Computer aided ship design

Part 1. Curve & Surface

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Advanced
Ship
Design
Automation
Laboratory

2.3 B[asis]-spline curves

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2.3.1 Definition of B-spline curves

2.3.1.1 Knots, spline curves

2.3.1.2 Definition of B-spline curves

2.3.1.3 Geometric meanings of cubic B-spline curve

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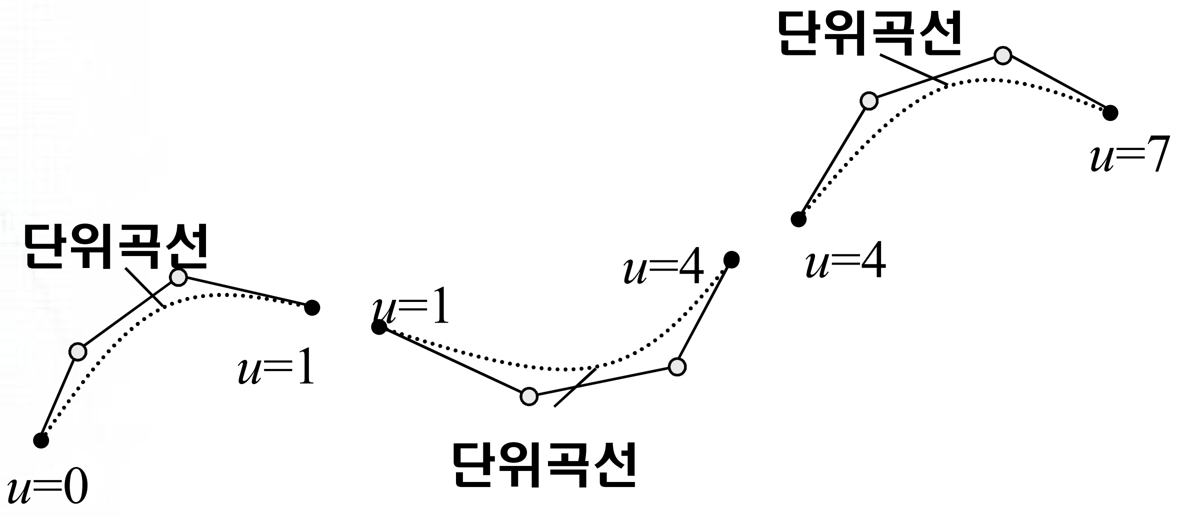
2.3.1.1 Knot & spline curves

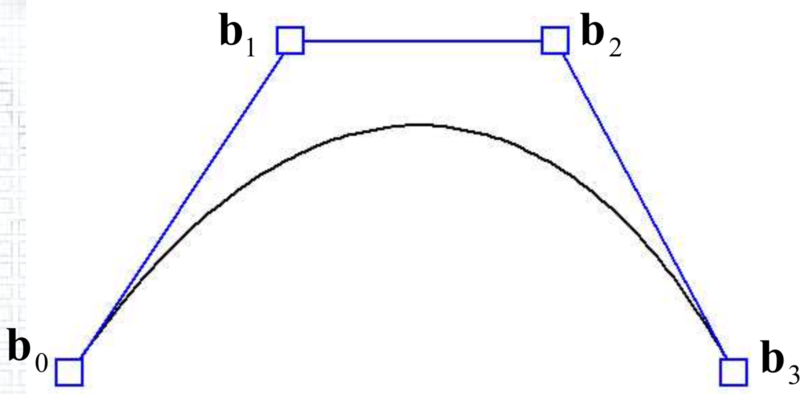


놋트 = {…, 0, 1, 4, 7, …}

- 단위 곡선들을 "부드럽게" 연결한 곡선: spline curve
- 단위 곡선을 묶는 매듭 : 놋트(knot)

2.3.1.2 Definition of B-spline curves

Cubic Bezier Curve



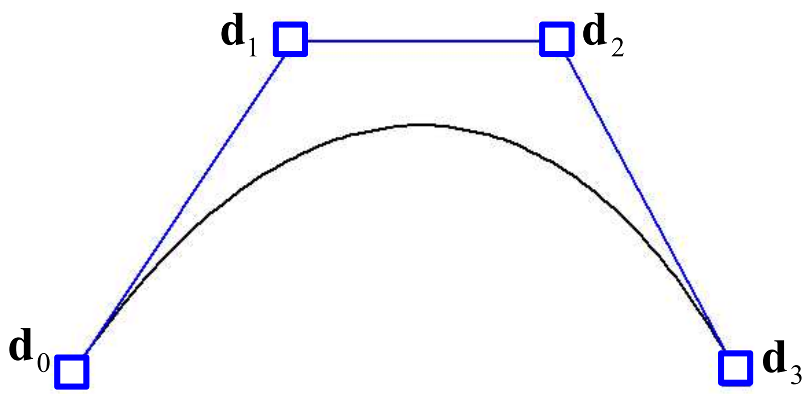
Given: $\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, t$

Find: points on curve at parameter t

$$\mathbf{r}(t) = \mathbf{b}_0 B_0^3(t) + \mathbf{b}_1 B_1^3(t) + \mathbf{b}_2 B_2^3(t) + \mathbf{b}_3 B_3^3(t)$$

Bernstein Polynomial Function

Cubic B-spline Curve



Given: $\mathbf{d_0}, \mathbf{d_1}, \mathbf{d_2}, \mathbf{d_3}, u$

Find: points on curve at parameter t

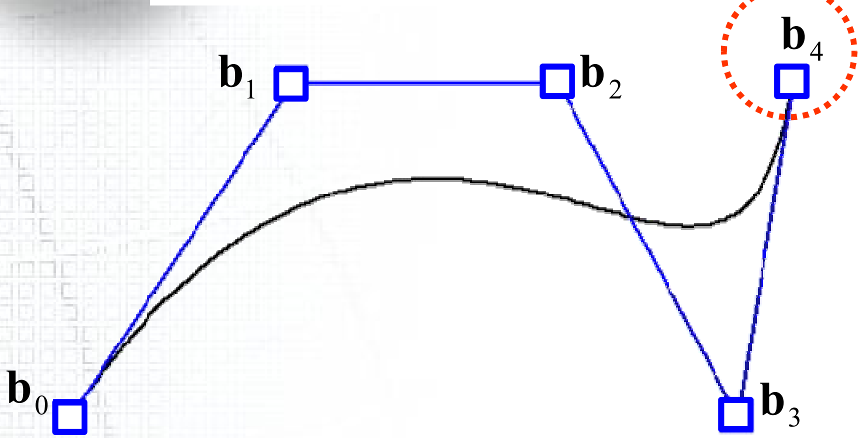
$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_3 N_3^3(u)$$

B-spline Basis Function
(Cox-de Boor Recursive Formula)

Control Point를 하나 더 추가하면 어떻게 될까?

2.3.1.2 Definition of B-spline curves

Bezier Curve of 4 degree



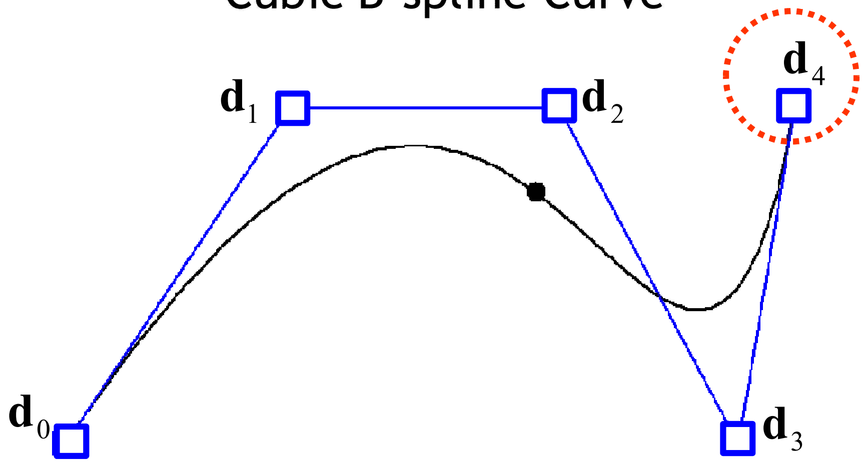
Given: $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, t$

Find: points on curve at parameter t

$$\mathbf{r}(t) = \mathbf{b}_0 B_0^4(t) + \mathbf{b}_1 B_1^4(t) + \mathbf{b}_2 B_2^4(t) + \mathbf{b}_3 B_3^4(t) + \mathbf{b}_4 B_4^4(t)$$

Bezier Curve를 사용할 경우
Control Point의 개수가 늘어나면
Curve의 차수도 늘어남

Cubic B-spline Curve



Given: $\mathbf{d}_0, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4, u$

Find: points on curve at parameter t

$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_3 N_3^3(u) + \mathbf{d}_4 N_4^3(u)$$

B-spline Curve를 사용할 경우
차수는 변경하지 않은 채
곡선 2개가 생성됨

2.3.1.2 Definition of B-spline curves

- Ex): Cubic B-spline curves

•Given: $\mathbf{d}_i$, u_j

•Find: $\mathbf{r}(u)$ (Points on curve at parameter u)

$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \cdots + \mathbf{d}_{D-1} N_{D-1}^3(u)$$

$\mathbf{d}_i$: de Boor points (control points), $i = 0, 1, \ldots, D\text{-}1$

$N_i^n(u)$: B-spline basis function of degree $n(=3)$

u_j : knots, $j = 0, 1, \ldots, K\text{-}1$

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \textbf{if } \ u_{i-1} \le u < u_i \\ 0 & \textbf{else} \end{cases} \qquad , \sum_{i=0}^{D-1} N_i^n(u) = 1$$

Cubic Bezier Curve

Given: $\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, t$

Find

$$\mathbf{r}(t) = \mathbf{b}_0 B_0^3(t) + \mathbf{b}_1 B_1^3(t) + \mathbf{b}_2 B_2^3(t) + \mathbf{b}_3 B_3^3(t)$$

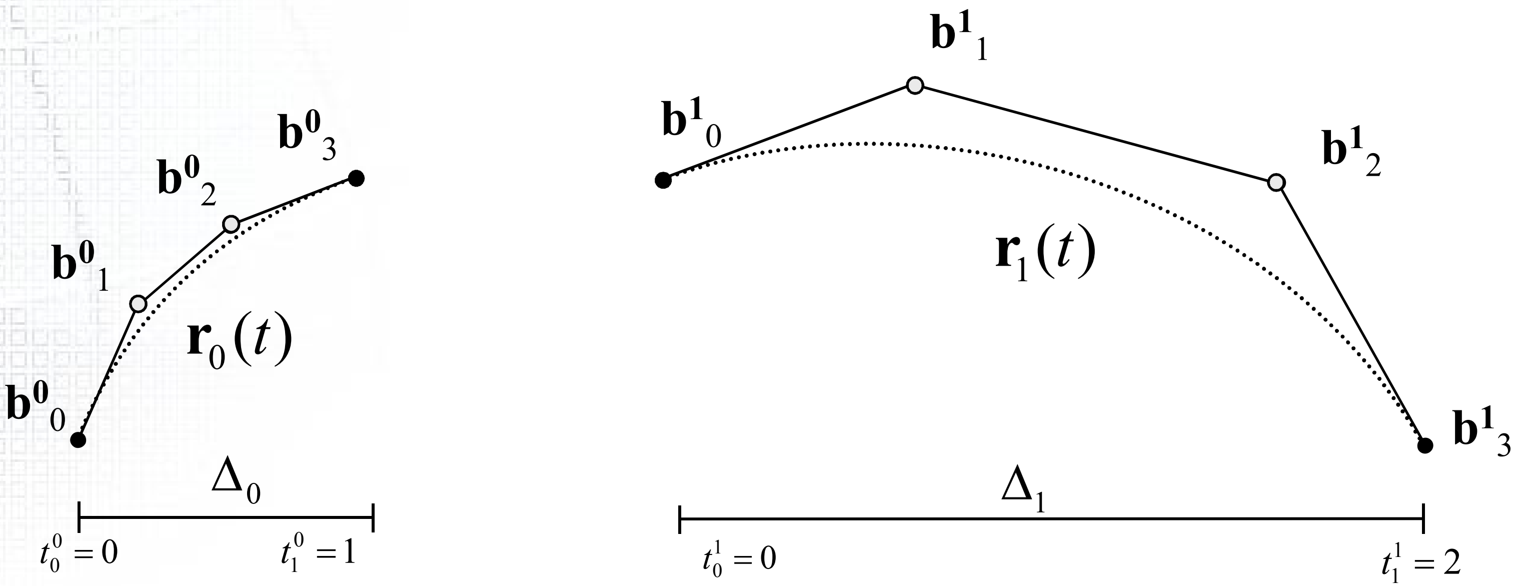
Bernstein polynomial function

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i},$$

$$\binom{n}{i} = {}_nC_i = \begin{cases} \dfrac{n!}{i!(n-i)!} & \textbf{if } 0 \le i \le n \\ \mathbf{0} & \textbf{else} \end{cases}$$

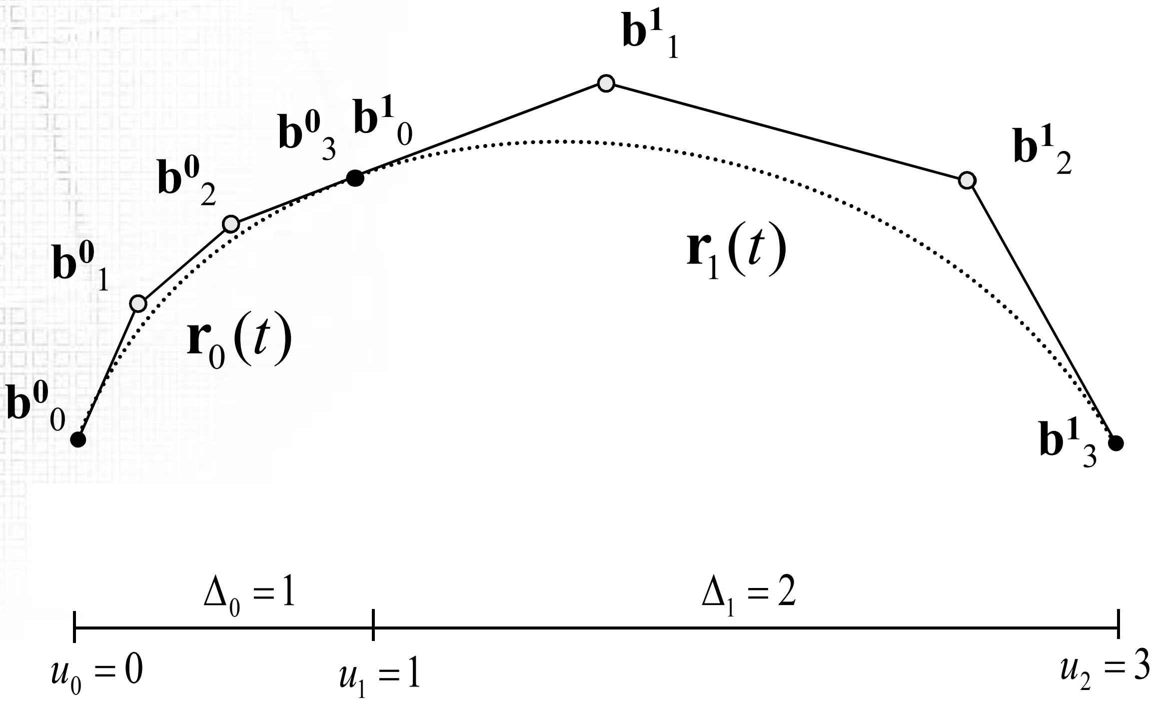
2.3.1.3 Geometric meanings of cubic B-spline curve (1)

- 'Cubic' B-spline curve consist of 'cubic' Bezier curves, which are connected with the C^2 continuity condition



2.3.1.3 Geometric meanings of cubic B-spline curve (1)

- 'Cubic' B-spline curve consist of 'cubic' Bezier curves, which are connected with the C^2 continuity condition



Assign new global parameter u to jointed curve

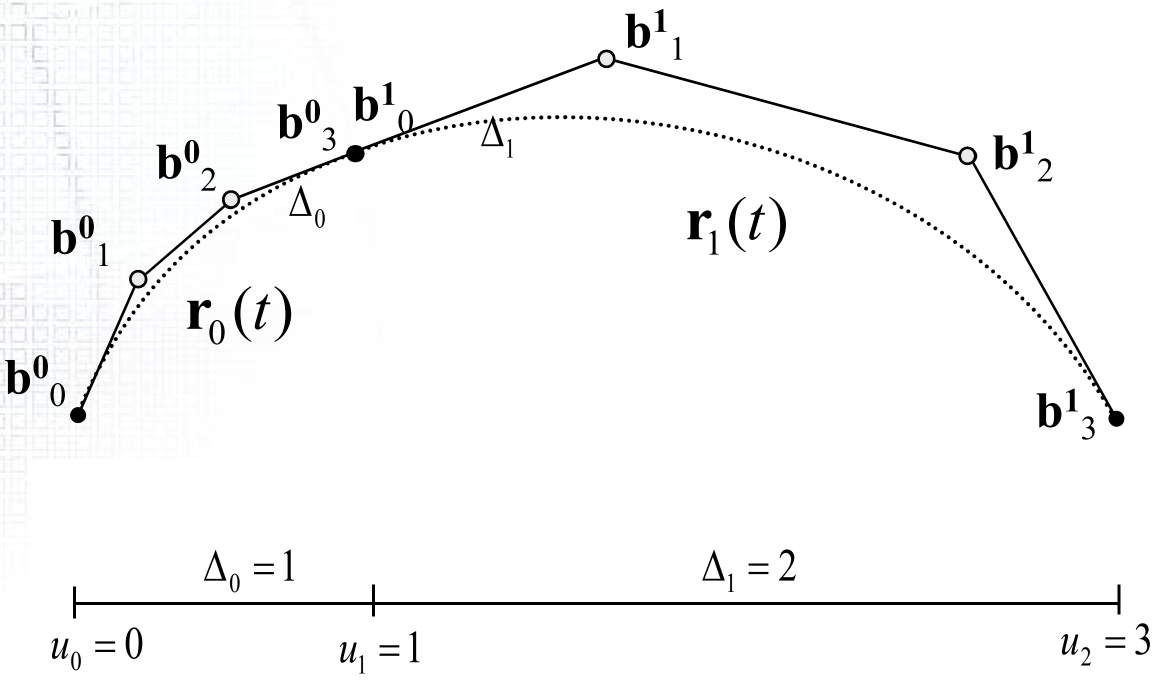
C^0 continuity condition

$$\mathbf{b}_3^0 = \mathbf{b}_0^1$$

2.3.1.3 Geometric meanings of cubic B-spline curve (1)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$

- 'Cubic' B-spline curve consist of 'cubic' Bezier curves, which are connected with the C^2 continuity condition



