Chapter 5. Long Waves

Wave length is long compared with water depth:

$$kh < \pi / 10$$
$$h / L < 1/20$$

From linear wave theory, for $kh << \pi / 10$,

$u_s = \dfrac{C\eta}{h}$; uniform over depth

$w_s = -C\left(1+\dfrac{z}{h}\right)\dfrac{\partial \eta}{\partial x}$; linearly varying vertically from zero at the bottom to a maximum at the surface

$\dfrac{(u_s)_{\max}}{(w_s)_{\max}} = \dfrac{1}{kh} >> 1$; almost horizontal motion

$p_s = \rho g(\eta - z)$; hydrostatic pressure

Depth-averaged conservation of mass

$$\int_{-h}^{\eta}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dz = 0$$

Using Leibniz rule of integration,

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} u dz - u(x,y,\eta)\frac{\partial \eta}{\partial x} - u(x,y,-h)\frac{\partial h}{\partial x}$$
$$+\frac{\partial}{\partial y}\int_{-h}^{\eta} v dz - v(x,y,\eta)\frac{\partial \eta}{\partial y} - v(x,y,-h)\frac{\partial h}{\partial y} + w(x,y,\eta) - w(x,y,-h) = 0$$

Introducing depth-averaged velocities:

$$U = \frac{1}{h+\eta}\int_{-h}^{\eta} u dz\,, \quad V = \frac{1}{h+\eta}\int_{-h}^{\eta} v dz$$

we have

$$\frac{\partial}{\partial x}\left[U(h+\eta)\right] + \frac{\partial}{\partial y}\left[V(h+\eta)\right] + w(x,y,\eta) - u(x,y,\eta)\frac{\partial \eta}{\partial x} - v(x,y,\eta)\frac{\partial \eta}{\partial y}$$

$$- w(x,y,-h) - u(x,y,-h)\frac{\partial h}{\partial x} - v(x,y,-h)\frac{\partial h}{\partial y} = 0$$

$$\therefore\ \frac{\partial}{\partial x}\left[U(h+\eta)\right] + \frac{\partial}{\partial y}\left[V(h+\eta)\right] + \frac{\partial \eta}{\partial t} = 0$$

Linearizing by assuming η, U, V are small,

$$\frac{\partial (Uh)}{\partial x} + \frac{\partial (Vh)}{\partial y} + \frac{\partial \eta}{\partial t} = 0$$

Depth-averaged equation of motion

$$x: \underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$

$$\downarrow \quad + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z}$$

Assuming horizontal velocity gradient is constant over depth so that τ_{xx}, τ_{yx} are also constant over depth and integrating over depth using Leibniz rule,

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{-h}^{\eta} udz - u(\eta)\frac{\partial \eta}{\partial t} - u(-h)\frac{\partial h}{\partial t} \\
&+ \frac{\partial}{\partial x}\int_{-h}^{\eta} u^2 dz - u(\eta)u(\eta)\frac{\partial \eta}{\partial x} - u(-h)u(-h)\frac{\partial h}{\partial x} \\
&+ \frac{\partial}{\partial y}\int_{-h}^{\eta} uvdz - u(\eta)v(\eta)\frac{\partial \eta}{\partial y} - u(-h)v(-h)\frac{\partial h}{\partial y} \\
&+ u(\eta)w(\eta) - u(-h)w(-h) = -g(h+\eta)\frac{\partial \eta}{\partial x} + \frac{1}{\rho}(h+\eta)\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right) + \frac{1}{\rho}\left(\tau_{zx}(\eta) - \tau_{zx}(-h)\right)
\end{aligned}$$

Introducing correction factors, β_{xx} and β_{xy}, such that

$$\beta_{xx}U^2(h+\eta) = \int_{-h}^{\eta} u^2 dz \quad \leftarrow \quad U(h+\eta) = \int_{-h}^{\eta} udz$$

$$\beta_{xy}UV(h+\eta) = \int_{-h}^{\eta} uvdz$$

we have

$$\frac{\partial}{\partial t}[U(h+\eta)]+\frac{\partial}{\partial x}\left[\beta_{xx}U^2(h+\eta)\right]+\frac{\partial}{\partial y}\left[\beta_{xy}UV(h+\eta)\right]$$
$$=-g(h+\eta)\frac{\partial\eta}{\partial x}+\frac{1}{\rho}(h+\eta)\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}\right)+\frac{1}{\rho}\left(\tau_{zx}(\eta)-\tau_{zx}(-h)\right)$$

$$(h+\eta)\frac{\partial U}{\partial t}+U\frac{\partial(h+\eta)}{\partial t}+(h+\eta)U\frac{\partial(\beta_{xx}U)}{\partial x}+\beta_{xx}U\frac{\partial\left(U(h+\eta)\right)}{\partial x}$$
$$+(h+\eta)V\frac{\partial(\beta_{xy}U)}{\partial y}+\beta_{xy}U\frac{\partial\left(V(h+\eta)\right)}{\partial y}$$
$$=-g(h+\eta)\frac{\partial\eta}{\partial x}+\frac{1}{\rho}(h+\eta)\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}\right)+\frac{1}{\rho}\left(\tau_{zx}(\eta)-\tau_{zx}(-h)\right)$$

$$\therefore\ \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-g\frac{\partial\eta}{\partial x}+\frac{1}{\rho}\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}\right)+\frac{\tau_{zx}(\eta)-\tau_{zx}(-h)}{\rho(h+\eta)}$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}=-g\frac{\partial\eta}{\partial y}+\frac{1}{\rho}\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}\right)+\frac{\tau_{zy}(\eta)-\tau_{zy}(-h)}{\rho(h+\eta)}$$

Assuming no friction (equivalent to Euler equation) and linearizing,

$$\frac{\partial U}{\partial t}=-g\frac{\partial\eta}{\partial x}$$
$$\frac{\partial V}{\partial t}=-g\frac{\partial\eta}{\partial y}$$

