# Nonlinear Optical Engineering

## Four-Wave Mixing

Yoonchan Jeong
School of Electrical Engineering, Seoul National University
Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

# Cross-Phase Modulation (XPM)

## Constitutive relations for E-field:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_o\mathbf{E} + \mathbf{P},$$

$$\mathbf{P} = \varepsilon_o\chi\mathbf{E} = \varepsilon_o\left(\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{EE} + \chi^{(3)}\mathbf{EEE} + \cdots\right)$$

## Suppose that there are two monochromatic waves:

$$\mathbf{E}(r,t) = \frac{1}{2}\hat{x}[E_1 \exp(-i\omega_1 t) + E_2 \exp(-i\omega_2 t)] + c.c.$$

## Induced nonlinear polarization:

$$\begin{aligned}\mathbf{P}_{NL}(r,t) = \varepsilon_0\chi^{(3)}\mathbf{EEE} = \frac{1}{2}\hat{x}\{&P_{NL}(\omega_1)\exp(-i\omega_1 t) + P_{NL}(\omega_1)\exp(-i\omega_2 t) \\ &+ P_{NL}(2\omega_1 - \omega_2)\exp[-i(2\omega_1 - \omega_2)t] \\ &+ P_{NL}(2\omega_2 - \omega_1)\exp[-i(2\omega_2 - \omega_1)t]\} + c.c \\ &+ \cdots\end{aligned}$$

$$\leftarrow \quad P_{NL}(\omega_1) = \chi_{eff}(|E_1|^2 + 2|E_2|^2)E_1$$

$$P_{NL}(\omega_2) = \chi_{eff}(|E_2|^2 + 2|E_1|^2)E_2$$

$$\chi_{eff} = \frac{3\varepsilon_0\chi^{(3)}_{xxxx}}{4}$$

# Nonlinear Pulse Propagation with XPM

NLSE:

$$\frac{\partial A}{\partial z}+\beta_1\frac{\partial A}{\partial t}+i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2}+\frac{\alpha}{2}A=i\gamma|A|^2A$$

Coupled NLSEs:

$$\frac{\partial A_1}{\partial z}+\beta_{11}\frac{\partial A_1}{\partial t}+i\frac{\beta_{21}}{2}\frac{\partial^2 A_1}{\partial t^2}+\frac{\alpha_1}{2}A_1=i\gamma_1(|A_1|^2+2|A_2|^2)A_1,$$

$$\frac{\partial A_2}{\partial z}+\beta_{12}\frac{\partial A_2}{\partial t}+i\frac{\beta_{22}}{2}\frac{\partial^2 A_2}{\partial t^2}+\frac{\alpha_2}{2}A_2=i\gamma_2(|A_2|^2+2|A_1|^2)A_2.$$

*Modulation instability?*

*Optical solitons?*

# 3rd-Order Susceptibility (1)

How many elements for $\chi_{ijkl}$? $\rightarrow$ *81 elements*

For isotropic media: $\rightarrow$ *21 nonzero elements*

$$\chi_{1111} = \chi_{2222} = \chi_{3333},$$
$$\chi_{1122} = \chi_{1133} = \chi_{2211} = \chi_{2233} = \chi_{3311} = \chi_{3322},$$
$$\chi_{1212} = \chi_{1313} = \chi_{2323} = \chi_{2121} = \chi_{3131} = \chi_{3232},$$
$$\chi_{1221} = \chi_{1331} = \chi_{2112} = \chi_{2332} = \chi_{3113} = \chi_{3223}.$$

