Chapter 7. Wave Statistics and Spectra

Wave height distribution

Rank wave heights by zero-crossing method

1 highest
2 2nd highest
$\vdots$
N lowest

$$H_{rms} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} H_i^2}$$

H_p = average of the highest pN waves

Take pN high waves $\rightarrow$ compute average

$H_{1/10}$ = average of the highest $N/10$ waves

$H_{1/3}$ = average of the highest $N/3$ waves

$H_1 = \overline{H}$ = average wave height

Probability, P

$$P(H \geq \hat{H}) = \frac{n}{N}$$

where n = number of waves whose height is equal to or greater than $\hat{H}$. Accordingly,

$$P(H < \hat{H}) = 1 - \frac{n}{N}$$

Artificial data

1. Single wave train

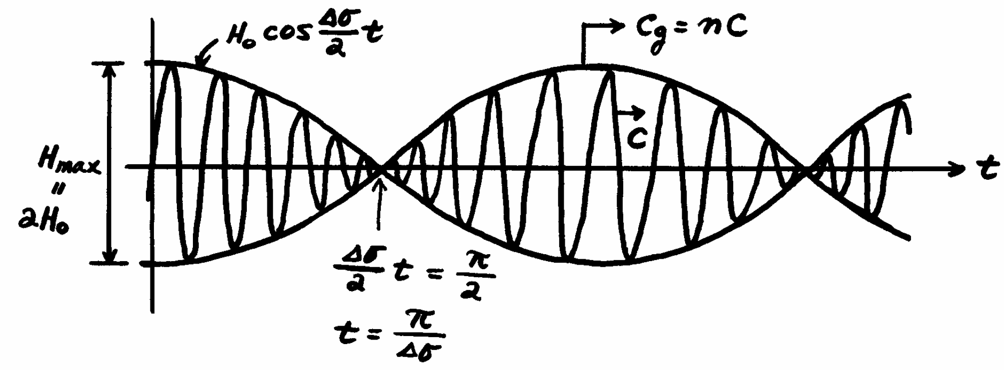
$$\eta = \frac{H_0}{2}\cos\sigma t$$

$$H_p = H_{rms} = H_0$$

2. Wave group

$$\eta = \frac{H_0}{2}\cos\left(\sigma - \frac{\Delta\sigma}{2}\right)t + \frac{H_0}{2}\cos\left(\sigma + \frac{\Delta\sigma}{2}\right)t$$
$$= H_0 \cos\sigma t \cos\frac{\Delta\sigma}{2}t$$
$$= \frac{H(t)}{2}\cos\sigma t$$

where $H(t) = 2H_0 \cos\frac{\Delta\sigma}{2}t$



$$H_p = \frac{1}{p\pi/\Delta\sigma}\int_0^{\frac{p\pi}{\Delta\sigma}} 2H_0\cos\left(\frac{\Delta\sigma}{2}t\right)dt = \frac{4}{p\pi}H_0\sin\frac{p\pi}{2}$$

$$H_{rms} = \sqrt{\frac{1}{\pi/\Delta\sigma}\int_0^{\frac{\pi}{\Delta\sigma}} 4H_0^2\cos^2\left(\frac{\Delta\sigma}{2}t\right)dt} = \sqrt{2}H_0$$

