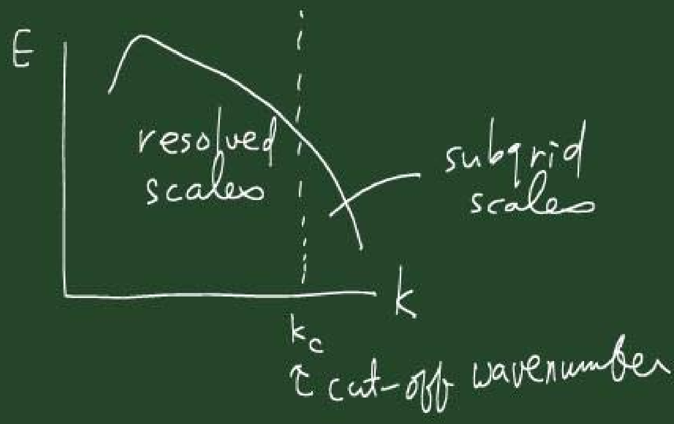
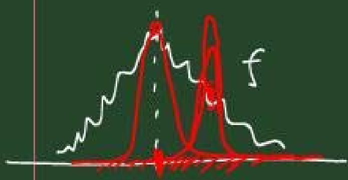
노트 제목 LES 2010-06-01



⊙ Filtering

- is an operation which damps out all the spatial fluctuations of flow variables smaller than a prescribed length scale (filter width)
- $\bar{f}(\underline{x}) = \int f(\underline{x}')\,\bar{G}(\underline{x}, \underline{x}')\,d\underline{x}' \xrightarrow{FT} \hat{\bar{f}}(k) = \hat{f}(k)\,\hat{\bar{G}}(k)$

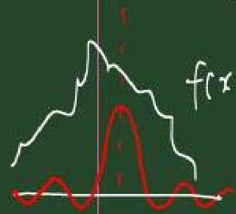
✓ Gaussian filter: $\bar{G}(x_i - x_i') = \left(\frac{6}{\pi\bar{\Delta}^2}\right)^{\frac{1}{2}} \exp\left(-\frac{6(x_i - x_i')^2}{\bar{\Delta}^2}\right)$

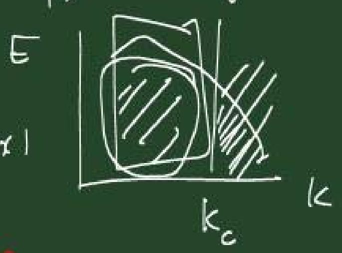


$\hat{\bar{G}}(k) = e^{-\frac{k^2\bar{\Delta}^2}{24}}$

$\bar{\Delta}$: filter width ($\simeq$ grid size)

∨ (Sharp) cut-off filter : $\bar{G}(x_i - x_i') = \dfrac{\sin[\pi(x_i - x_i')/\bar{\Delta}]}{\pi(x_i - x_i')}$



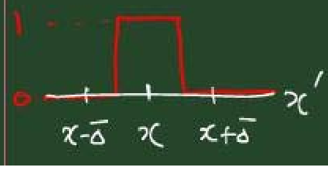


$\hat{\bar{G}}(k) = \begin{cases} 1 & \text{for } k \le k_c = \pi/\bar{\Delta} \\ 0 & \text{for } k > k_c \end{cases}$

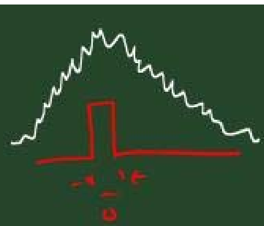
$f(x) \xrightarrow{FT} \hat{f}(k) \longrightarrow \hat{f}(k)\,\hat{\bar{G}}(k) \xrightarrow{IFT} \bar{f}(x)$

cut-off filter is used for spectral method.

∨ Box filter : $\bar{G}(x_i - x_i') = \begin{cases} 1 & \text{for } x_i - \frac{\bar{\Delta}}{2} < x_i' < x_i + \frac{\bar{\Delta}}{2} \\ 0 & \text{otherwise} \end{cases}$



$\hat{\bar{G}}(k) = \dfrac{\sin(\frac{1}{2}k\bar{\Delta})}{\frac{1}{2}k\bar{\Delta}}$



appropriate for FDM or FVM
or FEM

- Filtering the Navier-Stokes equation

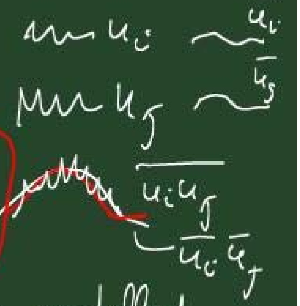
$\dfrac{\partial u_i}{\partial x_i} = 0 \Rightarrow \int \cdot\, \bar{G}\, d\underline{x}' \qquad \int \dfrac{\partial u_i}{\partial x_i} \cdot \bar{G}\, d\underline{x}' = \dfrac{\partial}{\partial x_i} \int u_i \bar{G}\, d\underline{x}'$

$\longrightarrow \boxed{\dfrac{\partial \bar{u}_i}{\partial x_i} = 0} \qquad = \dfrac{\partial}{\partial x_i} \bar{u}_i$

$\int \left(\dfrac{\partial u_i}{\partial t} + \dfrac{\partial}{\partial x_j} u_i u_j = -\dfrac{\partial p}{\partial x_i} + \dfrac{1}{Re} \dfrac{\partial}{\partial x_j} \dfrac{\partial u_i}{\partial x_j} \right) \cdot \bar{G}\, d\underline{x}'$