$\leftarrow$ *Centrosymmetry*

In addition:

$$\chi_{1111} = \chi_{1122} + \chi_{1212} + \chi_{1221}$$

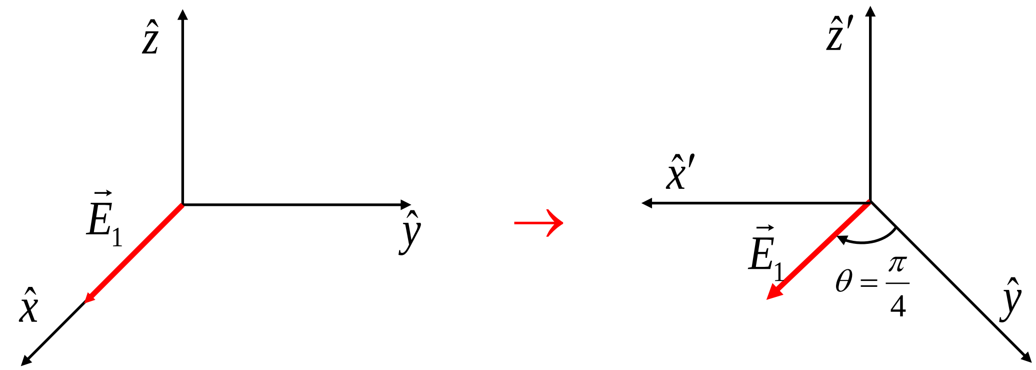
$\rightarrow$ In the compact form:

$$\chi_{ijkl} = \chi_{1122}\delta_{ij}\delta_{kl} + \chi_{1212}\delta_{ik}\delta_{jl} + \chi_{1221}\delta_{il}\delta_{jk}$$

# 3rd-Order Susceptibility (2)

## Rotation invariant:

$$\vec{P}_{NL} = \hat{x}\frac{3}{4}\varepsilon_0 \chi^{(3)}_{xxxx}\left|E_1^2\right|E_1 \;\; = \vec{P}'_{NL}$$




$$\vec{E}'_1 = \hat{x}'\frac{1}{\sqrt{2}}E_1 + \hat{y}'\frac{1}{\sqrt{2}}E_1 = \hat{x}'E'_{1,x} + \hat{y}'E'_{1,y}$$

$$P'_{NL,x'} = \frac{3}{4}\varepsilon_0(\chi^{(3)}_{xxxx}\left|E'^2_{1,x}\right|E_{1,x} + \chi^{(3)}_{xxyy}\left|E'^2_{1,y}\right|E_{1,x} + \chi^{(3)}_{xyxy}\left|E'^2_{1,y}\right|E_{1,x} + \chi^{(3)}_{xyyx}\left|E'^2_{1,y}\right|E_{1,x})$$

$$= \frac{3}{4}\varepsilon_0\left|E_1\right|^2 E_1 \frac{1}{2\sqrt{2}}(\chi^{(3)}_{xxxx} + \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx})$$

$$P'_{NL,y'} = \frac{3}{4}\varepsilon_0(\chi^{(3)}_{yyyy}\left|E'^2_{1,y}\right|E_{1,y} + \chi^{(3)}_{yyxx}\left|E'^2_{1,x}\right|E_{1,y} + \chi^{(3)}_{yxyx}\left|E'^2_{1,x}\right|E_{1,y} + \chi^{(3)}_{yxxy}\left|E'^2_{1,x}\right|E_{1,y})$$

$$= \frac{3}{4}\varepsilon_0\left|E_1\right|^2 E_1 \frac{1}{2\sqrt{2}}(\chi^{(3)}_{xxxx} + \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx})$$

$$\rightarrow \quad \chi_{xxxx} = \chi_{xxyy} + \chi_{xyxy} + \chi_{xyyx}$$

$$\vec{P}'_{NL} = \hat{x}'P'_{NL,x'} + \hat{y}'P'_{NL,y'}$$

$$\rightarrow \quad P'_{NL} = \sqrt{2}P'_{NL,x'} = \frac{3}{4}\varepsilon_0\left|E_1\right|^2 E_1 \frac{1}{2}(\chi^{(3)}_{xxxx} + \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx})$$

# Four-Wave Mixing (FWM)

Consider four optical waves oscillating at $\omega_1$, $\omega_2$, $\omega_3$, and $\omega_4$:

$$\mathbf{E}(r,t)=\frac{1}{2}\hat{x}\sum_{j=1}^{4}E_j\exp[i(k_jz-\omega_jt)]+c.c.$$

