh$ is used for the last step.

In shallow water, $n \to 1, \quad \therefore \; C_g = C$

In deep water, $n \to \frac{1}{2}, \quad \therefore \; C_{g_0} = \frac{C_0}{2}$

So far, we considered the dynamic view of the group velocity.

<u>Kinematic view of group velocity</u>

Consider two waves with slightly different frequencies and wave numbers, which gives a wave group.

$$\begin{aligned}\eta &= \eta_1 + \eta_2 \\ &= \frac{H}{2}\cos(k_1 x - \sigma_1 t) + \frac{H}{2}\cos(k_2 x - \sigma_2 t)\end{aligned}$$

where

$$\sigma_1 = \sigma - \frac{\Delta\sigma}{2}, \quad k_1 = k - \frac{\Delta k}{2}$$
$$\sigma_2 = \sigma + \frac{\Delta\sigma}{2}, \quad k_2 = k + \frac{\Delta k}{2}$$

such that

$$\frac{\Delta\sigma}{\sigma} << 1, \quad \frac{\Delta k}{k} << 1$$

Now, using the relationship

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

we have

$$\begin{aligned}\eta &= H\cos\left[\frac{1}{2}(k_1+k_2)x - \frac{1}{2}(\sigma_1+\sigma_2)t\right]\cos\left[\frac{1}{2}(k_1-k_2)x - \frac{1}{2}(\sigma_1-\sigma_2)t\right] \\ &= H\cos(kx-\sigma t)\cos\left(-\frac{1}{2}\Delta kx + \frac{1}{2}\Delta\sigma t\right) \\ &= H\cos(kx-\sigma t)\cos\left[\frac{1}{2}\Delta k\left(x - \frac{\Delta\sigma}{\Delta k}t\right)\right]\end{aligned}$$

In the last term of the above expression, the first part represents the individual wave propagating at the speed of $C = \sigma / k$, while the second part represents the modulation or envelope propagating at the speed of $C_g = \Delta\sigma / \Delta k$.

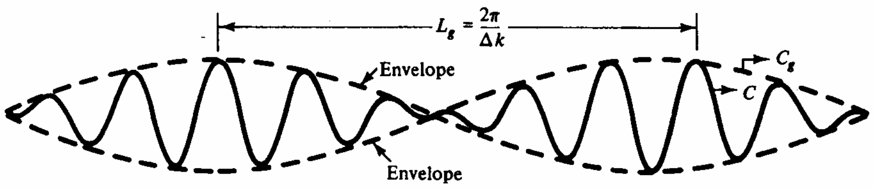


Figure 4.12 Characteristics of a "group" of waves.

The speed of propagation of envelope or wave group is

$$C_g \equiv \frac{\Delta\sigma}{\Delta k}$$

This is why we call C_g "group velocity".

4.8 Wave Transformation Due to Depth Change

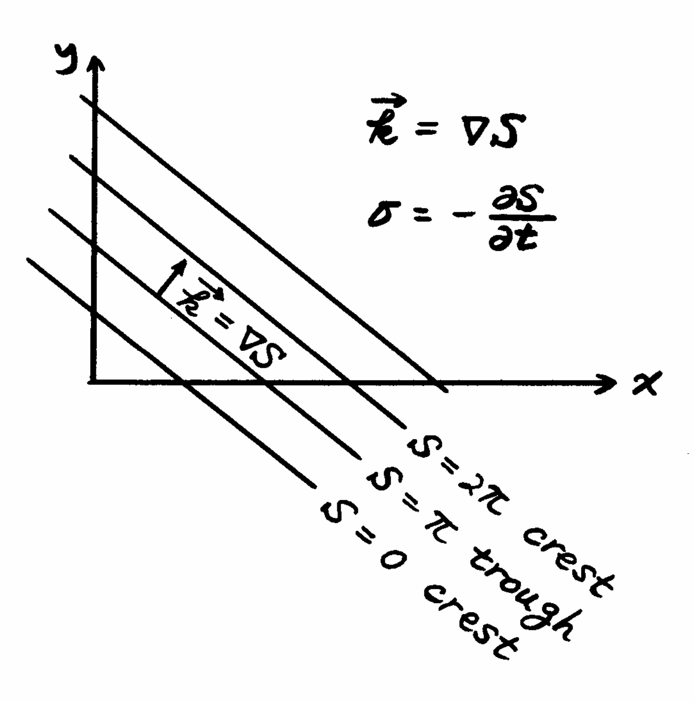
In this section, we deal with wave shoaling, refraction, and breaking.

<u>Conservation of waves equation</u>

$$\eta = \mathrm{Re}\left\{A e^{iS(x,y,t)}\right\}$$

where

$$S \equiv \text{scalar phase function} = \left(\vec{k} \cdot \vec{x} - \sigma t\right) = \left(k_x x + k_y y - \sigma t\right)$$



$\partial / \partial t$ and ∇ are independent operators so that the change of order does not make any difference. Thus

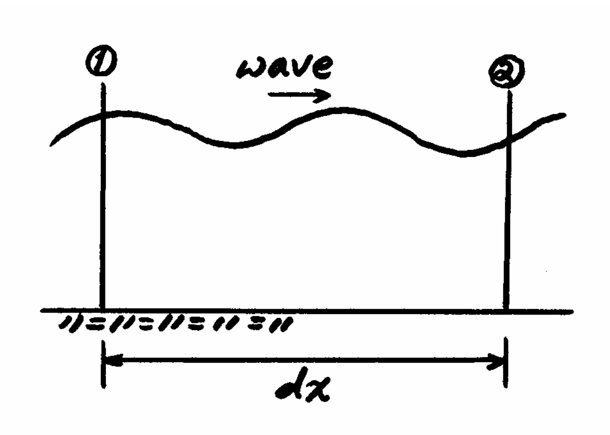
$$\frac{\partial}{\partial t}(\nabla S) - \nabla\left(\frac{\partial S}{\partial t}\right) = 0$$

or

$$\frac{\partial \vec{k}}{\partial t} + \nabla \sigma = 0$$

This is the equation for conservation of waves, which states that the temporal variation of wave number $\vec{k}$ should be balanced by the spatial variation of frequency σ.