On flat bottom, the continuity equation becomes

$$\frac{\partial\eta}{\partial t}+h\frac{\partial U}{\partial x}+h\frac{\partial V}{\partial y}=0$$

Differentiating with respect to t,

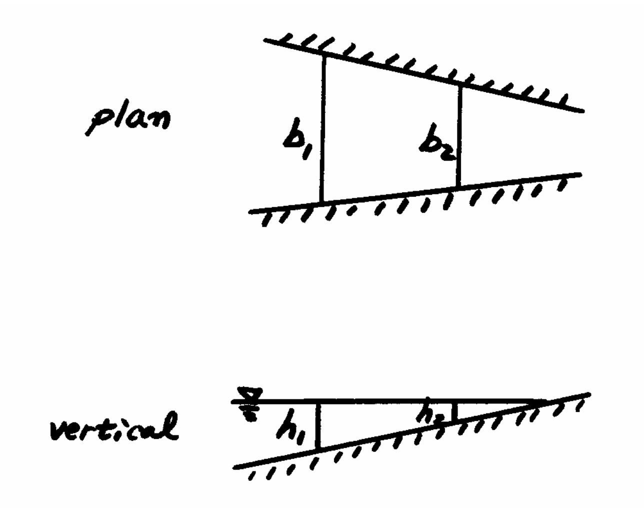
$$\frac{\partial^2 \eta}{\partial t^2} + h\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t}\right) + h\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial t}\right) = 0$$

Substituting the linearized momentum equation,

$$\frac{\partial^2 \eta}{\partial t^2} - gh\frac{\partial^2 \eta}{\partial x^2} - gh\frac{\partial^2 \eta}{\partial y^2} = 0$$

$$\frac{\partial^2 \eta}{\partial t^2} = C^2\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right); \quad \text{wave equation}$$

Energy flux



$$F = EC_g = \frac{1}{8}\rho g H^2 \sqrt{gh}$$

$$H_1^2 \sqrt{gh_1} b_1 = H_2^2 \sqrt{gh_2} b_2$$

$$\frac{H_2}{H_1} = \left(\frac{h_1}{h_2}\right)^{1/4} \left(\frac{b_1}{b_2}\right)^{1/2}$$

$$H_2 = H_1 \left(\frac{h_1}{h_2} \right)^{1/4} \left(\frac{b_1}{b_2} \right)^{1/2} = H_1 K_s K_r \quad \leftarrow \text{Green's law if } \; b_1 = b_2$$

Co-oscillating tide

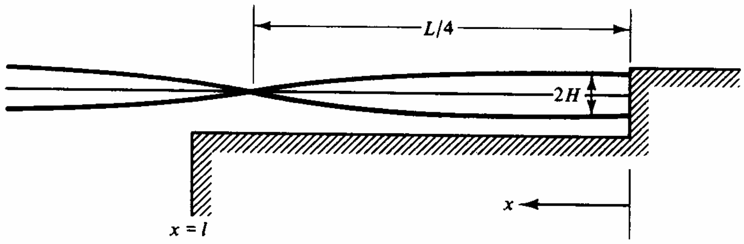


Figure 5.2 Co-oscillating tide in a channel of length *l*.

$$\eta(x,t) = \eta_i + \eta_r$$
$$= \frac{H}{2}\cos(kx + \sigma t) + \frac{H}{2}\cos(kx - \sigma t)$$
$$= H \cos kx \cos \sigma t$$

$$\eta(0,t) = H \cos \sigma t$$

$$\eta(0)_{\max} = H$$

$$\eta(l,t) = H \cos kl \cos \sigma t$$

$$\eta(l)_{\max} = H \cos kl$$

$$\frac{\eta(0)_{\max}}{\eta(l)_{\max}} = \frac{1}{\cos kl}$$

For the node to locate at $x = l$,

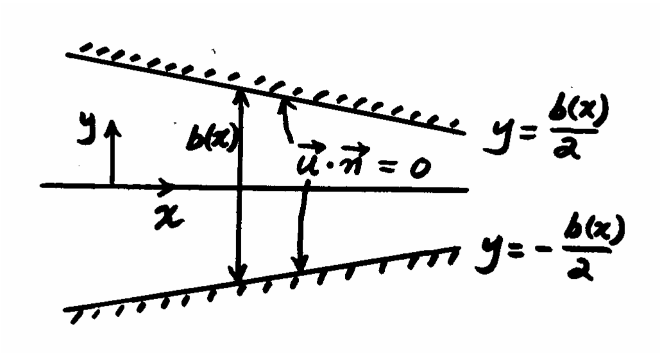
$$\cos kl = 0 \quad \text{for} \quad kl = \frac{\pi}{2}, \frac{3}{2}\pi, \cdots$$
$$l = \frac{L}{4}, \frac{3}{4}L, \cdots$$

Related topics:

1) Location of barrage (turbines and sluice gates) of tidal power plant

2) Wave reflection from perforated-wall caisson breakwaters

<u>Channel with variable cross-sections</u>



The equations describing the channel walls are

$$F(b/2) = y - \frac{b(x)}{2} = 0; \qquad F(-b/2) = y + \frac{b(x)}{2} = 0$$

Using the kinematic boundary conditions $\vec{u}\cdot\vec{n} = 0$ on channel walls and $\vec{n} = \nabla F / |\nabla F|$,

we have

$$-u(b/2)\frac{\partial(b/2)}{\partial x} + v(b/2) = 0; \qquad u(-b/2)\frac{\partial(b/2)}{\partial x} + v(-b/2) = 0$$

or

$$v(b/2) - v(-b/2) = u(b/2)\frac{\partial(b/2)}{\partial x} + u(-b/2)\frac{\partial(b/2)}{\partial x}$$

