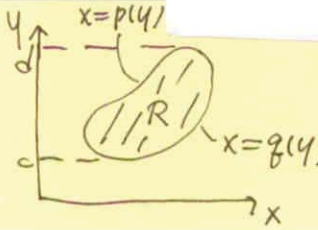
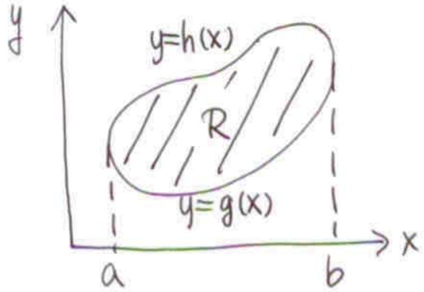


$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)\,dx \right] dy$$

ps 9.2. # 9, 17

~~9.3.~~ 10.3. Double Integrals   Optional



$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy \right] dx$$

* change of variables in double integrals

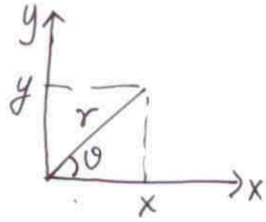
- $\displaystyle\int_a^b f(x)\,dx \overset{x\to u}{=} \int_\alpha^\beta f(x(u))\frac{dx}{du}\,du$

- $\displaystyle\iint_R f(x,y)\,dx\,dy \overset{(x,y)\to(u,v)}{=} \iint_{R^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$

Jacobian $\displaystyle J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

e.g. cartesian coordinates → polar coordinates

(x, y) → (r, θ)



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

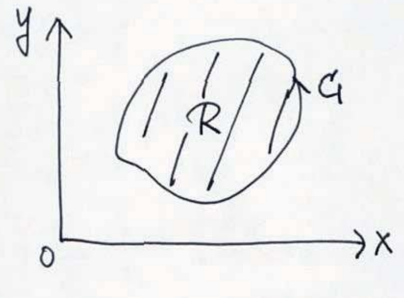
$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

10.4. Green's Theorem in the Plane

: Transformation between double integrals and line integrals

Theorem 1. Green's Theorem



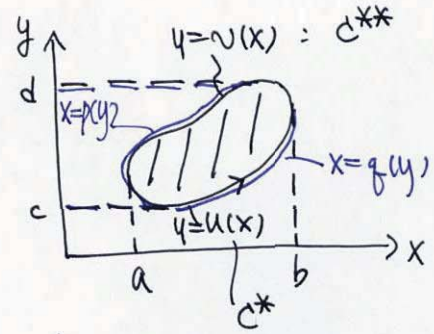
$$\vec{F} = F_1 \hat{i} + F_2 \hat{j}$$
$$= F_1(x,y)\,\hat{i} + F_2(x,y)\,\hat{j}$$

R: left to C

$$\boxed{\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C (F_1\,dx + F_2\,dy)}$$

vector form $\Rightarrow$ $\boxed{\iint_R (\nabla \times \vec{F}) \cdot \hat{k}\, dx\,dy = \oint_C \vec{F} \cdot d\vec{r}}$

Proof



$$\iint_R \frac{\partial F_1}{\partial y}\,dx\,dy = \int_a^b \left[\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y}\,dy\right] dx$$

$$= \int_a^b \left[F_1(x, v(x)) - F_1(x, u(x))\right] dx$$

$$= -\int_b^a F_1(x, v(x))\,dx - \int_a^b F_1(x, u(x))\,dx$$

$$= -\int_{C^{**}} F_1(x,y)\,dx - \int_{C^{*}} F_1(x,y)\,dx = -\oint_C F_1(x,y)\,dx$$

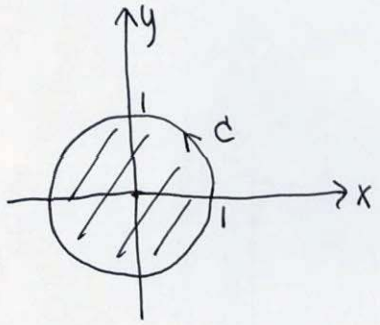
Similarly, $\iint_R \frac{\partial F_2}{\partial x}\,dx\,dy = \int_c^d \left[\int_{p(y)}^{q(y)} \frac{\partial F_2}{\partial x}\,dx \right] dy$

$$= \oint_C F_2(x,y)\,dy$$

$\therefore$ proven.