Assign new global parameter u to jointed curve

C^1 continuity condition

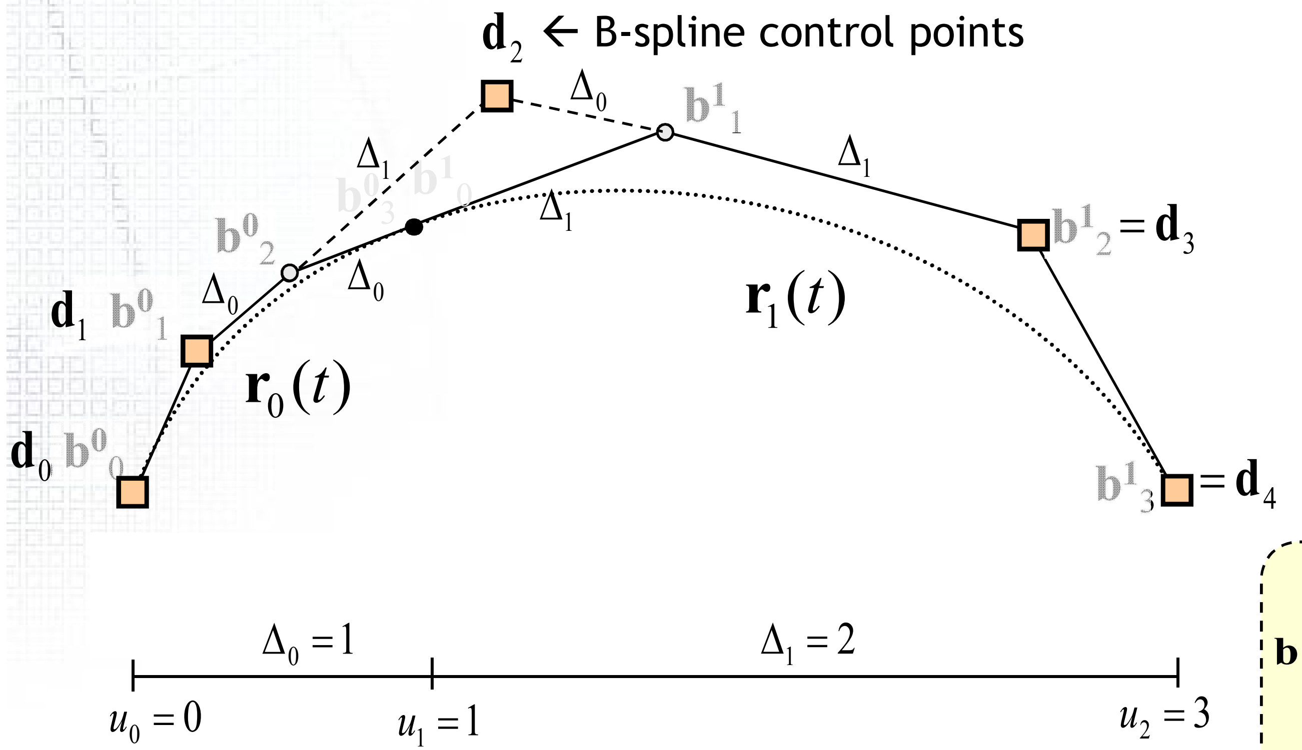
$$\mathbf{b}_3^0 = \mathbf{b}_0^1$$

$$\mathbf{b}_3^0 = \mathbf{b}_0^1 = \frac{\Delta_1}{\Delta_0 + \Delta_1}\mathbf{b}_2^0 + \frac{\Delta_0}{\Delta_0 + \Delta_1}\mathbf{b}_1^1$$

2.3.1.3 Geometric meanings of cubic B-spline curve (1)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$

- 'Cubic' B-spline curve consist of 'cubic' Bezier curves, which are connected with the C^2 continuity condition



Assign new global parameter u to jointed curve

C^2 continuity condition

$$\mathbf{b}_3^0 = \mathbf{b}_0^1$$

$$\mathbf{b}_3^0 = \mathbf{b}_0^1 = \frac{\Delta_1}{\Delta_0 + \Delta_1}\mathbf{b}_2^0 + \frac{\Delta_0}{\Delta_0 + \Delta_1}\mathbf{b}_1^1$$

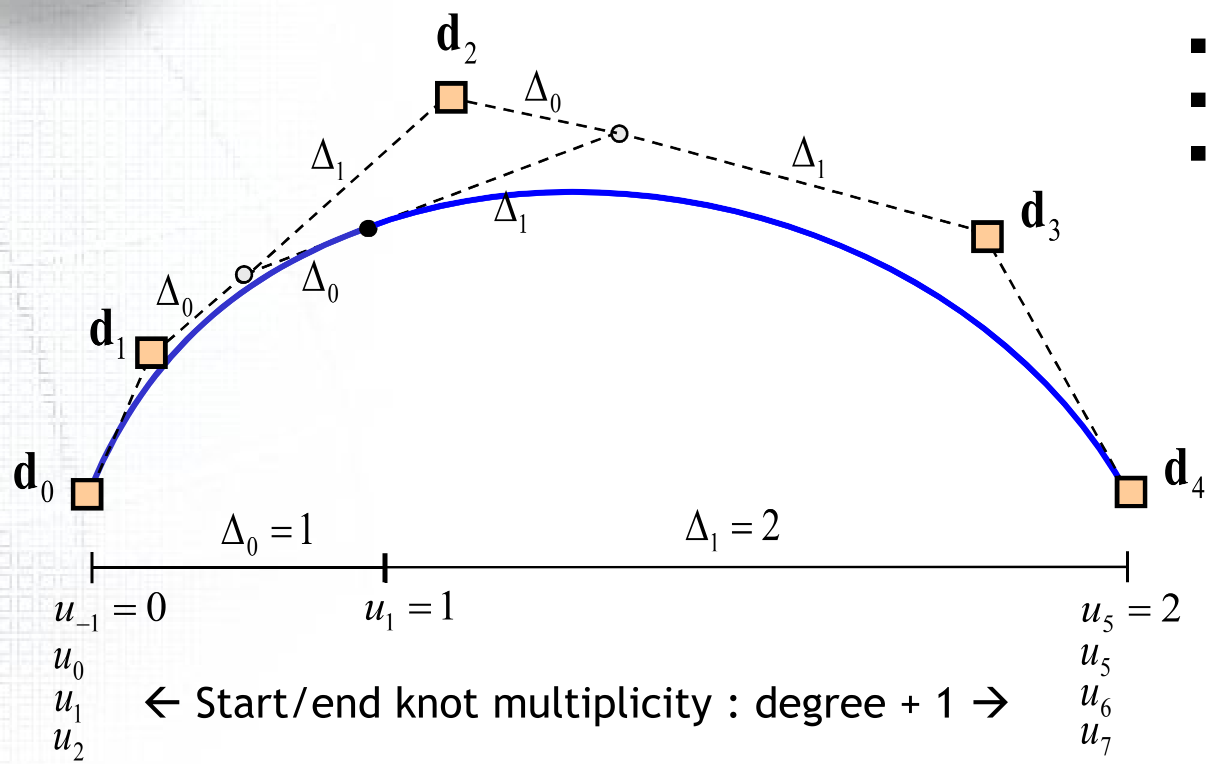
$$\mathbf{b}_2^0 = \{\frac{\Delta_1}{\Delta_0 + \Delta_1}\}\mathbf{d}_1 + \{\frac{\Delta_0}{\Delta_0 + \Delta_1}\}\mathbf{d}_2$$

$$\mathbf{b}_1^1 = \{\frac{\Delta_1}{\Delta_0 + \Delta_1}\}\mathbf{d}_2 + \{\frac{\Delta_0}{\Delta_0 + \Delta_1}\}\mathbf{d}_3$$

2.3.1.3 Geometric meanings of cubic B-spline curve (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$

- n: degree
- S: # of Bezier curve segments
- # of knot = (S-1) + 2(n+1)
- # of control points
  = 4 + (S-1) = (n+1) + (S-1)

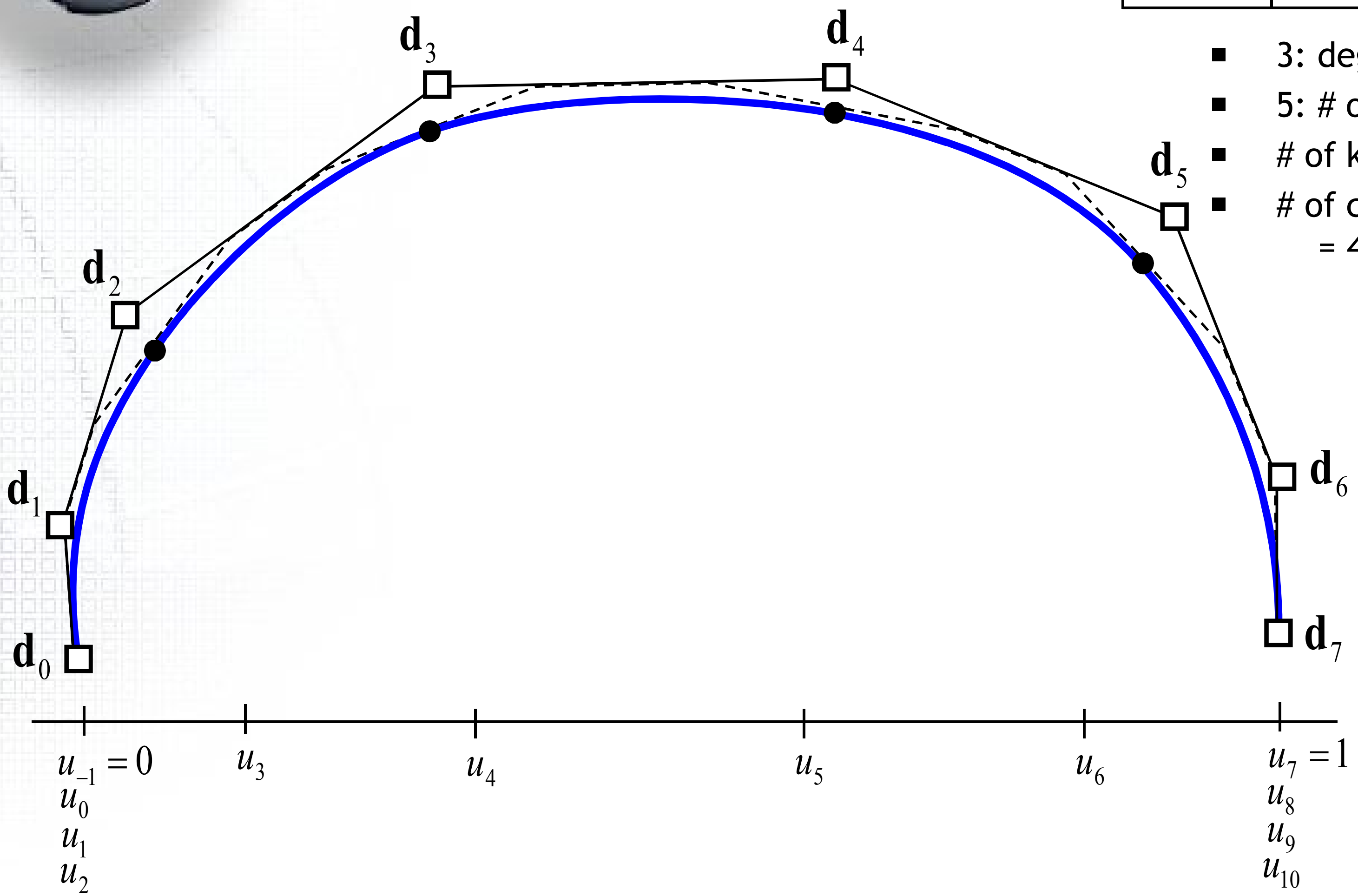


$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_3 N_3^3(u) + \mathbf{d}_4 N_4^3(u)$$

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u) \quad N_i^0(u) = \begin{cases} 1 & \textbf{if} \ \ u_{i-1} \le u < u_i \\ 0 & \textbf{else} \end{cases}, \sum_{i=0}^{D-1} N_i^n(u) = 1$$

2.3.1.3 Geometric meanings of cubic B-spline curve (3)

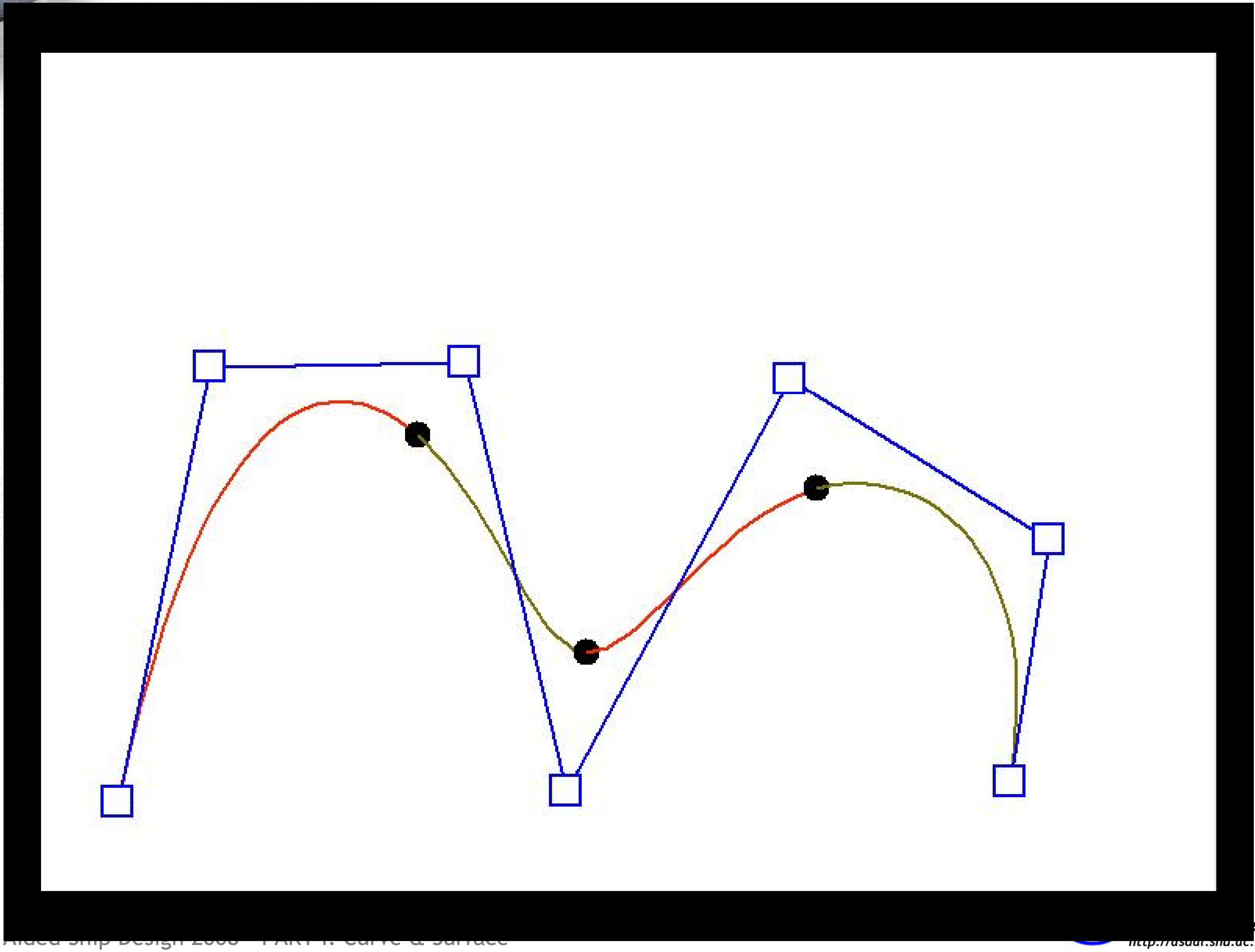
Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$



- 3: degree
- 5: # of Bezier curve segments
- # of knot = (5-1) + 2(3+1)
- # of control points = 4 + (5-1) = (3+1) + (5-1)

$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_3 N_3^3(u) + \mathbf{d}_4 N_4^3(u) + \mathbf{d}_5 N_5^3(u) + \mathbf{d}_6 N_6^3(u) + \mathbf{d}_7 N_7^3(u)$$