Integrating continuity equation over depth and channel width,

$$\int_{-h}^{\eta}\int_{-b/2}^{b/2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dydz = 0$$

$$\int_{-h}^{\eta}\left[\frac{\partial}{\partial x}\int_{-b/2}^{b/2} udy - u(b/2)\frac{\partial(b/2)}{\partial x} - u(-b/2)\frac{\partial(b/2)}{\partial x} + v(b/2) - v(-b/2) + b\frac{\partial w}{\partial z}\right]dz = 0$$

Using the kinematic boundary conditions on the walls and defining the cross-sectional average velocity,

$$\tilde{U} = \frac{1}{b}\int_{-b/2}^{b/2} u dy$$

we have

$$\int_{-h}^{\eta}\left(\frac{\partial(b\tilde{U})}{\partial x} + b\frac{\partial w}{\partial z}\right)dz = 0$$

$$\frac{\partial(b\tilde{U})}{\partial x}(h+\eta) + \int_{-h}^{\eta} b\frac{\partial w}{\partial z}dz = 0$$

$$\frac{\partial}{\partial x}\left[\tilde{U}(h+\eta)b\right] - b\tilde{U}\frac{\partial \eta}{\partial x} - b\tilde{U}\frac{\partial h}{\partial x} + bw(\eta) - bw(-h) = 0$$

Linearizing, assuming small bottom slope, and using LKFSBC,

$$\frac{\partial}{\partial x}(Uhb) + b\frac{\partial \eta}{\partial t} = 0\,; \quad \text{Continuity}$$

$$b\frac{\partial U}{\partial t} = -gb\frac{\partial \eta}{\partial x}\,; \quad \text{Equation of motion (same as before)}$$

Differentiating the continuity equation w.r.t. t,

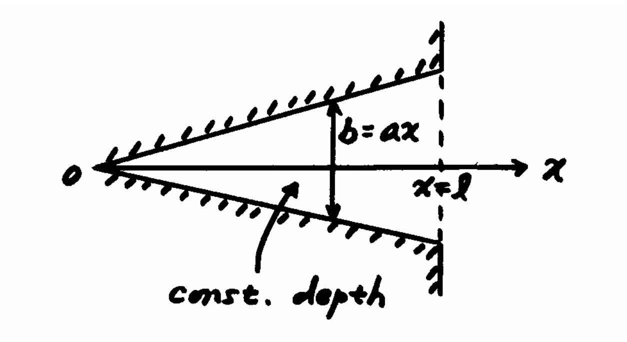
$$b\frac{\partial^2 \eta}{\partial t^2} + \frac{\partial}{\partial x}\left(hb\frac{\partial U}{\partial t}\right) = 0$$

Using the equation of motion,

$$b\frac{\partial^2 \eta}{\partial t^2} + \frac{\partial}{\partial x}\left(-ghb\frac{\partial \eta}{\partial x}\right) = 0$$

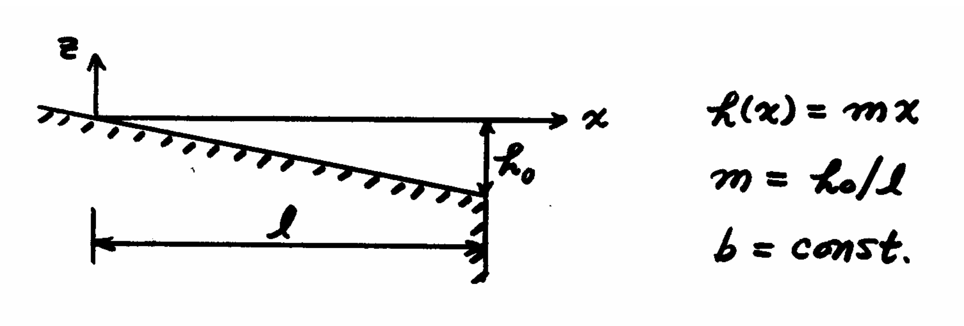
$$\frac{\partial^2 \eta}{\partial t^2} = \frac{g}{b}\frac{\partial}{\partial x}\left(hb\frac{\partial \eta}{\partial x}\right) = \frac{1}{b}\frac{\partial}{\partial x}\left(C^2 b\frac{\partial \eta}{\partial x}\right) \qquad \text{(5.36) in textbook}$$

Ex 5.1



$$\eta(x,t) = \frac{H}{2}\frac{J_0(kx)}{J_0(kl)}\cos\sigma t$$

Another example:



Assume long standing wave:

$$\eta(x,t) = Af(x)\cos\sigma t$$

Eq. (5.36) for constant b is

$$\frac{\partial^2\eta}{\partial t^2} - g\frac{\partial}{\partial x}\left(h\frac{\partial\eta}{\partial x}\right) = 0$$

Substituting the assumed solution,

$$\left[-\sigma^2 f - g\frac{\partial}{\partial x}\left(mx\frac{\partial f}{\partial x}\right)\right]\cos\sigma t = 0$$

$$-\sigma^2 f - gm\frac{df}{dx} - gmx\frac{d^2 f}{dx^2} = 0$$

$$\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} + \frac{\sigma^2}{gmx} f = 0$$

$$x^2\frac{d^2 f}{dx^2} + x\frac{df}{dx} + \frac{\sigma^2}{gm} xf = 0 \quad \leftarrow \text{Bessel equation}$$

The general solution to the Bessel equation is

$$f = C_1 J_0\left(2\sqrt{\frac{\sigma^2 x}{gm}}\right) + C_2 Y_0\left(2\sqrt{\frac{\sigma^2 x}{gm}}\right)$$