$$\therefore\ H_p = \frac{2\sqrt{2}}{p\pi} H_{rms} \sin\frac{p\pi}{2}$$

$$H_1 = \frac{2\sqrt{2}}{\pi} H_{rms} = 0.90 H_{rms}$$

$$H_{1/3} = 1.35 H_{rms}$$

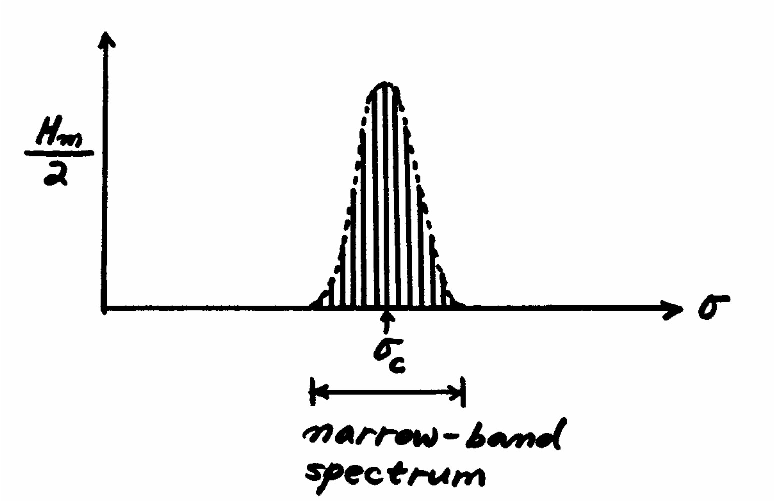
$$H_{\max} = H_{1/N} = \frac{2\sqrt{2}}{\pi / N} H_{rms} \sin\frac{\pi}{2N} = \frac{2\sqrt{2}N}{\pi} H_{rms} \frac{\pi}{2N} = \sqrt{2} H_{rms} \quad \text{or} \quad H_{\max} = 2H_0$$

<u>Narrow-banded spectrum</u>

$$\eta = \sum_{m=1}^{M} \frac{H_m}{2} \cos(\sigma_m t + \varepsilon_m) \quad \text{with central frequency } \sigma_c$$

$$\left| \frac{\sigma_m - \sigma_c}{\sigma_c} \right| << 1 \quad \text{for narrow-band spectrum}$$

The amplitude line spectrum looks like



$$\eta = \mathrm{Re}\left\{ \sum_{m=1}^{M} \frac{H_m}{2} e^{i(\sigma_m t + \varepsilon_m)} \right\} = \mathrm{Re}\left\{ e^{i\sigma_c t} \sum_{m=1}^{M} \frac{H_m}{2} e^{i[(\sigma_m - \sigma_c)t + \varepsilon_m]} \right\} = \mathrm{Re}\left\{ B(t) e^{i\sigma_c t} \right\}$$

where $B(t)$ = slowly varying envelope. Longuet-Higgins (1952) shows that $B(t)$ is governed by Rayleigh distribution:

$$P(H \geq \hat{H}) = e^{-(\hat{H}/H_{rms})^2}$$

From wave-by-wave method, we have

$$P(H \geq \hat{H}) = \frac{n}{N}$$

$$\therefore \ \frac{n}{N} = e^{-(\hat{H}/H_{rms})^2}$$

$$n = Ne^{-(\hat{H}/H_{rms})^2}$$

Given N and H_{rms}, then n waves are greater than or equal to $\hat{H}$.

$$\hat{H} = ?$$

$$\ln\frac{n}{N} = -(\hat{H}/H_{rms})^2$$

$$-\ln\frac{n}{N} = (\hat{H}/H_{rms})^2$$

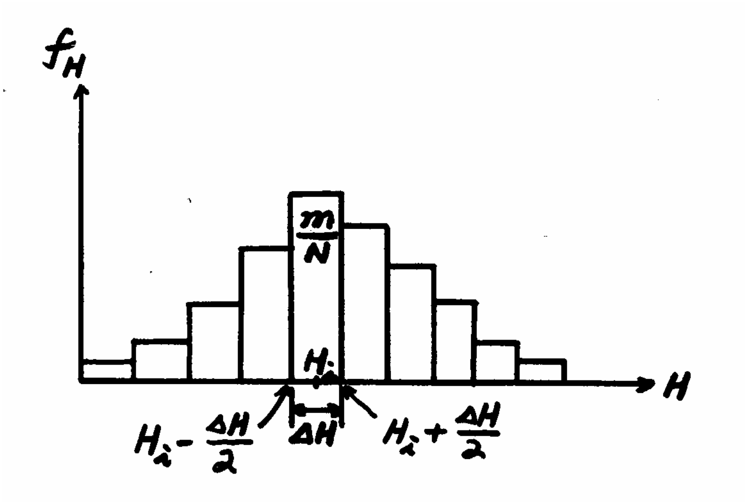
$$\ln\left(\frac{n}{N}\right)^{-1} = (\hat{H}/H_{rms})^2$$

$$\sqrt{\ln\frac{N}{n}} = \hat{H}/H_{rms}$$

$$\therefore \ \hat{H} = H_{rms}\sqrt{\ln\frac{N}{n}} = H_{rms}\sqrt{\ln\frac{1}{P}}$$