Example of B-spline Curve

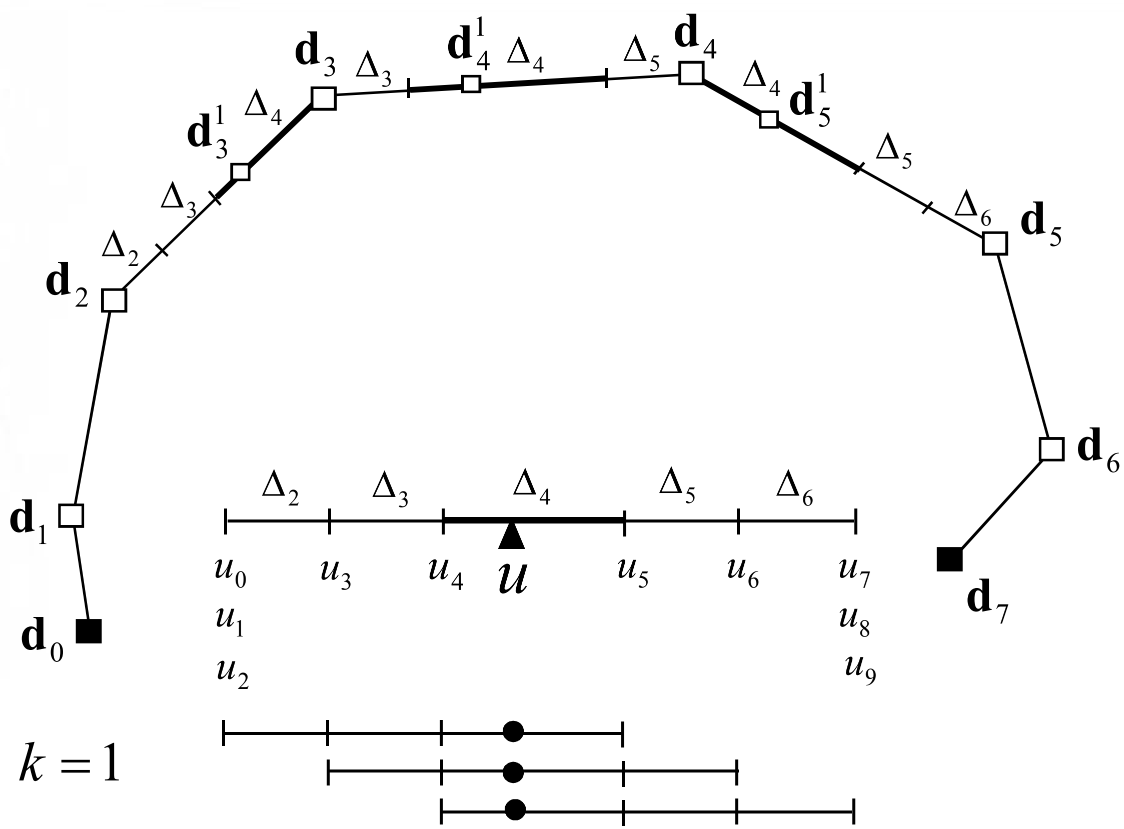


2.3.2 de Boor algorithm

2.3.2.1 de Boor algorithm

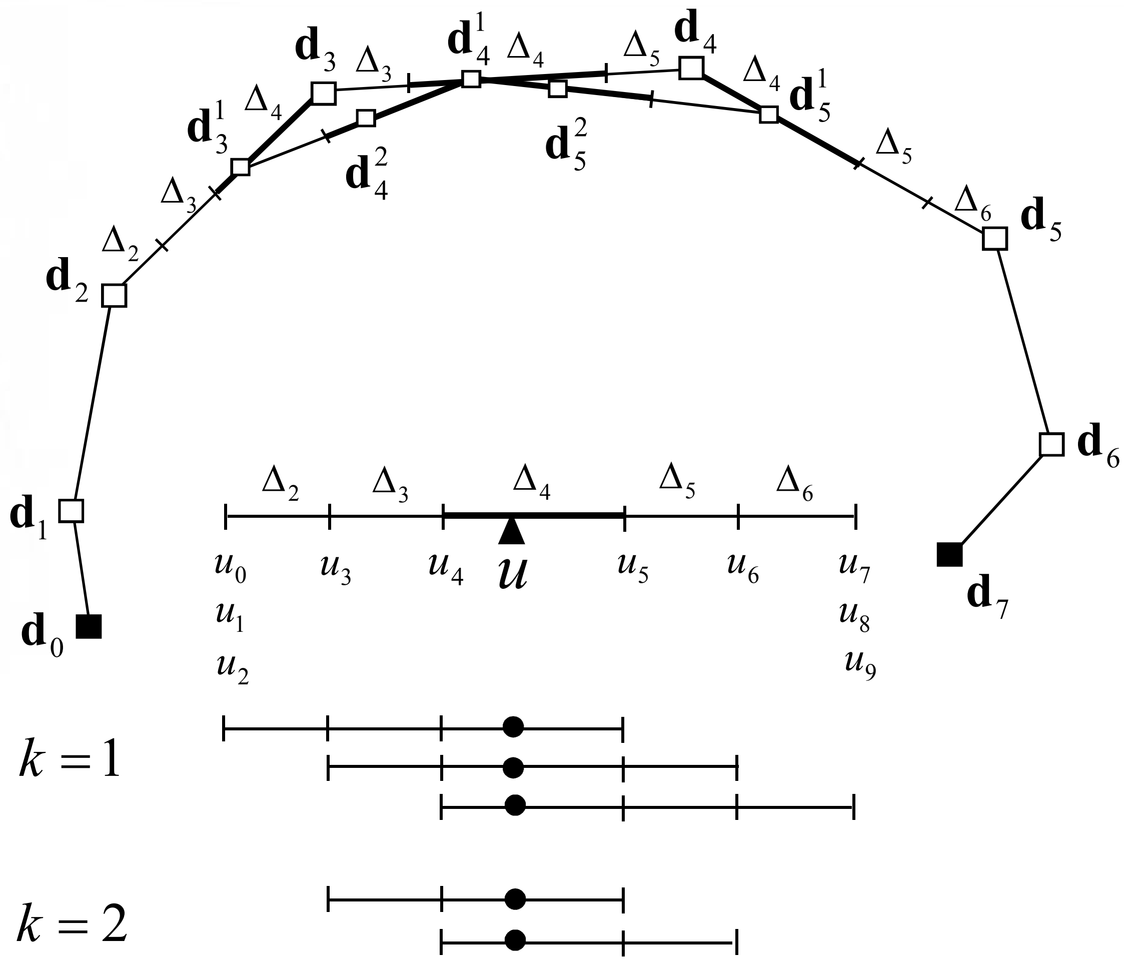
2.3.2.2 Relationship between de Boor algorithm & B-spline curves

2.3.2.1 de Boor Algorithm (1)



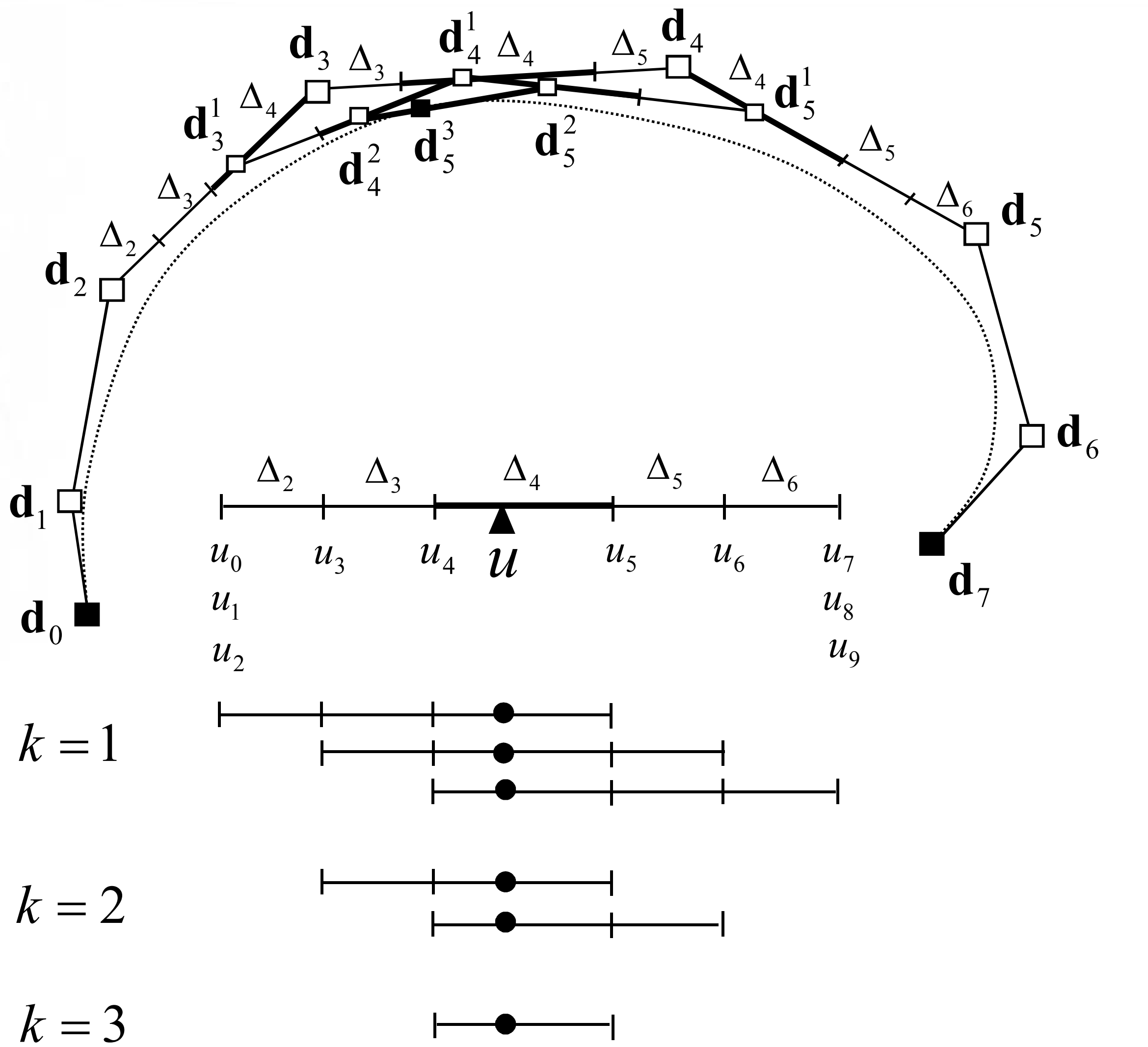
- Linear Interpolation 비율이 t:(1-t)로 일정했던 de Casteljau algorithm에 비하여 de Boor algorithm에서는 Linear Interpolation 비율이 변한다
- 이는 B-spline curve 를 구성하는 Bezier curve segment의 매개변수 간격이 서로 다르기 때문이다

2.3.2.1 de Boor Algorithm (2)



$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \cdots + \mathbf{d}_n N_n^3(u)$$

2.3.2.1 de Boor Algorithm (3)



$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \cdots + \mathbf{d}_n N_n^3(u)$$

2.3.2.2. Relationship between de Boor algorithm & B-spline curves

- de Boor 알고리즘 : "Constructive Approach"

Input: $\mathbf{d}_i$ (de Boor Points)

Processor: 구간별로 $\mathbf{d}_i$를 n번 순차적 'linear interpolation'

Output : n차 곡선상의 점

→ 'B-spline function'(Cox-de Boor recurrence formula) 형태로 표현 됨

$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \cdots + \mathbf{d}_n N_n^3(u)$$

2.3.3 B-spline basis function
(Cox-de Boor recurrence formula)

2.3.3.1 Cox-de Boor recurrence formula

2.3.3.2 B-spline curves

2.3.3.3 Relationship between de Boor algorithm & B-spline curves

2.3.3.4 Sample code of cubic B-spline curves

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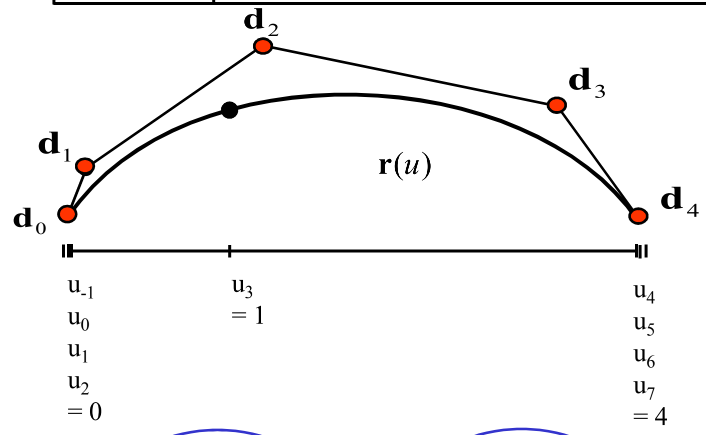
2.3.3.1 Cox-de Boor Recurrence Formula
(B-spline function) (1)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$

- 예: Cubic B-spline 곡선



$$\mathbf{r}(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix} = \sum_{i=0}^{D-1} \mathbf{d}_i N_i^n(u)$$

$$= \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_3 N_3^3(u) + \mathbf{d}_4 N_4^3(u)$$

- Cox-de Boor Recurrence Formula (B-spline function)

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

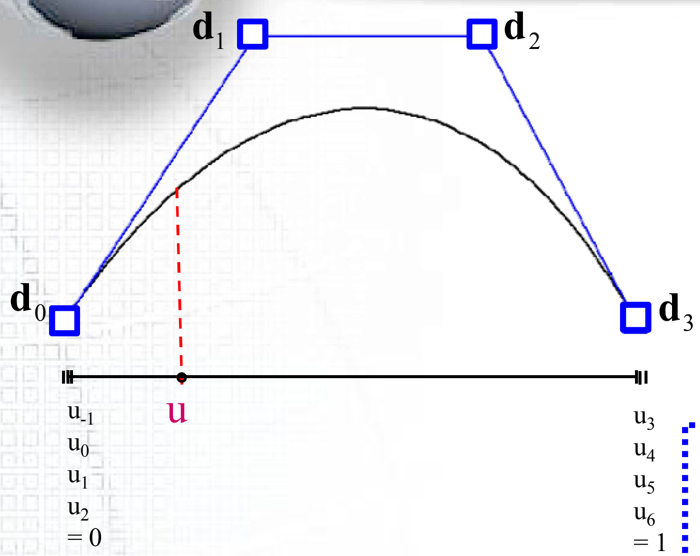
$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

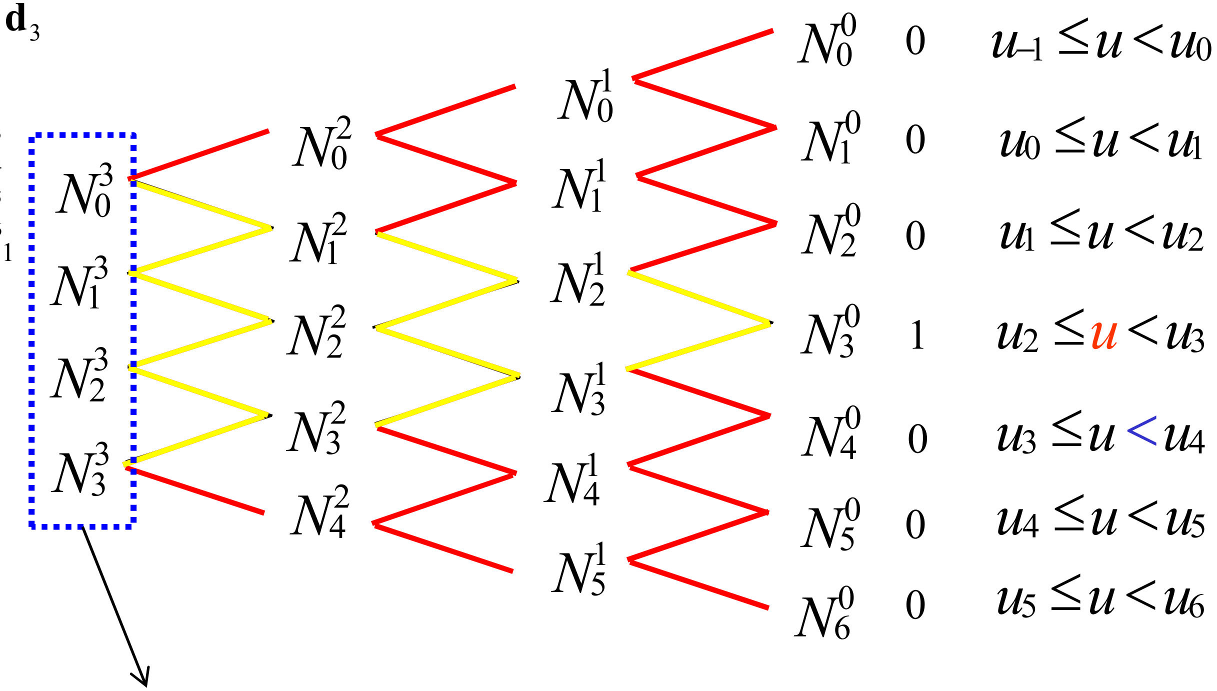
2.3.3.1 Cox-de Boor Recurrence Formula (B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

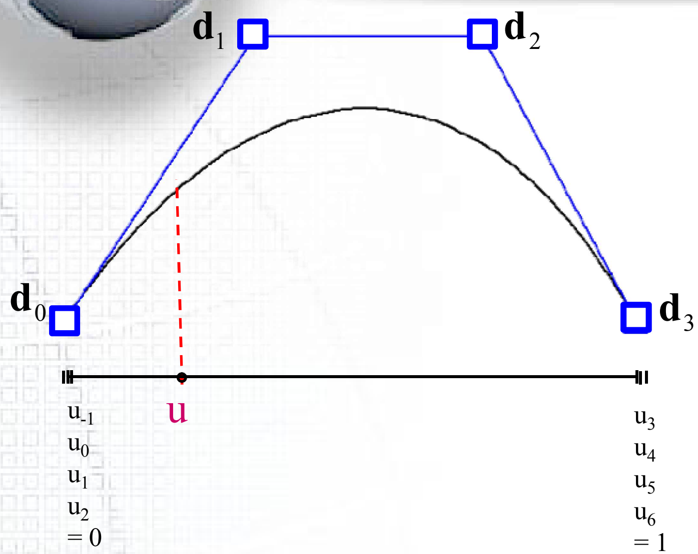




Parameter u에 대한 B-spline 곡선 상의 점을 구하기 위해서 위의 B-spline basis function을 계산해야 한다.

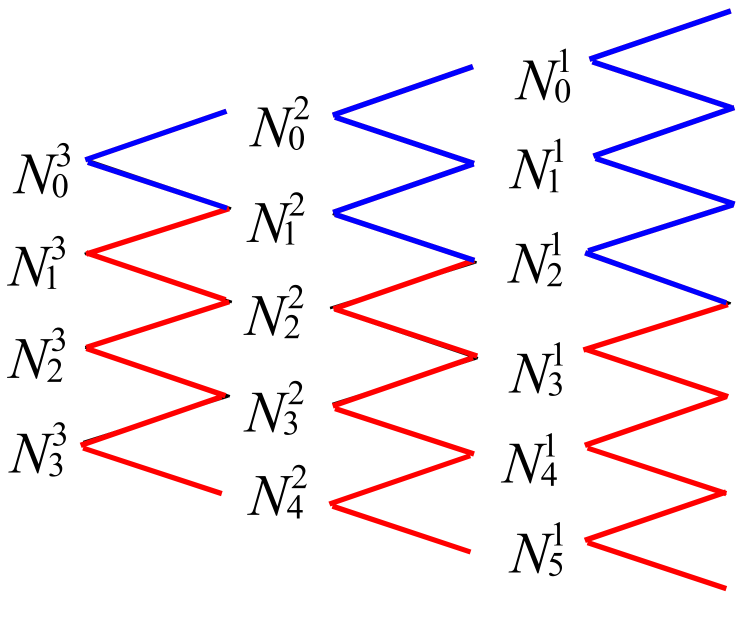
2.3.3.1 Cox-de Boor Recurrence Formula (B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$



$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$



N_0^0	0	$u_{-1} \le u < u_0$
N_1^0	0	$u_0 \le u < u_1$
N_2^0	0	$u_1 \le u < u_2$
N_3^0	1	$u_2 \le u < u_3$
N_4^0	0	$u_3 \le u < u_4$
N_5^0	0	$u_4 \le u < u_5$
N_6^0	0	$u_5 \le u < u_6$