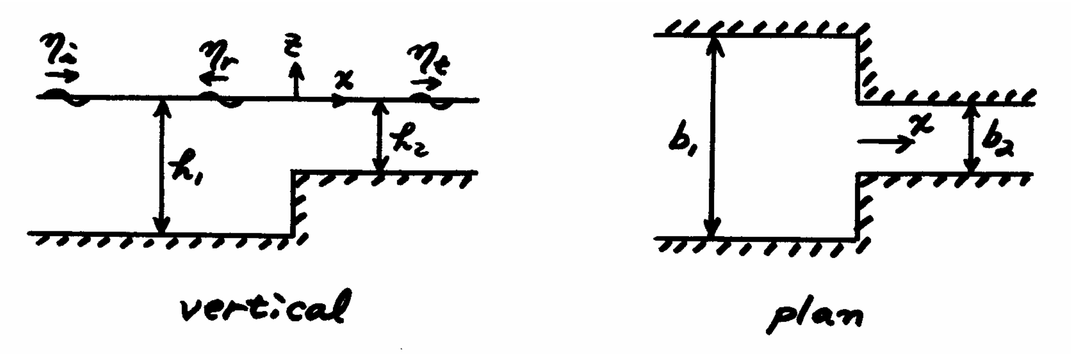
Since $Y_0 \to -\infty$ as $x \to 0$, C_2 must be zero. Thus

$$\eta = AJ_0\left(2\sqrt{\frac{\sigma^2 x}{gm}}\right)\cos\sigma t$$

Using the offshore boundary condition, $\eta(l,t) = (H/2)\cos\sigma t$,

$$\eta = \frac{H/2}{J_0\left(2\sqrt{\frac{\sigma^2 l}{gm}}\right)} J_0\left(2\sqrt{\frac{\sigma^2 x}{gm}}\right)\cos\sigma t$$

Step problem



Evanescent modes are important near the step ($-4h_1 < x < 4h_2$). Considering only the progressive modes,

$$\eta_1 = \eta_i + \eta_r = \frac{H_i}{2}\cos(k_1 x - \sigma t) + \frac{H_r}{2}\cos(k_1 x + \sigma t + \varepsilon_r)$$

$$\eta_2 = \eta_t = \frac{H_t}{2}\cos(k_2 x - \sigma t + \varepsilon_t)$$

The matching conditions at the step $(x = 0)$ are

$$\eta_1 = \eta_2 \quad \text{at} \quad x = 0 \quad \text{(dynamic)}$$

$$(Ubh)_1 = (Ubh)_2 \quad \text{at} \quad x = 0 \quad \text{(kinematic)}$$

Solving for the transmission and reflection coefficients,

$$\kappa_t = \frac{2}{1 + b_2 C_2 / b_1 C_1}$$

$$\kappa_r = \frac{1 - b_2 C_2 / b_1 C_1}{1 + b_2 C_2 / b_1 C_1}$$

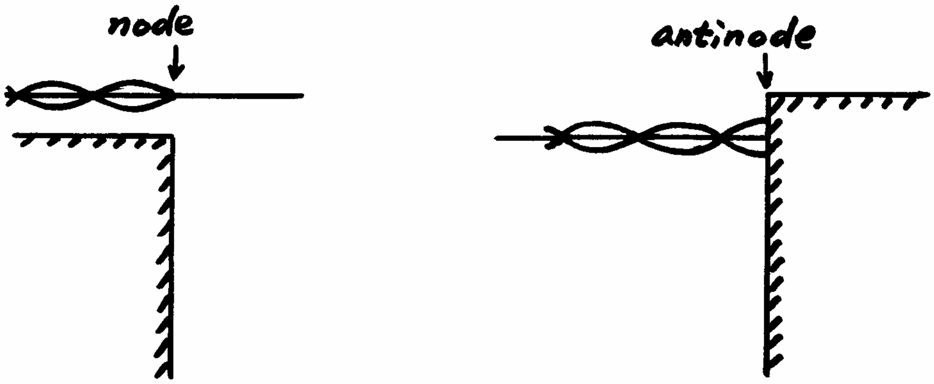
In the case of $b_1 = b_2$,

if $h_1 >> h_2$, $\kappa_r \to 1$: Pure standing wave in Region 1

$\kappa_t \to 2$: The wave of the same height as the standing wave is transmitted to Region 2

if $h_2 >> h_1$, $\kappa_t \rightarrow 0$: No transmission

$\kappa_r \rightarrow -1$: Full reflection but with $180°$ phase shift



Seiching

Seiching is an oscillation in a basin at its natural frequency instead of (known) tidal frequency

$$\eta = \frac{H}{2}\cos kx \cos \sigma t\,; \quad \sigma, k \text{ unknown}$$

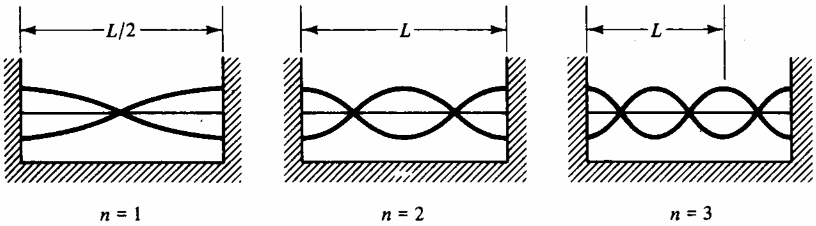


Figure 5.6 Standing waves in a simple rectangular basin. The first three modes are shown.

$$l = \frac{n}{2}L \;\Rightarrow\; L = \frac{2}{n}l$$

Using

$$C = \frac{L}{T} = \sqrt{gh}$$

we have

$$T = \frac{L}{\sqrt{gh}} = \frac{2l}{n\sqrt{gh}} \;\leftarrow \text{Merian formula}$$

$n = 1$: fundamental (or first) mode $\rightarrow$ most dominant

See Table 5.1 of textbook for variable cross-sections.