For example, $\hat{H} = 2.146H_{rms}$ for $n = 1$ and $N = 100$, or among 100 waves, one wave is greater than $2.146H_{rms}$.

Rayleigh probability density function



m out of N waves have height between $H_i - \Delta H / 2$ and $H_i + \Delta H / 2$

$$f_H \Delta H = \frac{m}{N} = P\left(H \le H_i + \frac{\Delta H}{2} \right) - P\left(H \le H_i - \frac{\Delta H}{2} \right) = dP(H \le H_i)$$

$$\therefore \ f_H = \frac{d}{dH} P(H \le H_i) = \frac{d}{dH}\left(1 - e^{-(H/H_{rms})^2}\right) = \frac{2H}{H_{rms}^2 e^{(H/H_{rms})^2}}$$

$$\overline{H} = \frac{\int_0^\infty H f_H dH}{\int_0^\infty f_H dH} = \int_0^\infty \frac{2H^2}{H_{rms}^2 e^{(H/H_{rms})^2}} dH = \int_0^\infty \frac{2H_{rms}^2 x^2}{H_{rms}^2 e^{x^2}} H_{rms} dx = \frac{\sqrt{\pi}}{2} H_{rms} = 0.886 H_{rms}$$

$$H_p = \frac{\int_{\hat{H}_p}^\infty H f_H dH}{\int_{\hat{H}_p}^\infty f_H dH} = \frac{\int_{\hat{H}_p}^\infty \frac{2H^2}{H_{rms}^2 e^{(H/H_{rms})^2}} dH}{\int_{\hat{H}_p}^\infty \frac{2H}{H_{rms}^2 e^{(H/H_{rms})^2}} dH} = \frac{\int_{\hat{H}_p/H_{rms}}^\infty H_{rms}^2 x^2 e^{-x^2} H_{rms} dx}{\int_{\hat{H}_p/H_{rms}}^\infty H_{rms} x e^{-x^2} H_{rms} dx}$$

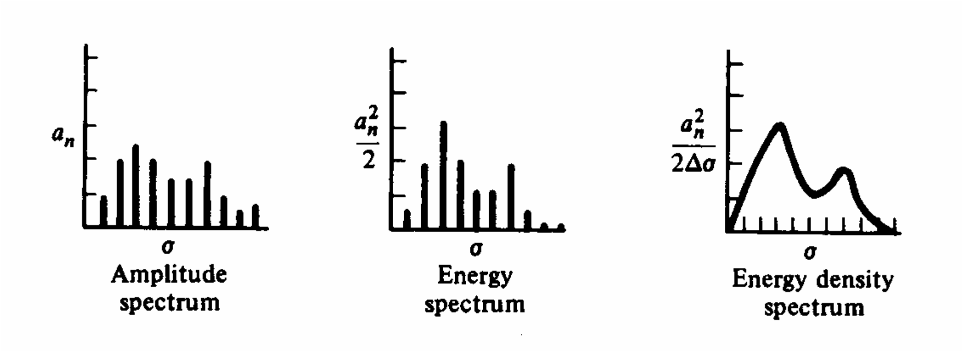
$$= H_{rms} \frac{\int_{\hat{H}_p/H_{rms}}^\infty x^2 e^{-x^2} dx}{\int_{\hat{H}_p/H_{rms}}^\infty x e^{-x^2} dx} = H_{rms} \left\{ \sqrt{\ln \frac{1}{p}} + \frac{\sqrt{\pi}}{2p} \mathrm{erfc}\left(\sqrt{\ln \frac{1}{p}} \right) \right\}$$

where erfc indicates complementary error function. Given H_{rms} and assuming Rayleigh distribution, H_p can be calculated by this relationship.