Consider 1-D case:



Number of waves coming in at section 1 per unit time = $\frac{1}{T} = \frac{\sigma}{2\pi}$.

$$\left(\frac{\sigma}{2\pi}\right)_2 = \left(\frac{\sigma}{2\pi}\right)_1 + \frac{\partial}{\partial x}\left(\frac{\sigma}{2\pi}\right)dx$$

$$-\frac{\partial}{\partial x}\left(\frac{\sigma}{2\pi}\right)dx = \left(\frac{\sigma}{2\pi}\right)_1 - \left(\frac{\sigma}{2\pi}\right)_2$$

Number of waves in dx = $\frac{dx}{L} = \frac{k}{2\pi}dx$. The rate at which the number of waves changes in dx = $\frac{\partial}{\partial t}\left(\frac{k}{2\pi}dx\right)$.

$$\therefore\ \frac{\partial}{\partial t}\left(\frac{k}{2\pi}dx\right) = -\frac{\partial}{\partial x}\left(\frac{\sigma}{2\pi}\right)dx$$

$$\frac{\partial k}{\partial t} + \frac{\partial \sigma}{\partial x} = 0$$ Conservation of waves equation in 1-D

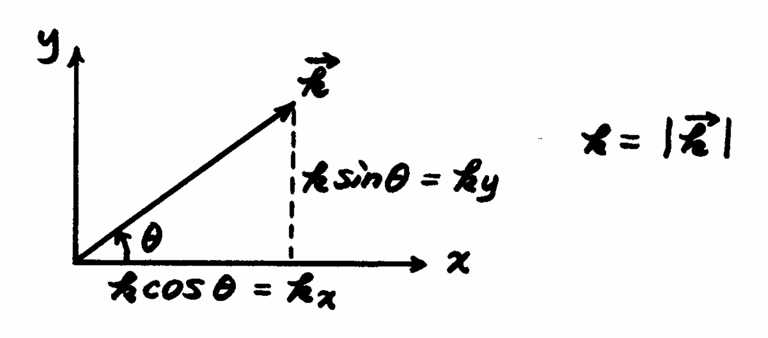
For a steady state,

$$\frac{\partial k}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial \sigma}{\partial x} = 0$$

that is, σ (or T) is constant along x in a steady state.

<u>Refraction</u>

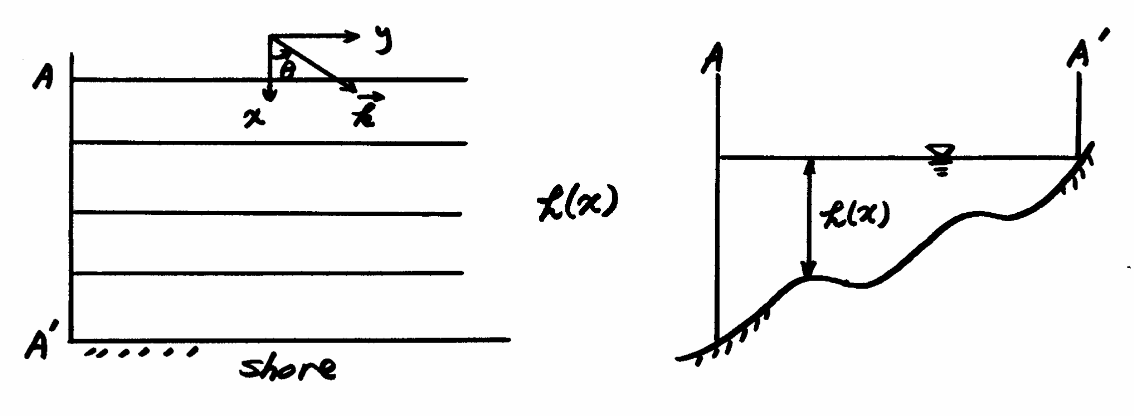
$\nabla \times \vec{k} = \nabla \times (\nabla S) = 0$ by vector identity (curl of gradient = 0)



$$\nabla \times \vec{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ k_x & k_y & 0 \end{vmatrix} = \left(\frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y}\right)\hat{k} = 0$$

$$\boxed{\frac{\partial (k \sin\theta)}{\partial x} - \frac{\partial (k\cos\theta)}{\partial y} = 0}$$

Simple example: straight and parallel contours



$$\frac{\partial}{\partial y} = 0 \quad \Rightarrow \quad k \sin\theta = \text{constant}$$

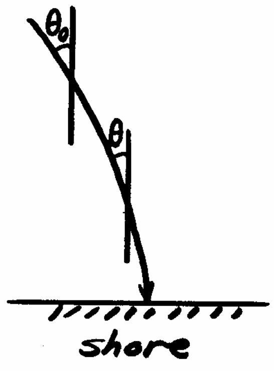
Dividing by σ,

$$\frac{\sin\theta}{C} = \text{constant}$$

or

$$\boxed{\frac{\sin\theta}{C} = \frac{\sin\theta_0}{C_0}} \quad \text{Snell's law}$$

Since $C < C_0$, $\theta < \theta_0$



Variable bathymetry:

$$\frac{\partial(k\sin\theta)}{\partial x}-\frac{\partial(k\cos\theta)}{\partial y}=0$$

Since $k=k(x,y)$ and $\theta=\theta(x,y)$,

$$k\cos\theta\frac{\partial\theta}{\partial x}+\sin\theta\frac{\partial k}{\partial x}+k\sin\theta\frac{\partial\theta}{\partial y}-\cos\theta\frac{\partial k}{\partial y}=0$$

This is a 1st-order nonlinear partial differential equation for θ, because k is given by $\sigma^2=gk\tanh kh$ for given σ and h. This equation was solved numerically by Noda et al. (1974).