Long waves with bottom friction

$$x: \quad \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -g\frac{\partial \eta}{\partial x} + \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right) + \frac{\tau_{zx}(\eta) - \tau_{zx}(-h)}{\rho(h+\eta)}$$

Assume no surface wind stress. Then, the linearized 1-D equation of motion with bottom friction becomes

$$\frac{\partial U}{\partial t} = -g\frac{\partial \eta}{\partial x} - \frac{\tau_{zx}(-h)}{\rho h}$$

For unidirectional open channel flow,

$$\tau_{zx}(-h) = \tau_b = \frac{\rho f}{8}U^2$$

where f = Darcy-Weisbach friction factor. For a bi-directional oscillatory flow,

$$\tau_b = \frac{\rho f}{8}U|U|$$

which is nonlinear. To linearize, we expand τ_b in a Fourier cosine series:

$$\tau_b = \frac{\rho f}{8}U|U| = \frac{\rho f}{8}U_m^2 \cos\sigma t|\cos\sigma t| = \frac{\rho f}{8}U_m^2\left[\sum_{n=0}^{\infty} a_n \cos n\sigma t\right]$$

Note that

$$\frac{1}{T}\int_0^T \cos(n\sigma t)\cos(m\sigma t)dt = 0 \quad \text{if} \quad m \neq n \quad \text{by orthogonality.}$$

Now,

$$\frac{1}{T}\int_0^T \cos\sigma t|\cos\sigma t|dt = \frac{1}{T}\int_0^T (a_0 + a_1\cos\sigma t + a_2\cos 2\sigma t + \cdots)dt$$

$$\therefore\ a_0 = \frac{1}{T}\int_0^T \cos\sigma t|\cos\sigma t|dt$$

To calculate a_1,

$$\frac{1}{T}\int_0^T \cos\sigma t|\cos\sigma t|\cos\sigma t dt = \frac{1}{T}\int_0^T (a_0 + a_1\cos\sigma t + a_2\cos 2\sigma t + \cdots)\cos\sigma t dt$$

$$= \frac{1}{T}\int_0^T a_1\cos^2\sigma t dt$$

$$= \frac{a_1}{2}$$

$$\therefore\ a_1 = \frac{2}{T}\int_0^T \cos\sigma t|\cos\sigma t|\cos\sigma t dt$$

Similarly,

$$a_n = \frac{2}{T}\int_0^T \cos\sigma t|\cos\sigma t|\cos n\sigma t dt$$

Evaluating the integrals,

$$a_0 = 0$$

$$a_1 = \frac{8}{3\pi}$$

$$a_2 = 0$$

$$a_3 = \frac{8}{15\pi} = \frac{a_1}{5}$$

$$a_4 = 0$$

$$\vdots$$

Keeping only the first term,

$$\tau_b \cong \frac{\rho f}{8} U_m^2 \frac{8}{3\pi} \cos \sigma t = \frac{\rho f}{3\pi} U_m U$$

which is now linear. The linearized equation of motion is now

$$\begin{aligned} \frac{\partial U}{\partial t} &= -g \frac{\partial \eta}{\partial x} - \frac{\tau_{zx}(-h)}{\rho h} \\ &= -g \frac{\partial \eta}{\partial x} - \frac{\rho f}{\rho h(3\pi)} U_m U \\ &= -g \frac{\partial \eta}{\partial x} - AU \end{aligned}$$

where

$$A = \frac{fU_m}{3\pi h} << 1$$

Continuity equation:

$$\frac{\partial \eta}{\partial t} + h \frac{\partial U}{\partial x} + U \frac{\partial h}{\partial x} = 0$$

On flat bottom,

$$\frac{\partial \eta}{\partial t} + h \frac{\partial U}{\partial x} = 0$$

Differentiating w.r.t. time,

$$\frac{\partial^2 \eta}{\partial t^2} + h \frac{\partial}{\partial x} \left(\frac{\partial U}{\partial t} \right) = 0$$

Substituting the equation of motion,

$$\frac{\partial^2 \eta}{\partial t^2} + h \frac{\partial}{\partial x} \left(-g \frac{\partial \eta}{\partial x} - AU \right) = 0$$

Using

$$h\frac{\partial U}{\partial x} = -\frac{\partial \eta}{\partial t} \quad \text{(from continuity equation)}$$

we have

$$\frac{\partial^2 \eta}{\partial t^2} + A\frac{\partial \eta}{\partial t} = gh\frac{\partial^2 \eta}{\partial x^2}$$

This is the wave equation with bottom friction. The second term is the friction term. Compare this with Eq. (5.23) of textbook.

Assume $\eta = a\exp[i(kx - \sigma t)]$. Substituting into the wave equation,

$$-\sigma^2 \eta - i\sigma A\eta + k^2 gh\eta = 0$$

The wave number, k, is complex. Define k_I as the wave number with no damping:

$$k_I = \frac{\sigma}{\sqrt{gh}} = \frac{2\pi}{L} \quad \rightarrow \quad k_I^2 = \frac{\sigma^2}{gh}$$

Then

$$-k_I^2 gh - \frac{ik_I^2 ghA}{\sigma} + k^2 gh = 0$$

$$-1 - \frac{iA}{\sigma} + \frac{k^2}{k_I^2} = 0$$

where

$$\frac{k}{k_I} = \frac{\text{wave number with damping}}{\text{wave number without damping}}$$

for which we have

$$\frac{k^2}{k_I^2} = 1 + \frac{iA}{\sigma}$$

Let $k = k_r + ik_i$. Then

$$\frac{k^2}{k_I^2} = \frac{k_r^2 + 2ik_r k_i - k_i^2}{k_I^2} = 1 + \frac{iA}{\sigma}$$

Imaginary part gives

$$\frac{2k_r k_i}{k_I^2} = \frac{A}{\sigma} \quad \rightarrow \quad \frac{k_i}{k_I} = \frac{A}{2\sigma \frac{k_r}{k_I}}$$

Real part gives

$$\frac{k_r^2 - k_i^2}{k_I^2} = 1$$

From these two equations,

$$\frac{k_r^2}{k_I^2} - \frac{\left(\frac{A}{2\sigma}\right)^2}{\left(\frac{k_r}{k_I}\right)^2} = 1$$

$$\left(\frac{k_r}{k_I}\right)^4 - \left(\frac{k_r}{k_I}\right)^2 - \left(\frac{A}{2\sigma}\right)^2 = 0$$

Solving the quadratic equation,

$$\left(\frac{k_r}{k_I}\right)^2 = \frac{1 \pm \sqrt{1+\left(\frac{A}{\sigma}\right)^2}}{2}$$

Note that $\sqrt{1+\varepsilon^2} \cong 1+\varepsilon^2/2$ if $\varepsilon << 1$. Since $A/\sigma << 1$, we have

$$\frac{k_r}{k_I} = \sqrt{\frac{1+\sqrt{1+\left(\frac{A}{\sigma}\right)^2}}{2}} \cong \sqrt{\frac{2+\frac{1}{2}\left(\frac{A}{\sigma}\right)^2}{2}} \cong 1+\frac{1}{8}\left(\frac{A}{\sigma}\right)^2$$

As bottom friction increases, $k_r\uparrow$, $L\downarrow$, and $C\downarrow$.