Wave spectrum

Single point measurement → frequency spectrum

$$\eta(t) = \sum_{n=0}^{\infty} a_n \cos(\sigma_n t - \varepsilon_n)$$



$$E = \frac{1}{8}\rho g H^2 = \frac{1}{2}\rho g a^2$$

Fourier transform

A periodic function $f(t)$ with a period T can be represented by a Fourier series:

$$f(t) = \sum_{n=0}^{\infty}(a_n \cos n\sigma t + b_n \sin n\sigma t)\,; \quad \sigma = \frac{2\pi}{T}$$

$$a_0 = \frac{1}{T}\int_t^{t+T} f(t)dt$$

$$b_0 = 0$$

$$a_n = \frac{2}{T}\int_t^{t+T} f(t)\cos n\sigma t dt, \quad n = 1, 2, \cdots, \infty$$

$$b_n = \frac{2}{T}\int_t^{t+T} f(t)\sin n\sigma t dt, \quad n = 1, 2, \cdots, \infty$$

Exponential form of Fourier series:

$$e^{in\sigma t} = \cos n\sigma t + i\sin n\sigma t\,; \quad e^{-in\sigma t} = \cos n\sigma t - i\sin n\sigma t$$

$$\cos n\sigma t = \frac{e^{in\sigma t} + e^{-in\sigma t}}{2}\,; \quad \sin n\sigma t = \frac{e^{in\sigma t} - e^{-in\sigma t}}{2i} = -\frac{i(e^{in\sigma t} - e^{-in\sigma t})}{2}$$

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty}\left\{a_n\left(\frac{e^{in\sigma t} + e^{-in\sigma t}}{2}\right) - ib_n\left(\frac{e^{in\sigma t} - e^{-in\sigma t}}{2}\right)\right\} \\ &= \sum_{n=0}^{\infty}\left\{\left(\frac{a_n - ib_n}{2}\right)e^{in\sigma t} + \left(\frac{a_n + ib_n}{2}\right)e^{-in\sigma t}\right\} \\ &= \sum_{n=-N}^{N} F(n)e^{in\sigma t} \end{aligned}$$

where

$$F(n) = \begin{cases} \dfrac{a_n - ib_n}{2} & \text{for} \quad n > 0 \\ a_0 & \text{for} \quad n = 0 \\ \dfrac{a_n + ib_n}{2} & \text{for} \quad n < 0 \end{cases}$$

$F(n)$ can also be obtained from the time series by

$$F(n) = \frac{1}{T}\int_t^{t+T} f(t)e^{-in\sigma t}dt$$

Thus, $f(t)$ and $F(n)$ are Fourier transform pair:

$$f(t) \text{ —— (Fourier transform)} \rightarrow F(n)$$
$$F(n) \text{ —— (inverse Fourier transform)} \rightarrow f(t)$$

Discrete Fourier transform:

$$F(n) = \frac{1}{N}\sum_{m=0}^{N-1} f(m\Delta t)e^{-in\sigma m\Delta t}, \quad n = 0, \pm 1, \pm 2, \cdots, \pm\frac{N}{2}$$

$$f(m\Delta t) = \sum_{n=-N/2}^{N/2} F(n) e^{in\sigma m\Delta t}, \quad m = 0, 1, 2, \cdots, (N-1)$$