A simpler way by Perlin and Dean (1983), Proc. Coastal Structures ’83, ASCE 988-999:

Defining

$$A(x,y)=k\sin\theta$$
$$B(x,y)=k\cos\theta$$

we have

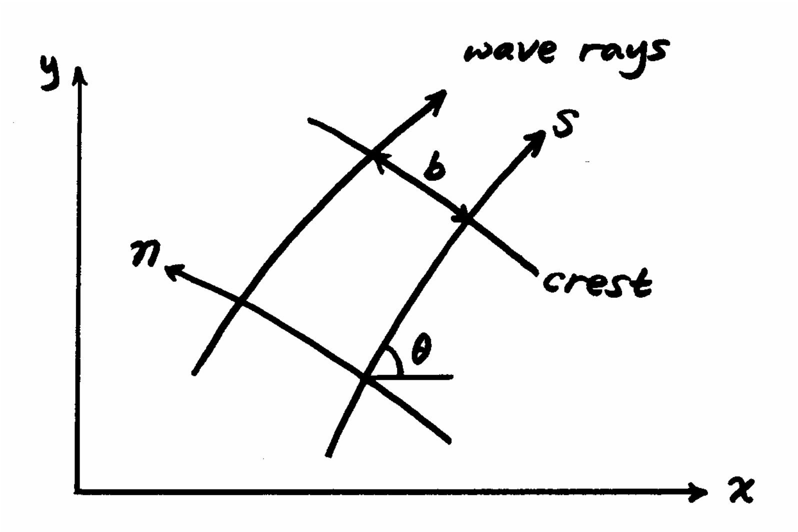
$$\frac{\partial B}{\partial y}=\frac{\partial A}{\partial x}\quad\leftarrow\text{ solved numerically}$$

$A^2 + B^2 = k^2 \;\Rightarrow\; \theta$ is calculated.

Wave height H is calculated by the similar way using the conservation of energy equation. Also see Dalrymple (1988). JWPCOE, ASCE, 114(4): 423-435, which includes wave-current interaction and nonlinearity.

Conservation of energy

Assume a steady state and no energy input or loss.



Assume wave energy is not transmitted across wave rays. Then

$$\frac{\partial}{\partial s}\left(bEC_g\right)=0$$

In Cartesian coordinates,

$$\frac{\partial}{\partial x}\left(EC_g\cos\theta\right)+\frac{\partial}{\partial y}\left(EC_g\sin\theta\right)=0$$

or

$$\boxed{\nabla\cdot\left(E\vec{C}_g\right)=0}$$ Conservation of energy equation

For straight and parallel contours, $\partial / \partial y = 0$. Therefore,

$$\frac{\partial}{\partial x}\left(EC_g \cos\theta\right) = 0$$

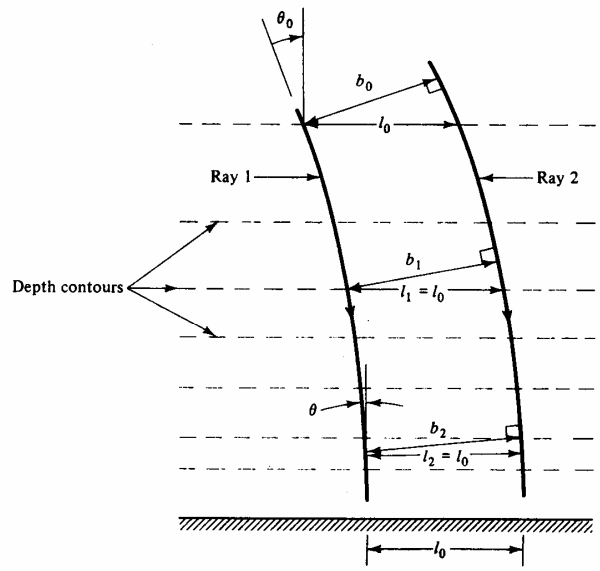


Figure 4.17 Characteristics of wave rays during refraction over idealized bathymetry.

$$\left(EC_g\right)_1 b_1 = \left(EC_g\right)_2 b_2$$

$$\frac{1}{8}\rho g H_1^2 C_{g1} b_1 = \frac{1}{8}\rho g H_2^2 C_{g2} b_2$$

$$H_2 = H_1 \sqrt{\frac{C_{g1}}{C_{g2}}}\sqrt{\frac{b_1}{b_2}} = H_1 \sqrt{\frac{C_{g1}}{C_{g2}}}\sqrt{\frac{\cos\theta_1}{\cos\theta_2}}$$

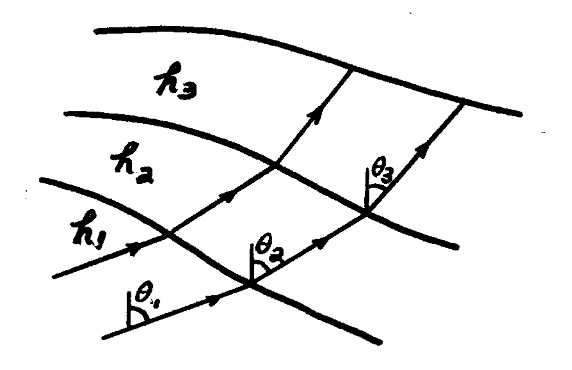
If position 1 is in deep water,

$$H_2 = H_0 \sqrt{\frac{C_0}{2C_{g2}}}\sqrt{\frac{b_0}{b_2}} = H_0 \sqrt{\frac{C_0}{2C_{g2}}}\sqrt{\frac{\cos\theta_0}{\cos\theta_2}} = H_0 K_s K_r$$

where K_s = shoaling coefficient, and K_r = refraction coefficient.

