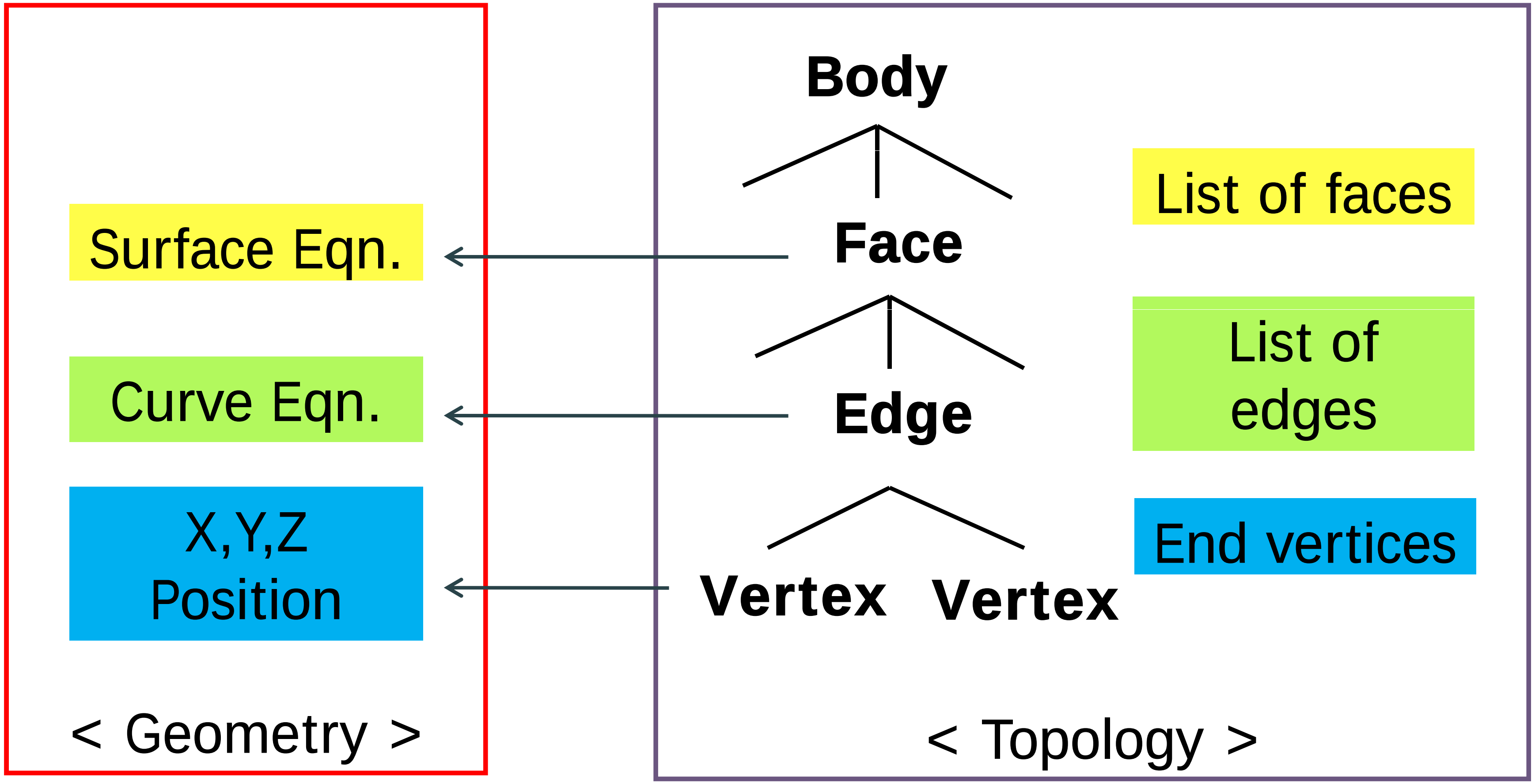
Theory of curves I

Human Centered CAD Lab.

