Ch.2 Equations of fluid motion

노트 제목 2010-03-04

- continuum Kn = mean free path (λ) / flow length scale (l) $\ll 1$
  Knudsen number

- Eulerian & Lagrangian fields

$$\underbrace{\frac{D}{Dt}}_{\text{material derivative}} \equiv \frac{\partial}{\partial t} + \underline{U} \cdot \nabla = \underbrace{\frac{\partial}{\partial t}}_{\text{time derivative}} + \underbrace{U_j \frac{\partial}{\partial x_j}}_{\text{convective derivative}}$$

- Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{U}) = 0 \qquad \underline{\nabla \cdot \underline{U} : \text{dilatation}}$$

For constant ρ, (incomp. flow) $\nabla \cdot \underline{U} = 0$

$\underline{U}$ is <u>solenoidal</u> or <u>divergence free</u>.

- momentum equation

τ_{ij} : stress tensor, $\tau_{ij} = \tau_{ji}$

$$\rho \frac{DU_j}{Dt} = \frac{\partial \tau_{ij}}{\partial x_i} - \rho g_j .$$

$$\left(= \frac{\partial U_j}{\partial t} + U_i \frac{\partial U_j}{\partial x_i} \right)$$

For constant-property Newtonian fluids,

$$\tau_{ij} = -\underset{\text{pressure}}{p}\,\delta_{ij} + \underset{\text{viscosity}}{\mu} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

•

$$\xrightarrow{(\partial U_i/\partial x_i = 0)} \quad \rho \frac{DU_j}{Dt} = \mu \nabla^2 U_j - \frac{\partial p}{\partial x_j} - \rho g_j$$

($\underline{g} = -\nabla\psi$ $\quad\psi$: gravitational potential)

$p' = p + \rho\psi$, $\quad\nu$: kinematic viscosity

$$\rightarrow \frac{DU_j}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \left(\frac{\mu}{\rho}\right)\nabla^2 U_j \qquad \text{Navier-Stokes equation}$$

$$\rightarrow \frac{D\underline{U}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{U} \quad \text{(drop prime)}$$

For inviscid fluid ($\mu = 0$), $\tau_{ij} = -p\delta_{ij}$

$$\rightarrow \frac{D\underline{U}}{Dt} = -\frac{1}{\rho}\nabla p \ : \text{Euler equation}$$

- Role of pressure

$$\nabla\cdot\left[\left(\underbrace{\frac{D}{Dt}}_{=\frac{\partial}{\partial t} + (\underline{U}\cdot\nabla)\underline{U}} - \nu\nabla^2\right)\underline{U} = -\frac{1}{\rho}\nabla p\right]$$

$$\rightarrow \left(\frac{D}{Dt} - \nu\nabla^2\right)\nabla\cdot\underline{U} = -\frac{1}{\rho}\nabla^2 p - \frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}$$

when $\nabla\cdot\underline{U} = 0$, $\boxed{\nabla^2 p = -\rho\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}}$.

At solid surface, $\frac{\partial p}{\partial n} = \mu\frac{\partial^2 U_n}{\partial n^2}$.

$$\left(\cancel{\frac{\partial u_m}{\partial m}} + \frac{\partial u_n}{\partial n} = 0 \quad \therefore \frac{\partial u_n}{\partial n} = 0 \right)$$

If p is a sol., $p + c$ is sol. too.

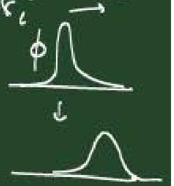
Using Green ft.,

$$p(\underline{x}, t) = \underbrace{p^{(h)}(\underline{x}, t)}_{\text{homo. sol.}} + \frac{\rho}{4\pi}\iiint_V \left(\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}\right)_{\underline{y}, t} \frac{d\underline{y}}{|\underline{x} - \underline{y}|}.$$

- Conserved passive scalars.

