11 Half-Range Expansions, Forced Oscillations

11.1 Half-Range Expansions

- In applications we often want to employ a Fourier series for a function f that is given only on some interval, say, $0 \le x \le L$.

Example 1. "Triangle" and its half-range expansions

$$\begin{aligned} f(x) &= \frac{2k}{L}x \qquad \text{if } 0 \le x \le \frac{L}{2} \\ f(x) &= \frac{2k}{L}(L-x) \qquad \text{if } \frac{L}{2} \le x \le L \end{aligned}$$

Solution.

- (a) Even periodic extension

$$\begin{aligned} a_0 &= \frac{1}{L}[\frac{2k}{L}\int_0^{L/2} x dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)dx] \\ &= \frac{1}{L}\cdot\frac{2k}{L}[\frac{1}{2}\frac{L^2}{4} + L\cdot(L-\frac{L}{2}) - \frac{1}{2}\cdot(L^2-\frac{L^2}{4})] \\ &= \frac{k}{2} \end{aligned}$$

$$a_n = \frac{2}{L}[\frac{2k}{L}\int_0^{L/2} x\cdot\cos\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\cdot\cos\frac{n\pi x}{L}dx]$$

- 1st integral in the above equation:

$$\begin{aligned} \int_0^{L/2} x\cdot\cos\frac{n\pi x}{L}dx &= \frac{Lx}{n\pi}\sin\frac{n\pi x}{L}|_0^{L/2} - \frac{L}{n\pi}\int_o^{L/2}\sin\frac{n\pi x}{L}dx \\ &= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}|_0^{L/2} \\ &= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}(\cos\frac{n\pi}{2} - 1) \end{aligned}$$

- 2nd integral in the above equation:

$$\begin{aligned} \int_{L/2}^{L}(L-x)\cdot\cos\frac{n\pi x}{L}dx &= \frac{L}{n\pi}(L-x)\sin\frac{n\pi x}{L}|_{L/2}^{L} + \frac{L}{n\pi}\int_o^{L/2}\sin\frac{n\pi x}{L}dx \\ &= 0 - \frac{L}{n\pi}(L-\frac{L}{2})\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}|_{L/2}^{L} \\ &= -\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}(\cos n\pi - \cos\frac{n\pi}{2}) \end{aligned}$$

$$\begin{aligned}\therefore a_n &= \frac{4k}{L^2}[\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}(\cos\frac{n\pi}{2} - 1) - \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}(\cos n\pi - \cos\frac{n\pi}{2})] \\ &= \frac{4k}{n^2\pi^2}(2\cos\frac{n\pi}{2} - \cos n\pi - 1)\end{aligned}$$

- When n is odd,

$$2\cos\frac{n\pi}{2} - \cos n\pi - 1 = 0 - (-1) - 1 = 0$$

- When $n = 4, 8, 12, 16, \cdots$,

$$2\cos\frac{n\pi}{2} - \cos n\pi - 1 = 2 \cdot 1 - 1 - 1 = 0$$

- Thus,

$$a_2 = -\frac{16k}{2^2\pi^2}, \quad a_6 = -\frac{16k}{6^2\pi^2}, \quad a_{10} = -\frac{16k}{10^2\pi^2}, \cdots$$

$$\therefore f(x) = \frac{k}{2} - \frac{16k}{\pi^2}(\frac{1}{2^2}\cos\frac{2\pi x}{L} + \frac{1}{6^2}\cos\frac{6\pi x}{L} + \cdots)$$

- (b) Odd periodic extension

$$b_n = \frac{2}{L}[\int_0^{L/2}\frac{2k}{L}x \cdot \sin\frac{n\pi x}{L}dx + \int_{L/2}^{L}\frac{2k}{L}(L-x) \cdot \sin\frac{n\pi x}{L}dx]$$

- 1st integral in the above equation:

$$\begin{aligned}\int_0^{L/2} x \cdot \sin\frac{n\pi x}{L}dx &= -\frac{Lx}{n\pi}\cos\frac{n\pi x}{L}|_0^{L/2} + \frac{L}{n\pi}\int_o^{L/2}\cos\frac{n\pi x}{L}dx \\ &= -\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}|_0^{L/2} \\ &= -\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\end{aligned}$$

- 2nd integral in the above equation:

$$\begin{aligned}\int_{L/2}^{L}(L-x) \cdot \sin\frac{n\pi x}{L}dx &= -\frac{L}{n\pi}(L-x)\cos\frac{n\pi x}{L}|_{L/2}^{L} - \frac{L}{n\pi}\int_o^{L/2}\cos\frac{n\pi x}{L}dx \\ &= \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}|_{L/2}^{L} \\ &= \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\end{aligned}$$

$$\begin{aligned}\therefore b_n &= \frac{4k}{L^2}(-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\cos\frac{n\pi}{2} + \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}) \\ &= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}\end{aligned}$$

$$\therefore f(x) = \frac{8k}{\pi^2}(\frac{1}{1^2}\sin\frac{\pi x}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L} + \frac{1}{5^2}\sin\frac{5\pi x}{L} - \cdots)$$

11.2 Forced Oscillations

- Equation of motion

$$r(t) - ky - cy' = my''$$
$$\therefore my'' + cy' + ky = r(t)$$

- RLC-circuit

$$LI'' + RI' + \frac{1}{C}I = E'(t)$$

Example 2. Forced oscillations under a nonsinusoidal periodic driving force.