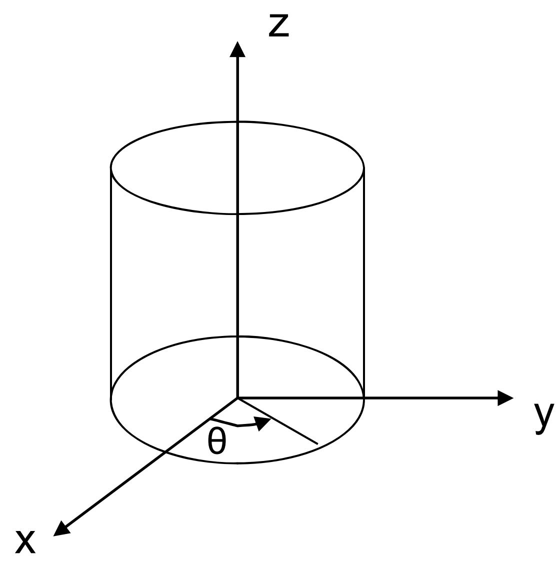
B-Rep Structure – review

Geometry vs. Topology



Types of curve equations

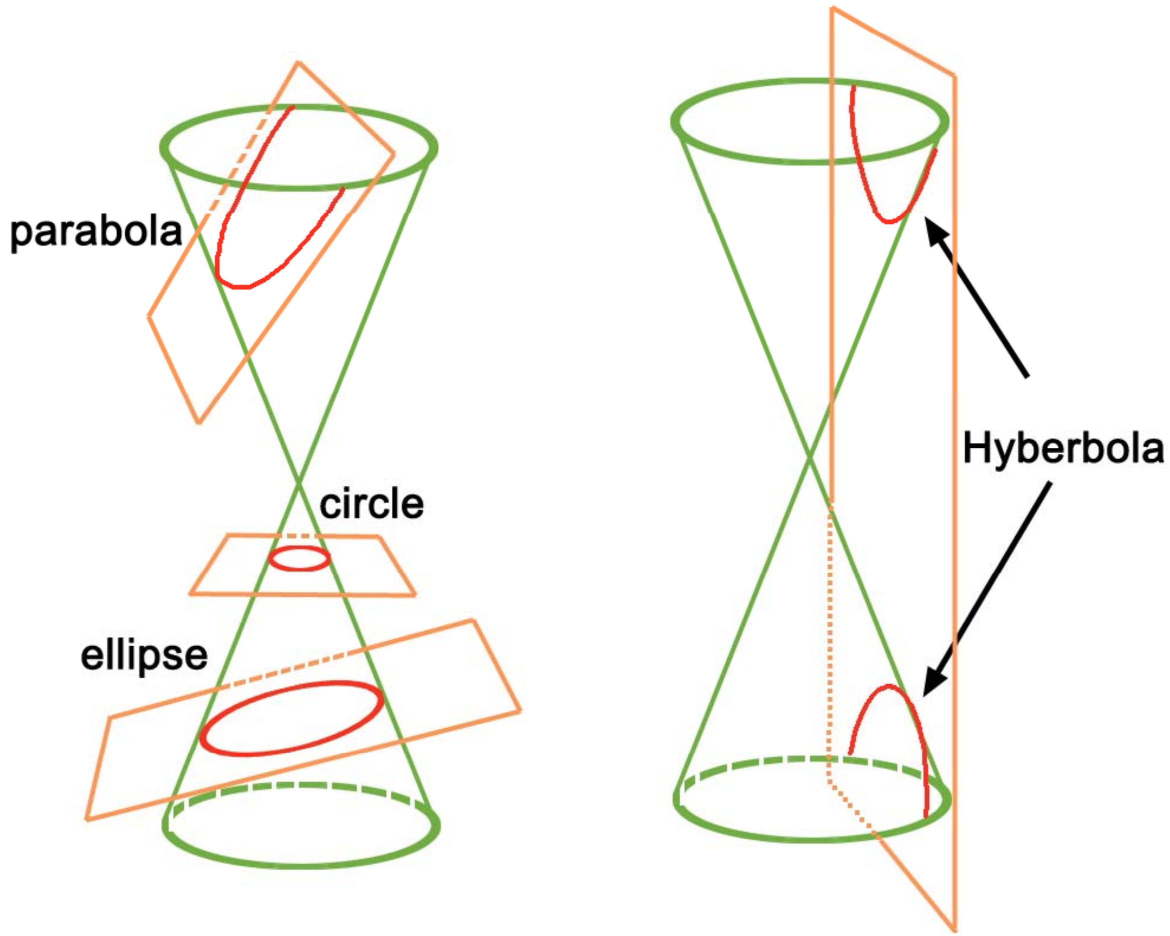
- Parametric equation
  - x=x(t), y=y(t), z=z(t)
  - Ex) x=Rcosθ, y=Rsinθ, z=0 ($0 \le \theta \le 2\pi$)
- Implicit nonparametric
  - $x^2 + y^2 - R^2 = 0, \quad z = 0$
  - F(x, y, z)=0, G(x, y, z)=0
  - Intersection of two surfaces
  - Ambiguous independent parameters
- Explicit nonparametric
  - $y = \pm\sqrt{R^2 - x^2}, \quad z = 0$
  - Should choose proper neighboring point during curve generation