$\phi(\underline{x}, t)$: conserved passive scalar

$\frac{\partial \phi}{\partial t} + U_i \frac{\partial \phi}{\partial x_i}$



In a constant-property flow, $\frac{D\phi}{Dt} = \Gamma \nabla^2 \phi$.

Γ : (const & uniform) diffusivity

ϕ is conserved because there is no source and sink in the equation.

ϕ is passive because it does not change the flow.

ϕ : temperature $\rightarrow$ Γ : thermal diffusivity
$Pr = \nu/\Gamma$ (Prandtl number)

concentration $\rightarrow$ Γ : molecular diffusivity
$Sc = \nu/\Gamma$ (Schmidt number)

boundedness : if the initial & boundary values of ϕ lie within a given range
$\phi_{min} \le \phi \le \phi_{max}$,
then $\phi(\underline{x}, t)$ for all $(\underline{x}, t)$ also lies in this range.
i.e., values of ϕ greater than ϕ_{max} or less than ϕ_{min} cannot occur.

- Vorticity equation

turbulent flow $\rightarrow$ rotational $\rightarrow$ non-zero vorticity

vorticity $\underline{\omega} = \nabla \times \underline{U}$: twice the rotation rate

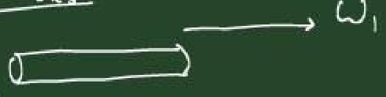
$\nabla \times$ [N-S eq.]

$\rightarrow \frac{D\underline{\omega}}{Dt} = \boxed{\underline{\omega} \cdot \nabla \underline{U}} + \nu \nabla^2 \underline{\omega}$

vortex stretching : increase $|\underline{\omega}|$

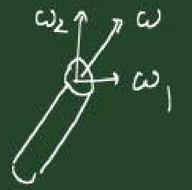
$$\frac{D\omega_1}{Dt} = \omega_1 \frac{\partial U_1}{\partial x_1} + \omega_2 \frac{\partial U_1}{\partial x_2} + \omega_3 \frac{\partial U_1}{\partial x_3} + \nu \nabla^2 \omega_1$$

$\frac{\partial U_1}{\partial x_1} > 0$



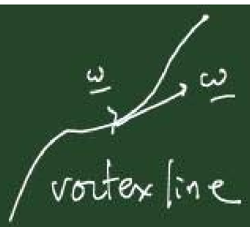
vortex stretching   vortex tilting

$\frac{\partial U_1}{\partial x_2} > 0$



For the material line : $\frac{d\underline{s}}{dt} = (\underline{s} \cdot \nabla)\underline{U}$

$\therefore$ In inviscid flow, the vorticity vector behaves in the same way as an infinitesimal material line element. $\rightarrow$ Helmholtz theorem.



In 2-D $(1, 2) \Rightarrow (U_1, U_2 \neq 0,\ \omega_3 \neq 0)$

$$\frac{D\omega_3}{Dt} = \omega_1 \underbrace{\frac{\partial U_3}{\partial x_1}}_{0} + \omega_2 \underbrace{\frac{\partial U_3}{\partial x_2}}_{0} + \omega_3 \underbrace{\frac{\partial U_3}{\partial x_3}}_{0} + \nu \nabla^2 \omega_3$$

$\rightarrow$ no vortex stretching $\rightarrow$ decays

- Rates of strain and rotation

$$\underbrace{\frac{\partial U_i}{\partial x_j}}_{\text{velocity gradient tensor}} = \underbrace{\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)}_{S_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)}_{\Omega_{ij}}$$

S_{ij} " strain rate tensor (symmetric ") Ω_{ij} " rotation rate tensor (anti-symmetric ")

$$\tau_{ij} = -p\delta_{ij} + 2\mu S_{ij} \quad (\text{not a ft. of } \Omega_{ij})$$

$$\omega_i = -\varepsilon_{ijk}\Omega_{jk} \quad (\text{not a ft. of } S_{ij})$$

$$\Omega_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k$$