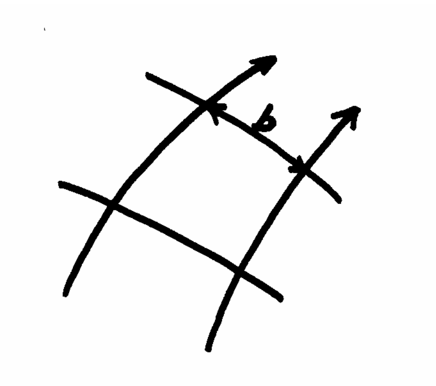
Ray tracing method

1) Graphical method: Use Snell's law over two adjacent bottom contours.



$$\frac{\sin\theta_1}{C_1} = \frac{\sin\theta_2}{C_2} \quad \rightarrow \quad \theta_2, \quad \frac{\sin\theta_2}{C_2} = \frac{\sin\theta_3}{C_3} \quad \rightarrow \quad \theta_3, \quad \cdots$$

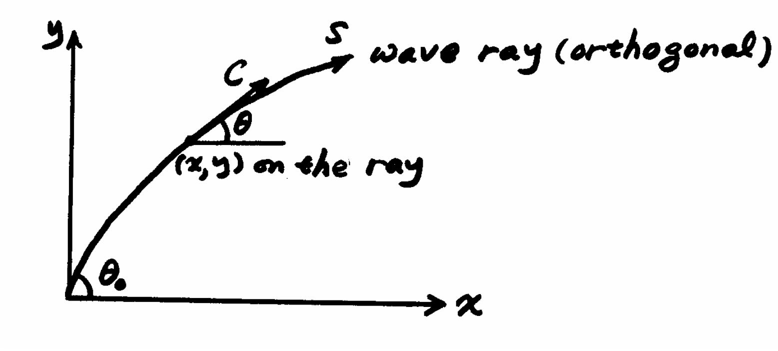
Draw $\left\{ \begin{matrix} \text{orthogonals} \\ \text{crests} \end{matrix} \right\}$ $\rightarrow$ refraction diagram



Measure b $\Rightarrow$ Calculate $K_r = \sqrt{\frac{b_0}{b}}$ $\Rightarrow$ $H = H_0 K_r K_s$

2) Numerical method

Munk and Arthur (1952)
Griswold (1963), Journal of Geophysical Research, 68(6): 1715-1723.

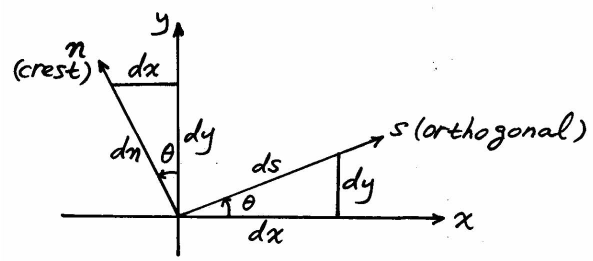


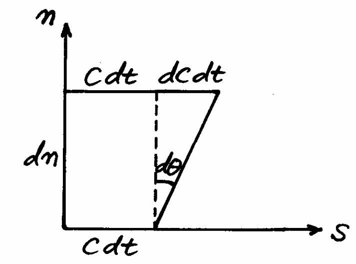
Let t = travel time along a ray of a wave moving with speed $C(x,y)$. Let the wave is located at $\left[x(t), y(t)\right]$ at time t. Then

$$\frac{dx}{dt} = C\cos\theta \qquad (1)$$

$$\frac{dy}{dt} = C\sin\theta \qquad (2)$$

$$\theta(x,y) = ?$$





$$dx = ds\cos\theta = -dn\sin\theta$$

$$dy = ds\sin\theta = dn\cos\theta$$

$$ds = Cdt$$

$$d\theta \cong \tan d\theta = -\frac{dCdt}{dn} = -\frac{1}{C}\frac{\partial C}{\partial n}ds$$

$$\frac{\partial\theta}{\partial s}=-\frac{1}{C}\frac{\partial C}{\partial n}=-\frac{1}{C}\left(\frac{\partial C}{\partial x}\frac{\partial x}{\partial n}+\frac{\partial C}{\partial y}\frac{\partial y}{\partial n}\right)=-\frac{1}{C}\left(-\sin\theta\frac{\partial C}{\partial x}+\cos\theta\frac{\partial C}{\partial y}\right)$$

$$=\frac{1}{C}\left(\sin\theta\frac{\partial C}{\partial x}-\cos\theta\frac{\partial C}{\partial y}\right)$$

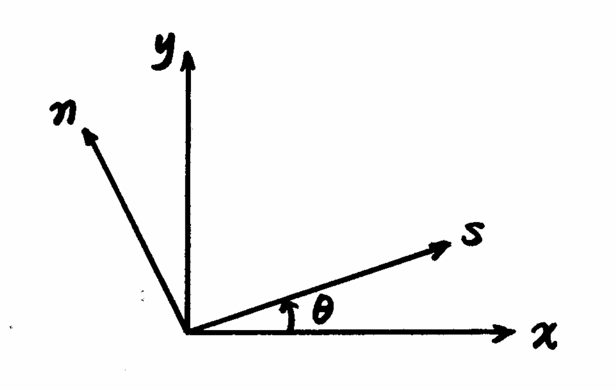
$$\frac{\partial\theta}{\partial t}=\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial t}=C\frac{\partial\theta}{\partial s}=\sin\theta\frac{\partial C}{\partial x}-\cos\theta\frac{\partial C}{\partial y} \qquad (3)$$

Solve (1), (2), and (3) to find a ray using a numerical method (ray by ray).

Let $\beta=b/b_0$ = ray separation factor such that

$$K_r=\sqrt{\frac{b_0}{b}}=\beta^{-1/2}$$

β can be found analytically (Munk and Arthur, 1952)



$$x=s\cos\theta-n\sin\theta$$
$$y=s\sin\theta+n\cos\theta$$

By chain rule,

$$\frac{\partial}{\partial s}=\frac{\partial}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial}{\partial y}\frac{\partial y}{\partial s}=\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y} \qquad (4.112b)$$

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial}{\partial y}\frac{\partial y}{\partial n} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y} \tag{4.112c}$$

$$\frac{\partial\theta}{\partial s} = \cos\theta\frac{\partial\theta}{\partial x} + \sin\theta\frac{\partial\theta}{\partial y} = \frac{1}{k}\cos\theta\frac{\partial k}{\partial y} - \frac{1}{k}\sin\theta\frac{\partial k}{\partial x} = \frac{1}{k}\frac{\partial k}{\partial n} = -\frac{1}{C}\frac{\partial C}{\partial n} \tag{4.113}$$