Induced nonlinear polarization:

$$\mathbf{P}_{NL}=\varepsilon_0\chi^{(3)}\mathbf{EEE}=\frac{1}{2}\hat{x}\sum_{j=1}^{4}P_j\exp[i(k_jz-i\omega_jt)]+c.c$$

$$+\cdots$$

For $\omega_4$:

$$P_4=\frac{3\varepsilon_0\chi^{(3)}_{xxxx}}{4}[|E_4|^2E_4+2(|E_1|^2+|E_2|^2+|E_3|^2)E_4$$

$$+2E_1E_2E_3\exp(i\theta_+)+2E_1E_2E_3^*\exp(i\theta_-)+\cdots],$$

$$\rightarrow \begin{array}{l}\theta_+=(k_1+k_2+k_3-k_4)z-(\omega_1+\omega_2+\omega_3-\omega_4)t\\ \boxed{\theta_-=(k_1+k_2-k_3-k_4)z-(\omega_1+\omega_2-\omega_3-\omega_4)t}\end{array}$$

*Easy for phase-matching*

$\rightarrow$ *Four-wave mixing*

# Phase-Matching Condition

## Energy conservation:

$$\omega_3 + \omega_4 - \omega_1 - \omega_2 = 0$$

$$\omega_3 + \omega_4 - 2\omega_1 = 0 \leftarrow \omega_1 = \omega_2$$

## Momentum conservation:

$$\Delta k = k_3 + k_4 - k_1 - k_2$$
$$= (n_3\omega_3 + n_4\omega_4 - n_1\omega_1 - n_2\omega_2)/c = 0$$

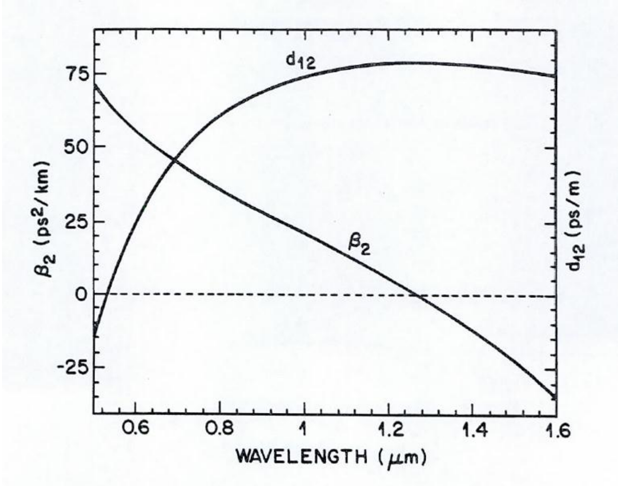
$$\Delta k = k_3 + k_4 - 2k_1$$
$$= (n_3\omega_3 + n_4\omega_4 - 2n_1\omega_1)/c = 0 \leftarrow \omega_1 = \omega_2$$

*Is this it?*

*Recall:* $\beta(\omega) = n(\omega)\frac{\omega}{c} = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 + \cdots,$

$$\text{where} \quad \beta_m = \left(\frac{d^m\beta}{d\omega_m}\right)_{\omega=\omega_0} \quad (m = 1, 2, 3, \ldots)$$

*Consider work-off length!*




Source: Nonlinear Fiber Optics, G. P. Agrawal

# Coupled-Wave Equations

## Complex-field amplitudes :

$$E_j(\mathbf{r}) = F_j(x,y)A_j(z),$$

## Coupled-wave equations :

$$\frac{dA_1}{dz} = \frac{in_2'\omega_1}{c}[(f_{11}|A_1|^2 + 2\sum_{j\neq 1}|A_j|^2)A_1 + 2f_{1234}A_2^*A_3A_4e^{i\Delta kz}$$

$$\frac{dA_2}{dz} = \frac{in_2'\omega_2}{c}[(f_{22}|A_2|^2 + 2\sum_{j\neq 2}|A_j|^2)A_2 + 2f_{2134}A_1^*A_3A_4e^{i\Delta kz}$$