Conic curves

- Curves obtained by intersecting a cone with a plane
- Circle (circular arc), ellipse, hyperbola, parabola
  - Ex) Circle (circular arc)
    - Circle in xy-plane with center (x_c, y_c) and radius R
    - $x = R\cos\theta + x_c$
    - $y = R\sin\theta + y_c$
    - $z = 0$
- Points on the circle are generated by incrementing θ by $\triangle\theta$ from 0, points are connected by line segments
- Equation of a circle lying on an arbitrary plane can be derived by transformation

Conic curves – cont’



Hermite curves

- Parametric eq. is preferred in CAD systems
  - Polynomial form of degree 3 is preferred :
    - C2 continuity is guaranteed when two curves are connected

$$\therefore \mathbf{P}(u) = [x(u)\ v(u)\ z(u)] = \mathbf{a}_0 + \mathbf{a}_1\ u + \mathbf{a}_2\ u^2 + \mathbf{a}_3\ u^3 \qquad (1)$$

$$(0 \le u \le 1) : \text{algebraic eq.}$$

- Impossible to predict the shape change from change in coefficients $\Rightarrow$ not intuitive
  - Bad for interactive manipulation

Hermite curves – cont'

- Apply Boundary conditions to replace algebraic coefficients

- Use $\mathbf{P}_{(0)},\ \mathbf{P}_{(1)},\ \mathbf{P}'_{(0)},\ \mathbf{P}'_{(1)} \Rightarrow$ Substitute in Eq(1)
  $\mathbf{P}_0,\ \mathbf{P}_1,\ \mathbf{P}'_0,\ \mathbf{P}'_1$

$$\left.\begin{aligned}
\mathbf{P}_{(0)} &= \mathbf{P}_0 = \mathbf{a}_0 \\
\mathbf{P}_{(1)} &= \mathbf{P}_1 = \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \\
\mathbf{P}'_{(0)} &= \mathbf{P}'_0 = \mathbf{a}_1 \\
\mathbf{P}'_{(1)} &= \mathbf{P}'_1 = \mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_3
\end{aligned}\right\} \quad (2)$$

Hermite curves – cont'

- Solve for $\mathbf{a}_0,\ \mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3$ in Eq (2)

$$\left.\begin{aligned}
\mathbf{a}_0 &= \mathbf{P}_0 \\
\mathbf{a}_1 &= \mathbf{P}_0' \\
\mathbf{a}_2 &= -3\mathbf{P}_0 + 3\mathbf{P}_1 - 2\mathbf{P}_0' - \mathbf{P}_1' \\
\mathbf{a}_3 &= 2\mathbf{P}_0 - 2\mathbf{P}_1 + \mathbf{P}_0' - \mathbf{P}_1'
\end{aligned}\right\} \quad (3)$$

Hermite curves – cont'

- Substitute (3) into (1)

$$\mathbf{P}(u) = \underbrace{\begin{bmatrix} 1-3u^2+2u^3 & 3u^2-2u^3 & u-2u^2+u^3 & -u^2+u^3 \end{bmatrix}}_{} \underbrace{\begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}'_0 \\ \mathbf{P}'_1 \end{bmatrix}}_{\text{geometric coefficientt}}$$

$$\Uparrow$$

Hermite curve equation

- It is possible to predict the curve shape change from the change in P_0, P_1, P_0', P_1' to some extent

Hermite curves – cont’