From $u_2 \le u < u_3$,

we can get $N_0^0(u) = 0$

$N_1^0(u) = 0$

$N_2^0(u) = 0$

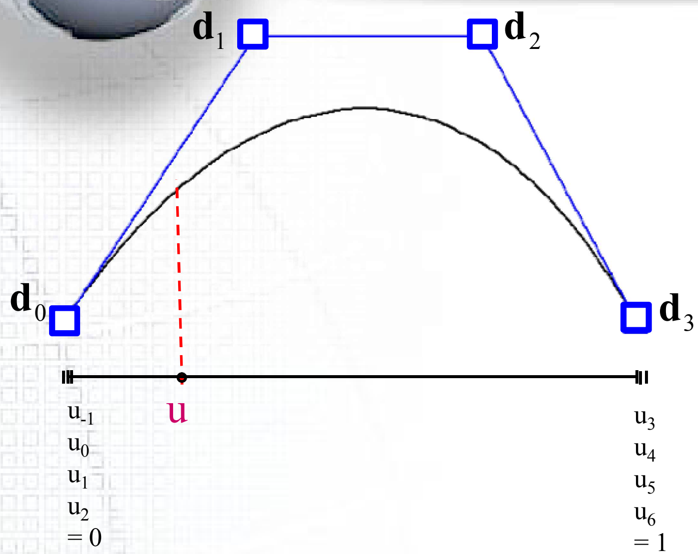
$N_3^0(u) = 1$

2.3.3.1 Cox-de Boor Recurrence Formula
(B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$

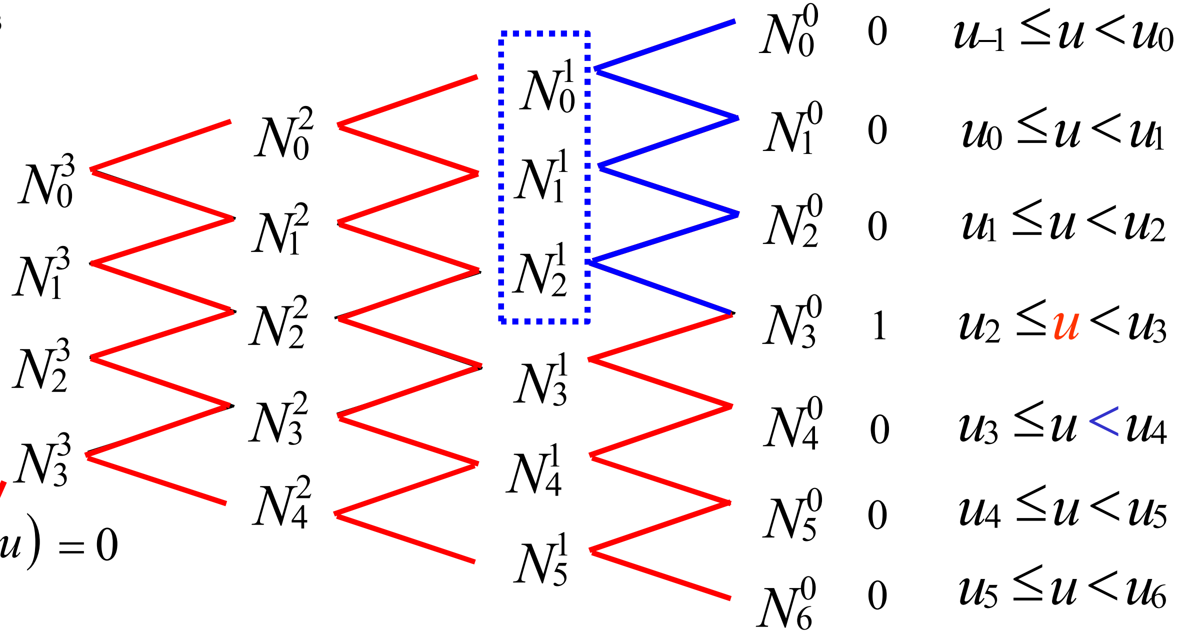
$$N_i^n(u) = \frac{u-u_{i-1}}{u_{i+n-1}-u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n}-u}{u_{i+n}-u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$



$N_0^0(u)=0,\quad N_1^0(u)=0$

$N_2^0(u)=0,\quad N_3^0(u)=1$



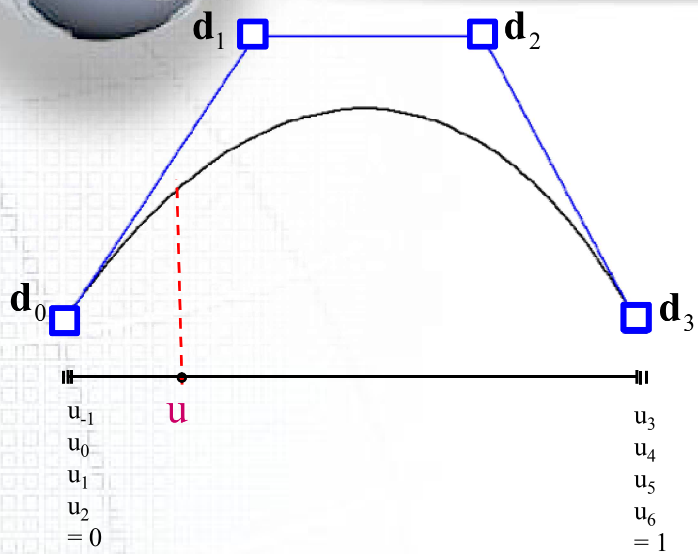
$$N_0^1(u) = \frac{u-u_{-1}}{u_0-u_{-1}} N_0^0(u) + \frac{u_1-u}{u_1-u_0} N_1^0(u) = 0$$

$$N_1^1(u) = \frac{u-u_0}{u_1-u_0} N_1^0(u) + \frac{u_2-u}{u_2-u_1} N_2^0(u) = 0$$

$$N_2^1(u) = \frac{u-u_1}{u_2-u_1} N_2^0(u) + \frac{u_3-u}{u_3-u_2} N_3^0(u) = \frac{u_3-u}{u_3-u_2} = 1-u$$

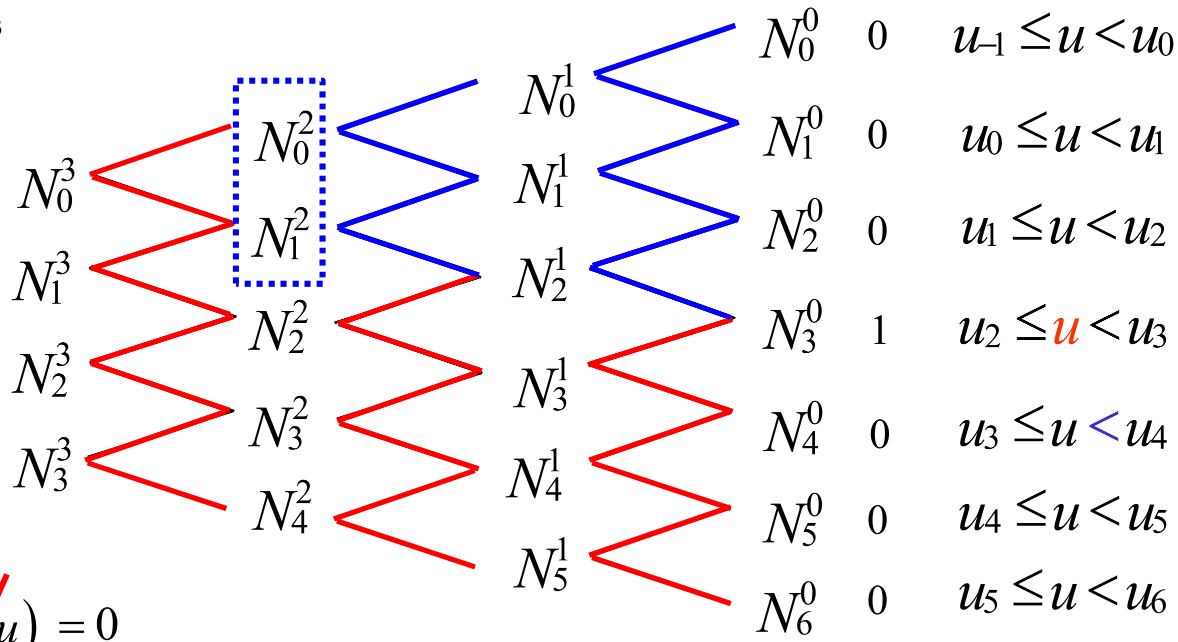
2.3.3.1 Cox-de Boor Recurrence Formula
(B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$



$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$



$N_0^0(u) = 0,\quad N_1^0(u) = 0$

$N_2^0(u) = 0,\quad N_3^0(u) = 1$

$N_0^1(u) = 0,\quad N_1^1(u) = 0,\quad N_2^1(u) = 1 - u$

$$N_0^2(u) = \frac{u - u_{-1}}{u_1 - u_{-1}} N_0^1(u) + \frac{u_2 - u}{u_2 - u_0} N_1^1(u) = 0$$

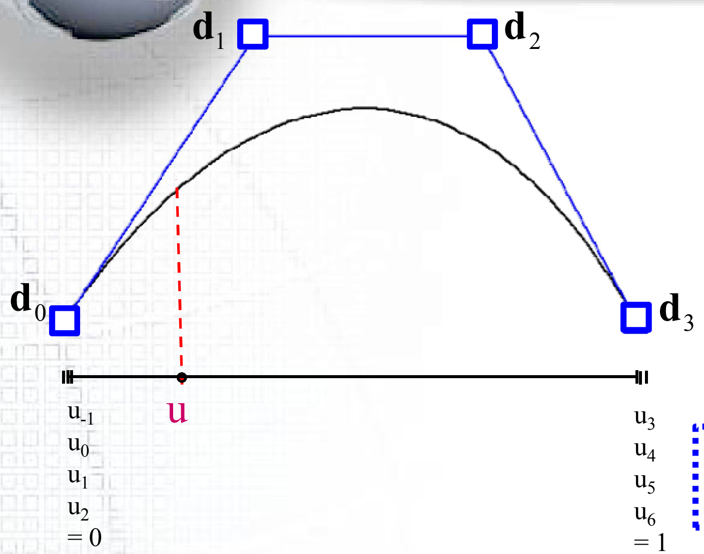
$$N_1^2(u) = \frac{u - u_0}{u_2 - u_0} N_1^1(u) + \frac{u_3 - u}{u_3 - u_1} N_2^1(u) = \frac{u_3 - u}{u_3 - u_1}(1 - u) = (1 - u)^2$$

2.3.3.1 Cox-de Boor Recurrence Formula (B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

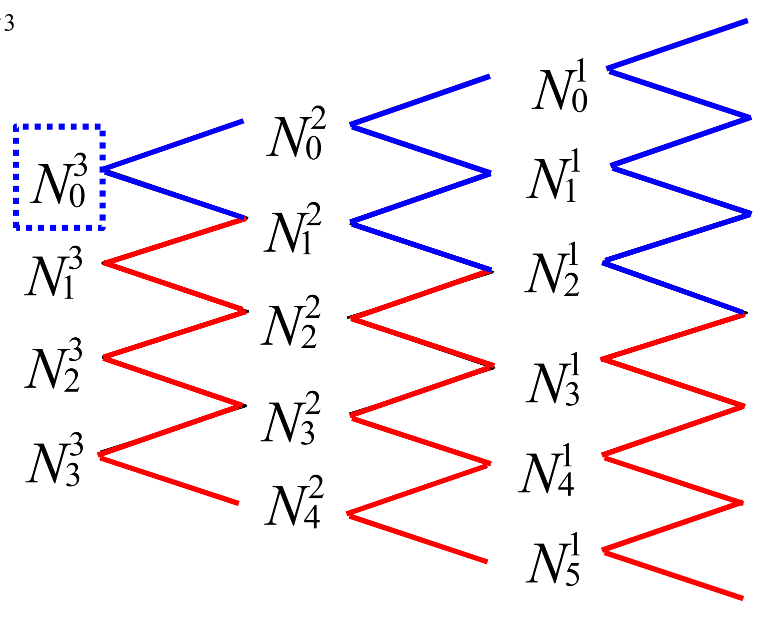


$N_0^0(u) = 0, \quad N_1^0(u) = 0$

$N_2^0(u) = 0, \quad N_3^0(u) = 1$

$N_0^1(u) = 0, \quad N_1^1(u) = 0, \quad N_2^1(u) = 1 - u$

$N_0^2(u) = 0, \; N_1^2(u) = (1-u)^2$



N_0^0	0	$u_{-1} \le u < u_0$
N_1^0	0	$u_0 \le u < u_1$
N_2^0	0	$u_1 \le u < u_2$
N_3^0	1	$u_2 \le u < u_3$
N_4^0	0	$u_3 \le u < u_4$
N_5^0	0	$u_4 \le u < u_5$
N_6^0	0	$u_5 \le u < u_6$

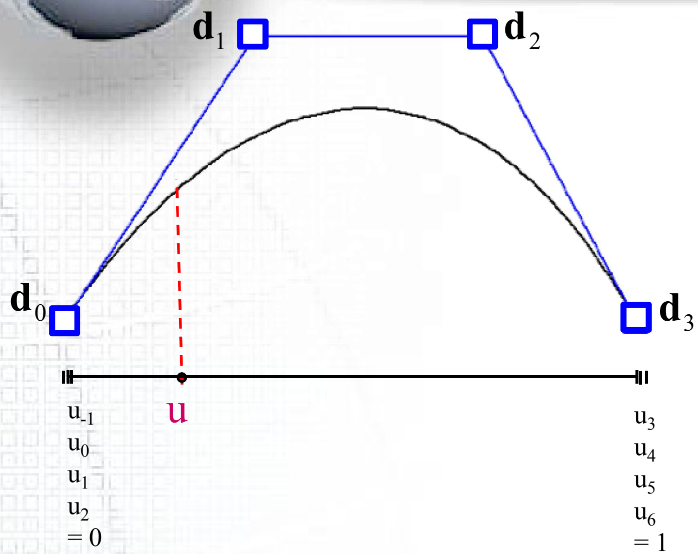
$$N_0^3(u) = \frac{u - u_{-1}}{u_2 - u_{-1}} N_0^2(u) + \frac{u_3 - u}{u_3 - u_0} N_1^2(u) = \frac{u_3 - u}{u_3 - u_0}(1-u)^2 = (1-u)^3$$

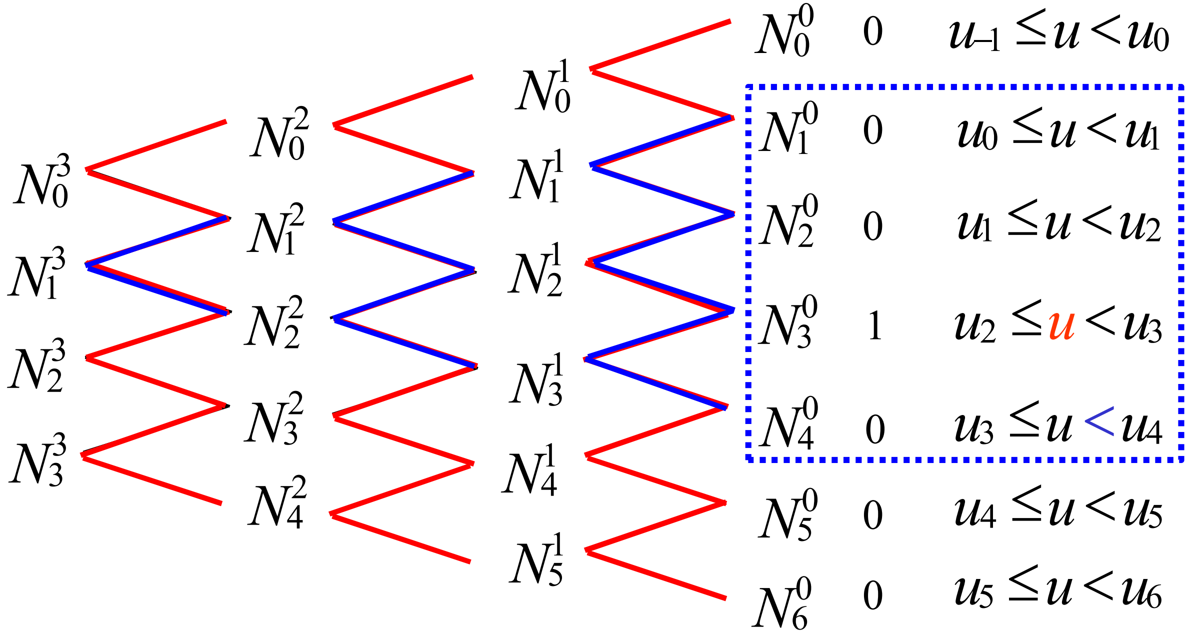
2.3.3.1 Cox-de Boor Recurrence Formula
(B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$





From $u_2 \le u < u_3$,

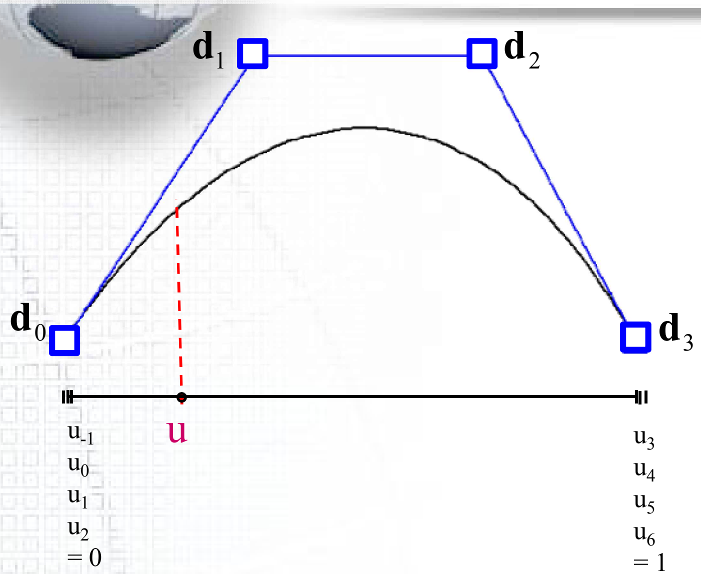
we can get
$$N_1^0(u) = 0$$
$$N_2^0(u) = 0$$
$$N_3^0(u) = 1$$
$$N_4^0(u) = 0$$

2.3.3.1 Cox-de Boor Recurrence Formula (B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$

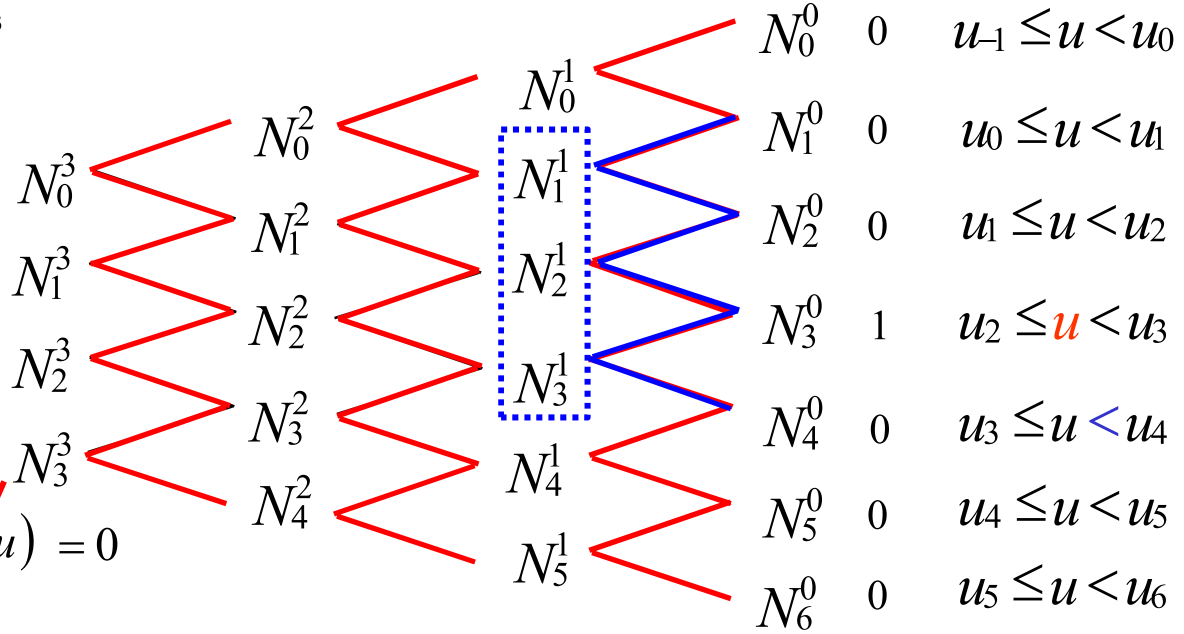
$$N_i^n(u) = \frac{u-u_{i-1}}{u_{i+n-1}-u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n}-u}{u_{i+n}-u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$



