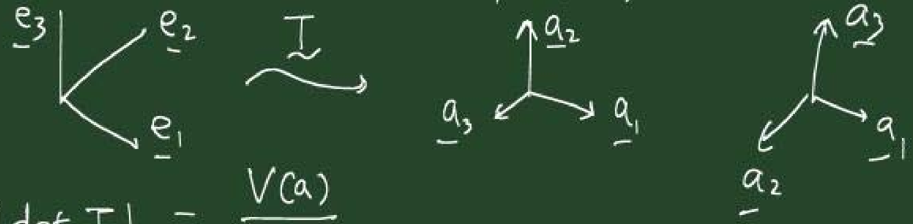
노트 제목 2010-03-09

- Determinant of $\underset{\sim}{T} \equiv \det \underset{\sim}{T}$

$(\underline{e}_1, \underline{e}_2, \underline{e}_3) \longrightarrow (\underline{a}_1, \underline{a}_2, \underline{a}_3)$



$$|\det \underset{\sim}{T}| \equiv \frac{V(a)}{V(e)}$$

sign of $\det \underset{\sim}{T} = \begin{cases} + & \text{if } \underline{e}_i \text{ and } \underline{a}_i \text{ are of same handedness} \\ - & \text{〃 different} \end{cases}$

$$V(e) = |\underline{e}_1 \times \underline{e}_2 \cdot \underline{e}_3|$$

$$V(a) = |\underline{a}_1 \times \underline{a}_2 \cdot \underline{a}_3|$$

$$\therefore \det \underset{\sim}{T} = \frac{\underline{a}_1 \times \underline{a}_2 \cdot \underline{a}_3}{\underline{e}_1 \times \underline{e}_2 \cdot \underline{e}_3}$$

what is $\det \underset{\sim}{T}$ in terms of T_{ij}?

choose for $\underline{e}_i$ the particular basis $\hat{i}_i$

Then $\underline{a}_i = \underset{\sim}{T}\,\hat{i}_i = (T_{mn}\,\hat{i}_m \otimes \hat{i}_n)\,\hat{i}_i = T_{mi}\,\hat{i}_m$

$$\underline{a}_1 \times \underline{a}_2 \cdot \underline{a}_3 = (T_{m1}\,\hat{i}_m \times T_{n2}\,\hat{i}_n) \cdot T_{p3}\,\hat{i}_p$$

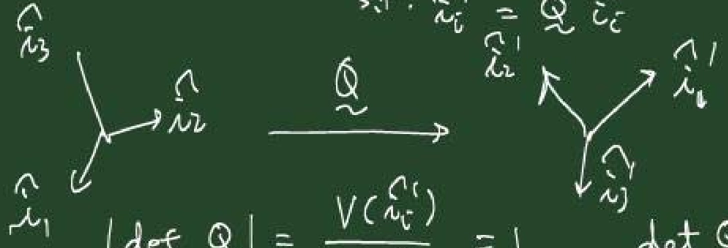
$$= \varepsilon_{mnp}\, T_{m1} T_{n2} T_{p3}$$

$$\hat{i}_1 \times \hat{i}_2 \cdot \hat{i}_3 = \epsilon_{123} = 1$$

$$\therefore \det \underset{\sim}{T} = \varepsilon_{mnp}\, T_{m1} T_{n2} T_{p3} = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix}$$

• Change of basis (transformation of tensor components)

Given $\hat{i}_i$ and $\hat{i}'_i$, $\exists$ an orthogonal tensor $\underset{\sim}{Q}$ s.t. $\hat{i}'_i = \underset{\sim}{Q}\,\hat{i}_i$



$$\left|\det \underset{\sim}{Q}\right| = \frac{V(\hat{i}'_i)}{V(\hat{i}_i)} = 1. \qquad \det \underset{\sim}{Q} = \pm 1$$

proper orthogonal transformation if $\det \underset{\sim}{Q} = +1$: rotation

improper " " " -1 : reflection

⊙ Vectors and tensors are invariant under change of basis.

· vectors $\underline{u} = u_i \hat{i}_i = u'_i \hat{i}'_i \qquad \hat{i}'_i = \underset{\sim}{Q}\,\hat{i}_i$

$$\underbrace{(u'_i \hat{i}'_i)\cdot \hat{i}'_j}_{u'_j} = u_i \hat{i}_i \cdot \hat{i}'_j = u_i \underline{\hat{i}_i \cdot \underset{\sim}{Q}\,\hat{i}_j} = u_i \underline{Q_{ij}}$$

$$\Rightarrow \quad u'_j = u_i Q_{ij}$$

Inversely, $u'_j Q_{mj} = u_i Q_{ij} Q_{mj} = u_i \delta_{im} = u_m$

$$\Rightarrow \quad u_m = u'_j Q_{mj}$$

· tensors $\underset{\sim}{T} = T_{ij}\, \hat{i}_i \otimes \hat{i}_j = T'_{ij}\, \hat{i}'_i \otimes \hat{i}'_j$