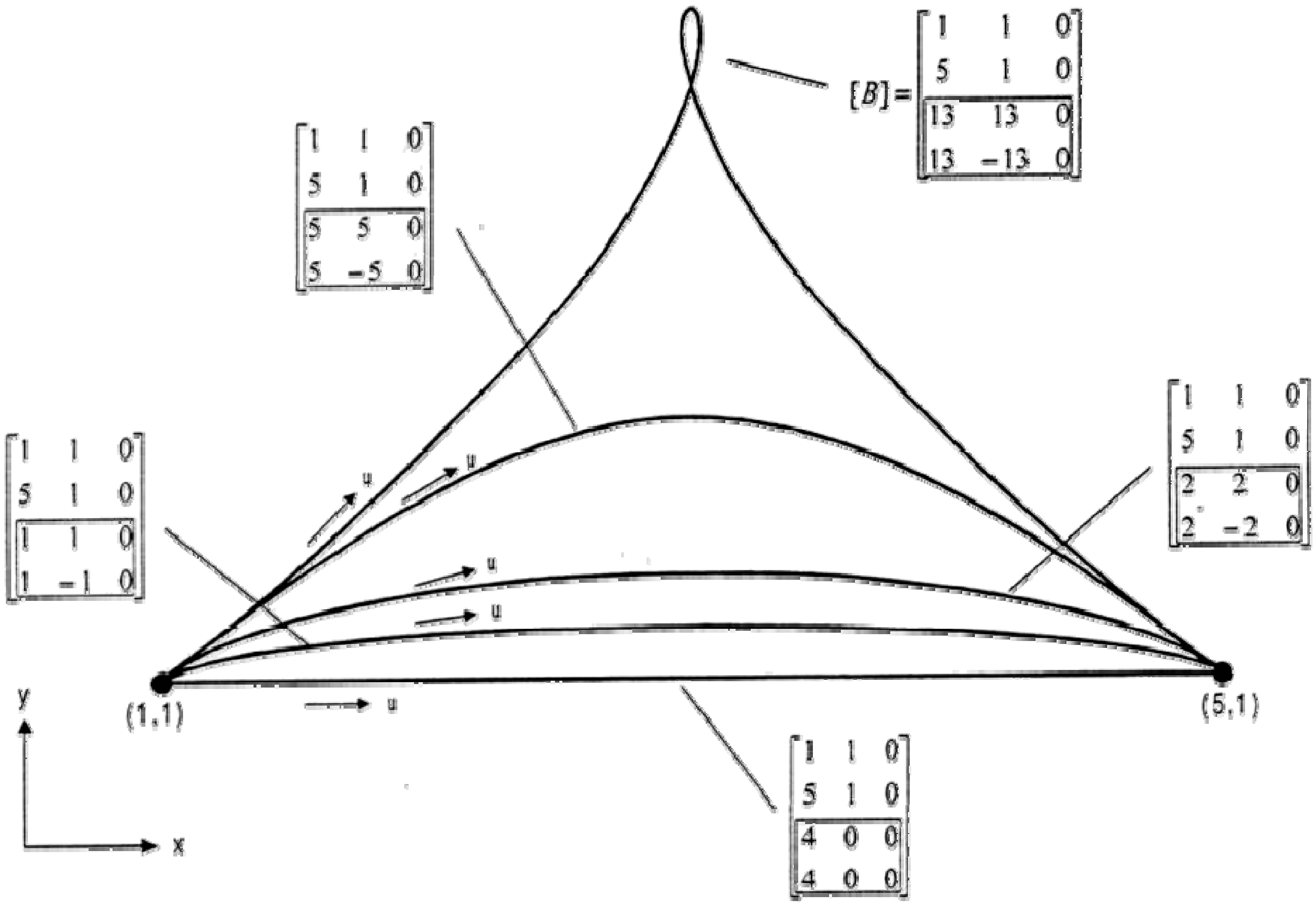


Figure 6.2 Effect of P_0' and P_1' on curve shape

Hermite curves – cont’

- $1-3u^2+2u^3,\ 3u^2-2u^3,\ u-2u^2+u^3,\ -u^2+u^3$

  determine the curve shape by blending the effects of P_0, P_1, P_0', P_1' → blending function

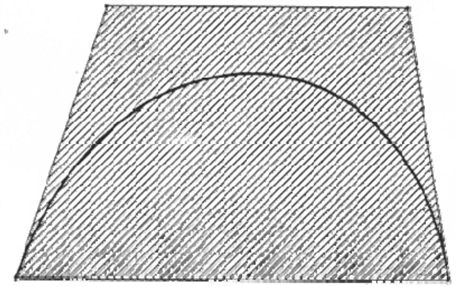
Bezier curves

- It is difficult to realize a curve in one's mind by changing size and direction of P_0', P_1' in Hermite curves
- Bezier curves
  - Invented by Bezier at Renault
  - Use polygon that enclose a curve approximately
  - control polygon, control point

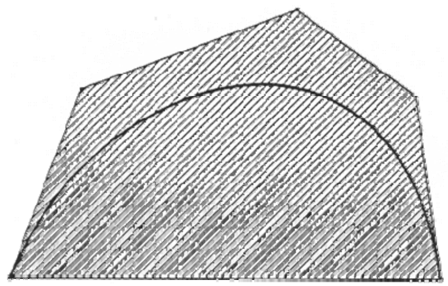
Bezier curves – cont’