$N_1^0(u)=0,\quad N_2^0(u)=0$

$N_3^0(u)=1,\quad N_4^0(u)=0$



$$N_1^1(u) = \frac{u-u_0}{u_1-u_0} N_1^0(u) + \frac{u_2-u}{u_2-u_0} N_2^0(u) = 0$$

$$N_2^1(u) = \frac{u-u_1}{u_2-u_1} N_2^0(u) + \frac{u_3-u}{u_3-u_2} N_3^0(u) = \frac{u_3-u}{u_3-u_2} = 1-u$$

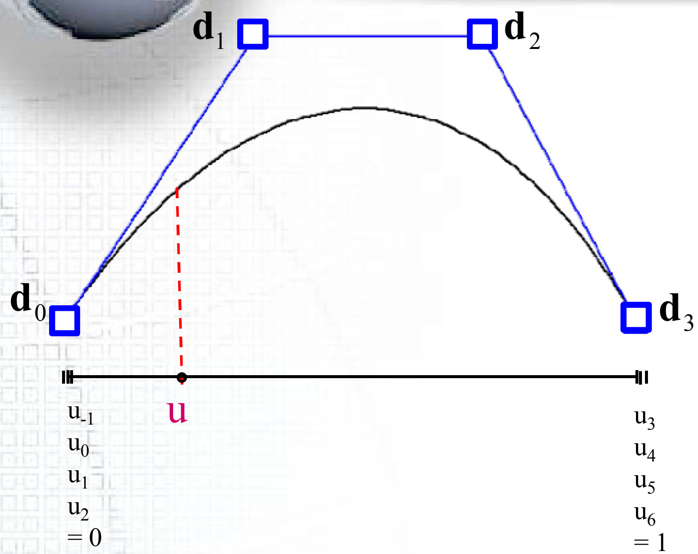
$$N_3^1(u) = \frac{u-u_1}{u_3-u_2} N_3^0(u) + \frac{u_4-u}{u_4-u_3} N_4^0(u) = \frac{u-u_1}{u_3-u_2} = u$$

2.3.3.1 Cox-de Boor Recurrence Formula
(B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

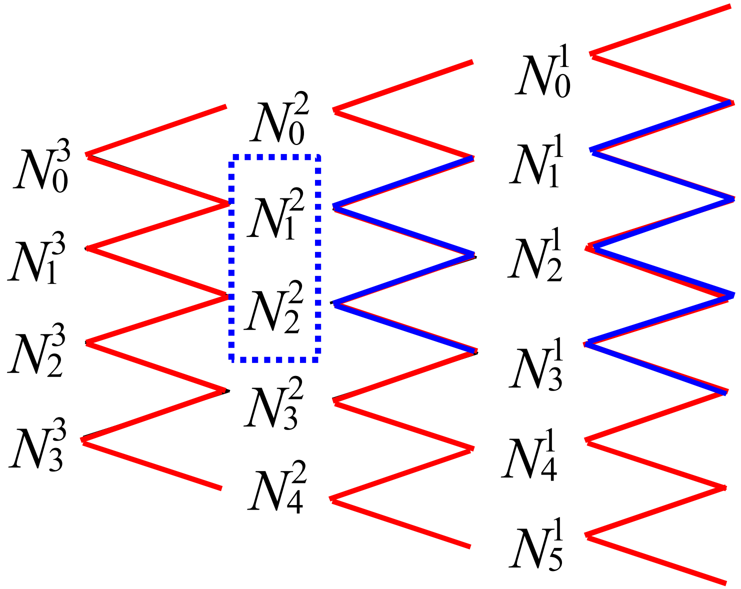
$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$



$N_1^0(u) = 0, \quad N_2^0(u) = 0$

$N_3^0(u) = 1, \quad N_4^0(u) = 0$

$N_1^1(u) = 0, \quad N_2^1(u) = 1 - u, \quad N_2^1(u) = u$



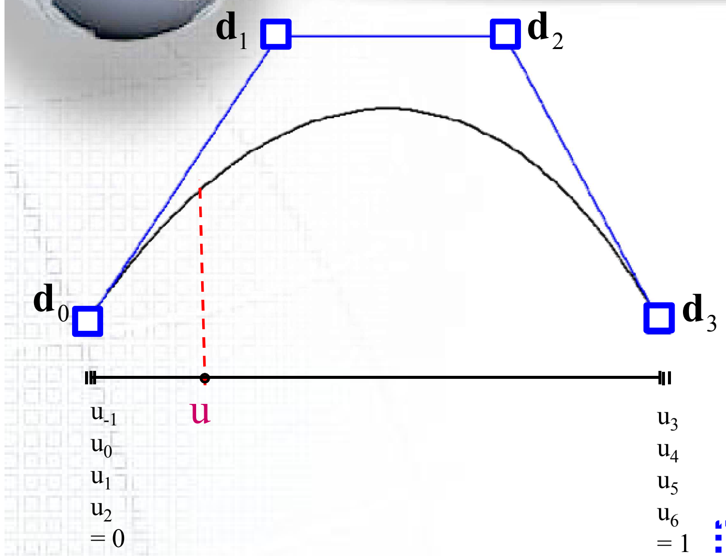
N_0^0	0	$u_{-1} \le u < u_0$
N_1^0	0	$u_0 \le u < u_1$
N_2^0	0	$u_1 \le u < u_2$
N_3^0	1	$u_2 \le u < u_3$
N_4^0	0	$u_3 \le u < u_4$
N_5^0	0	$u_4 \le u < u_5$
N_6^0	0	$u_5 \le u < u_6$

$$N_1^2(u) = \frac{u - u_0}{u_2 - u_0} N_1^1(u) + \frac{u_3 - u}{u_3 - u_1} N_2^1(u) = \frac{u_3 - u}{u_3 - u_1}(1 - u) = (1 - u)^2$$

$$N_2^2(u) = \frac{u - u_1}{u_3 - u_1} N_2^1(u) + \frac{u_4 - u}{u_4 - u_2} N_3^1(u) = \frac{u - u_1}{u_3 - u_1}(1 - u) + \frac{u_4 - u}{u_4 - u_2} u = 2u(1 - u)$$

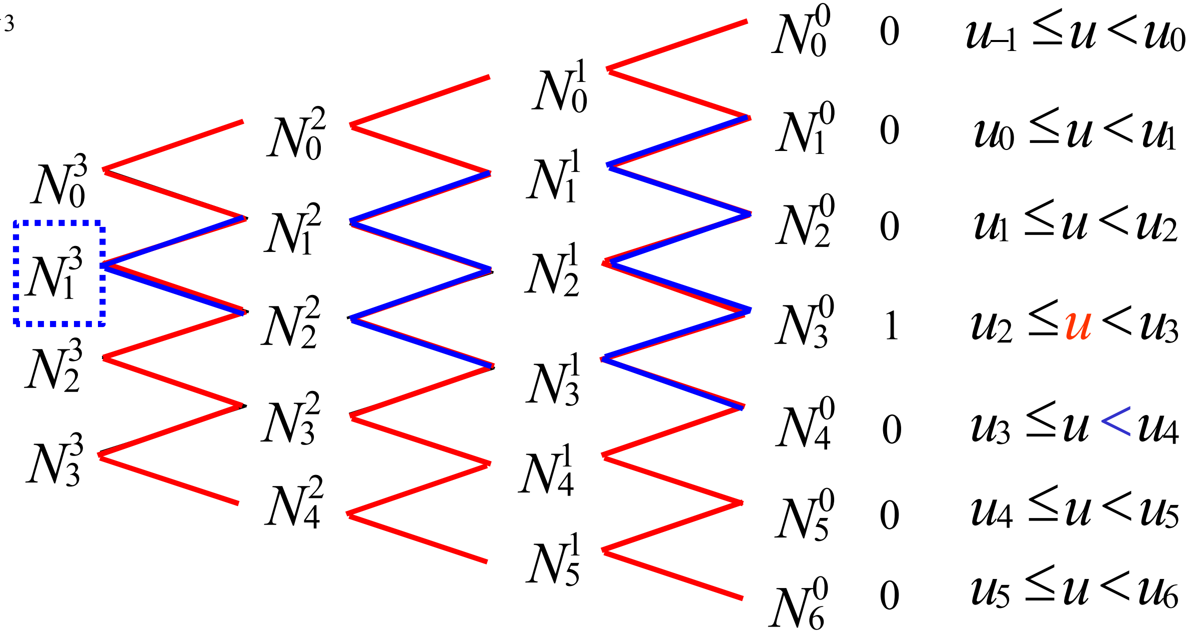
2.3.3.1 Cox-de Boor Recurrence Formula
(B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$



$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$



$$N_1^0(u) = 0, \quad N_2^0(u) = 0$$

$$N_3^0(u) = 1, \quad N_4^0(u) = 0$$

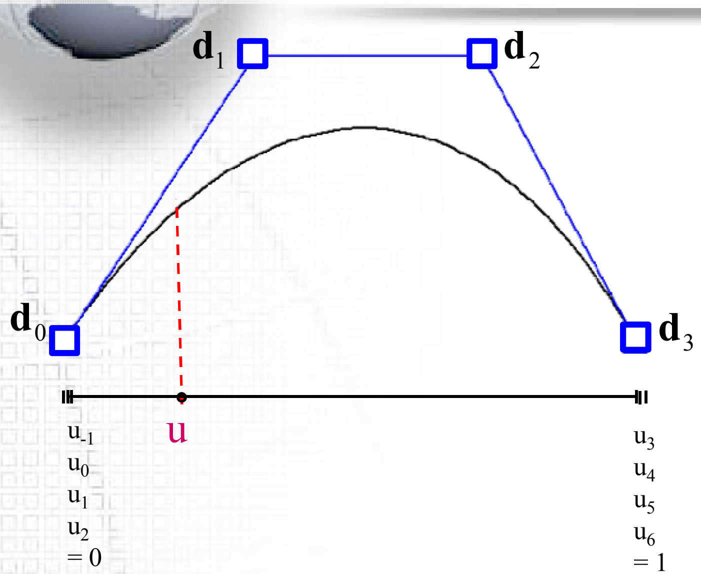
$$N_0^1(u) = 0, \quad N_1^1(u) = 0, \quad N_2^1(u) = 1 - u$$

$$N_1^2(u) = (1-u)^2, \quad N_2^2(u) = 2u(1-u)$$

$$N_1^3(u) = \frac{u - u_0}{u_3 - u_0} N_1^2(u) + \frac{u_4 - u}{u_4 - u_1} N_2^2(u) = \frac{u - u_0}{u_3 - u_0}(1-u)^2 + \frac{u_4 - u}{u_4 - u_1} \cdot 2u(1-u) = 3u(1-u)^2$$

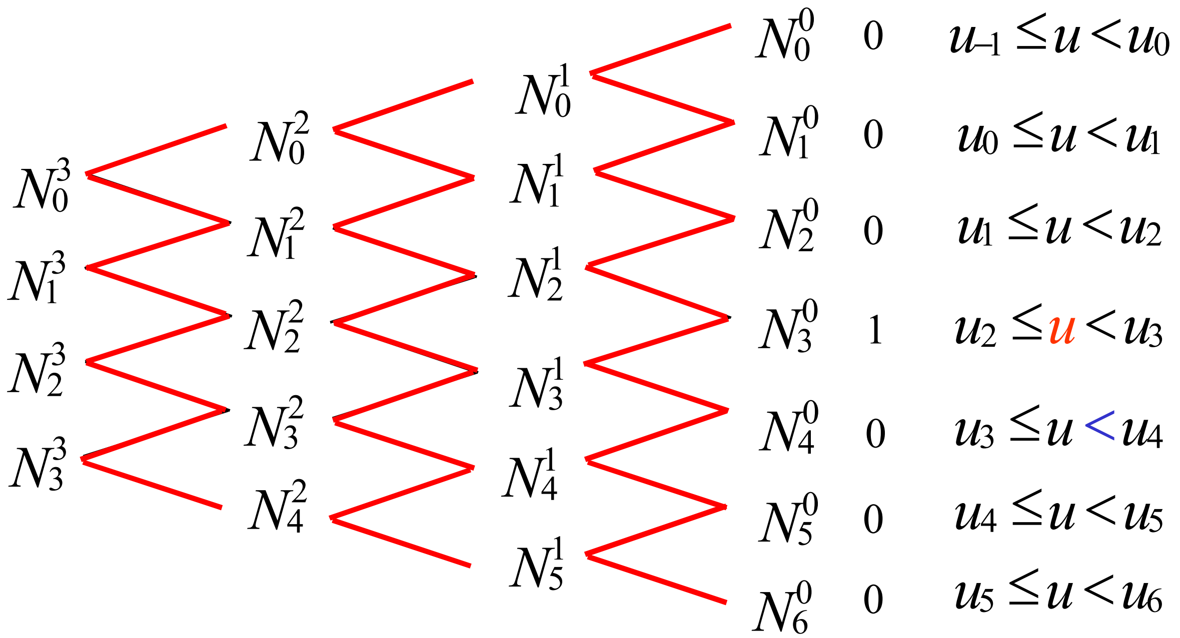
2.3.3.1 Cox-de Boor Recurrence Formula (B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$



$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$



$N_2^3(u)$, $N_2^3(u)$ 도 동일하게 계산

$$N_0^3(u) = (1-u)^3$$

$$N_1^3(u) = 3u(1-u)^2$$

$$N_2^3(u) = 3u^2(1-u)$$

$$N_3^3(u) = 3u^3$$

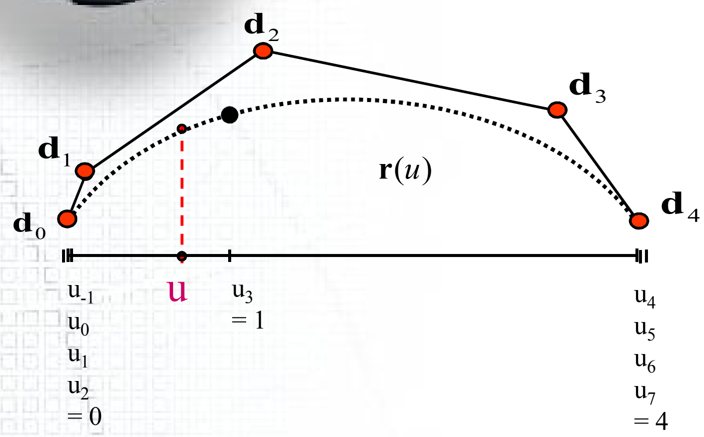
$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(t) + \mathbf{d}_2 N_2^3(t) + \mathbf{d}_3 N_3^3(t)$$

$$\mathbf{r}(u) = (1-u)^3 \mathbf{d}_0 + 3u(1-u)^2 \mathbf{d}_1 + 3u^2(1-u)\mathbf{d}_2 + u^3 \mathbf{d}_3$$

→3차 Bezier Curve 곡선식과 동일

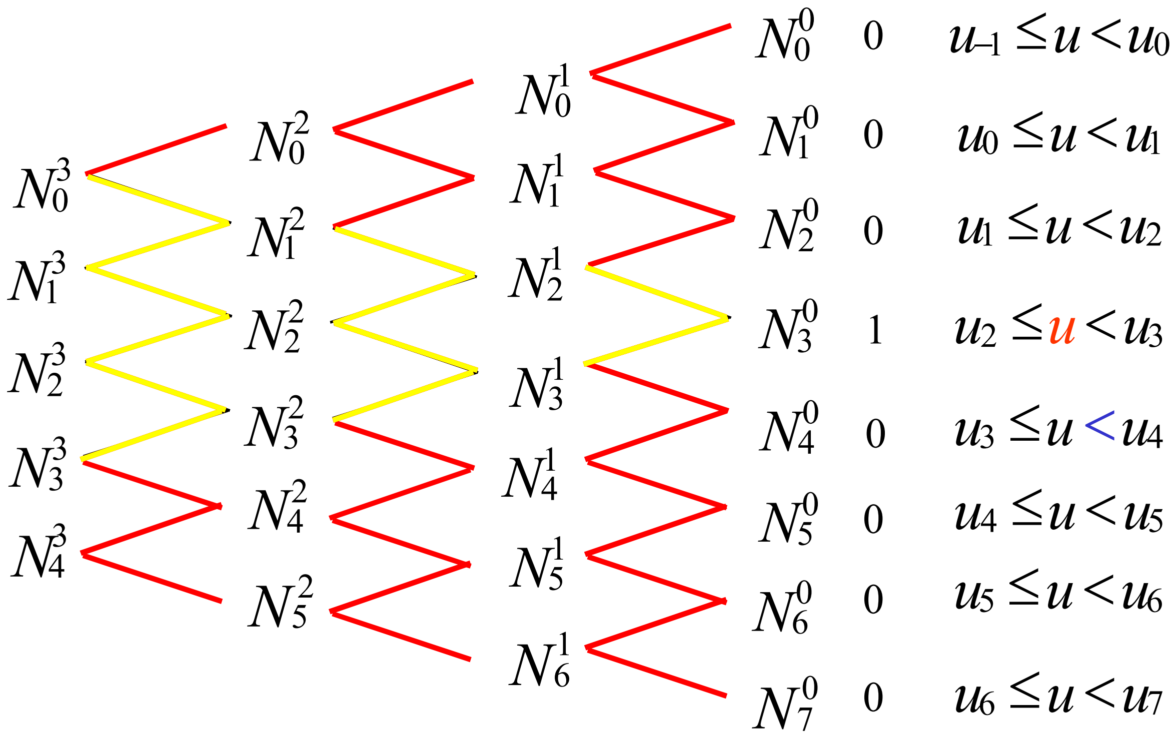
2.3.3.1 Cox-de Boor Recurrence Formula
(B-spline function) (2)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$



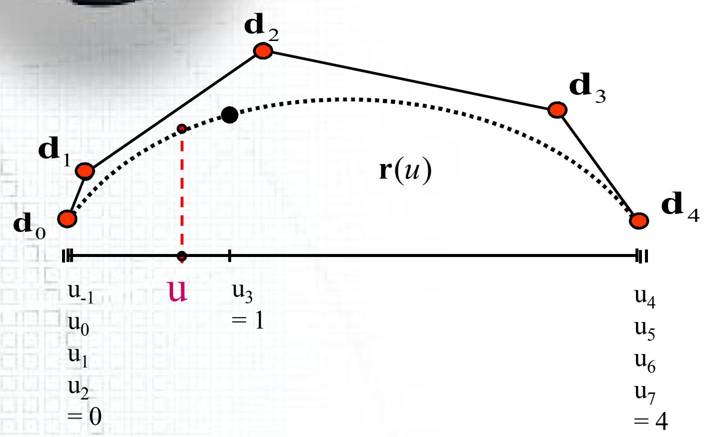
$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$