$\longrightarrow \dfrac{\partial \bar{u}_i}{\partial t} + \dfrac{\partial}{\partial x_j} \overline{u_i u_j} = -\dfrac{\partial \bar{p}}{\partial x_i} + \dfrac{1}{Re} \dfrac{\partial}{\partial x_j} \dfrac{\partial \bar{u}_i}{\partial x_j}$

$$\overline{u_i u_j} = \bar{u}_i \bar{u}_j + \underbrace{(\overline{u_i u_j} - \bar{u}_i \bar{u}_j)}_{\tau_{ij}}$$



$$\rightarrow \boxed{\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\bar{u}_i \bar{u}_j = -\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{Re}\nabla^2 \bar{u}_i - \frac{\partial \tau_{ij}}{\partial x_j}}$$

$\tau = 2\mu S_{ij}$

$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j \Leftarrow$ must be modelled.

- Smagorinsky eddy viscosity model (1963)
  - isotropic turbulence

$$\boxed{\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2(C_s\bar{\Delta})^2 |\bar{S}| \bar{S}_{ij}}$$

$\bar{\Delta}$: filter width

where $\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$ and $|\bar{S}| = \sqrt{2\bar{S}_{ij}\bar{S}_{ij}}$

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2\nu_T \bar{S}_{ij}, \quad \nu_T = (C_s\bar{\Delta})^2|\bar{S}|$$

eddy viscosity

✓ $C_s = 0.17$: Smagorinsky constant

0.1 ~ 0.3 depending on the flow type.

✓ Damping function is required near the wall

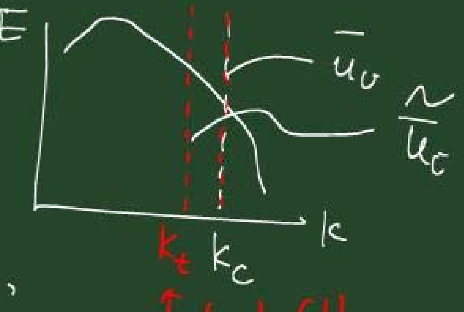
$$C_s\bar{\Delta} \rightarrow C_s\bar{\Delta}\left[1 - \exp(y^+/A^+)\right]$$

Kim & Moin (1982) : turb. channel flow

prob: C_s ?

- Dynamic subgrid-scale model (Germano et al. 1991)

use two filters



Apply two filters to the N-S eq,

$\widetilde{(\text{N-S})}$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial x_i} = 0$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{\bar{u}}_i \tilde{\bar{u}}_j\right) = -\frac{\partial \tilde{\bar{p}}}{\partial x_i} + \frac{1}{Re}\nabla^2 \tilde{\bar{u}}_i - \frac{\partial T_{ij}}{\partial x_j}$$

where $T_{ij} = \widetilde{\overline{u_i u_j}} - \tilde{\bar{u}}_i \tilde{\bar{u}}_j$.

Let $L_{ij} = \widetilde{\bar{u}_i \bar{u}_j} - \tilde{\bar{u}}_i \tilde{\bar{u}}_j$; computable

$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$ $\qquad$ $\tilde{\tau}_{ij} = \widetilde{\overline{u_i u_j}} - \widetilde{\bar{u}_i \bar{u}_j}$

① $L_{ij} = T_{ij} - \tilde{\tau}_{ij}$; Germano identity

Modelling : $\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2C\bar{\Delta}^2 |\bar{S}| \bar{S}_{ij}$ —②

$T_{ij} - \frac{1}{3}\delta_{ij} T_{kk} = -2C\tilde{\Delta}^2 |\tilde{\bar{S}}| \tilde{\bar{S}}_{ij}$ —③

②,③ → ①

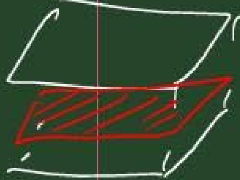
$$L_{ij} - \frac{1}{3}\delta_{ij} L_{kk} = -2C\tilde{\Delta}^2 |\tilde{\bar{S}}| \tilde{\bar{S}}_{ij} + 2\,\widetilde{C\bar{\Delta}^2 |\bar{S}| \bar{S}_{ij}}$$

$$\approx -2C\Big[\underbrace{\tilde{\Delta}^2|\tilde{\bar{S}}|\tilde{\bar{S}}_{ij} - \widetilde{\bar{\Delta}^2|\bar{S}|\bar{S}_{ij}}}_{M_{ij}}\Big]$$

Lilly (1992)

Least square error

$$Q = \left(L_{ij} - \frac{1}{3}\delta_{ij}L_{kk} + 2CM_{ij}\right)^2$$



$$\rightarrow \frac{\partial Q}{\partial C} = 0 \rightarrow \boxed{C_{(x,y,z,t)} = -\frac{1}{2}\frac{L_{ij}M_{ij}}{M_{ij}M_{ij}}}$$

probs : C can be negative → computation blows up

→ requires averaging over homogeneous directions(s) and/or ad hoc clipping.

$$C = -\frac{1}{2}\frac{\langle L_{ij}M_{ij}\rangle_h}{\langle M_{ij}M_{ij}\rangle_h}$$

→ hinders application of dynamic model to complex geometries.

Alternative SGS models for flows in complex geometry

Ghosal et al. (1995) JFM

Meneveau et al. (1996) JFM/PoF

→ Park et al. (2006) PoF

Vreman model (2004, PoF) — Smagorinsky model

↓ $\tau_{ij} = 0$ for simple laminar shear flow.

↓ $\tau_{ij} \neq 0$ even for laminar shear flow.