$$(T'_{ij}\, \hat{i}'_i \otimes \hat{i}'_j)\, \hat{i}'_m = (T_{ij}\, \hat{i}_i \otimes \hat{i}_j)\, \hat{i}'_m$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$(T'_{im}\, \hat{i}'_i)\cdot \hat{i}'_k = (T_{ij}\, \hat{i}_i\, Q_{jm})\cdot \hat{i}'_k$$

$$\rightarrow T'_{km} = T_{ij} Q_{jm} Q_{ik} \qquad \delta_{js}$$

$$T'_{km} Q_{pk} Q_{sm} = T_{ij} \overbrace{Q_{jm} \underbrace{Q_{ik} Q_{pk}}_{\delta_{ip}} Q_{sm}} = T_{ps}$$

$$\therefore T_{km} = T'_{ij} Q_{ki} Q_{mj}$$

In general,

$$T'_{ij\cdots k} = T_{pq\cdots s} Q_{pi} Q_{qj} \cdots Q_{sk}$$

$$\text{or } T_{pq\cdots s} = T'_{ij\cdots k} Q_{pi} Q_{qj} \cdots Q_{sk}$$

- "tr" and "det" are invariant operations.

$$\operatorname{tr} \underset{\sim}{T} = T_{ii} = T'_{ii}$$

$$\det \underset{\sim}{T} = |T_{ij}| = |T'_{ij}|$$

- Isotropic tensor: components remain unchanged under all proper change of basis.

$$\hat{i}'_i = \underset{\sim}{Q}\, \hat{i}_i \ , \ \det \underset{\sim}{Q} = +1$$

$$T'_{ij\cdots k} = T_{ij\cdots k} \qquad \frac{\partial u_i}{\partial x_j}$$

examples:

$$T_{ij} = \alpha \delta_{ij} \rightarrow T'_{ij} = T_{ij}$$

$$T_{ijk} = \alpha \epsilon_{ijk} \rightarrow T'_{ijk} = T_{ijk}$$

$$T_{ijkl} = a \delta_{ij} \delta_{kl} + b \delta_{ik} \delta_{jl} + c \delta_{il} \delta_{jk}$$

$$T_{ijklm} = a_1 \epsilon_{ijk} \delta_{lm} + a_2 \epsilon_{ikl} \delta_{jm} + \cdots$$

$$- \quad - \quad - \quad -$$

$$+ a_{10} \epsilon_{ikm} \delta_{jl} .$$

- $\det(\underset{\sim}{T} - t\underset{\sim}{1}) = -t^3 + \mathrm{I}\,t^2 - \mathrm{II}\,t + \mathrm{III}$

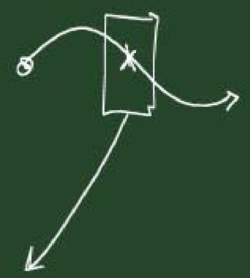
where

$$\left.\begin{aligned} \mathrm{I} &= \operatorname{trace} \underset{\sim}{T} \\ \mathrm{II} &= \tfrac{1}{2}\left[(\operatorname{tr} \underset{\sim}{T})^2 - \operatorname{tr} \underset{\sim}{T}^2 \right] \\ \mathrm{III} &= \det \underset{\sim}{T} \end{aligned}\right\} \text{principal invariants.}$$

Part 1. Governing Equations

1. Basic conservation laws.

- continuum
- Eulerian & Lagrangian coordinates
- Material derivative



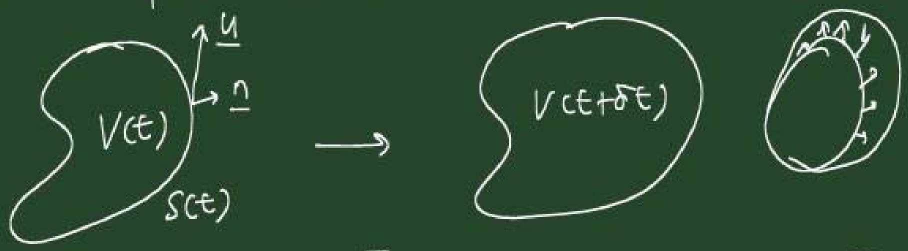
$$\frac{D\alpha}{Dt} = \frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{u} + \frac{\partial \alpha}{\partial y}\underbrace{\frac{\partial y}{\partial t}}_{v} + \frac{\partial \alpha}{\partial z}\underbrace{\frac{\partial z}{\partial t}}_{w}$$

↑ material derivative

$$= \frac{\partial \alpha}{\partial t} + (\underline{u} \cdot \nabla)\alpha = \frac{\partial \alpha}{\partial t} + \underbrace{u_j \frac{\partial \alpha}{\partial x_j}}_{\text{convection term}}$$