- Passes through 1st and last vertex of control polygon
- Tangent vector at the starting point is in the direction of 1st segment of control polygon
- Tangent vector at the ending point is in the direction of the last segment
  - Useful feature for smooth connection of two Bezier curves
- The n-th derivative at starting or ending point is determined by the first or last (n+1) vertices of control polygon
- Bezier curve resides completely inside its convex hull
  - Useful property for efficient calculation of intersection points

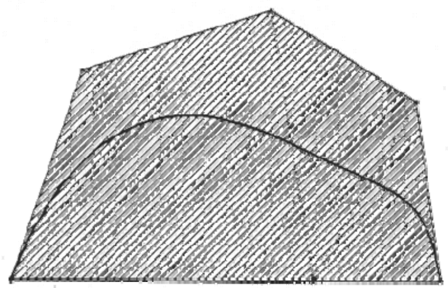
Bezier curves – cont’



(a)



(b)



(c)

Bezier curves – cont'

$$\mathbf{P}(u) = \sum_{i=0}^{n} \binom{n}{i} u^{i} (1-u)^{n-i} \mathbf{P}_{i} \qquad (0 \le u \le 1)$$

↑ Control Point

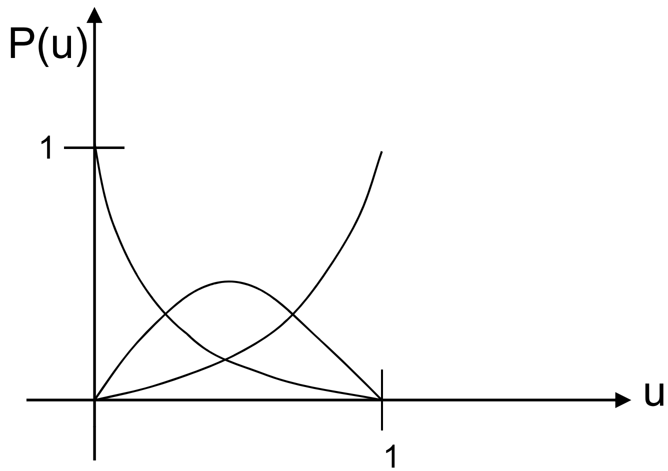
$$\mathbf{P}(u) = (1-u)\mathbf{P}_0 + u\mathbf{P}_1$$

: Straight line from P0 to P1 satisfies the desired qualities including convex hull property