Pump fields

$$\frac{dA_3}{dz} = \frac{in_2'\omega_3}{c}[(f_{33}|A_3|^2 + 2\sum_{j\neq 3}|A_j|^2)A_3 + 2f_{3412}A_1A_2A_4^*e^{-i\Delta kz}$$

$$\frac{dA_4}{dz} = \frac{in_2'\omega_4}{c}[(f_{44}|A_4|^2 + 2\sum_{j\neq 4}|A_j|^2)A_4 + 2f_{4312}A_1A_2A_3^*e^{-i\Delta kz}$$

Signal & idler

$\rightarrow$ *NDPA*

$$f_{jk} = \frac{\langle|F_j|^2|F_k|^2\rangle}{\langle|F_j|^2\rangle\langle|F_k|^2\rangle}$$

$$f_{ijkl} = \frac{\langle F_i^*F_j^*F_kF_l\rangle}{[\langle|F_i|^2\rangle\langle|F_j|^2\rangle\langle|F_k|^2\rangle\langle|F_l|^2\rangle]^{1/2}}$$

*Effective phase-matching condition:*

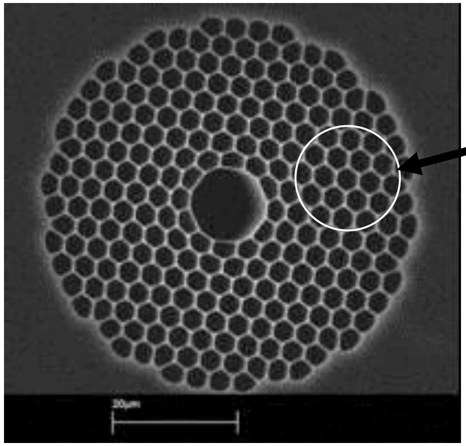
$$\kappa = \Delta k + 2\gamma P_0 = 0$$

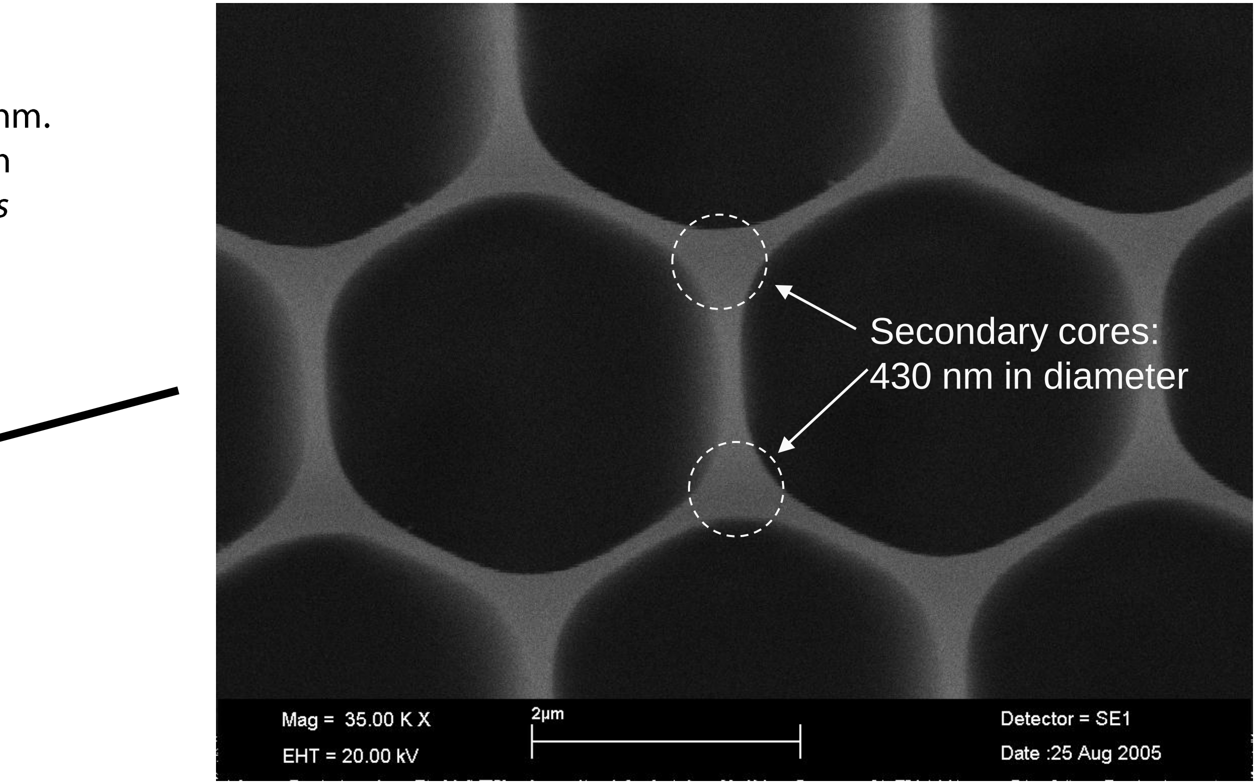
$$\leftarrow \quad P_0 = P_1 + P_2$$

# Four-Wave-Mixing in an Optical Fiber

## Fiber Parameters:

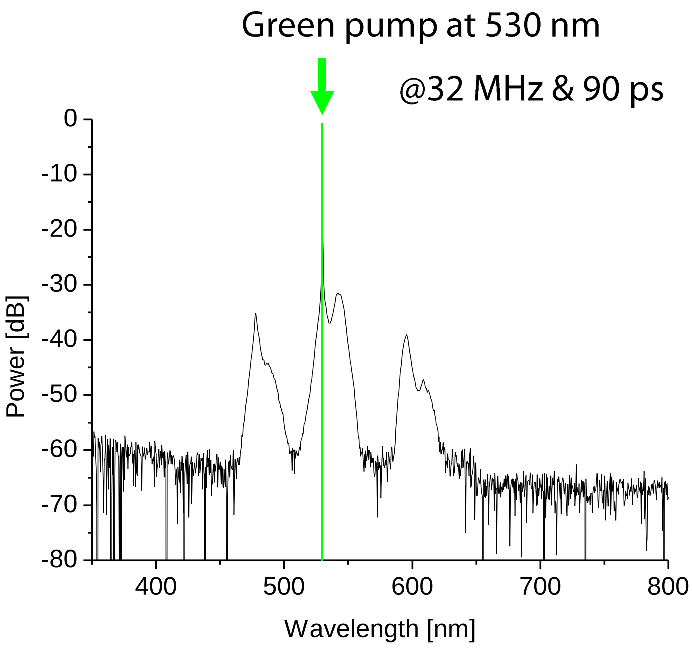
Structure: 7-point core defect, 7½ ring structure.
Hole-to-hole spacing: 3.6 μm
Hole diameter relative size (d/Λ): 0.95
Air filling factor: ~87%
Core diameter: 11.6 μm.
Fundamental bandgap: 1570 nm.
Higher order: 458, 505, 560 nm
*(Fabricated from the resources outside the project)*