On the other hand,

$$\frac{k_i}{k_I} = \frac{A}{2\sigma\left[1+\frac{1}{8}\left(\frac{A}{\sigma}\right)^2\right]} \cong \frac{A}{2\sigma}$$

Now

$$\eta = ae^{i(kx-\sigma t)} = ae^{i(k_r x + ik_i x - \sigma t)} = ae^{-k_i x}e^{i(k_r x - \sigma t)}$$

Thus the wave amplitude decays exponentially with x.

$$\frac{\eta(x+L)}{\eta(x)} = \frac{\exp\left[-k_i(x+L)\right]}{\exp(-k_i x)} = \exp(-k_i L) = \exp(-2\pi k_i / k_r)$$

$$= \exp\left[-2\pi \frac{\frac{A}{2\sigma}}{1+\frac{1}{8}\left(\frac{A}{\sigma}\right)^2}\right]$$

$$\cong \exp(-\pi A/\sigma)$$

Storm surge

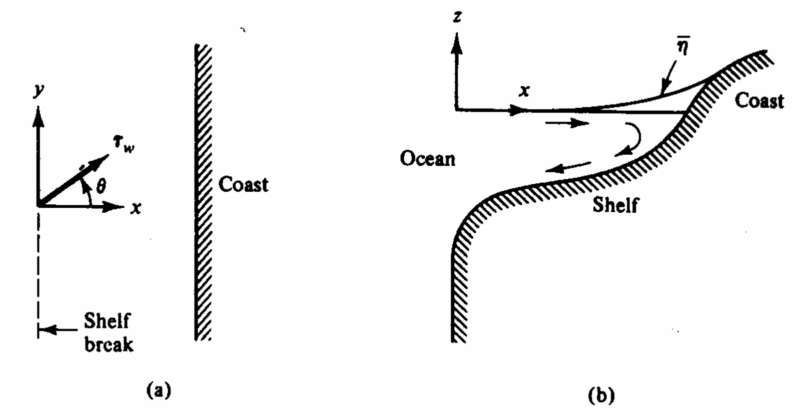


Figure 5.11 (a) Plan and (b) cross-sectional view of the coast.

$$\tau_w = \rho k W^2$$

where ρ = density of water, W = wind speed, and k = wind friction factor given by Van Dorn (1953) as

$$k = \begin{cases} 1.2\times10^{-6}, & |W| \le W_c \\ 1.2\times10^{-6} + 2.25\times10^{-6}(1 - W_c/|W|)^2, & |W| > W_c \end{cases}$$

with W_c = 5.6 m/s. The x- and y-components of τ_w are

$$\tau_{wx} = \tau_w \cos\theta$$
$$\tau_{wy} = \tau_w \sin\theta$$

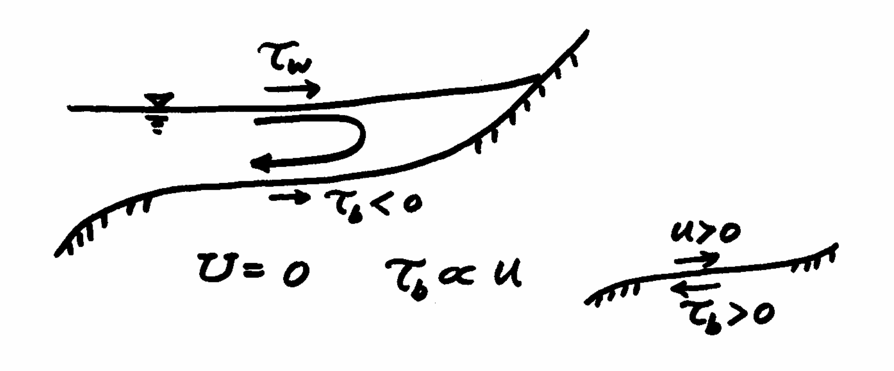
Linear equations:

x-momentum: $$\frac{\partial U}{\partial t} = -g\frac{\partial \eta}{\partial x} + \frac{1}{\rho(h+\eta)}\left[\tau_{zx}(\eta) - \tau_{zx}(-h)\right]$$

y-momentum: $$\frac{\partial V}{\partial t} = -g\frac{\partial \eta}{\partial y} + \frac{1}{\rho(h+\eta)}\left[\tau_{zy}(\eta) - \tau_{zy}(-h)\right]$$

continuity eq: $$\frac{\partial (Uh)}{\partial x} + \frac{\partial (Vh)}{\partial y} + \frac{\partial \eta}{\partial t} = 0$$

Simple case: $\theta = 0$ (2-D problem) and steady state



Equation of motion gives

$$0 = -g \frac{\partial \eta}{\partial x} + \frac{1}{\rho(h+\eta)} \left(\tau_w - \tau_b \right)$$