2.3.3.1 Cox-de Boor Recurrence Formula
(B-spline function) (2)

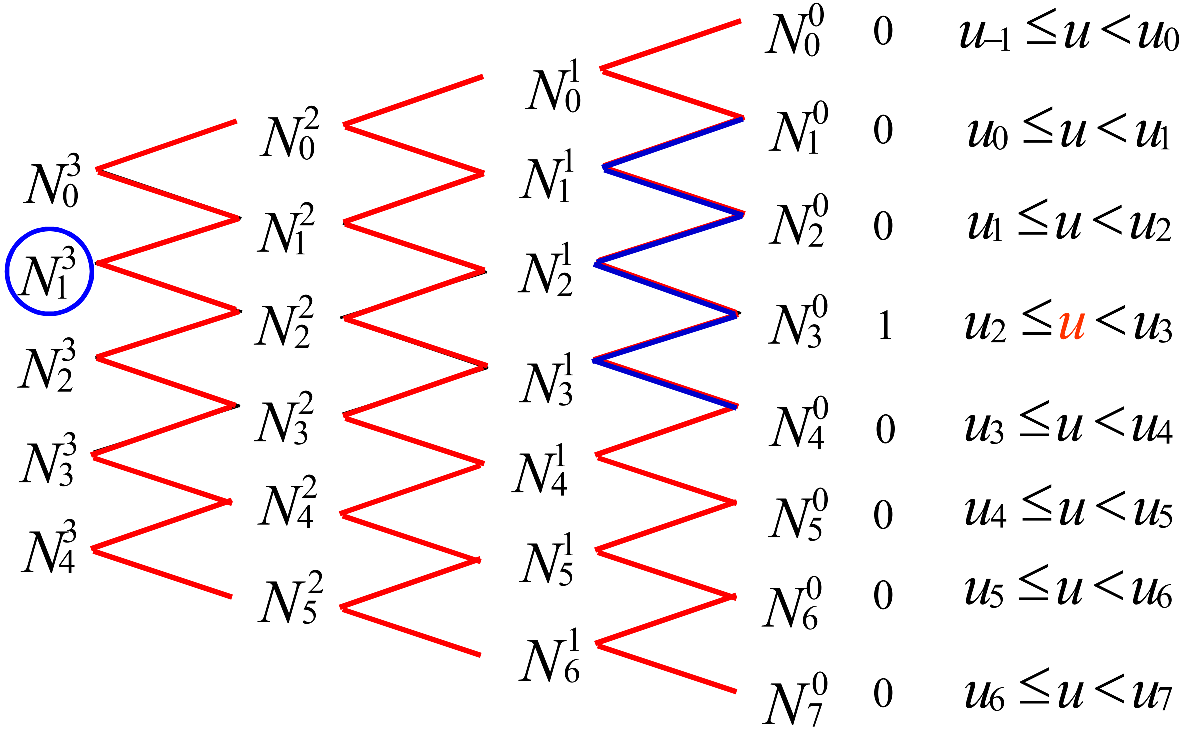
Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$



$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

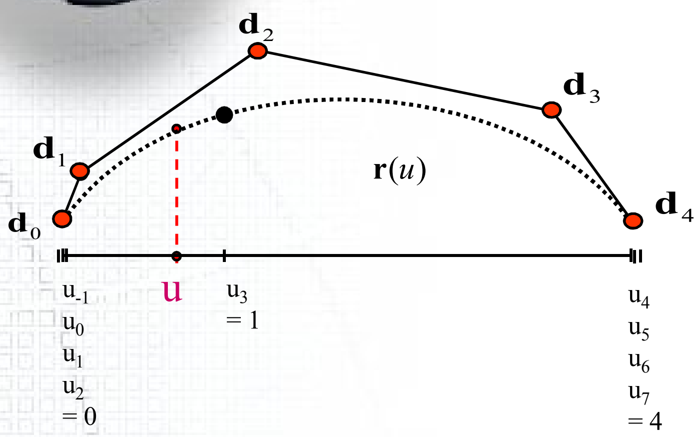
$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

$N_1^1 = 0 \quad N_2^1 = \dfrac{u_2 - u}{u_3 - u_2} \quad N_3^1 = \dfrac{u - u_2}{u_3 - u_2}$



2.3.3.1 Cox-de Boor Recurrence Formula
(B-spline function) (3)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$



$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

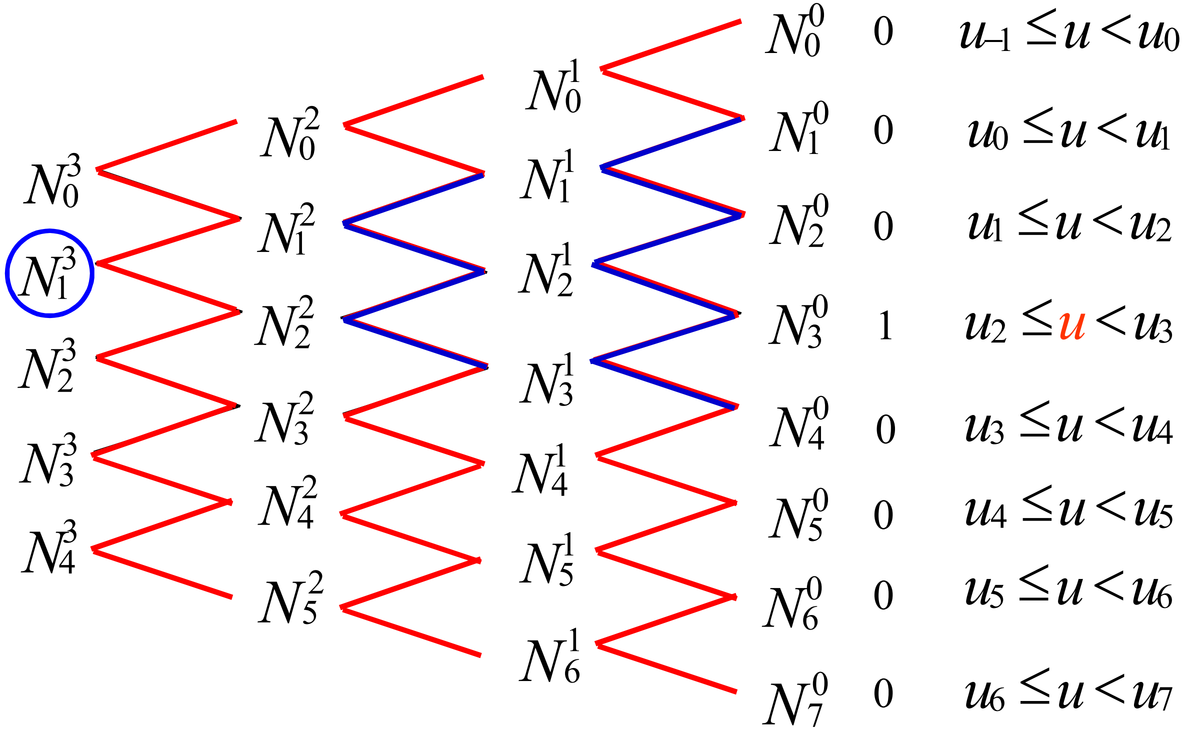
$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

$$N_1^1 = 0 \qquad N_2^1 = \frac{u_2 - u}{u_3 - u_2} \qquad N_3^1 = \frac{u - u_2}{u_3 - u_2}$$

$$N_1^2 = \frac{u_3 - u}{u_3 - u_1} N_2^1 = \frac{u_3 - u}{u_3 - u_1} \cdot \frac{u_2 - u}{u_3 - u_2}$$

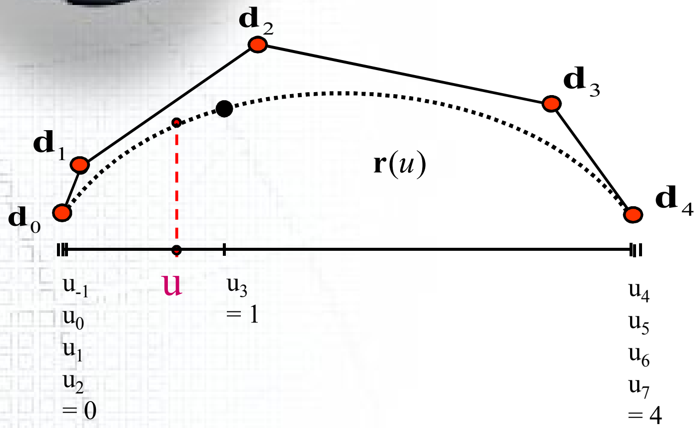
$$N_2^2 = \frac{u - u_1}{u_3 - u_1} N_2^1 + \frac{u_4 - u}{u_4 - u_2} N_3^1$$

$$= \frac{u - u_1}{u_3 - u_1} \cdot \frac{u_2 - u}{u_3 - u_2} + \frac{u_4 - u}{u_4 - u_2} \cdot \frac{u - u_2}{u_3 - u_2}$$



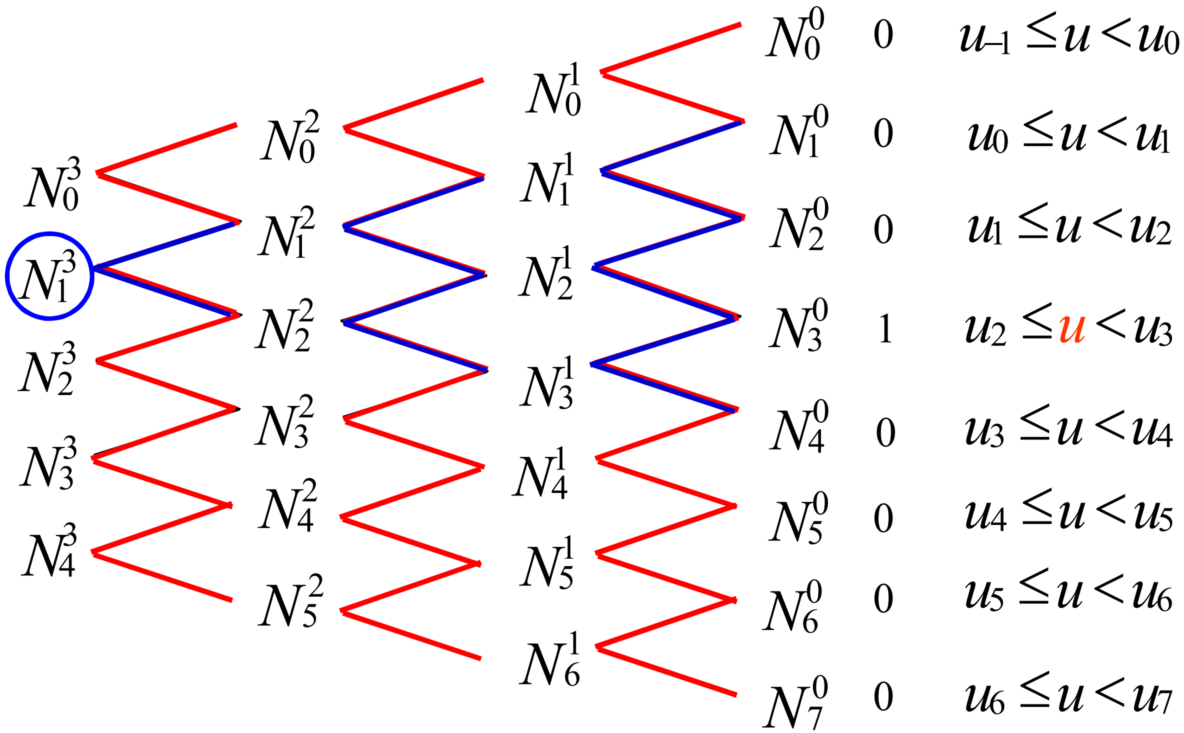
2.3.3.1 Cox-de Boor Recurrence Formula
(B-spline function) (4)

Given	B-spline Control Point $\mathbf{d}_i$ Parameter u B-spline Basis Func. $N_i^n(u)$
Find	B-spline Curve $\mathbf{r}(u)$



$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$



$$N_1^1 = 0 \qquad N_2^1 = \frac{u_2 - u}{u_3 - u_2} \qquad N_3^1 = \frac{u - u_2}{u_3 - u_2}$$

$$N_1^2 = \frac{u_3 - u}{u_3 - u_1} N_2^1 = \frac{u_3 - u}{u_3 - u_1} \cdot \frac{u_2 - u}{u_3 - u_2}$$

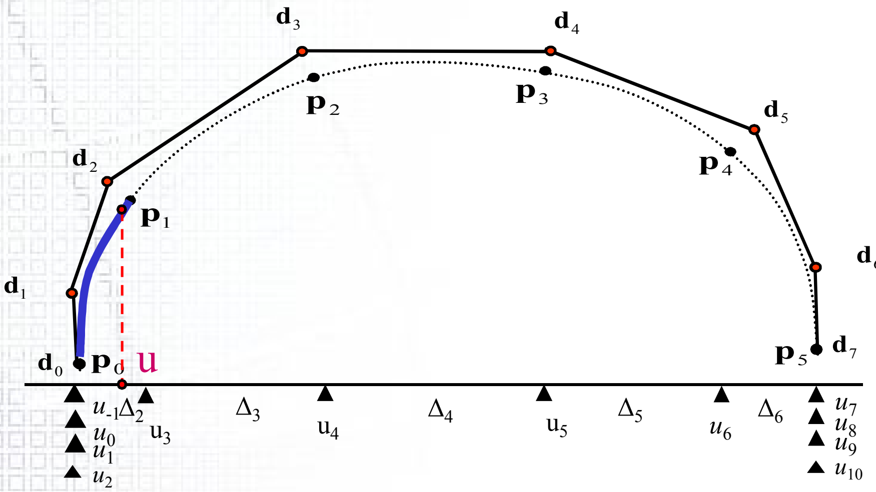
$$N_2^2 = \frac{u - u_1}{u_3 - u_1} N_2^1 + \frac{u_4 - u}{u_4 - u_2} N_3^1$$

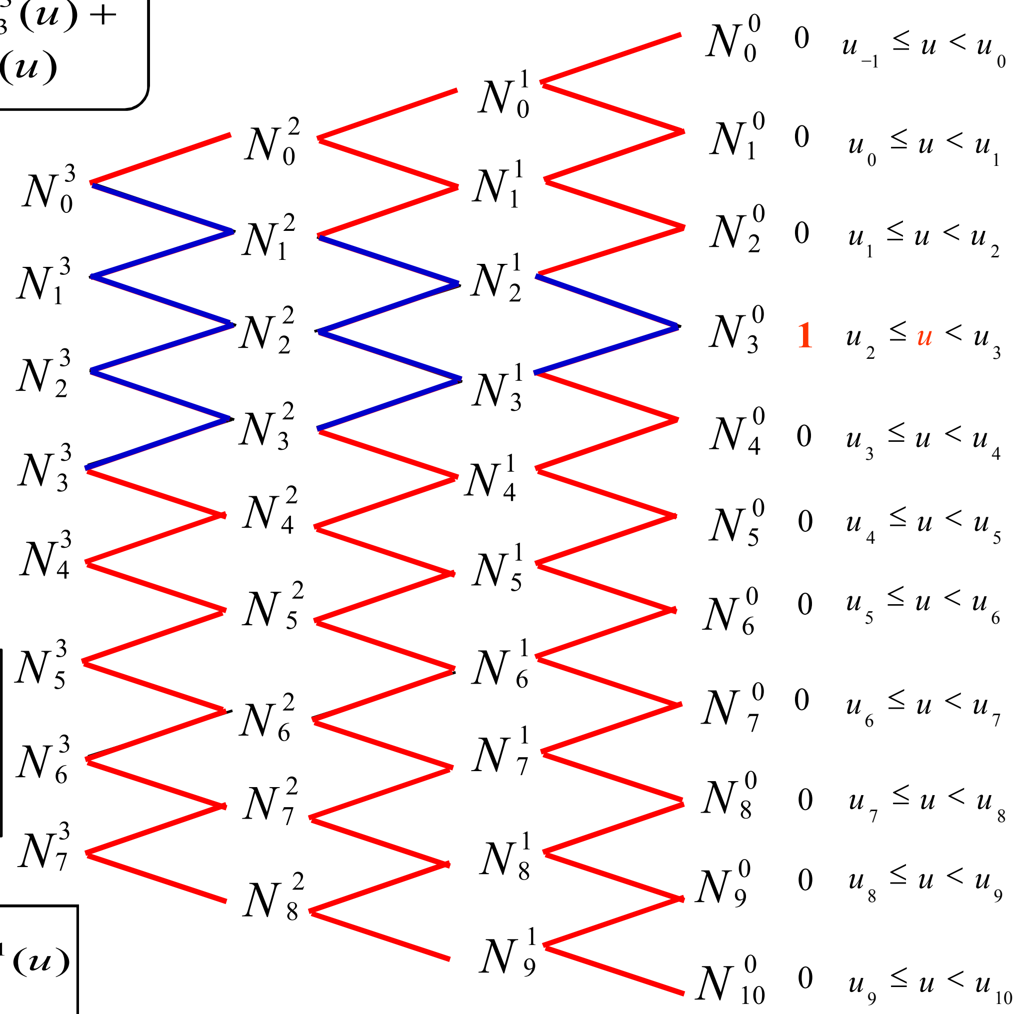
$$= \frac{u - u_1}{u_3 - u_1} \cdot \frac{u_2 - u}{u_3 - u_2} + \frac{u_4 - u}{u_4 - u_2} \cdot \frac{u - u_2}{u_3 - u_2}$$

$$N_1^3 = \frac{u - u_0}{u_3 - u_2} N_1^2 + \frac{u_4 - u}{u_4 - u_1} N_2^2 = \frac{u - u_0}{u_3 - u_2} \cdot \frac{u_3 - u}{u_3 - u_1} \cdot \frac{u_2 - u}{u_3 - u_2} + \frac{u_4 - u}{u_4 - u_1} \cdot \frac{u - u_1}{u_3 - u_1} \cdot \frac{u_2 - u}{u_3 - u_2} + \frac{u_4 - u}{u_4 - u_1} \cdot \frac{u_4 - u}{u_4 - u_2} \cdot \frac{u - u_2}{u_3 - u_2}$$

2.3.3.2 B-spline curves (1)

$$\mathbf{r}(u)=\mathbf{d}_0N_0^3(u)+\mathbf{d}_1N_1^3(u)+\mathbf{d}_2N_2^3(u)+\mathbf{d}_3N_3^3(u)+\mathbf{d}_4N_4^3(u)+\mathbf{d}_5N_5^3(u)+\mathbf{d}_6N_6^3(u)+\mathbf{d}_7N_7^3(u)$$





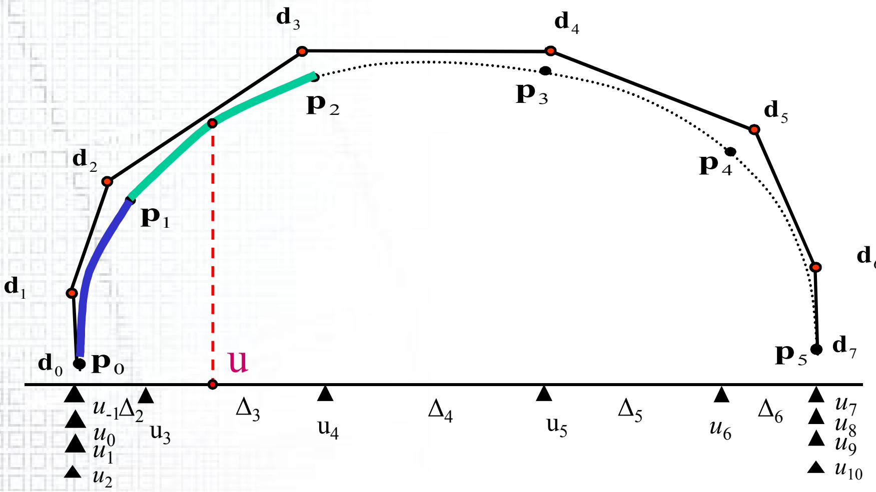
$$\mathbf{r}(u)=\mathbf{d}_0N_0^3(u)+\mathbf{d}_1N_1^3(u)+\mathbf{d}_2N_2^3(u)+\mathbf{d}_4N_4^3(u)$$
$$+\mathbf{d}_4\cancel{N_4^3}(u)+\mathbf{d}_5\cancel{N_5^3}(u)+\mathbf{d}_6\cancel{N_6^3}(u)+\mathbf{d}_7\cancel{N_7^3}(u)$$

$$N_i^n(u)=\frac{u-u_{i-1}}{u_{i+n-1}-u_{i-1}}N_i^{n-1}(u)+\frac{u_{i+n}-u}{u_{i+n}-u_i}N_{i+1}^{n-1}(u)$$