$$\mathbf{P}(u) = (1-u)^2 \mathbf{P}_0 + 2(1-u)u\mathbf{P}_1 + u^2\mathbf{P}_2$$

$$\Rightarrow (1-u)^2 + 2(1-u)u + u^2 = 1$$

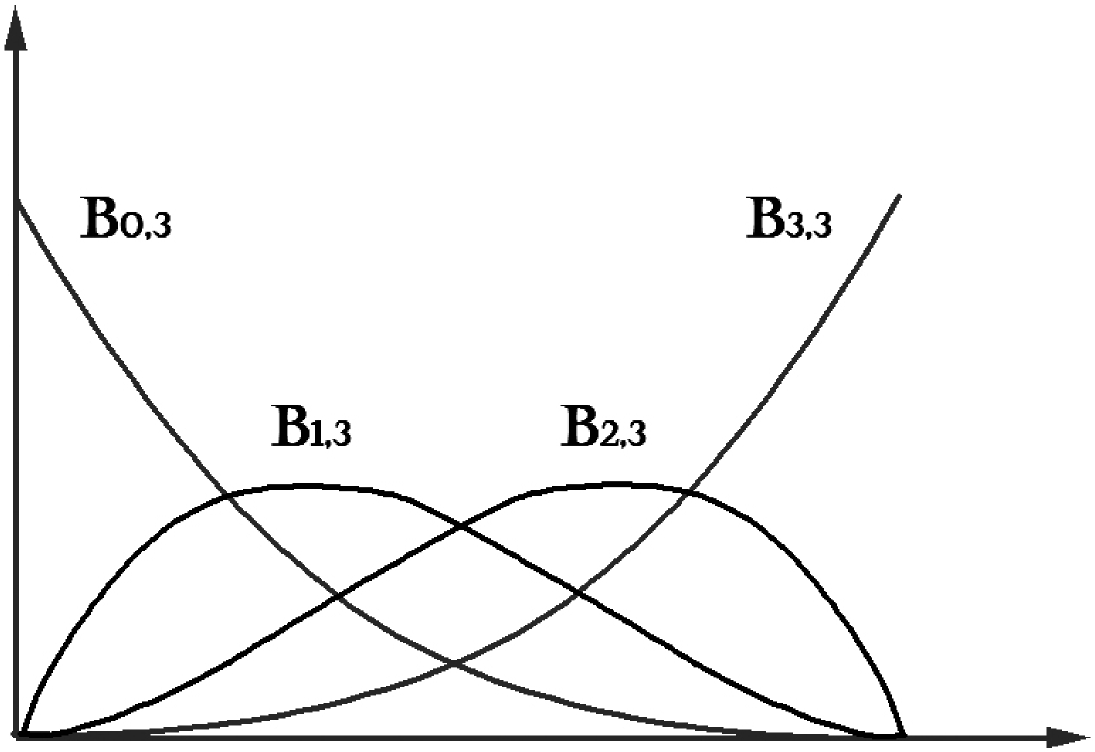
satisfies the desired qualities



Bezier curves – cont'

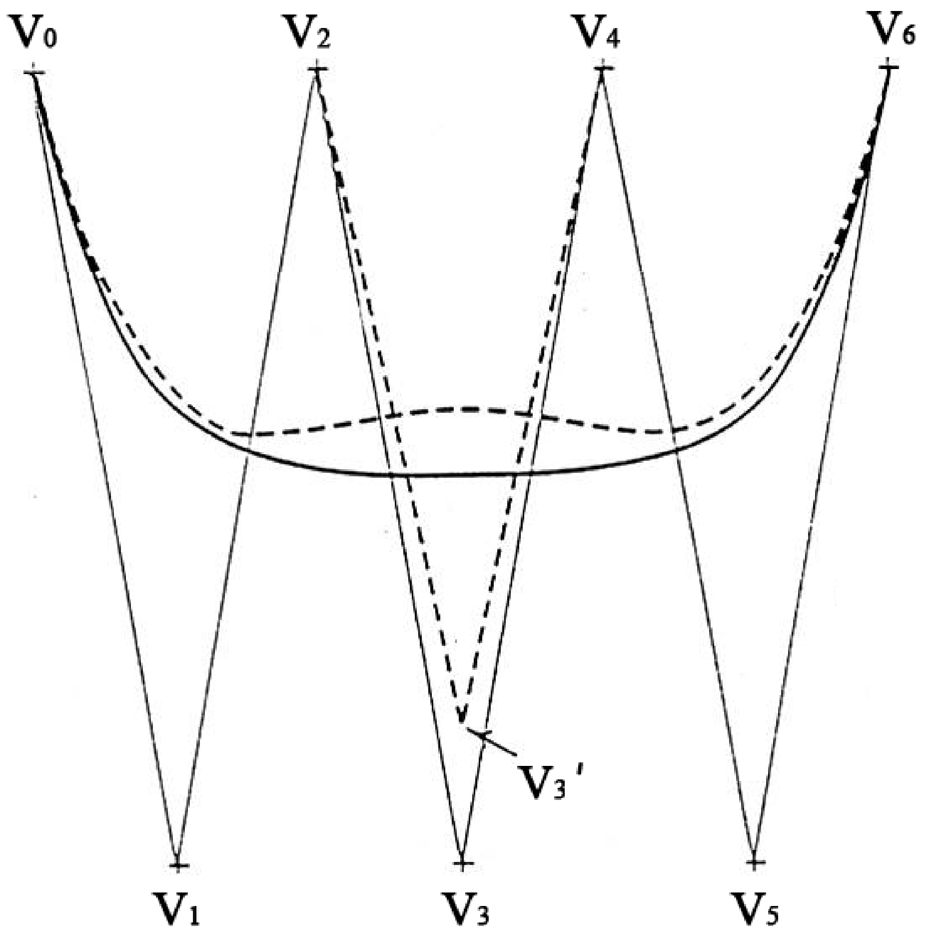
- Highest term is u^n for the curve defined by (n+1) control points
  - Polynomial of degree n
- Degree of curve is determined by number of control points
- Large number of control points are needed to represent a curve of complex shape → high degree is necessary.
  - Heavy computation, oscillation
  - Better to connect multiple Bezier curves
- Global modification property (not local modification)
  - Difficult to result a curve of desired shape by modifying portions

Blending functions in Bezier curve



for degree 3

Bezier curves – cont'



Bezier Curve does NOT have local modification property