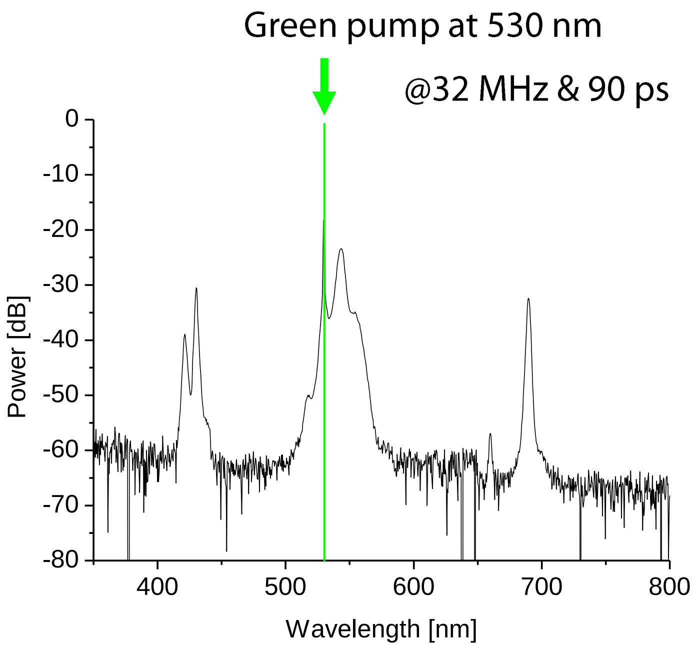




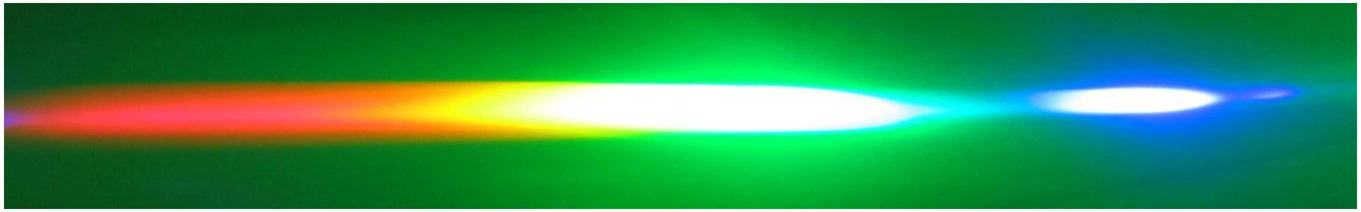

*Secondary cores to be investigated for four-wave mixing!*

# RGB Generation via FWM in an Optical Fiber



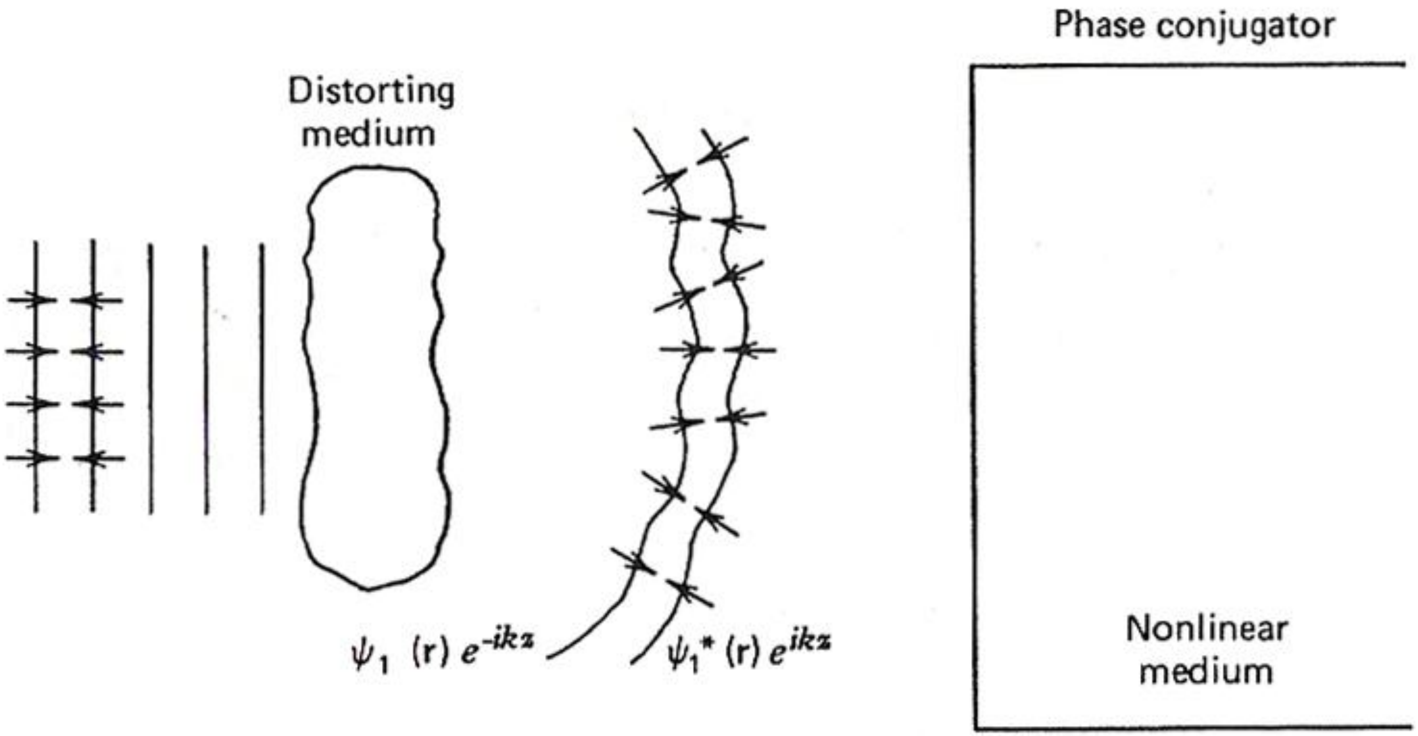




Diffracted output image



# Phase Conjugation

## Complex conjugation of an input field via FWM:




Source: *Optical Waves in Crystals*, A. Yariv and P. Yeh