$$\sigma = \frac{2\pi}{N\Delta t} = \frac{2\pi}{T}$$

<u>Covariance (or correlation) function</u>

$$C_{ij}(\tau) = \frac{1}{T}\int_t^{t+T} f_i(t) f_j(t+\tau) dt$$

$i = j$: auto-correlation function

$i \neq j$: cross-correlation function

$$\begin{aligned}
C_{ij}(\tau) &= \frac{1}{T}\int_t^{t+T} f_i(t) \sum_{n=-N/2}^{N/2} F_j(n) e^{in\sigma(t+\tau)} dt \\
&= \sum_{n=-N/2}^{N/2} F_j(n) \frac{1}{T}\int_t^{t+T} f_i(t) e^{in\sigma t} dt e^{in\sigma\tau} \\
&= \sum_{n=-N/2}^{N/2} F_j(n) F_i(-n) e^{in\sigma\tau} \\
&= \sum_{n=-N/2}^{N/2} F_j(n) F_i^*(n) e^{in\sigma\tau} \\
&= \sum_{n=-N/2}^{N/2} \left|F_j(n)\right| \left|F_i(n)\right| e^{i(\varepsilon_j - \varepsilon_i)} e^{in\sigma\tau}
\end{aligned}$$

$i = j$: $\quad C_{ii}(\tau) = \sum_{n=-N/2}^{N/2} \left|F_i(n)\right|^2 e^{in\sigma\tau}$

$$\left|F_i(n)\right|^2 = \frac{1}{T}\int_t^{t+T} C_{ii}(\tau) e^{-in\sigma\tau} d\tau \equiv \Phi_{ii}(n)$$

↑

power spectrum or energy spectrum

$$\Phi_{ij}(n) = \frac{1}{T}\int_t^{t+T} C_{ij}(\tau) e^{-in\sigma\tau} d\tau = F_i^*(n) F_j(n)$$

↑

cross-spectrum, which will be used in directional spectrum analysis.

Calculation of power or cross spectrum:

• Old method:

1. Calculate correlation functions $C_{ij}(\tau)$.

2. Calculate Fourier transform of $C_{ij}(\tau)$ to obtain power or cross spectrum.

• New method: Calculate Fourier coefficients $F_i(n)$ directly using FFT.

<u>Continuous spectrum</u>

$$|F(\sigma_n)|^2 \Delta\sigma = |F(n)|^2$$

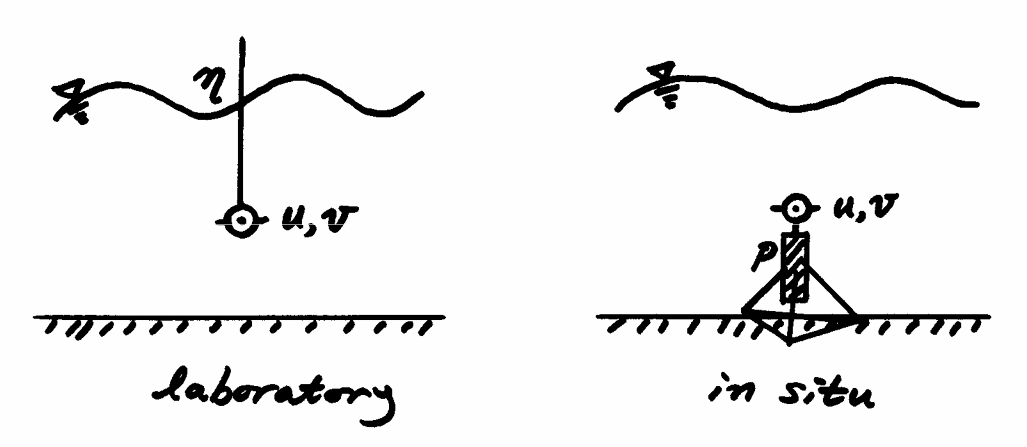
↑	↑
continuous spectrum	discrete spectrum
↑	↑
energy density spectrum	energy spectrum

$$\int_0^\infty |F(\sigma)|^2 d\sigma = \sum_{n=1}^{\infty} |F(n)|^2 \quad = \text{total energy}$$

Directional wave spectrum

$$\eta(t) = \sum_{n=-N/2}^{N/2} \int_0^{2\pi} F(n,\theta) e^{in\sigma t} d\theta$$

$$\eta - u - v \qquad \leftarrow \text{equivalent} \rightarrow \qquad p - u - v$$