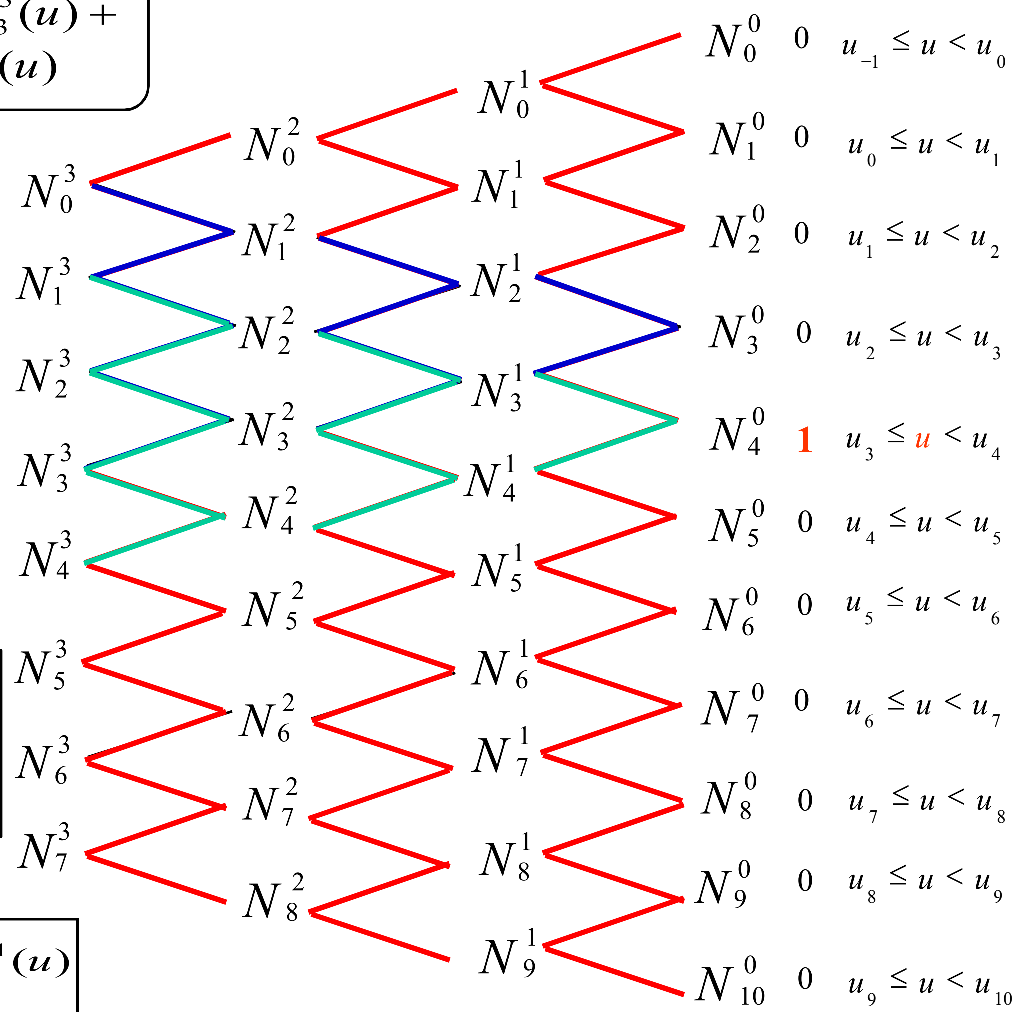
$$N_i^0(u)=\begin{cases}1 & \text{if } u_{i-1}\le u<u_i\\ 0 & \text{else}\end{cases}$$

2.3.3.2 B-spline curves (2)

$$\mathbf{r}(u)=\mathbf{d}_0N_0^3(u)+\mathbf{d}_1N_1^3(u)+\mathbf{d}_2N_2^3(u)+\mathbf{d}_3N_3^3(u)+\mathbf{d}_4N_4^3(u)+\mathbf{d}_5N_5^3(u)+\mathbf{d}_6N_6^3(u)+\mathbf{d}_7N_7^3(u)$$



$$\mathbf{r}(u)=\mathbf{d}_0\cancel{N_0^3}(u)+\mathbf{d}_1N_1^3(u)+\mathbf{d}_2N_2^3(u)+\mathbf{d}_4N_4^3(u)+\mathbf{d}_4N_4^3(u)+\mathbf{d}_5\cancel{N_5^3}(u)+\mathbf{d}_6\cancel{N_6^3}(u)+\mathbf{d}_7\cancel{N_7^3}(u)$$

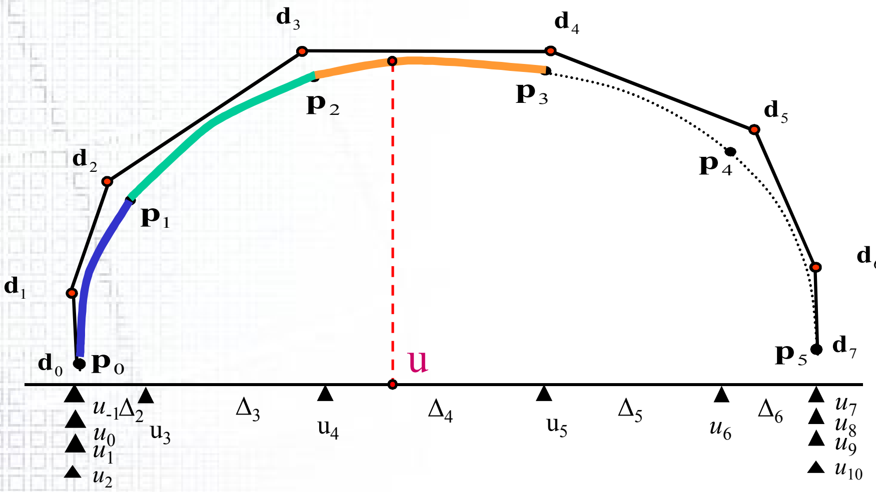


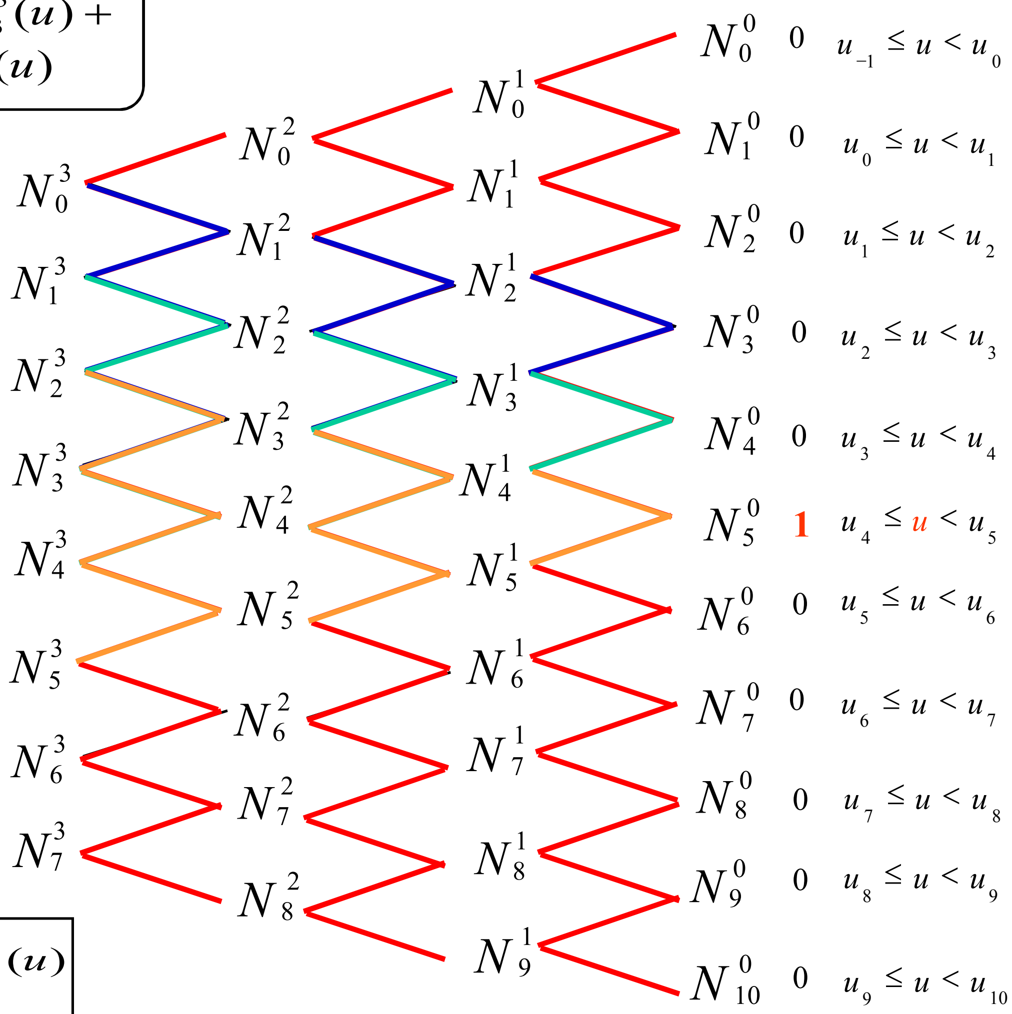
$$N_i^n(u)=\frac{u-u_{i-1}}{u_{i+n-1}-u_{i-1}}N_i^{n-1}(u)+\frac{u_{i+n}-u}{u_{i+n}-u_i}N_{i+1}^{n-1}(u)$$

$$N_i^0(u)=\begin{cases}1 & \text{if } u_{i-1}\le u<u_i\\ 0 & \text{else}\end{cases}$$

2.3.3.2 B-spline curves (3)

$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_3 N_3^3(u) + \mathbf{d}_4 N_4^3(u) + \mathbf{d}_5 N_5^3(u) + \mathbf{d}_6 N_6^3(u) + \mathbf{d}_7 N_7^3(u)$$





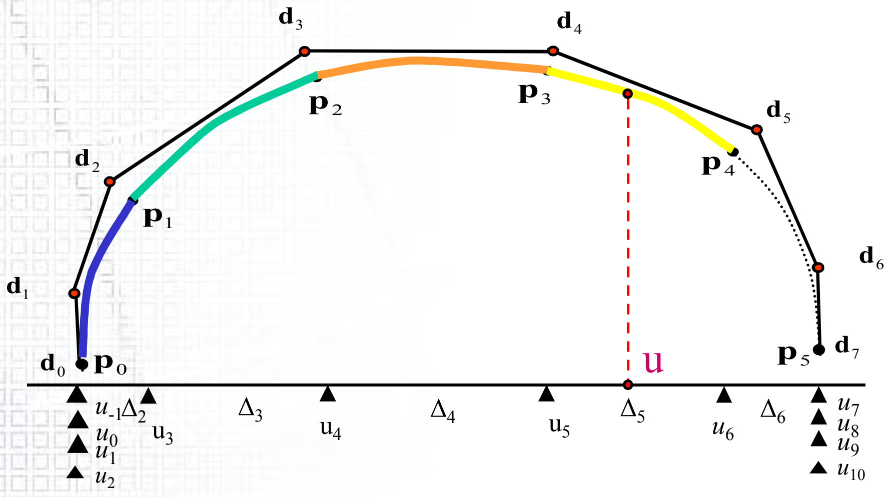
$$\mathbf{r}(u) = \mathbf{d}_0 \cancel{N_0^3}(u) + \mathbf{d}_1 \cancel{N_1^3}(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_4 N_4^3(u) + \mathbf{d}_4 N_4^3(u) + \mathbf{d}_5 N_5^3(u) + \mathbf{d}_6 \cancel{N_6^3}(u) + \mathbf{d}_7 \cancel{N_7^3}(u)$$

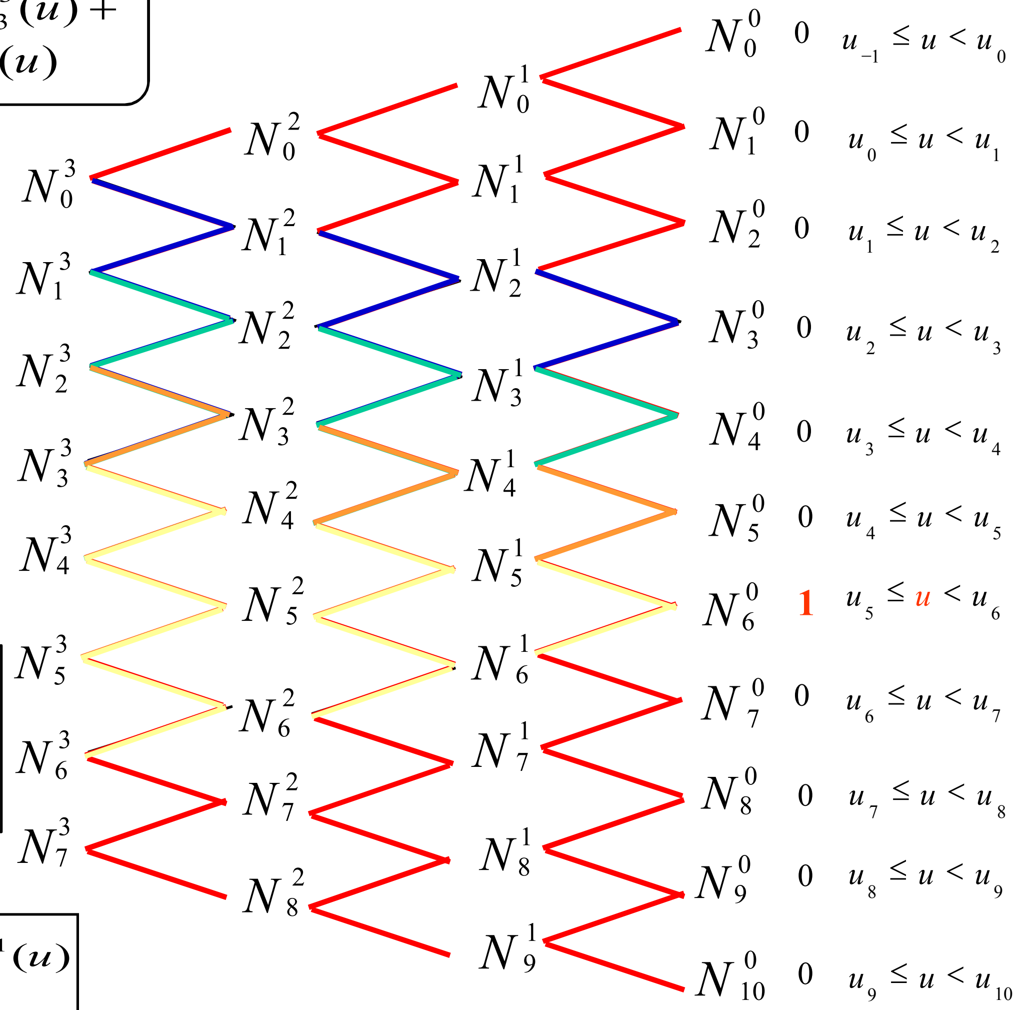
$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

2.3.3.2 B-spline curves (4)

$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_3 N_3^3(u) + \mathbf{d}_4 N_4^3(u) + \mathbf{d}_5 N_5^3(u) + \mathbf{d}_6 N_6^3(u) + \mathbf{d}_7 N_7^3(u)$$





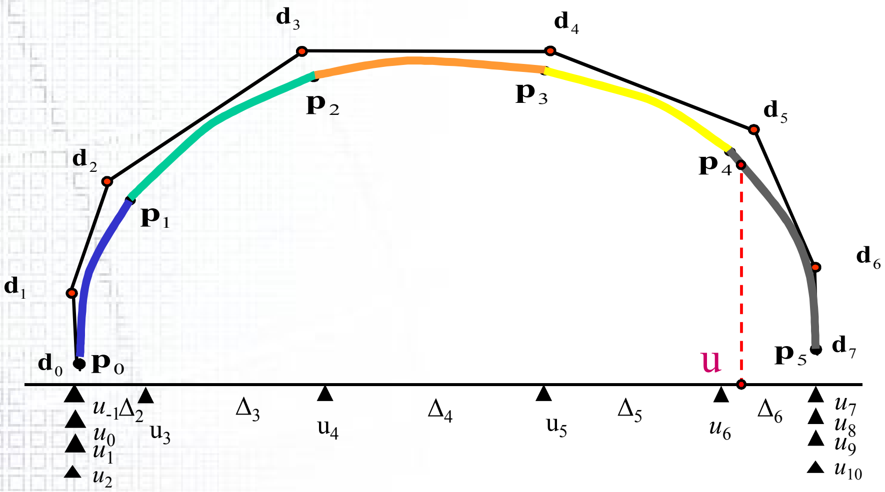
$$\mathbf{r}(u) = \mathbf{d}_0 \cancel{N_0^3}(u) + \mathbf{d}_1 \cancel{N_1^3}(u) + \mathbf{d}_2 \cancel{N_2^3}(u) + \mathbf{d}_4 N_4^3(u) + \mathbf{d}_4 N_4^3(u) + \mathbf{d}_5 N_5^3(u) + \mathbf{d}_6 N_6^3(u) + \mathbf{d}_7 \cancel{N_7^3}(u)$$

