Second-order Linear ODEs

# Chap. 2 Linear Differential Equations of Second and Higher Order

* linear / nonlinear $\quad y', \; yy' \; \cdots$

order $\quad y', \; y'', \; yy''$

## 2.1. Homogeneous Linear ~~Equations~~ ODEs of Second order

– Linear diff. eq of 2nd-order

$$y'' + p(x)y' + q(x)y = r(x) \quad : \text{standard form ("1}\cdot y'')$$

$r(x) = 0$ : homogeneous

$r(x) \neq 0$ : nonhomogeneous

$y'' + 4y = e^{-x}\sin x$ : linear, 2nd-order, nonhomogeneous

$(1-x^2)y'' - 2xy' + 6y = 0$ : linear " , homogeneous

$x(y''y + y'^2) + 2y'y = 0$ : nonlinear. " , "

$y'' = \sqrt{y'^2 + 1}$ : " , " , nonhomogeneous

* Superposition principle for homogeneous, linear differential eq : $\boxed{y'' + p(x)y' + q(x)y = 0.}$

: solutions : $y_1, y_2$ $\Rightarrow$ linear combinations $c_1y_1 + c_2y_2 \Rightarrow$ sol. : $\boxed{y = c_1y_1 + c_2y_2}$ Gen. sol.

pf.) $y_1'' + py_1' + qy_1 = 0, \quad y_2'' + py_2' + qy_2 = 0 \quad //$

$y = c_1y_1 + c_2y_2 \Rightarrow (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2)$ $(= 0 ?)$

$\Rightarrow c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2)$

$= 0\cdot c_1 + 0\cdot c_2 = 0.$ $\quad c_1, c_2$: arbitrary constants

valid only for homogeneous & linear

Ex. 1* $\quad y'' - y = 0$ : lin, hom

$y_1 = e^x, \quad y_2 = e^{-x}$

$y = c_1 e^x + c_2 e^{-x} \quad$ O

Ex. 2. $\quad y'' + y = 1$ : lin. nonhom.

$y_1 = 1 + \cos x, \quad y_2 = 1 + \sin x$

$y = 2(1+\cos x) \;/\; y = (1+\cos x) + (1+\sin x) \quad$ X

Ex. 3. $\quad y''y - xy' = 0$ : nonlinear,

$y_1 = x^2. \quad y_2 = 1$

$y = -x^2 \;/\; x^2 + 1 \quad$ X

## § Initial Value Problem

1st-order diff. eq. $\quad$ I.C. $\quad y(x_0) = y_0$. $\quad$ "1"

2nd-order diff eq $\quad$ I.C. $\quad y(x_0) = K_0$ & $y'(x_0) = K_1$ $\quad$ "2"

Ex. 4* $\quad y'' - y = 0, \quad y(0) = 4, \quad y'(0) = -2$ : IVP

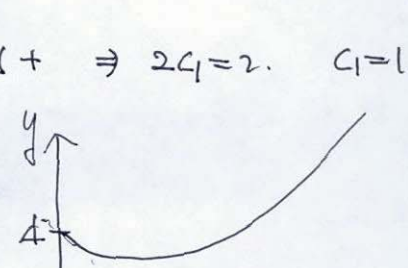
$y_1 = e^x, \quad y_2 = e^{-x}$

$y = c_1 e^x + c_2 e^{-x}. \qquad y' = c_1 e^x - c_2 e^{-x}$

$y(0) = c_1 + c_2 = 4$

$y'(0) = c_1 - c_2 = -2$ $\quad$ + $\Rightarrow 2c_1 = 2. \quad c_1 = 1. \quad c_2 = 3.$

$\therefore y = e^x + 3e^{-x}.$



what if we take $y_1 = e^x$, $y_2 = \ell e^x$ ?

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 e^x + c_2 \ell e^x$$
$$= (c_1 + c_2 \ell) e^x = \tilde{c} e^x$$

only one arbitrary const.

$$\left.\begin{aligned} y(0) &= c_1 + c_2 \ell = 4 \\ y'(0) &= c_1 + c_2 \ell = -2 \end{aligned}\right\} \quad //$$

$e^x$, $\ell e^x$ : proportional.

