14 Review of Fourier Analysis, Partial Differential Equations

14.1 Fourier Series/Transform/Integrals

- orthogonality of eigenfunctions of the S-L problem → orthogonality of trigonometric system → Euler formulas for the Fourier series
- odd/even → Fourier sine/cosine series, Half-range expansion
- extension to nonperiodic functions → Fourier integral
- Fourier transform (sine/cosine)
- *Recall that Fourier series are deeply connected with periodic phenomena involving ordinary differential equations (ODEs). Now we will see they are extremely useful to find the solutions for more complex differential equations called partial differential equations (PDEs).*

14.2 Basic Concepts of Partial Differential Equations

- PDE : diff eqn involving partial derivatives of two or more independent variables
- The second order differential equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F$$

Discriminant $B^2 - AC$:

$$\begin{aligned} B^2 - AC &= 0 : && \text{Parabolic} \\ B^2 - AC &> 0 : && \text{Hyperbolic} \\ B^2 - AC &< 0 : && \text{Elliptic} \end{aligned}$$

Example 1. Important linear partial differential equations of the second order
- 1-dimensional wave equation

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Discriminant: $B^2 - AC = 0 - 1 \cdot (-c^2) = c^2 > 0$: hyperbolic
- 1-dimensional heat equation

$$\frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial x^2}$$

Discriminant: $B^2 - AC = 0 - 0 \cdot c^2 = 0$: parabolic
- 2-dimensional Laplace equation (steady state heat equation with no heat generation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Discriminant: $B^2 - AC = 0 - 1 \cdot 1 = -1, 0$: elliptic
- 2-dimensional Poisson equation (steady state heat equation with heat generation)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)$$

- 2-dimensional wave equation

$$\frac{\partial^2 u}{\partial^2 t} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

- 3- dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Theorem 1 Fundamental Theorem (Superposition or linearity principle)
If u_1 and u_2 are any solutions of a linear homogeneous partial differential equation in some region R, then

$$u = c_1 u_1 + c_2 u_2$$

with any constants c_1 and c_2 is also a solution of that equation in R.
Example 2 Find a solution $u(x, y)$ of the partial differential equation

$$u_{xx} - u = 0$$

solution)

$$u(x, y) = A(y) \cdot e^x + B(y) \cdot e^{-x}$$

Example 3 Solve the partial differential equation

$$u_{xy} = -u_x$$

solution)

$$u_x = P, \quad \Rightarrow \quad P_y = -P$$

$$\frac{P_y}{P} = -1 \quad \Longrightarrow \quad \ln P = -y + \tilde{c}(x)$$

$$\therefore P = c(x) \cdot e^{-y}$$

$$u(x, y) = f(x)e^{-y} + g(y) \quad \text{where} \quad f(x) = \int c(x)dx$$

14.3 Modeling; Vibrating String; Wave Equation

Physical Assumptions
1. Homogeneous, perfectly elastic string.
2. Gravitational force is negligible, compared to the tension.
3. String performs small transverse motions in a vertical plane.

Derivation of the Differential Equation from Forces
Force balance

$$x: \quad T_1 \cos\alpha = T_2 \cos\beta = T = \text{const}$$

$$y: \quad T_2 \cdot \cos\beta - T_1 \cdot \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\bigg|_x \qquad \text{and} \qquad \tan\beta = \left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x}$$

- By Taylor expansion around x,

$$\frac{\partial u}{\partial x}\bigg|_{x+\Delta x} = \left(\frac{\partial u}{\partial x}\right)\bigg|_x + \frac{\partial^2 u}{\partial x^2}\bigg|_x \Delta x + \cdots$$

$$\left[\left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\bigg|_x\right] = \left[\frac{\partial u}{\partial x}\bigg|_x + \frac{\partial^2 u}{\partial x^2}\bigg|_x \Delta x - \frac{\partial u}{\partial x}\bigg|_x\right] = \frac{\rho\Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \cdot \frac{\partial^2 u}{\partial x^2}$$

- One-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \left(c^2 = \frac{T}{\rho}\right)$$

- Unit of c

$$[c^2] = \left[\frac{T}{\rho}\right] = \left[\frac{\text{kg} \cdot \text{m/s}^2}{\text{kg/m}}\right] = [\text{m/s}]^2$$

$$\therefore [c] = [\text{m/s}] \qquad \text{(propagation velocity)}$$