Ex. 1. $F_1 = y^2 - 7y$, $\quad F_2 = 2xy + 2x$

$C: \ x^2 + y^2 = 1$



$$I_1 = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2y + 2 - (2y - 7) = 9$$

$$I_1 = 9 \iint_R dx\,dy = 9\pi$$

$$I_2 = \oint_C (F_1\,dx + F_2\,dy) = \int_0^{2\pi} \left[(\sin^2 t - 7\sin t)(-\sin t)\,dt + (2\sin t\cos t + 2\cos t)\cos t\,dt \right]$$

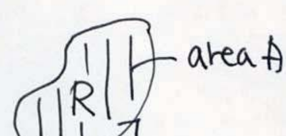
$\begin{pmatrix} x = \cos t \\ y = \sin t \end{pmatrix} \ t: 0 \to 2\pi$

$$= 9\pi$$

$\therefore I_1 = I_2$

Ex. 2 If we set $\begin{cases} F_1 = 0, \ F_2 = x & : \ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 \\ F_1 = -y, \ F_2 = 0 & : \qquad '' \qquad = 1 \end{cases}$

Green's theorem : $\iint_R dx\,dy = \oint_C x\,dy = \text{Area}$



$\iint_R dx\,dy = \oint_C -y\,dx = A$

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

ellipse $\rightarrow$ $A = \pi ab$: reading
$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.\right)$

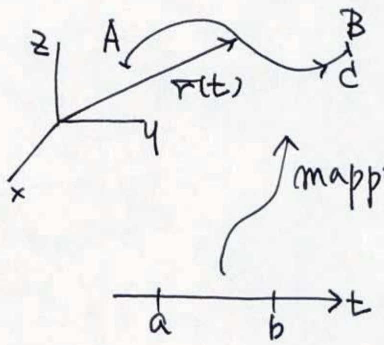
Ex. 3/4 reading

ps 9.4. # 5, 9,

~~9~~ 10.5. Surfaces for Surface Integrals

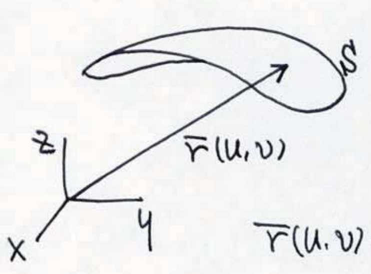
* Representation of surfaces

cf) curve:



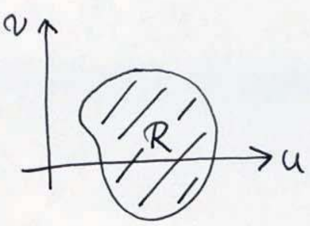
: parametric representation

surface:

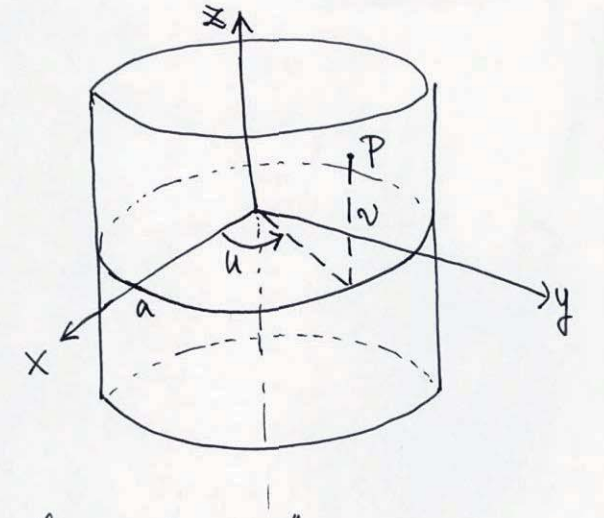


$$\begin{cases} z = f(x,y) \\ g(x,y,z) = 0 \end{cases}$$

$$\bar{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$



Ex.1. Cylinder : $x^2+y^2=a^2$. $-1 \le z \le 1$

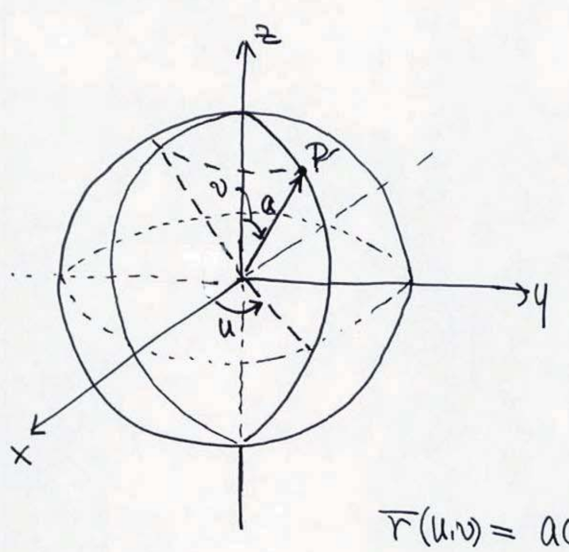


"cylindrical coordinates"