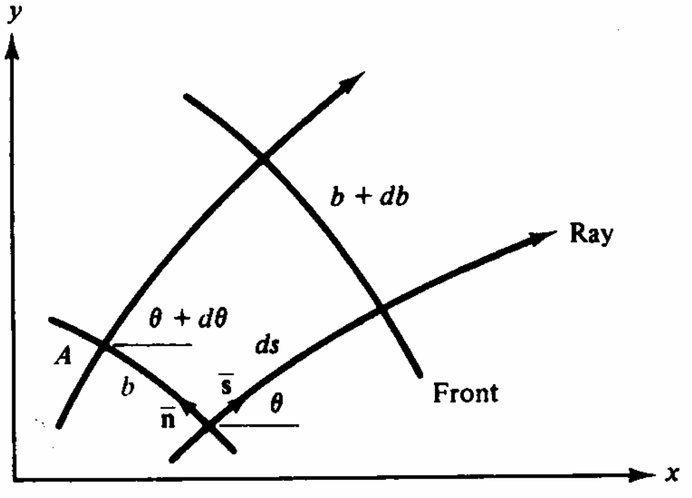


Figure 4.20 Schematic diagram showing adjacent rays.

$$\frac{ds}{dt} = C, \quad \frac{dx}{dt} = \frac{ds\cos\theta}{dt} = C\cos\theta, \quad \frac{dy}{dt} = \frac{ds\sin\theta}{dt} = C\sin\theta$$

At the point A,

$$d\theta = \frac{\partial\theta}{\partial n}b, \quad d\theta = \frac{db}{ds}$$

$$\frac{1}{b}\frac{\partial b}{\partial s} = \frac{\partial\theta}{\partial n}$$

$$\frac{1}{\beta b_0}\frac{\partial \beta b_0}{\partial s} = \frac{\partial\theta}{\partial n}$$

$$\frac{1}{\beta}\frac{\partial\beta}{\partial s} = \frac{\partial\theta}{\partial n} \tag{4.120b}$$

Now

$$\frac{\partial}{\partial n}\frac{\partial \theta}{\partial s}-\frac{\partial}{\partial s}\frac{\partial \theta}{\partial n}=\frac{\partial}{\partial n}\left(\cos\theta\frac{\partial\theta}{\partial x}+\sin\theta\frac{\partial\theta}{\partial y}\right)-\frac{\partial}{\partial s}\left(-\sin\theta\frac{\partial\theta}{\partial x}+\cos\theta\frac{\partial\theta}{\partial y}\right)$$

$$=\left(-\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial y}\right)\left(\cos\theta\frac{\partial\theta}{\partial x}+\sin\theta\frac{\partial\theta}{\partial y}\right)$$

$$-\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)\left(-\sin\theta\frac{\partial\theta}{\partial x}+\cos\theta\frac{\partial\theta}{\partial y}\right)$$

$$=\sin^2\theta\left(\frac{\partial\theta}{\partial x}\right)^2-\sin\theta\cos\theta\frac{\partial^2\theta}{\partial x^2}-\sin\theta\cos\theta\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial y}-\sin^2\theta\frac{\partial^2\theta}{\partial x\partial y}$$

$$-\cos\theta\sin\theta\frac{\partial\theta}{\partial y}\frac{\partial\theta}{\partial x}+\cos^2\theta\frac{\partial^2\theta}{\partial x\partial y}+\cos^2\theta\left(\frac{\partial\theta}{\partial y}\right)^2+\sin\theta\cos\theta\frac{\partial^2\theta}{\partial y^2}$$

$$+\cos^2\theta\left(\frac{\partial\theta}{\partial x}\right)^2+\sin\theta\cos\theta\frac{\partial^2\theta}{\partial x^2}+\sin\theta\cos\theta\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial y}-\cos^2\theta\frac{\partial^2\theta}{\partial x\partial y}$$

$$-\sin\theta\cos\theta\frac{\partial\theta}{\partial y}\frac{\partial\theta}{\partial x}+\sin^2\theta\frac{\partial^2\theta}{\partial x\partial y}+\sin^2\theta\left(\frac{\partial\theta}{\partial y}\right)^2-\sin\theta\cos\theta\frac{\partial^2\theta}{\partial y^2}$$

$$=\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2$$

$$=\left(\frac{\partial\theta}{\partial s}\right)^2+\left(\frac{\partial\theta}{\partial n}\right)^2$$

$$=\frac{1}{C^2}\left(\frac{\partial C}{\partial n}\right)^2+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial s}\right)^2$$

On the other hand, directly from (4.113) and (4.120b),

$$\frac{\partial}{\partial n}\frac{\partial \theta}{\partial s}-\frac{\partial}{\partial s}\frac{\partial \theta}{\partial n}=\frac{\partial}{\partial n}\left(-\frac{1}{C}\frac{\partial C}{\partial n}\right)-\frac{\partial}{\partial s}\left(\frac{1}{\beta}\frac{\partial\beta}{\partial s}\right)$$

$$=\frac{1}{C^2}\left(\frac{\partial C}{\partial n}\right)^2-\frac{1}{C}\frac{\partial^2 C}{\partial n^2}+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial s}\right)^2-\frac{1}{\beta}\frac{\partial^2\beta}{\partial s^2}$$

Comparing the two results,

$$\frac{1}{C}\frac{\partial^2 C}{\partial n^2}+\frac{1}{\beta}\frac{\partial^2 \beta}{\partial s^2}=0$$

$$\frac{\partial^2 \beta}{\partial s^2}+\frac{1}{C}\frac{\partial^2 C}{\partial n^2}\beta=0$$

