노트 제목 2010-03-25

2\. Bernoulli equation

① Inviscid fluid
② conservative body force
③ steady or irrotational

} momentum eq. $\downarrow$ single scalar eq. $\Downarrow$ Bernoulli eq.

For inviscid fluid with conservative body force,

$$\rho \frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}\cdot\nabla)\underline{u} = -\nabla p + \rho \nabla G$$

$$(\underline{u}\cdot\nabla)\underline{u} = \nabla\left(\frac{1}{2}\underline{u}\cdot\underline{u}\right) - \underline{u}\times\underline{\omega}$$

$$\rightarrow \frac{\partial \underline{u}}{\partial t} + \nabla\left(\frac{1}{2}\underline{u}\cdot\underline{u}\right) - \underline{u}\times\underline{\omega} = -\frac{1}{\rho}\nabla p + \nabla G$$

$$\left(d\underline{l}\cdot\left(\frac{1}{\rho}\nabla p\right) = \frac{1}{\rho}d\underline{l}\cdot\nabla p = \frac{1}{\rho}dp = d\int\frac{dp}{\rho} = d\underline{l}\cdot\nabla\int\frac{dp}{\rho} \right)$$

$$\rightarrow \frac{\partial \underline{u}}{\partial t} + \nabla\left(\int\frac{dp}{\rho} + \frac{1}{2}\underline{u}\cdot\underline{u} - G\right) = \underline{u}\times\underline{\omega}$$

i) steady flow $\quad \nabla\left(\int\frac{dp}{\rho} + \frac{1}{2}\underline{u}\cdot\underline{u} - G\right) = \underline{u}\times\underline{\omega}$

$$\underline{u}\cdot\nabla\left(\quad '' \quad\right) = \underline{u}\cdot(\underline{u}\times\underline{\omega}) = 0$$

material derivative

Lagrangian approach → follow the particle → stream-line

$\therefore \int \frac{dp}{\rho} + \frac{1}{2}\underline{u}\cdot\underline{u} - G = \text{const. along each streamline}$

Bernoulli eq.

ii) irrotational flow $\underline{\omega} = \nabla \times \underline{u} = 0 \rightarrow \underline{u} = \nabla\phi$

velocity potential

$$\frac{\partial \underline{u}}{\partial t} + \nabla\left(\int \frac{dp}{\rho} + \frac{1}{2}\underline{u}\cdot\underline{u} - G\right) = 0, \quad \frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}\nabla\phi$$

$$\rightarrow \nabla\left(\frac{\partial\phi}{\partial t} + \int \frac{dp}{\rho} + \frac{1}{2}\nabla\phi\cdot\nabla\phi - G\right) = 0$$

$$d\underline{l}\cdot\left(\quad // \quad \right) = 0$$

$$\rightarrow d\left(\quad // \quad \right) = 0$$

$$\rightarrow \frac{\partial\phi}{\partial t} + \int \frac{dp}{\rho} + \frac{1}{2}\nabla\phi\cdot\nabla\phi - G = F(t) \quad \text{unsteady Bernoulli eq.}$$

if steady, $\int \frac{dp}{\rho} + \frac{1}{2}\nabla\phi\cdot\nabla\phi - G = \text{const}$ all over the place.

3. Crocco's equation

* Isentropic flows are irrotational and irrotational flows are isentropic.

for steady flow of inviscid fluid
no body force
constant stagnation enthalpy

For inviscid, no body force

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\nabla p$$

$$\frac{\partial \underline{u}}{\partial t} + \nabla\left(\frac{1}{2}\underline{u}\cdot\underline{u}\right) - \underline{u}\times\underline{\omega} = -\frac{1}{\rho}\nabla p$$

$$\left(\begin{array}{l} \text{Gibbs eq. } Tds = dh - \frac{1}{\rho}dp \\ \qquad\qquad \rightarrow T\nabla s = \nabla h - \frac{1}{\rho}\nabla p \end{array}\right)$$

$$\rightarrow \frac{\partial \underline{u}}{\partial t} + \nabla\left(\frac{1}{2}\underline{u}\cdot\underline{u}\right) - \underline{u}\times\underline{\omega} = T\nabla s - \nabla h$$

$$\rightarrow \boxed{\underline{u}\times\underline{\omega} + T\nabla s = \nabla\left(h + \frac{1}{2}\underline{u}\cdot\underline{u}\right) + \frac{\partial \underline{u}}{\partial t}}$$

Crocco's eq.

$h_0 \equiv h + \frac{1}{2}\underline{u}\cdot\underline{u}$

stagnation enthalpy

Thermal energy eq. $\rho\frac{De}{Dt} = -p\nabla\cdot\underline{u} + \nabla\cdot(k\nabla T) + \underline{\Phi}$

$$\rightarrow \rho\frac{Dh}{Dt} = \frac{Dp}{Dt} + \underbrace{\nabla\cdot(k\nabla T) + \underline{\Phi}}_{=0 \text{ for adiabatic}} \quad \text{(Prob. 3.1)}$$

$$\underline{u}\cdot\left(\rho\frac{D\underline{u}}{Dt} = -\nabla p\right) \rightarrow \rho\frac{D}{Dt}\left(\frac{1}{2}\underline{u}\cdot\underline{u}\right) = -\underline{u}\cdot\nabla p$$

$$\rightarrow \rho\frac{D}{Dt}\left(h + \frac{1}{2}\underline{u}\cdot\underline{u}\right) = \frac{Dp}{Dt} - \underline{u}\cdot\nabla p$$

$$\rightarrow \rho\frac{Dh_0}{Dt} = \frac{\partial p}{\partial t}$$

for steady flow, $\frac{\partial p}{\partial t} = 0$

$\rightarrow \frac{Dh_0}{Dt} = 0$: h_0 is const. along each streamline

For Steady flow (Crocco eq.)