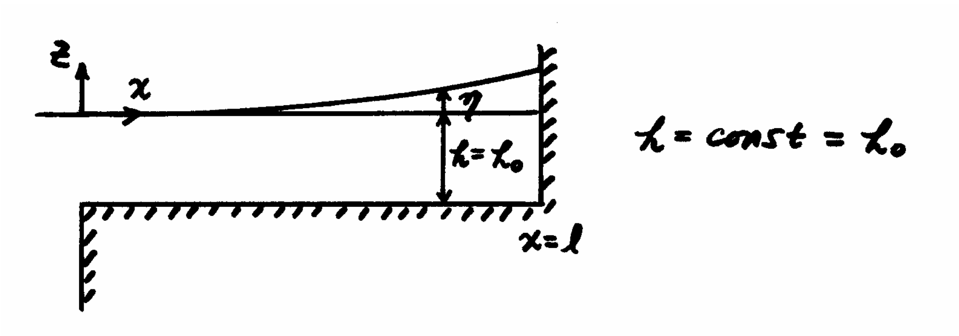
Let

$$\tau_w - \tau_b = n \tau_w ; \quad n = 1.15 \sim 1.3 \quad \text{(Note that } \tau_b \text{ is negative)}$$

Then

$$\frac{\partial \eta}{\partial x} = \frac{n \tau_w}{\rho g (h+\eta)}$$

Constant depth:



$$(h_0 + \eta)\frac{\partial \eta}{\partial x} = \frac{n\tau_w}{\rho g}$$

$$\frac{1}{2}\frac{d(h_0 + \eta)^2}{dx} = \frac{n\tau_w}{\rho g}$$

$$(h_0 + \eta)^2 = \frac{2n\tau_w}{\rho g}x + C$$

Using the boundary condition, $\eta = 0$ at $x = 0$, we have $C = h_0^2$. Therefore,

$$(h_0 + \eta)^2 = \frac{2n\tau_w}{\rho g}x + h_0^2$$

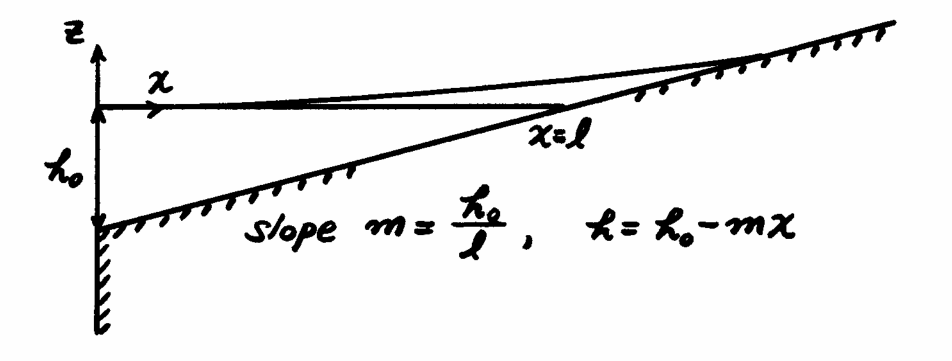
$$\eta = \sqrt{\frac{2n\tau_w}{\rho g}x + h_0^2} - h_0$$

$$\frac{\eta}{h_0} = \sqrt{1 + \frac{2n\tau_w l}{\rho g h_0^2}\frac{x}{l}} - 1$$

Letting $(n\tau_w l)/(\rho g h_0^2) = A$, we have

$$\frac{\eta}{h_0} = \sqrt{1 + \frac{2Ax}{l}} - 1 \cong A\frac{x}{l} \quad \text{if} \quad A << 1$$

Sloping bottom:



$$(h+\eta)\frac{d(h+\eta)}{dx}=\frac{n\tau_w}{\rho g}+(h+\eta)\frac{dh}{dx}$$

Using $dh/dx=-h_0/l$,

$$\frac{(h+\eta)d(h+\eta)}{\frac{n\tau_w}{\rho g}-(h+\eta)\frac{h_0}{l}}=dx$$

$$\frac{(h+\eta)d(h+\eta)}{\frac{h_0^2}{l}\left(\frac{n\tau_w l}{h_0^2\rho g}-\frac{h+\eta}{h_0}\right)}=dx$$

Recalling $(n\tau_w l)/(\rho g h_0^2)=A$ and integrating after substitution of $(h+\eta)/h_0-A=u$,

$$l\left[\left(A-\frac{h+\eta}{h_0}\right)-A\ln\left(\frac{h+\eta}{h_0}-A\right)\right]=x+C$$

Using the boundary condition, $\eta=0$ and $h=h_0$ at $x=0$, we have

$$C=l\left[(A-1)-A\ln(1-A)\right]$$

Therefore,

$$x = l\left[\left(A - \frac{h+\eta}{h_0}\right) - A\ln\left(\frac{h+\eta}{h_0} - A\right) - (A-1) + A\ln(1-A)\right]$$

$$\frac{x}{l} = 1 - \frac{h+\eta}{h_0} - A\ln\left(\frac{\frac{h+\eta}{h_0} - A}{1-A}\right)$$

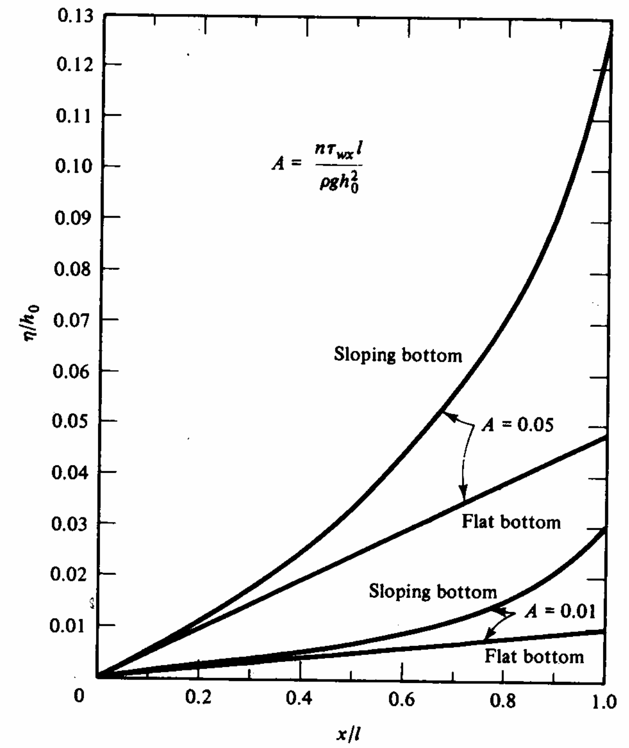


Figure 5.12 Dimensionless storm surge versus dimensionless distance of a continental shelf for two cases of dimensionless wind shear stress.