where

$$\begin{aligned}
\frac{\partial^2 C}{\partial n^2} &= \frac{\partial}{\partial n}\left(-\sin\theta\frac{\partial C}{\partial x}+\cos\theta\frac{\partial C}{\partial y}\right)\\
&= -\cos\theta\frac{\partial\theta}{\partial n}\frac{\partial C}{\partial x}-\sin\theta\frac{\partial^2 C}{\partial x\partial n}-\sin\theta\frac{\partial\theta}{\partial n}\frac{\partial C}{\partial y}-\cos\theta\frac{\partial^2 C}{\partial y\partial n}\\
&= -\left(\cos\theta\frac{\partial C}{\partial x}+\sin\theta\frac{\partial C}{\partial y}\right)\frac{\partial\theta}{\partial n}-\sin\theta\frac{\partial}{\partial x}\left(-\sin\theta\frac{\partial C}{\partial x}+\cos\theta\frac{\partial C}{\partial y}\right)\\
&\qquad +\cos\theta\frac{\partial}{\partial y}\left(-\sin\theta\frac{\partial C}{\partial x}+\cos\theta\frac{\partial C}{\partial y}\right)\\
&= -\frac{\partial C}{\partial s}\frac{\partial\theta}{\partial n}+\sin\theta\cos\theta\frac{\partial\theta}{\partial x}\frac{\partial C}{\partial x}+\sin^2\theta\frac{\partial^2 C}{\partial x^2}+\sin^2\theta\frac{\partial\theta}{\partial x}\frac{\partial C}{\partial y}-\sin\theta\cos\theta\frac{\partial^2 C}{\partial x\partial y}\\
&\quad -\cos^2\theta\frac{\partial\theta}{\partial y}\frac{\partial C}{\partial x}-\sin\theta\cos\theta\frac{\partial^2 C}{\partial x\partial y}-\sin\theta\cos\theta\frac{\partial\theta}{\partial y}\frac{\partial C}{\partial y}+\cos^2\theta\frac{\partial^2 C}{\partial y^2}\\
&= -\frac{\partial C}{\partial s}\frac{\partial\theta}{\partial n}+\cos\theta\frac{\partial C}{\partial x}\left(\sin\theta\frac{\partial\theta}{\partial x}-\cos\theta\frac{\partial\theta}{\partial y}\right)+\sin^2\theta\frac{\partial^2 C}{\partial x^2}\\
&\quad +\sin\theta\frac{\partial C}{\partial y}\left(\sin\theta\frac{\partial\theta}{\partial x}-\cos\theta\frac{\partial\theta}{\partial y}\right)-2\sin\theta\cos\theta\frac{\partial^2 C}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 C}{\partial y^2}\\
&= -\frac{\partial C}{\partial s}\frac{\partial\theta}{\partial n}-\frac{\partial\theta}{\partial n}\left(\cos\theta\frac{\partial C}{\partial x}+\sin\theta\frac{\partial C}{\partial y}\right)+\sin^2\theta\frac{\partial^2 C}{\partial x^2}-2\sin\theta\cos\theta\frac{\partial^2 C}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 C}{\partial y^2}\\
&= -2\frac{\partial C}{\partial s}\frac{\partial\theta}{\partial n}+\sin^2\theta\frac{\partial^2 C}{\partial x^2}-2\sin\theta\cos\theta\frac{\partial^2 C}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 C}{\partial y^2}\\
&= -2\left(\cos\theta\frac{\partial C}{\partial x}+\sin\theta\frac{\partial C}{\partial y}\right)\frac{1}{\beta}\frac{\partial\beta}{\partial s}+\sin^2\theta\frac{\partial^2 C}{\partial x^2}-2\sin\theta\cos\theta\frac{\partial^2 C}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 C}{\partial y^2}
\end{aligned}$$

Now, we have an ODE for $\beta(s)$:

$$\frac{d^2\beta}{ds^2} + p\frac{d\beta}{ds} + q\beta = 0$$

where

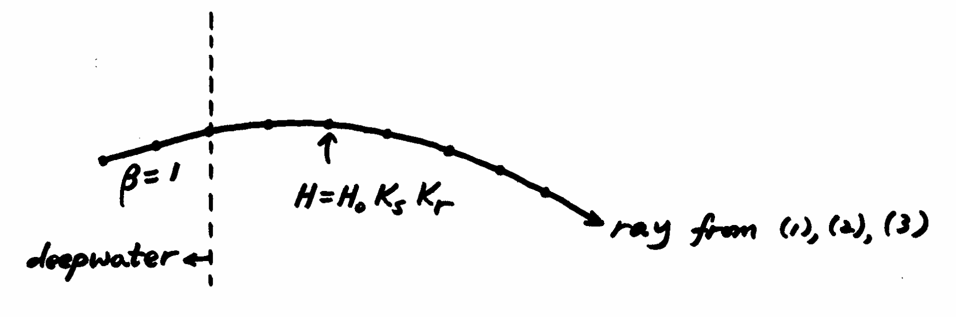
$$p(s) = -\frac{2}{C}\left(\cos\theta\frac{\partial C}{\partial x} + \sin\theta\frac{\partial C}{\partial y}\right)$$

$$q(s) = \frac{1}{C}\left(\sin^2\theta\frac{\partial^2 C}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 C}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 C}{\partial y^2}\right)$$

Initial condition:

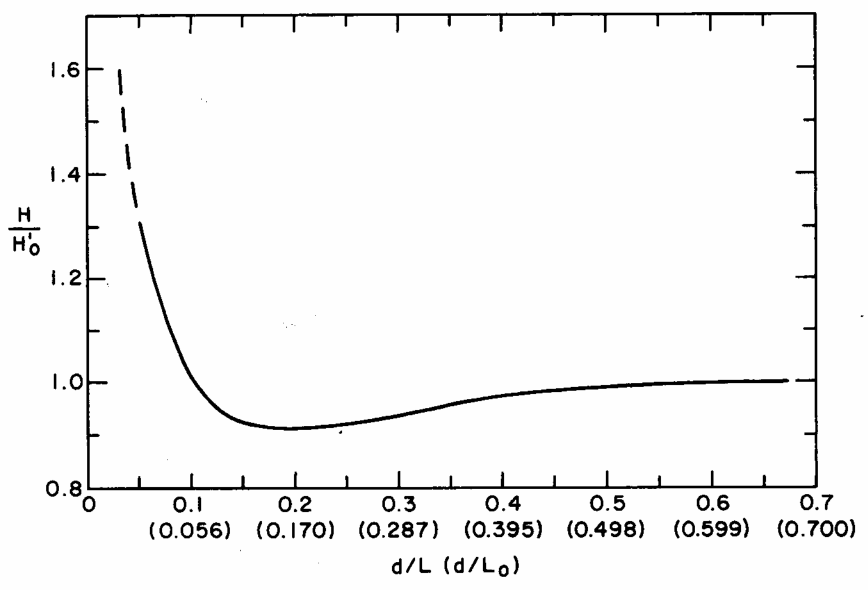
$$\beta = 1,\ \frac{d\beta}{dt} = 0 \quad \text{at} \quad t = 0 \quad \text{in deep water}$$