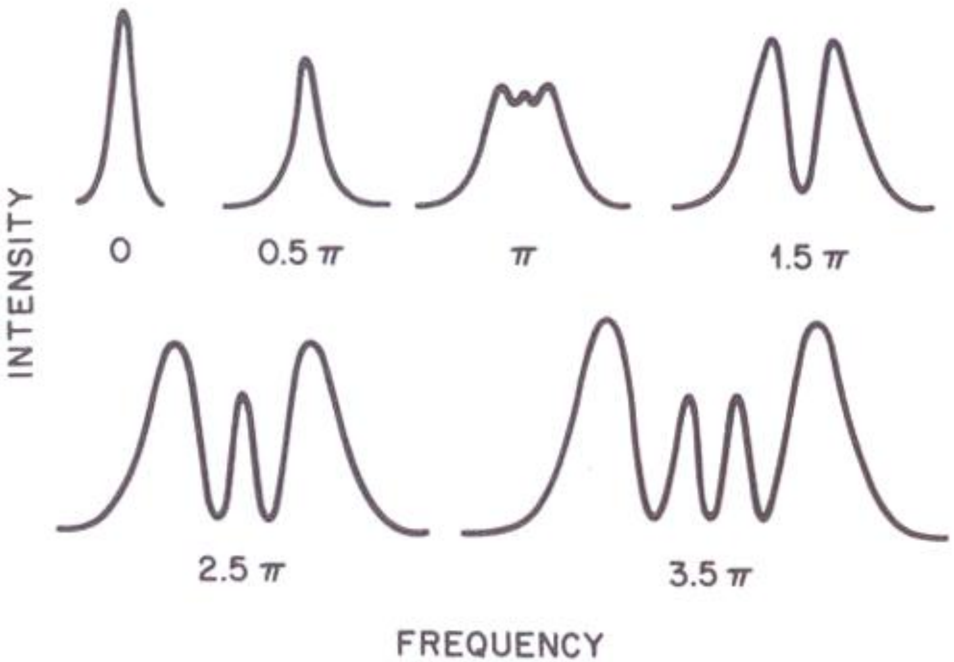
*Recall:*

*Phase conjugation*

$$\frac{dA_4}{dz} = \frac{in_2'\omega_4}{c}[(f_{44}|A_4|^2 + 2\sum_{j\neq 4}|A_j|^2)A_4 + 2f_{4312}A_1A_2A_3^*e^{-i\Delta kz}$$

# Nonlinear Phase Modulation

For Gaussian pulses:




Source: Nonlinear Fiber Optics, G. P. Agrawal

*SPM or XPM?*

$$P_1 = \frac{3\varepsilon_0 \chi_{xxxx}^{(3)}}{4}[|E_1|^2 E_1 + 2(|E_2|^2 + |E_3|^2 + |E_4|^2)E_1 ?$$