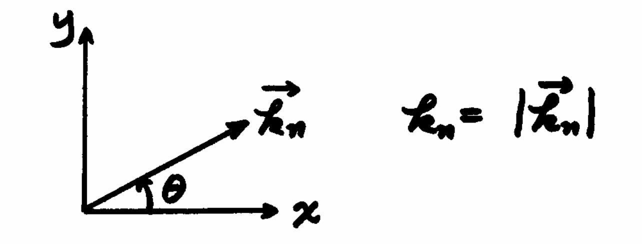


$$u(t) = \sum_{n=-N/2}^{N/2} \int_0^{2\pi} \frac{gk_n \cos\theta}{n\sigma} K_p(z) F(n,\theta) e^{in\sigma t} d\theta$$

$$v(t) = \sum_{n=-N/2}^{N/2} \int_0^{2\pi} \frac{gk_n \sin\theta}{n\sigma} K_p(z) F(n,\theta) e^{in\sigma t} d\theta$$

where

$$K_p(z) = \frac{\cosh k_n(h+z)}{\cosh k_n h}$$



Correlation function: $C_{\eta\eta}(\tau) = \sum_{n=-N/2}^{N/2} \int_0^{2\pi} \left|F(n,\theta)\right|^2 e^{in\sigma\tau} d\theta$

Energy spectrum: $\Phi_{\eta\eta}(n) = \frac{1}{T}\int_t^{t+T} C_{\eta\eta}(\tau) e^{-in\sigma\tau} d\tau = \int_0^{2\pi} \left|F(n,\theta)\right|^2 d\theta$

In summary,

$$\Phi_{\eta\eta}(n) = \int_0^{2\pi} \left|F(n,\theta)\right|^2 d\theta$$

$$\Phi_{uu}(n) = K_n^2 \int_0^{2\pi} \cos^2\theta \left|F(n,\theta)\right|^2 d\theta$$

$$\Phi_{vv}(n) = K_n^2 \int_0^{2\pi} \sin^2\theta \left|F(n,\theta)\right|^2 d\theta$$

$$\Phi_{u\eta}(n) = K_n \int_0^{2\pi} \cos\theta \left|F(n,\theta)\right|^2 d\theta$$

$$\Phi_{v\eta}(n) = K_n \int_0^{2\pi} \sin\theta \left|F(n,\theta)\right|^2 d\theta$$

$$\Phi_{uv}(n) = K_n^2 \int_0^{2\pi} \cos\theta \sin\theta \left|F(n,\theta)\right|^2 d\theta$$

for $-N/2 \le n \le N/2$, and where

$$K_n = \frac{g k_n K_p(z)}{n\sigma}$$

Express the directional spectrum as a Fourier series (Longuet-Higgins et al., 1963):

$$\begin{aligned}\left|F(n,\theta)\right|^2 &= \sum_{m=0}^{\infty} \left(A_m(n)\cos m\theta + B_m(n)\sin m\theta\right) \\ &= A_0 + A_1\cos\theta + B_1\sin\theta + A_2\cos 2\theta + B_2 \sin 2\theta + \cdots\end{aligned}$$

Now,

$$
\begin{aligned}
\Phi_{\eta\eta}(n) &= \int_0^{2\pi} \left|F(n,\theta)\right|^2 d\theta \\
&= \int_0^{2\pi} \left(A_0 + A_1 \cos\theta + B_1 \sin\theta + A_2 \cos 2\theta + B_2 \sin 2\theta + \cdots\right) d\theta \\
&= 2\pi A_0(n)
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{uu}(n) &= \int_0^{2\pi} K_n^2 \cos^2\theta \left|F(n,\theta)\right|^2 d\theta \\
&= K_n^2 \int_0^{2\pi} \cos^2\theta \left(A_0 + A_1 \cos\theta + B_1 \sin\theta + A_2 \cos 2\theta + B_2 \sin 2\theta + \cdots\right) d\theta \\
&= K_n^2 \int_0^{2\pi} \left(A_0 \cos^2\theta + A_1 \cos^3\theta + B_1 \cos^2\theta \sin\theta + A_2 \cos^2\theta \cos 2\theta + \cdots\right) d\theta \\
&= K_n^2 \left(A_0(n)\pi + \frac{A_2(n)}{2}\pi \right)
\end{aligned}
$$