which means that no refraction occurs, i.e. $K_r = 1$, in deep water



A ray tracing method gives θ and H along rays not at grid points. So, it is difficult to use the results from ray tracing method as input data for numerical models (e.g. sediment transport model) using finite difference method. We need interpolation. Furthermore, infinite wave heights are calculated at the ray-crossing points. To resolve these problems, finite-difference refraction models have been developed, e.g. Noda et al. (1974), Perlin and Dean (1983), Dalrymple (1988).

<u>Wave breaking</u>



(1) Finite, constant depth

Miche (1944):

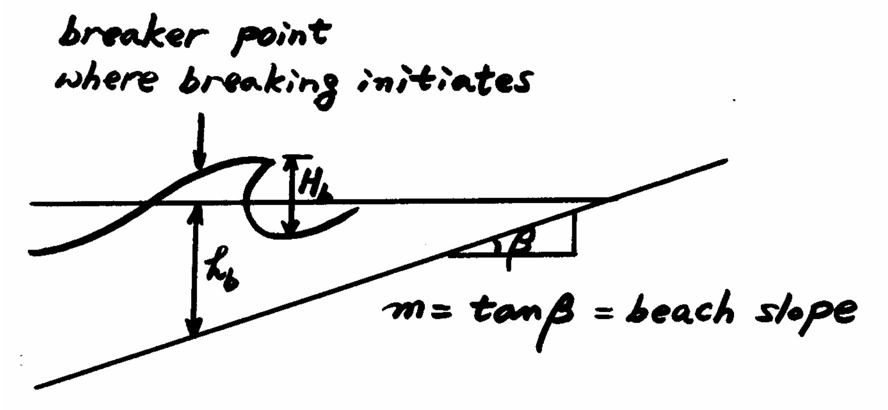
$$\left(\frac{H}{L}\right)_b = 0.142 \tanh\left(\frac{2\pi h}{L}\right)$$

As $h/L \to 0$, $H/h = 0.89$ at breaking.

But, Nelson (1983, 6th Australian Conf. on Coastal and Ocean Eng.) found based on experimental results that

$$\left(\frac{H}{h}\right)_b \le 0.55 \quad \text{on a constant depth}$$

(2) Shoaling waves in shallow water



where h_b = breaking depth, and H_b = breaker height.

$$\frac{H_b}{h_b} = \kappa\left(m, \frac{H_b}{gT^2}\right)$$ as in Fig. 12.7 of textbook,

where m = beach slope.

Surf similarity parameter based on H_b (Battjes, 1974, Proc. 14th Coastal Eng. Conf., ASCE, 466-480):

$$\xi_b = \frac{m}{\sqrt{H_b / L_0}}$$

Breaker types depending on ξ_b :

Spilling	$\xi_b \le 0.4$	smaller beach slope and steeper wave
Plunging	$0.4 \le \xi_b \le 2$	

Collapsing $2 \le \xi_b \le 3.5$

Surging $3.5 \le \xi_b$ larger beach slope and milder wave

For other breaker height formulas, see Rattanapitikon and Shibayama (2000) Coastal Eng. Journal, 42(4): 389-406.

Analyzing many laboratory results, Weggel (1972) proposed

$$\kappa = b(m) - a(m)\frac{H_b}{gT^2}$$

where

$$a(m) = 43.8\left(1.0 - e^{-19m}\right)$$

$$b(m) = 1.56\left(1.0 + e^{-19.5m}\right)^{-1}$$

On a beach with straight and parallel contours,

$$H = H_0\sqrt{\frac{C_0}{2C_g}}\sqrt{\frac{\cos\theta_0}{\cos\theta}} = H_0 K_s K_r$$

For breaking wave in shallow water, $C_g \cong \sqrt{gh_b}$ and $\theta_b \cong 0$. Therefore

$$H_b = \kappa h_b \cong H_0\left(\frac{C_0}{2\sqrt{gh_b}}\cos\theta_0\right)^{1/2}$$

$$\kappa^2 h_b^2 = H_0^2\frac{C_0}{2\sqrt{gh_b}}\cos\theta_0$$

$$h_b^{5/2} = \frac{1}{g^{1/2}\kappa^2}\frac{H_0^2 C_0 \cos\theta_0}{2}$$

$$h_b = \frac{1}{g^{1/5}\kappa^{4/5}}\left(\frac{H_0^2 C_0 \cos\theta_0}{2}\right)^{2/5}$$

$$H_b = \kappa h_b = \left(\frac{\kappa}{g}\right)^{1/5}\left(\frac{H_0^2 C_0 \cos\theta_0}{2}\right)^{2/5}$$

For a plane beach with $h = mx$, where x = distance from shoreline,

$$mx_b = \frac{1}{g^{1/5}\kappa^{4/5}}\left(\frac{H_0^2 C_0 \cos\theta_0}{2}\right)^{2/5}$$

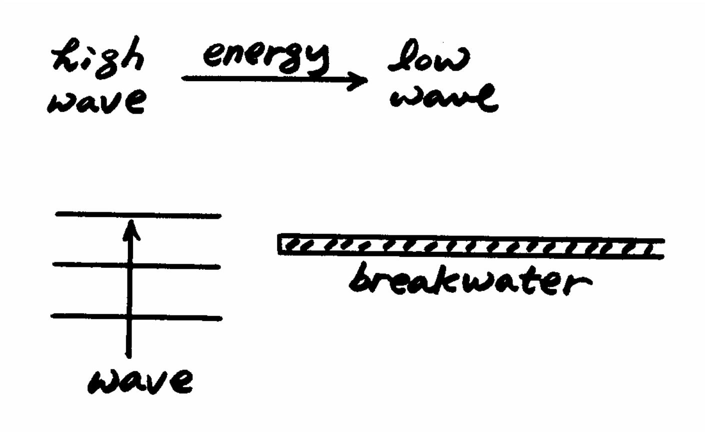
$$x_b = \frac{1}{mg^{1/5}\kappa^{4/5}}\left(\frac{H_0^2 C_0 \cos\theta_0}{2}\right)^{2/5}$$