Given : $m = 1 \times 10^{-3}$ kg, $c = 0.02$ g/sec, $k = 25$ g/sec^2.

$$y'' + 0.02y' + 25y = r(t)$$

where $r(t)$ measures in g·cm/sec^2.

$$\begin{aligned} r(t) &= t + \pi/2 \qquad \text{if } -\pi < t < 0 \\ r(t) &= -t + \pi/2 \qquad \text{if } 0 < t < \pi \end{aligned}$$

- $r(t)$: even function

Find the steady-state solution $y(t)$.

Solution.

- The steady-state solution is the particular solution of the equation.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^\pi r(t)dt = \frac{1}{\pi}\int_0^\pi (-t + \frac{\pi}{2})dt = \frac{1}{\pi}[-\frac{t^2}{2} + \frac{\pi}{2}t]|_0^\pi \\ &= \frac{1}{\pi}(-\frac{\pi^2}{2} + \frac{\pi^2}{2}) = 0 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi (-t + \frac{\pi}{2})\cos ntdt = -\frac{2}{\pi}\int_0^\pi t\cos ntdt + \int_0^\pi \cos ntdt \\ &= -\frac{2}{\pi}\cdot\frac{1}{n}t\cdot\sin nt|_0^\pi + \frac{2}{\pi n}\int_0^\pi \sin ntdt = -\frac{2}{\pi n^2}\cos nt|_0^\pi \\ &= \frac{2}{\pi n^2}(1 - \cos n\pi) \end{aligned}$$

- If n is even, $1 - \cos n\pi = 0$.
- If n is odd, $1 - \cos n\pi = 2$.

$$\therefore a_n = \frac{4}{\pi n^2} \qquad n = 1, 3, 5, 7, 9, \cdots$$

$$\therefore r(t) = \frac{4}{\pi}(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots)$$

$$y'' + 0.02y' + 25y = \frac{4}{n^2\pi}\cos nt \qquad (n = 1, 3, 5, \cdots) \tag{1}$$

- Steady-state solution $y_n(t)$ of (1)

$$\begin{aligned} y_n(t) &= A_n \cos nt + B_n \sin nt \\ y_n'(t) &= -nA_n \sin nt + nB_n \cos nt \\ y_n''(t) &= -n^2 A_n \cos nt - n^2 B_n \sin nt \end{aligned} \quad (2)$$

- By substitution (2) into (1),

$$-n^2 A_n \cos nt - n^2 B_n \sin nt - 0.02nA_n \sin nt + 0.02nB_n \cos nt$$
$$+25A_n \cos nt + 25B_n \sin nt = \tfrac{4}{n^2\pi} \cos nt$$

$$(25A_n - n^2 A_n + 0.02nB_n) \cos nt + (25B_n - n^2 B_n - 0.02nA_n) \sin nt = \frac{4}{n^2\pi} \cos nt$$

$$25A_n - n^2 A_n + 0.02nB_n = \tfrac{4}{n^2\pi} \quad (3)$$
$$-0.02nA_n + 25B_n - n^2 B_n = 0 \quad (4)$$

- From (4)

$$B_n = \frac{0.02n}{25 - n^2} A_n$$

- Substituting this into (3)

$$(25 - n^2)A_n + \frac{0.02n}{25 - n^2} A_n = \frac{4}{n^2\pi}$$

$$\begin{aligned} \therefore A_n &= \frac{4(25 - n^2)}{n^2\pi[(25 - n^2)^2 + (0.02n)^2]} \\ \therefore B_n &= \frac{0.08}{n\pi[(25 - n^2)^2 + (0.02n)^2]} \end{aligned}$$

- Amplitude of (2)

$$C_n = \sqrt{A_n^2 + B_n^2}, \qquad D = (25 - n^2)^2 + (0.02n)^2$$

$$\therefore C_n = \frac{4}{n^2\pi D}[(25 - n^2)^2 + (0.02n)^2]^{1/2} = \frac{4}{n^2\pi\sqrt{D}}$$

- Numerical values are $C_1 = 0.0011$, $C_3 = 0.0088$, $C_5 = 0.5100$, $C_7 = -.0011$, $C_9 = 0.0003$, $\cdots$.
- In conclusion, the steady state motion is almost a harmonic oscillation whose frequency equals five times that of the exciting force.