$$\underbrace{\underline{u}\times\underline{\omega}} + T\nabla s = \nabla h_0$$

$(\underline{u}\times\underline{\omega}) \perp$ streamline

h_0 is const. along streamline

$\nabla h_0 \perp$ streamline

$\therefore$ should be perpendicular to the streamline

$$\Rightarrow U\Omega + T\frac{ds}{dn} = \frac{dh_0}{dn}$$

n : perpendicular to the streamline

Ω : mag. of $|\underline{\omega}|$

If h_0 is constant everywhere,

$$U\Omega + T\frac{ds}{dn} = 0$$

if $s = \text{const}$ (isentropic), $\Omega = 0$ (irrotational)

if $\Omega = 0$, $s = \text{const}$ (isentropic)

4. Vorticity eq.

For a fluid of constant density and viscosity,

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\underline{u}$$

$$\rightarrow \frac{\partial \underline{u}}{\partial t} + \nabla\left(\frac{1}{2}\underline{u}\cdot\underline{u}\right) - \underline{u}\times(\nabla\times\underline{u}) = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\underline{u}$$

$\nabla\times($ 〃 $)$

$$\rightarrow \frac{\partial \underline{\omega}}{\partial t} - \underbrace{\nabla \times (\underline{u} \times \underline{\omega})} = \nu \nabla^2 \underline{\omega}$$

$$= \underline{u}(\cancel{\nabla \cdot \underline{\omega}}) - \underline{\omega}(\cancel{\nabla \cdot \underline{u}}) - (\underline{u} \cdot \nabla)\underline{\omega} + (\underline{\omega} \cdot \nabla)\underline{u}$$

$$\nabla \cdot \underline{\omega} = 0, \quad \nabla \cdot \underline{u} = 0$$

$$\rightarrow \boxed{\frac{\partial \underline{\omega}}{\partial t} + (\underline{u} \cdot \nabla)\underline{\omega} = (\underline{\omega} \cdot \nabla)\underline{u} + \nu \nabla^2 \underline{\omega}}$$ vorticity equation

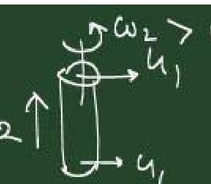
convection | stretching tilting | diffusion

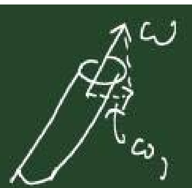
$$\frac{D\omega_1}{Dt} = \underline{\omega_1 \frac{\partial u_1}{\partial x_1}} + \underline{\omega_2 \frac{\partial u_1}{\partial x_2}} + \omega_3 \frac{\partial u_1}{\partial x_3} + \nu \nabla^2 \omega_1$$

$\omega_1 > 0$ $\quad \frac{\partial u_1}{\partial x_1} > 0$ $\quad \omega_1 \frac{\partial u_1}{\partial x_1} > 0$ stretching term

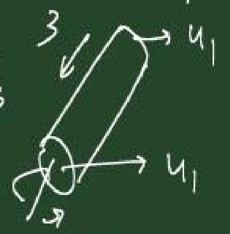
$V = A\ell = \text{const}$ stretching

$\omega_2 \frac{\partial u_1}{\partial x_2}$ $\quad \omega_2 > 0$ $\quad \frac{\partial u_1}{\partial x_2} > 0$ tilting term

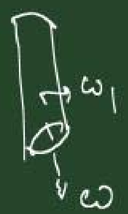




$\omega_3 \frac{\partial u_1}{\partial x_3}$ $\quad \frac{\partial u_1}{\partial x_3} > 0$ $\quad$ //



$\omega_3 > 0$



For 2D, $(\underline{\omega} \cdot \nabla)\underline{u} = 0$ $\quad$ For 2D, $\omega_3 \neq 0$

$$\rightarrow \frac{\partial \underline{\omega}}{\partial t} + (\underline{u} \cdot \nabla)\underline{\omega} = \nu \nabla^2 \underline{\omega}$$

$$\underline{\omega_3} \underbrace{\frac{\partial u_1}{\partial x_3}}_{=0} = \omega_3 \underbrace{\frac{\partial u_2}{\partial x_3}}_{=0} = 0$$

no pressure in the vorticity eq.
→ knowledge on press. is not needed.

p ? $\quad \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\underline{u}$

$\nabla\cdot\Big($ $\qquad$ " $\qquad$ $\Big)$

$\rightarrow \quad \nabla^2\left(\frac{p}{\rho}\right) = -\nabla\cdot(\underline{u}\cdot\nabla)\underline{u}$

$\rightarrow \quad \frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{p}{\rho}\right) = -u_{i,j}u_{j,i}$ Poisson eq.