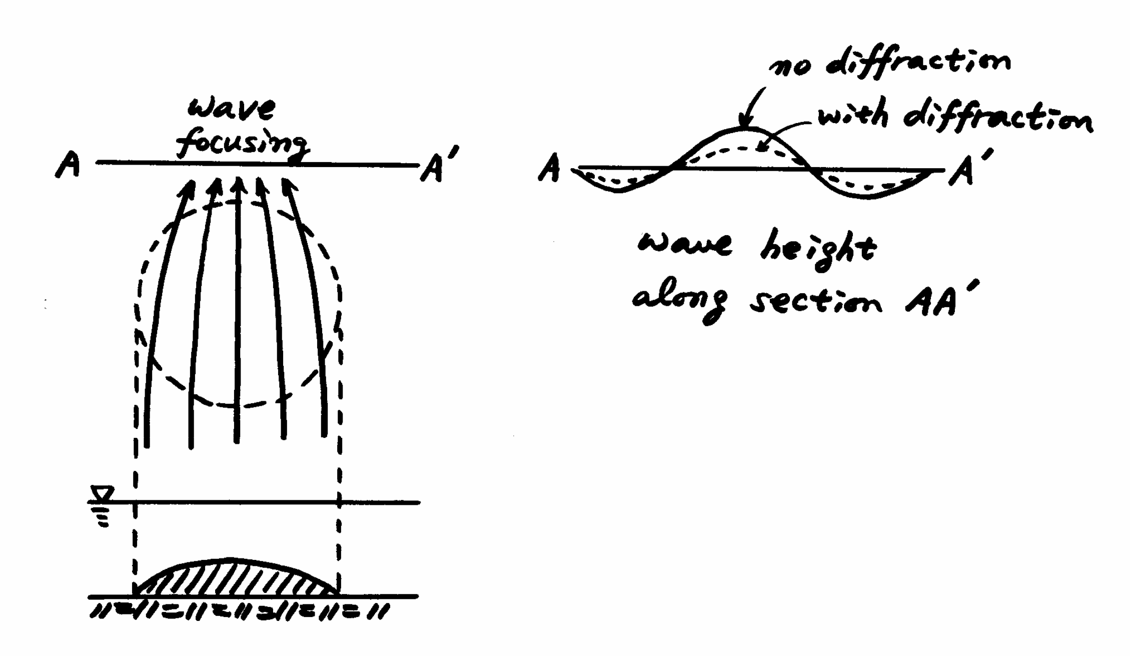
where x_b = distance from shoreline to breaker point.

4.9 Wave Diffraction

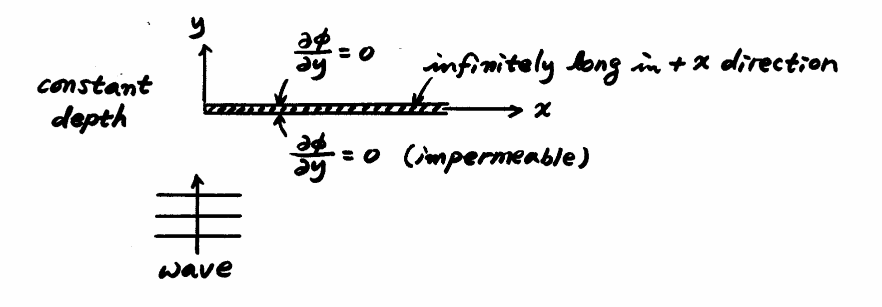
High energy region →(Energy transfer)→ Low energy region

Examples:





Diffraction behind a breakwater: Penney and Price (1952) Philosophical Transaction of Royal Society of London, A244: 236-253.



BBC, DFSBC, and KFSBC are same as before. Far offshore from breakwater,

$$\phi = -\frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\sin(ky-\sigma t) \quad \text{as} \quad y \to -\infty$$

Assume

$$\phi = \mathrm{Im}\left\{F(x,y)Z(z)e^{-i\sigma t}\right\}$$

where

$$Z(z) = \cosh k(h+z)$$

Substituting into the Laplace equation,

$$\nabla^2\phi = e^{-i\sigma t}\left(\frac{\partial^2 ZF}{\partial x^2} + \frac{\partial^2 ZF}{\partial y^2} + F\frac{\partial^2 Z}{\partial z^2}\right) = 0$$

$$Z\frac{\partial^2 F}{\partial x^2} + Z\frac{\partial^2 F}{\partial y^2} + Fk^2 Z = 0$$

$$\therefore \; \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + k^2 F = 0 \quad \text{Helmholtz equation for} \quad F(x,y)$$

The solution is given by Eq. (4.135) of textbook.

4.10 Combined Refraction-Diffraction

Assume locally

$$\phi = F(x, y)\frac{\cosh k(h+z)}{\cosh kh}e^{-i\sigma t}$$

Berkoff (1972), Proc. 13th International Conference on Coastal Eng., 471-490:

$$\boxed{\nabla_h \cdot \left(CC_g \nabla_h F\right) + \sigma^2 \frac{C_g}{C} F = 0} \qquad \text{mild-slope equation}$$

The mild-slope equation is an elliptic equation, which gives a boundary value problem in horizontal space and can be solved when all the boundary conditions are specified.

Radder (1979, Journal of Fluid Mechanics, 95: 159-176) proposed the parabolic approximation, which gives an initial value problem so that the parabolic equation can be solved by marching toward shore, with given offshore boundary condition.

Also, Copeland (1985, Coastal Engineering, 9: 125-149) proposed a hyperbolic equation, and Massel (1993, Coastal Engineering, 19: 97-126) proposed an extended refraction-diffraction equation, which includes higher-order bottom effect terms compared with the mild-slope equation.