- Control volumes

- Reynolds transport theorem



$$\frac{D}{Dt}\int_{V(t)} \alpha(t)\, dV = \lim_{\delta t\to 0} \frac{1}{\delta t}\left[\int_{V(t+\delta t)} \alpha(t+\delta t)\, dV - \int_{V(t)} \alpha(t)\, dV\right]$$

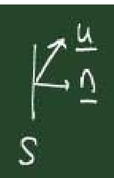
$$= \lim_{\delta t\to 0} \frac{1}{\delta t}\left[\int_{V(t+\delta t)} \alpha(t+\delta t)\, dV - \int_{V(t)} \alpha(t+\delta t)\, dV\right]$$

$$+ \frac{1}{\delta t}\left[\int_{V(t)} \alpha(t+\delta t)\, dV - \int_{V(t)} \alpha(t)\, dV\right]$$

$$= \lim_{\delta t\to 0} \frac{1}{\delta t}\int_{V(t+\delta t)-V(t)} \alpha(t+\delta t)\, dV + \int_{V(t)} \frac{\partial \alpha}{\partial t}\, dV$$

$$V(t+\delta t) = V(t) + \underline{\underline{\delta V(t)}}$$

$$= \underline{u}\cdot\underline{n}\,\delta t\,\delta S$$



$$\therefore \frac{D}{Dt}\int_{V(t)} \alpha(t)\, dV = \lim_{\delta t\to 0} \int_{S(t)} \alpha(t+\delta t)\, \underline{u}\cdot\underline{n}\, dS + \int_{V(t)} \frac{\partial \alpha}{\partial t}\, dV$$

$$\boxed{= \int_{S(t)} \alpha(t)\, \underline{u}\cdot\underline{n}\, dS + \int_{V(t)} \frac{\partial \alpha}{\partial t}\, dV}$$

Reynolds transport theorem.

Gauss theorem : $\int_{S(t)} \alpha(t)\, \underline{u}\cdot\underline{n}\, dS = \int_{V(t)} \nabla\cdot(\alpha \underline{u})\, dV$

then, $\frac{D}{Dt}\int_{V(t)} \alpha\, dV = \int_{V(t)} \left(\frac{\partial \alpha}{\partial t} + \nabla\cdot(\alpha\underline{u})\right) dV$

$$= \int_{V(t)} \left(\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial x_j}(\alpha u_j)\right) dV$$

- Conservation of mass ($\alpha = \rho$) ρ: density

$$\frac{D}{Dt}\int_V \rho \, dV = 0 = \int_{V(t)} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\, dV$$

$$\rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \quad : \text{continuity eq.}$$

incompressible flow ($\rho = \text{const}$) $\rightarrow \dfrac{\partial u_j}{\partial x_j} = 0$.

- Conservation of momentum ($\alpha = \underline{u}$)

$\underline{f}$: body force per unit mass

$\rightarrow$ net external force acting on a mass of volume V

$$= \int_V \rho \underline{f} \, dV$$

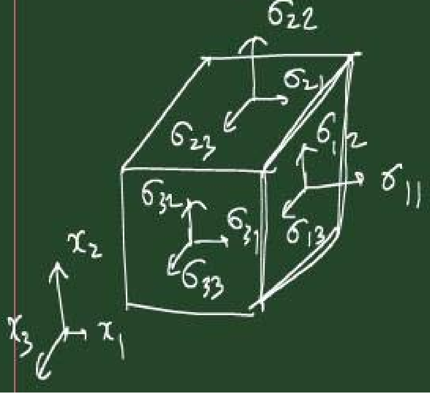
$\underline{P}$: surface force per unit area

$\rightarrow$ net external force acting on the surface S

$$= \int_S \underline{P} \, dS$$

Newton's 2nd law : $\sum \underline{F} = \frac{D}{Dt}(m\underline{U})$

$$\therefore \frac{D}{Dt}\int_V \rho \underline{u} \, dV = \int_S \underline{P} \, dS + \int_V \rho \underline{f} \, dV$$



stress σ_{ij}

j: stress acting in the x_j direction

i: stress component acting on the plane x_i = const then $P_j = \sigma_{ij} n_i$ Gauss theorem

$$\frac{D}{Dt}\int_V \rho u_j \, dV = \int_S \sigma_{ij} n_i \, dS + \int_V \rho f_j \, dV$$

$$\int_V \left[\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_k}(\rho u_j u_k)\right] dV = \int_V \frac{\partial \sigma_{ij}}{\partial x_i} \, dV + \int_V \rho f_j \, dV$$

$$\rightarrow \frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_k}(\rho u_j u_k) = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j \quad : \text{mtm eq.}$$

$$\rightarrow \rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j \quad (\text{using cont. eq.})$$

σ_{ij} ? $\sigma_{ij} \sim \frac{\partial u_i}{\partial x_j}$

↳ constitutive eq.