$$\bar{r}(u,v) = a\cos u\,\hat{i} + a\sin u\,\hat{j} + v\hat{k}$$

$$0 \le u \le 2\pi, \quad -1 \le v \le 1$$

Ex.2 Sphere: $x^2+y^2+z^2=a^2$



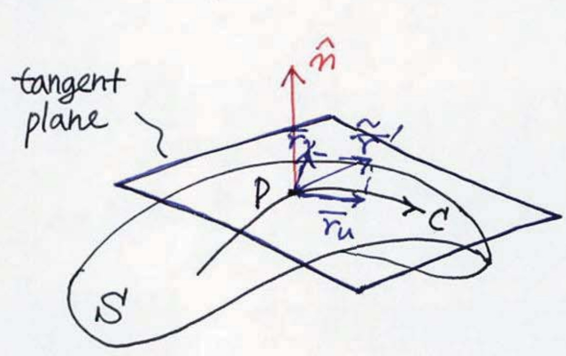
$$\begin{cases} x = a\sin v\cos u \\ y = a\sin v\sin u \\ z = a\cos v \end{cases}$$

$0 \le u \le 2\pi$

$0 \le v \le \pi$

$$\bar{r}(u,v) = a\cos u\sin v\,\hat{i} + a\sin u\sin v\,\hat{j} + a\cos v\,\hat{k}$$

* Tangential plane and surface normal



S: $\bar{r} = \bar{r}(u,v)$

C (on S) : $\tilde{r} = \bar{r}(u(t), v(t))$

tangent vector of C

$$\tilde{r}'(t) = \frac{d\tilde{r}}{dt} = \frac{\partial \bar{r}}{\partial u}u' + \frac{\partial \bar{r}}{\partial v}v'$$

$$= \bar{r}_u u' + \bar{r}_v v'$$

$$\bar{N} = \bar{r}_u \times \bar{r}_v$$

$$\hat{n} = \frac{1}{|\bar{N}|}\bar{N} = \frac{1}{|\bar{r}_u \times \bar{r}_v|}\,\bar{r}_u \times \bar{r}_v$$

Recall. $S: \; g(x,y,z) = 0.$

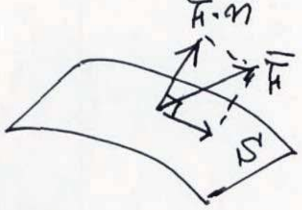
$$\hat{n} = \frac{1}{|\nabla g|}\nabla g$$

ps 9.5 # 9, 13, 29

~~9~~ 10.6. Surface Integrals

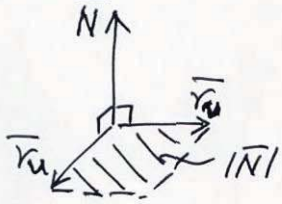
$$I = \iint_S \bar{F}\cdot\hat{n}\, dA = \iint_R \bar{F}(\bar{r}(u,v))\cdot \bar{N}(u,v)\, du\,dv$$

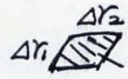
$\bar{N} = \bar{r}_u \times \bar{r}_v$



I = "flux" integral

$\hat{n}\,dA = \bar{N}\,du\,dv$?





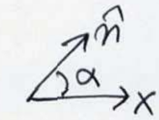
$$\Delta A = \Delta r_1 \Delta r_2 \sin\theta$$
$$= \frac{\partial r}{\partial u}\Delta u \cdot \frac{\partial r}{\partial v}\Delta v \cdot \sin\theta$$

$$dA = |\bar{r}_u \times \bar{r}_v|\, du\,dv$$

$$\therefore \hat{n}\,dA = \hat{n}|\bar{N}|\,du\,dv$$
$$= \bar{N}\,du\,dv$$

In terms of direction cosines:

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$



$\hat{n}\cdot\hat{i} = \cos\alpha$