11.3 Approximations by Trigonometric Polynomials

- $f(x)$: a periodic function of period 2π that can be represented by a Fourier series.

$$f(x) \approx a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \tag{5}$$

- Best approximation to f

$$F(x) = A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx) \tag{6}$$

- Total square error of F relative to the function f on the interval $-\pi \le x \le \pi$

$$E = \int_{-\pi}^{\pi}(f - F)^2 dx \ge 0 \tag{7}$$

$$E = \int_{-\pi}^{\pi} f^2 dx - 2\int_{-\pi}^{\pi} fF dx + \int_{-\pi}^{\pi} F^2 dx \tag{8}$$

- When we square (5), orthogonality of trigonometric functions are:

$$\int_{-\pi}^{\pi} \cos mx \cdot \sin nx dx = 0 \qquad \text{for all } m, n$$

$$\begin{aligned} \int_{-\pi}^{\pi} \cos^2 nx dx &= \frac{1}{2}\int_{-\pi}^{\pi}(1 + \cos 2nx)dx = \pi \\ \int_{-\pi}^{\pi} \sin^2 nx dx &= \frac{1}{2}\int_{-\pi}^{\pi}(1 - \cos 2nx)dx = \pi \end{aligned}$$

- 3rd integral in (8):

$$\begin{aligned} \int_{-\pi}^{\pi} F^2 dx &= \int_{-\pi}^{\pi}(A_0^2 + A_1^2\cos^2 x + B_1^2 \sin^2 x + \cdots + A_N^2 \cos^2 Nx + B_N^2 \sin^2 Nx + \cdots \\ &+ 2A_0A_1 \cos x + \cdots 2A_1B_1 \cos x \sin x + \cdots + 2A_NB_N \cos Nx \sin Nx)dx \\ &= \pi(2A_0^2 + A_1^2 + \cdots + A_N^2 + B_1^2 + \cdots + B_N^2) \end{aligned}$$

- 2nd integral in (8)

$$\int_{-\pi}^{\pi} fF dx = \int_{-\pi}^{\pi}(fA_0 + fA_1 \cos x + fB_1 \sin x + \cdots + fA_N \cos Nx + fB_N \sin Nx)dx$$

- Revoking Euler's formulas in Sec.10.2

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \cos nx dx,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \sin nx dx)$$

$$\int_{-\pi}^{\pi} fF dx = \pi(2A_0a_0 + A_1a_1 + \cdots + A_Na_N + B_1b_1 + \cdots + B_Nb_N)$$

$$\begin{aligned} E = \int_{-\pi}^{\pi} f^2 dx \quad &- \quad 2\pi \left[2A_0a_0 + \sum_{n=1}^{N}(A_na_n + B_nb_n) \right] \\ &+ \quad \pi \left[2A_0^2 + \sum_{n=1}^{N}(A_n^2 + B_n^2) \right] \end{aligned} \tag{9}$$

If we take $A_n = a_n$ and $B_n = b_n$n assuming that F is a Fourier series.

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2) \right] \tag{10}$$

- (9)-(10): using $A_n^2 - 2A_na_n + a_n^2 = (A_n - a_n)^2$,

$$E - E^* = \pi \left[2(A_0 - a_0)^2 + \sum_{n=1}^{N} \left\{ (A_n - a_n)^2 + (B_n - b_n)^2 \right\} \right]$$

$E - E^* \geq 0$ thus $E \geq E^*$
$E = E^*$ if and only if $A_0 = a_0, \cdots, B_n = b_n$

Theorem 1 [Minimum square error]

The total square error of F in (6) (with fixed N) relative to f on the interval $-\pi \leq x \leq \pi$ is minimum if and only if the coefficients of F in (6) are the Fourier coefficients of f. This minimum value E^* is given by (10).
- Hence with increasing N the partial sums of the Fourier series of f yield better and better approximations to f.
- **Bessel inequality**
Since $E^* \geq 0$

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x) dx \tag{11}$$

- Parseval's identity

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x) dx \tag{12}$$

Example 3. Square error for the sawtooth wave

$$f(x) = x + \pi \qquad (-\pi < x < \pi)$$

$$F(x) = \pi + 2\sin x - \sin 2x + \frac{2}{3}\sin 3x$$

$$
\begin{aligned}
E^* &= \int_{-\pi}^{\pi} (x+\pi)^2 dx - \pi \left[2\pi^2 + 2^2 + 1^2 + \left(\frac{2}{3}\right)^2 \right] \\
&= \frac{8}{3}\pi^3 - \pi \left(2\pi^2 + \frac{49}{9} \right) \approx 3.567
\end{aligned}
$$