Similarly,

$$\Phi_{u\eta}(n) = K_n A_1(n)\pi$$

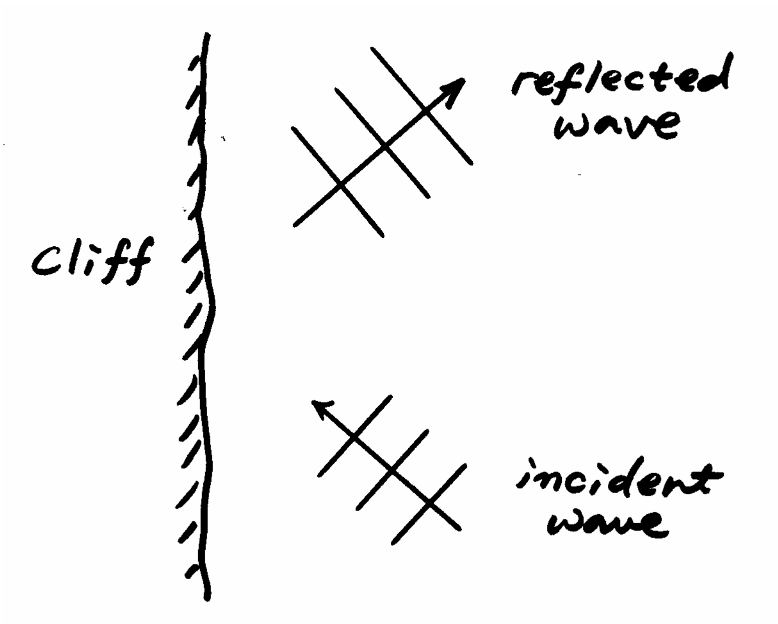
$$\Phi_{v\eta}(n) = K_n B_1(n)\pi$$

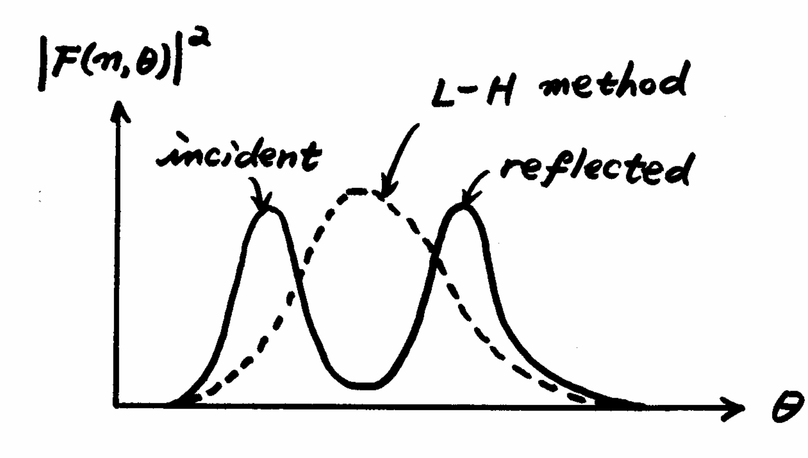
$$\Phi_{uv}(n) = K_n^2 \frac{B_2(n)}{2}\pi$$

Now, we have 5 equations for 5 unknowns, A_0, A_1, A_2, B_1, B_2.

More gages $\rightarrow$ more coefficients $\rightarrow$ more accurate.

Longuet-Higgins method cannot resolve combined waves from different directions.





For more information on other methods for directional spectrum analysis, refer to Oh et al. (1992), “A study on the estimation of directional wave spectrum using arrays”, Ocean Research (KORDI), 14(2): 111-129.