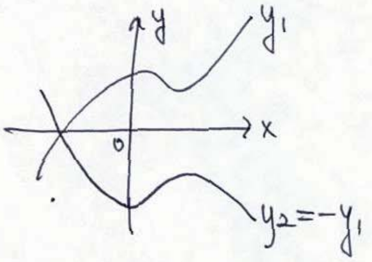
* $y_1$ and $y_2$ are called proportional on $I$ if

$$y_1 = k y_2 \quad \text{or} \quad y_2 = \ell y_1$$

holds for all $x$ on $I$, where $k$ and $\ell$ are numbers, zero or not.

e.g. $y_1 = x^2 + 3x$. $y_2 = 2x^2 + 6x$

$\langle y_2 = 2y_1 \rangle$




* $y_1(x)$ and $y_2(x)$ are $\boxed{\text{linearly independent}}$ on an interval $I$ where they are defined if

$k_1 y_1(x) + k_2 y_2(x) = 0$ on $I$ implies $k_1 = 0$ and $k_2 = 0$.

<u>otherwise,</u> $y_1$ and $y_2$ are linearly dependent.

$k_1 \neq 0$ or $k_2 \neq 0$ : $k_1 y_1 + k_2 y_2 = 0$ : $y_1 = -\frac{k_2}{k_1} y_2$ or $y_2 = -\frac{k_1}{k_2} y_1$

$y_1, y_2$: proportional

Definition

$y'' + p(x)y' + q(x)y = 0$

- $y_1, y_2$ : linearly independent solutions $\Rightarrow$ basis

- $y = c_1 y_1 + c_2 y_2$. ($c_1, c_2$ : arbitrary const.)

  : general solution

* Definition of a basis of

$$y'' + p(x)y' + q(x)y = 0. \quad \text{on an interval } I \quad \cdots (*)$$

: A basis of solutions of Eq (1) is a pair of $y_1, y_2$ of linearly independent solutions

Ex. 6. $y'' + y = 0$

$y_1 = \cos x$, $y_2 = \sin x$ : basis. (lin. indep.) $\frac{y_1}{y_2} = \cot x \neq \text{const}$

∴ General solution : $y = c_1 y_1 + c_2 y_2$
$= c_1 \cos x + c_2 \sin x$.

§ Reduction of order

: How to obtain a basis if one solution is known

$$\underset{\text{(2nd order)}}{y'' + py' + qy = 0} \xrightarrow[\text{known}]{\text{one sol. "}y_1\text{"}} \text{1st-order DE} \rightarrow \text{"}y_2\text{"}$$

- $y_1$ : known solution
- set $y_2 = uy_1$ $\quad u(x) = ?$ $\quad y_2$ : linearly independent solution of $y_1$

$$y_2' = u'y_1 + uy_1'$$
$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0$$

$$u''y_1 + u'(2y_1' + py_1) + u(\underbrace{y_1'' + py_1' + qy_1}_{=0}) = 0$$

$$u'' + \frac{2y_1' + py_1}{y_1} u' = 0. \qquad \text{set } u' = U$$

then $u'' = U'$

$$U' + \frac{2y_1' + py_1}{y_1} U = 0$$ : "1st" order

$$\frac{dU}{dx} = -\left(\frac{2y_1'}{y_1} + p\right) U$$

$$\frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p\right) dx$$

$$\ln|U| = -2\ln|y_1| - \int p\,dx + \cancel{c}$$ unnecessary

$$U = \frac{1}{y_1^2} \exp\left(-\int p\,dx\right)$$

$u' = U \quad \therefore u = \int U\,dx \quad \therefore y_2 = u y_1 = y_1 \int U\,dx$

$$\therefore y_2 = y_1 \int \frac{1}{y_1^2} \exp\left(-\int p\,dx\right) dx$$

Ex. 17*  $x^2 y'' - xy' + y = 0.$

one solution $y_1 = x$ known

standard form: $\underbrace{y'' - \frac{1}{x}}_{p} y' + \underbrace{\frac{1}{x^2}}_{q} y = 0.$

$$U = \frac{1}{x^2} \exp\left(+\int \frac{1}{x} dx\right) = \frac{1}{x^2} \exp(\ln|x|) = \frac{1}{x^2}|x| = \frac{1}{x}$$

$y_2 = x \int \frac{1}{x} dx = x \ln|x|$   $x > 0: \; y_2 = x \ln x$

$x < 0: \; y_2 = x \ln(-x)$

Gen. sol.
$$\begin{cases} y = c_1 x + c_2 x \ln x. & x > 0 \\ y = c_1 x + c_2 x \ln(-x) & x < 0 \end{cases}$$