Ch.1 Scalars, vectors and tensors

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노트 제목 2010-03-02

scalar: no sense of direction

vector: not only a magnitude but also a direction

tensor: ?

• Cartesian vectors and tensors: their algebra

In ordinary 3-D space, the system defined by three mutually orthogonal directions with equal units of measurement is called Cartesian.

✓ addition of vectors

✓ scalar multiplication

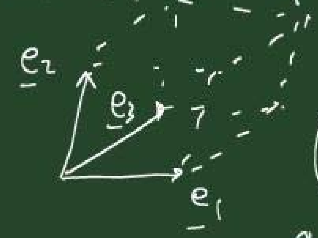
✓ unit vector: a vector of unit length

⊙ Bases, linear forms

vector components $\underline{a} = (a_1, a_2, a_3)$

base vectors: 3 noncoplanar vector components
(3 dimensional vector space)

basis: set of base vectors



$$\underline{a} = \alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 + \alpha_3 \underline{e}_3$$

$$\left(\alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 + \alpha_3 \underline{e}_3 = 0 \longleftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \right)$$

$$\underline{a} = \alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 + \alpha_3 \underline{e}_3 = \sum_{i=1}^{3} \alpha_i \underline{e}_i$$

$$\equiv \alpha_i \underline{e}_i \quad \text{(summation convention)}$$

$$a^2 = |\underline{a}|^2 = a_1^2 + a_2^2 + a_3^2 = a_i a_i$$

equation: $a_i + b_{ij} c_j + d_{kl} g_{kli} = 0 \quad (i, j, k, l = 1, 2, \cdots, N)$

$$\underbrace{b_{ij} c_j}_{= \sum_{j=1}^{N} b_{ij} c_j} \qquad \underbrace{d_{kl} g_{kli}}_{= \sum_{k=1}^{N} \sum_{l=1}^{N} d_{kl} g_{kli}}$$

stands for N separate equations

$$\begin{cases} a_1 + \sum_{j=1}^{N} b_{1j} c_j + \sum_{k=1}^{N} \sum_{l=1}^{N} d_{kl} g_{kl1} = 0 \\ a_2 + \cdots \\ \vdots \\ a_N + \quad - \quad - \quad = 0 \end{cases}$$

Live script (free suffix) must occur <u>only once</u> in each term of equation : i

Dummy script (repeated suffix) occurs <u>twice</u> in one term and is to be summed over the range : j, k, l

$N = 3$

b_i : b_1, b_2, b_3 3 components, components of a vector

A_{ij} : $\begin{matrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{matrix}$ 9 components, components of second-order tensor

$b_i A_{ij} = b_1 A_{1j} + b_2 A_{2j} + b_3 A_{3j}$: contracted product

✓ Contracted product of A_{ij} and B_{ij}

① single contracted product

$$C_{ik} = A_{ij} B_{jk} = A_{i1} B_{1k} + A_{i2} B_{2k} + A_{i3} B_{3k}$$ 2 live scripts, 9 components

② doubly contracted product

$$A_{ij} B_{ji} = A_{11} B_{11} + A_{12} B_{21} + \cdots$$ no live script, 1 component

- Vector $\underline{b} = (b_1, b_2, b_3)$

  b_i : components of a vector

  $b_i c_i$: scalar

  $b_i c_j$: $\begin{matrix} b_1c_1 & b_1c_2 & b_1c_3 \\ b_2c_1 & - & - \\ - & - & - \end{matrix}$ 9 components

  tensor product

  open product of two vectors

  called dyad $\underline{b} \otimes \underline{c}$

- Kronecker delta and alternating ϵ

$$\delta_{ij} \equiv \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \quad (\delta_{\alpha\alpha} = 1)$$

$$B_j = A_i \delta_{ij} = A_1 \delta_{1j} + A_2 \delta_{2j} + A_3 \delta_{3j} = A_j$$

$$\epsilon_{ijk} \equiv \begin{cases} 0 & \text{if any two of } i, j, k \text{ are the same} \\ 1 & \text{if } ijk \text{ is an even permutation of } 1, 2, 3 \\ -1 & \text{if } \quad " \quad \text{odd} \quad " \quad " \end{cases}$$

$\epsilon_{112} = 0 \quad \epsilon_{111} = 0 \quad \epsilon_{322} = 0 \quad \cdots$ (21 components)

$\epsilon_{123} = 1, \quad \epsilon_{231} = 1, \quad \epsilon_{312} = 1$

$\epsilon_{321} = -1, \quad \epsilon_{213} = -1, \quad \epsilon_{132} = -1$

⊙ Symmetry and anti-symmetry

$\alpha_{ij} = \alpha_{ji}$ symmetric

$A_{lmn} = A_{lnm}$ " in the indices n & m.

$$\begin{cases} \alpha_{ij} = -\alpha_{ji} & \text{anti-symmetric or skew} \\ A_{lmn} = -A_{lnm} & \text{" in the indices } n \,\&\, m. \end{cases}$$

- The doubly contracted product of symmetric and antisymmetric tensors vanishes.

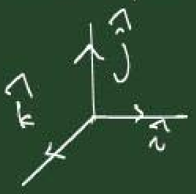
pf. $S_{ij} = S_{ji}$, $A_{ij} = -A_{ji}$

$$\phi = A_{ij} S_{ji} = A_{ij} S_{ij} = (-A_{ji}) S_{ij} = -A_{ji} S_{ij}$$
$$= -A_{ij} S_{ji} = -\phi \qquad \therefore \phi = 0$$

(same if $\phi = A_{ij} S_{ij}$)

- Scalar product : $\underline{a} \cdot \underline{b}$

(inner, dot) (positive definite : $\underline{a} \cdot \underline{a} \geq 0$)



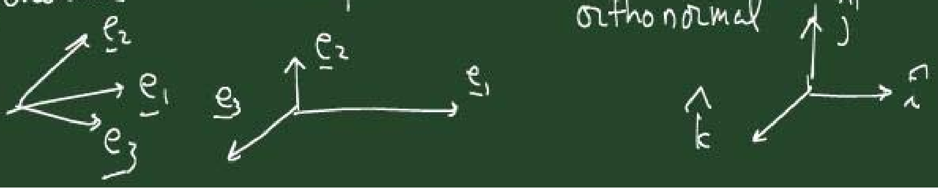
$$\underline{a} \cdot \underline{b} = (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \cdot (b_x \hat{i} + b_y \hat{j} + b_z \hat{k})$$
$$= a_x b_x + a_y b_y + a_z b_z = a_i b_i \quad (i = 1, 2, 3)$$

- Orthogonality

A basis is orthogonal if $\underline{e}_i \cdot \underline{e}_j = 0 \quad i \neq j$

" orthonormal " $\quad " = \left.\begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}\right\} \equiv \delta_{ij}$

general orthogonal orthonormal



Any vector $\underline{a}$ in the n-dimensional vector space can be expressed as a unique linear combination of n base vectors

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + \cdots + a_n \underline{e}_n = a_i \underline{e}_i$$

⊙ Gram-Schmidt Orthonormalization

Given any basis ($\underline{e}_i$) ($i=1,2,\cdots,n$), we can generate an orthonormal basis ($\hat{i}_i$) of same dimension.