$$\frac{\partial\eta}{\partial x} = \frac{n\tau_w}{\rho g(h+\eta)} \cong \frac{n\tau_w}{\rho g h}$$

If τ_w = constant, $\frac{\partial\eta}{\partial x}\uparrow$ as $h\downarrow$

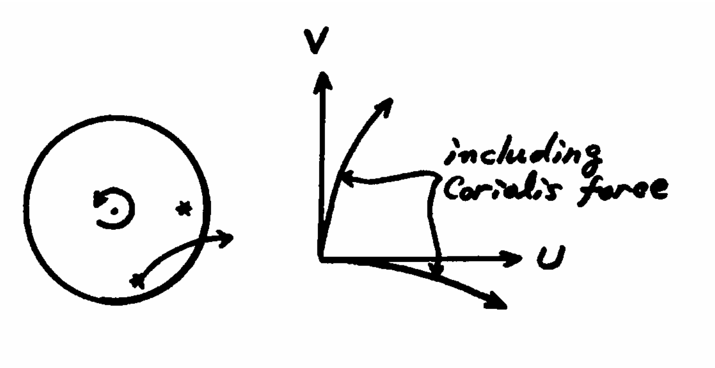
Including Coriolis force,

$$\frac{\partial U}{\partial t} = -g\frac{\partial \eta}{\partial x} + \frac{1}{\rho(h+\eta)}\left[\tau_{zx}(\eta) - \tau_{zx}(-h)\right] + f_C V$$
$$\frac{\partial V}{\partial t} = -g\frac{\partial \eta}{\partial y} + \frac{1}{\rho(h+\eta)}\left[\tau_{zy}(\eta) - \tau_{zy}(-h)\right] - f_C U$$

where $f_C = 2\omega \sin\phi$ = Coriolis parameter

ω = earth's rotation speed = $2\pi / 24\text{hrs} = 7.27\times 10^{-5}$ rad/s

ϕ = latitude (positive in northern hemisphere, negative in southern hemisphere)



Assume a long straight coastline so that $\partial\eta/\partial y = 0$ and $U = 0$ everywhere, which is true only in equilibrium state. Also assume that winds start to blow at $t = 0$. Then the y-momentum equation gives

$$\frac{\partial V}{\partial t} = \frac{\tau_w \sin\theta - \frac{\rho f}{8}V^2}{\rho(h+\eta)}$$

If $h >> \eta$,

$$\frac{\partial V}{\partial t} + \frac{fV^2}{8h} = \frac{\rho k W^2 \sin\theta}{\rho h}$$

Assume $V = A\tanh Bt$ so that $dV/dt = AB\operatorname{sech}^2 Bt$. Then

$$AB\operatorname{sech}^2 Bt + \frac{f}{8h}A^2 \tanh^2 Bt = \frac{kW^2 \sin\theta}{h}$$

$$AB\left(\operatorname{sech}^2 Bt + \frac{fA}{8Bh}\tanh^2 Bt\right) = \frac{kW^2 \sin\theta}{h}$$

We have one equation for two unknowns A and B. To solve this equation, let $fA/8Bh = 1$. Then

$$AB\left(\frac{1}{\cosh^2 Bt} + \frac{\sinh^2 Bt}{\cosh^2 Bt}\right) = \frac{kW^2 \sin\theta}{h}$$

Using the relationship $\cosh^2 x - \sinh^2 x = 1$,

$$AB = \frac{kW^2 \sin\theta}{h}$$

$$A = \frac{kW^2 \sin\theta}{hB}$$

Recalling $fA/8Bh = 1$,

$$B = \frac{fA}{8h} = \frac{fkW^2 \sin\theta}{8h^2 B}$$

$$B = \sqrt{\frac{fkW^2 \sin\theta}{8h^2}}$$

and

$$A = \frac{kW^2 \sin\theta}{\sqrt{\frac{fkW^2 \sin\theta}{8}}} = \sqrt{\frac{8k^2W^4 \sin^2\theta}{fkW^2 \sin\theta}} = \sqrt{\frac{8kW^2 \sin\theta}{f}}$$

Finally

$$V = \sqrt{\frac{8k\sin\theta}{f}}W \tanh\left(\sqrt{\frac{fk\sin\theta}{8}}\frac{Wt}{h}\right)$$

When steady state is achieved after a long time,

$$V(t \to \infty) = V_S = \sqrt{\frac{8k\sin\theta}{f}}W$$

Now

$$V = V_S \tanh\left(\frac{V_S f}{8h}t\right)$$

Introducing V into x -momentum equation,

$$\frac{\partial\eta}{\partial x} = \frac{n\tau_w \cos\theta}{\rho g(h+\eta)} + \frac{f_C}{g}V_S \tanh\left(\frac{V_S f}{8h}t\right)$$

$$(h+\eta)\frac{\partial(h+\eta)}{\partial x} = (h+\eta)\left[\frac{\partial h}{\partial x} + \frac{f_C}{g}V_S \tanh\left(\frac{V_S f}{8h}t\right)\right] + \frac{n\tau_w \cos\theta}{\rho g}$$

Solving for $(h+\eta)$ with the assumption of constant slope, i.e. $\partial h / \partial x = -h_0 / l$,

$$\frac{x}{l^*} = 1 - \frac{h+\eta}{h_0} - A^* \ln\left(\frac{\frac{h+\eta}{h_0} - A^*}{1-A^*}\right)$$

where l^* and A^* are given by Eq. (5.108) of textbook.