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

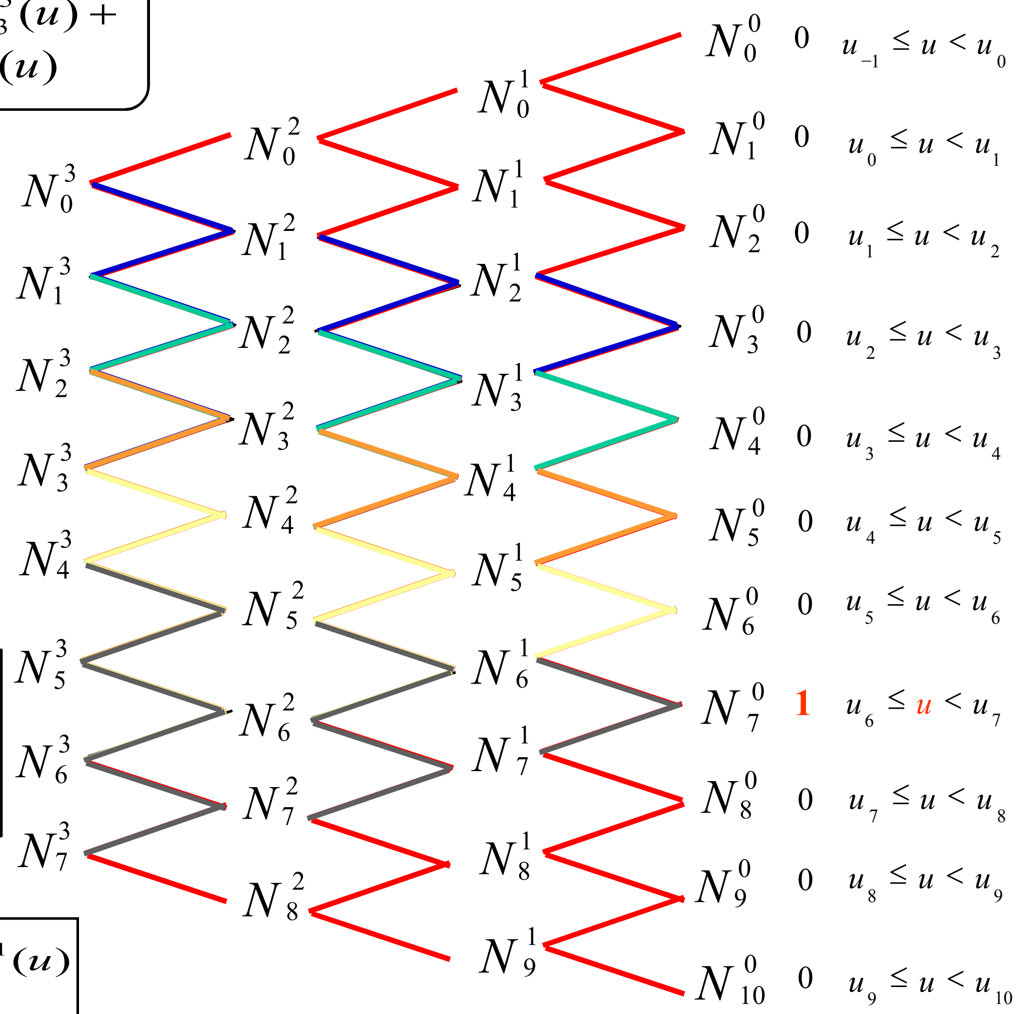
$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

2.3.3.2 B-spline curves (5)

$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_3 N_3^3(u) + \mathbf{d}_4 N_4^3(u) + \mathbf{d}_5 N_5^3(u) + \mathbf{d}_6 N_6^3(u) + \mathbf{d}_7 N_7^3(u)$$



$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_4 N_4^3(u) + \mathbf{d}_4 N_4^3(u) + \mathbf{d}_5 N_5^3(u) + \mathbf{d}_6 N_6^3(u) + \mathbf{d}_7 N_7^3(u)$$



$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

2.3.3.3 Relationship between de Boor algorithm & B-spline curves

- de Boor 알고리즘 : "Constructive Approach"

  Input: $\mathbf{d}_i$ (de Boor Points)

  Processor: 구간별로 $\mathbf{d}_i$를 n번 순차적 'linear interpolation'

  Output : n차 곡선상의 점

  → 'B-spline function'(Cox-de Boor recurrence formula) 형태로 표현 됨

- B-spline 곡선식: "B-spline function evaluation Approach"

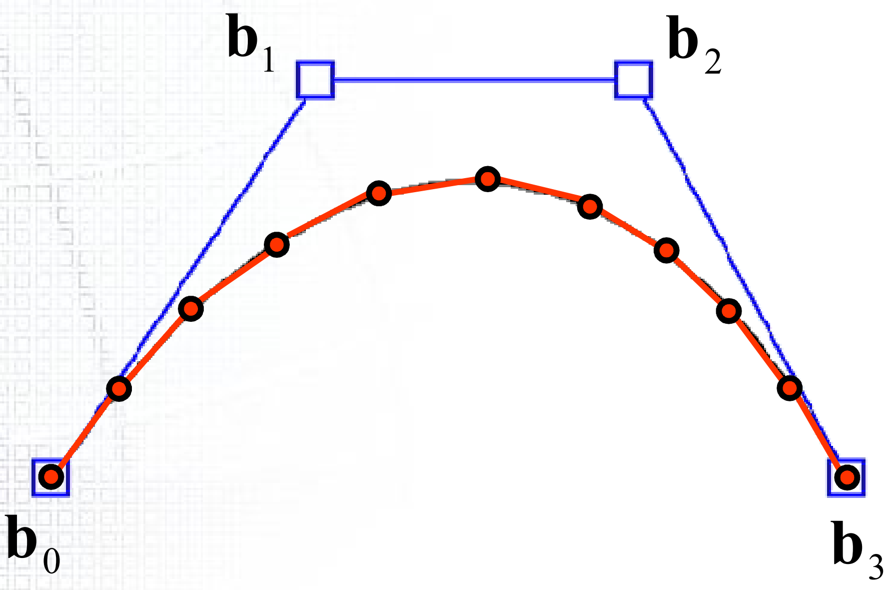
  Input: $\mathbf{d}_i$ (de Boor Points)

  Processor: 공간 상의 점 $\mathbf{d}_i$와 B-spline function을 "blending"하여 함수 값을 계산하면 곡선상의 점을 구할 수 있음

  Output: B-spline function과 $\mathbf{d}_i$의 혼합 함수 형태로 표현

2.3.3.4 Programming B-spline Curve class

Cubic B-spline 예시



$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_3 N_3^3(u)$$

1) B-spline Curve의 구성
- Degree
- Control Point

<u>Member Variables of B-spline Curve Class</u>

int n: degree of B-spline Curve
Vector* m_ControlPoint: Control Point
int m_nControlPoint: the number of Control Point

2) B-spline Basis Function 계산
(Cox-de Boor Recurrence Formula)

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

3) B-spline Curve 작도
- 곡선을 Line Segment로 나누어 작도
- Parameter u를 u_{min}~u_{max}까지 n등분하여 각 u에 대한 곡선 상의 점을 구함
- 위에서 구한 점을 직선으로 연결하여 곡선을 표현

2.3.3.4 Sample code of Cubic B-spline Curve (1)

```
#ifndef __CubicBspline_h__
#define __CubicBspline_h__

#include "vector.h“

class CubicBsplineCurve {
public:
    Vector* m_ControlPoint;    int m_nControlPoint;
    double* m_Knot; int m_nKnot;
    int m_nDegree;

    CubicBsplineCurve();
    ~CubicBsplineCurve();

    void SetControlPoint(Vector* pControlPoint, int nControlPoint);
    void SetKnot(double* pKnot, int nKnot);
    Vector CalcPoint(double u);
    double N(int d, int i, double u);             // B-spline basis function
};
#endif
```

Member Variables of B-spline Curve Class

int n: degree of B-spline Curve
Vector* m_ControlPoint: Control Point
int m_nControlPoint: the number of Control Point

2.3.3.4 Sample code of Cubic B-spline Curve (2)

```
CubicBsplineCurve::CubicBsplineCurve () {
   m_ControlPoint = 0;        m_Knot = 0;
   m_nControlPoint = 0;       m_nKnot = 0;       int m_nDegree =3;
}
CubicBsplineCurve::~CubicBsplineCurve () {
   if(m_ControlPoint)  delete[] m_ControlPoint;
   if(m_Knot)  delete[] m_Knot;
}
void CubicBsplineCurve::SetControlPoint(Vector* pControlPoint, int nControlPoint) {
   m_ControlPoint = new Vector[nControlPoint];
   for(int i=0; i < nControlPoint; i++) {
      m_ControlPoint[i] = pControlPoint[i];
   }
}
void CubicBsplineCurve::SetKnot(double* pKnot, int nKnot){
   m_Knot = new double[nKnot];
   for(int i=0; i < nKnot; i++) {
     m_Knot[i] = pKnot[i];
   }
}
```

2.3.3.4 Sample code of Cubic B-spline Curve (3)

```
Vector CubicBsplineCurve::CalcPoint(double u)
{
    Vector PointOnCurve(0,0,0);
    if ( t < m_Knot[0] || t > m_Knot[m_nKnot-1] ) {
        return PointOnCurve;
    }
    for(int i = 0; i < m_nControlPoint; i++){
        PointOnCurve = PointOnCurve + m_ControlPoint[i] * N(m_nDegree, i, u);
    }
    return PointOnCurve;
}
```

Get points on curve at parameter *u*

$$\mathbf{r}(u) = \mathbf{d}_0 N_0^3(u) + \mathbf{d}_1 N_1^3(u) + \mathbf{d}_2 N_2^3(u) + \mathbf{d}_3 N_3^3(u)$$

2.3.3.4 Sample code of Cubic B-spline Curve (4)

```
double CubicBsplineCurve:: N(int d, int i, double u) {
    // Find Span k
    // U i-1 <= U < U i  → k = i

    if( d == 0 ) {
        // return 0 or 1;
    } else {
        // return Cox de-Boor recurrence formula
    }
}
```

B-spline Basis Function 계산

(Cox-de Boor Recurrence Formula)

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

2.3.4 C^1 and C^2 continuity condition

2.3.4.1 1st Derivatives of Cubic Bezier Curves at Junction point

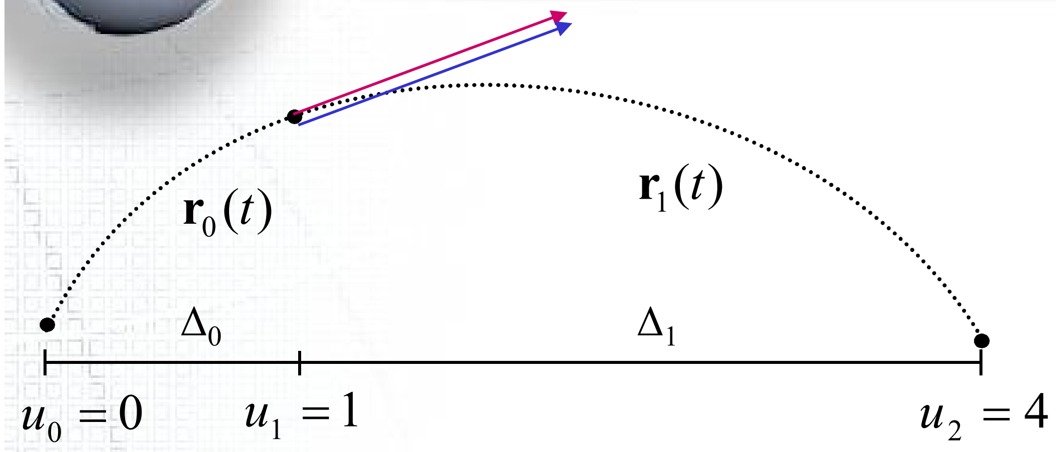
2.3.4.2 C^1 continuity condition of composite curves

2.3.4.3 2nd Derivatives of Cubic Bezier Curves

2.3.4.4 C^2 continuity condition of composite curves

Advanced
Ship
Design
Automation
Laboratory

2.3.4.1 1st Derivatives of Cubic Bezier Curves at Junction point



$$t = \frac{u - u_i}{u_{i+1} - u_i} = \frac{u - u_i}{\Delta_i}$$

t는 [0,1] 구간의 국부매개변수('local parameter')

$$\frac{d\mathbf{r}(u(t))}{du} = \frac{d\mathbf{r}_i(t)}{dt}\frac{dt}{du} = \frac{1}{\Delta_i}\frac{d\mathbf{r}_i(t)}{dt}$$

$\frac{d\mathbf{r}(u)}{du}$ 의 $u_0 \le \mathrm{u} \le u_1$ 에서의 미분 값

$$t = \frac{u - u_0}{u_1 - u_0} = \frac{u - u_0}{\Delta_0}$$

t는 [0,1] 구간의 국부 매개 변수

$$\frac{d\mathbf{r}(u)}{du} = \frac{d\,\mathbf{r}_0(u(t))}{dt}\frac{dt}{du} = \frac{1}{\Delta_0}\frac{d\,\mathbf{r}_0(t)}{dt}$$

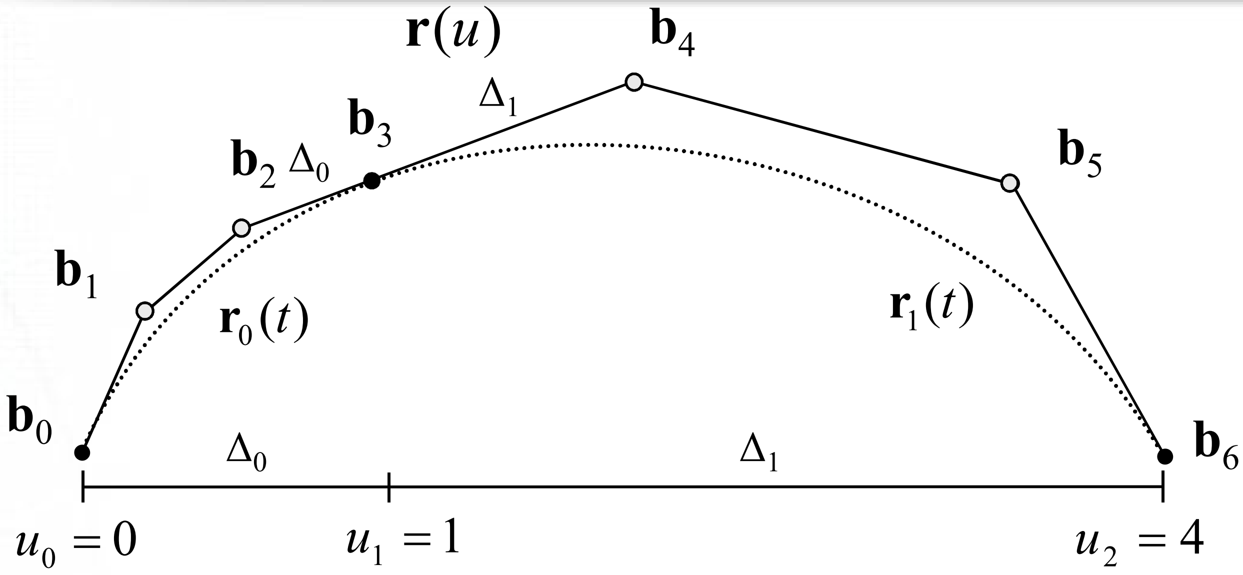
$\frac{d\mathbf{r}(u)}{du}$ 의 $u_1 \le \mathrm{u} \le u_2$ 에서의 미분 값

$$t = \frac{u - u_1}{u_2 - u_1} = \frac{u - u_1}{\Delta_1}$$

t는 [0,1] 구간의 국부 매개 변수

$$\frac{d\mathbf{r}(u)}{du} = \frac{d\,\mathbf{r}_1(t)}{dt}\frac{dt}{du} = \frac{1}{\Delta_1}\frac{d\,\mathbf{r}_1(t)}{dt}$$

2.3.4.2 C¹ continuity condition of composite curves



$\mathbf{r}(u=u_1) = \mathbf{r}_0(t=1) = \mathbf{r}_1(t=0)$ 연결 점에서 C¹ 조건을 만족 해야 하므로

$$\left.\frac{d\mathbf{r}(u)}{du}\right|_{u_1=1} = \frac{1}{\Delta_0}\left.\frac{d\mathbf{r}_0(t)}{dt}\right|_{t=1} = \frac{1}{\Delta_0}3(\mathbf{b}_3-\mathbf{b}_2)$$

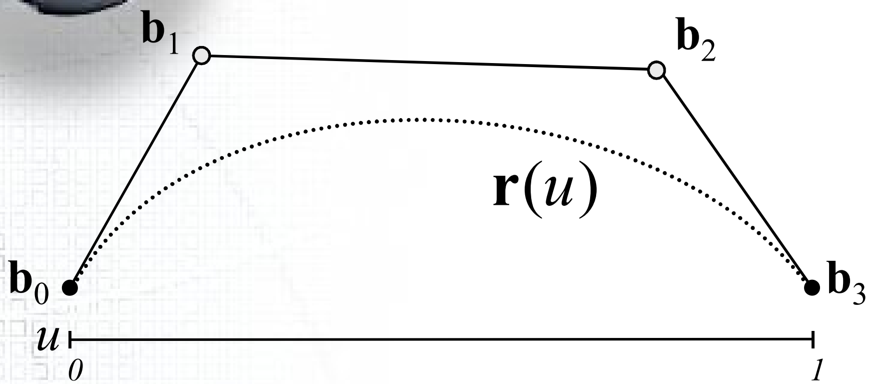
$$= \frac{1}{\Delta_1}\left.\frac{d\mathbf{r}_1(t)}{dt}\right|_{t=0} = \frac{1}{\Delta_1}\cdot 3(\mathbf{b}_4-\mathbf{b}_3)$$

$$(\mathbf{b}_3-\mathbf{b}_2):(\mathbf{b}_4-\mathbf{b}_3) = \Delta_0 : \Delta_1$$

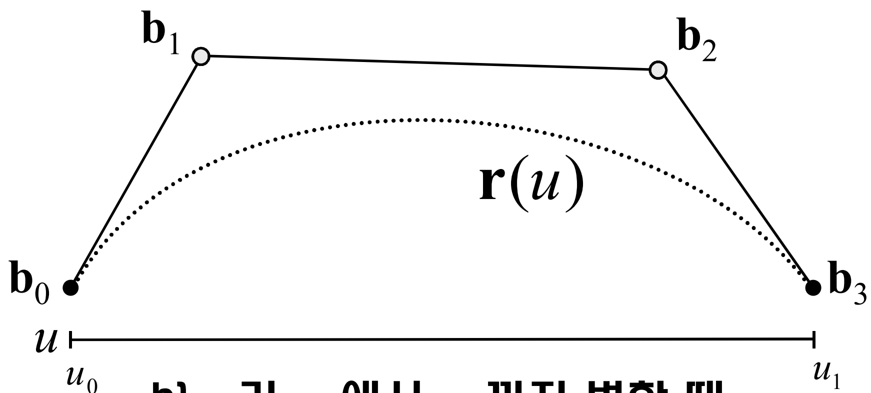
$$\mathbf{b}_3 = \frac{\Delta_1}{\Delta}\mathbf{b}_2 + \frac{\Delta_0}{\Delta}\mathbf{b}_4$$

parameter *u*를 시간이라고 생각하면, 1차 미분 계수는 곡선상을 지나는 점의 속도라고 생각할 수 있다.
연결점 $\mathbf{b}_3$에서 1차 미분 계수가 연속이라면 그 점에서 속도가 연속이어야 한다는 의미이다.
그러므로 시간 간격이 Δ_0 에서 Δ_1 으로 변하면 즉, 시간 간격이 변하면,
그 거리도 비례하여 변하여야 연결점에서 속도가 연속이다.!!!

2.3.4.3 2nd Derivatives of Cubic Bezier Curves



a) u 가 0 에서 1까지 변할 때



b) u 가 u_0 에서 u_1 까지 변할 때

n차 Bezier곡선 2차미분

$$\frac{d^2\mathbf{r}(u)}{du^2} = n(n-1)\sum_{i=0}^{n-2}(\mathbf{b}_{i+2} - 2\mathbf{b}_{i+1} + \mathbf{b}_i)B_i^{n-2}$$

3차 Bezier곡선 2차미분

$$\frac{d^2\mathbf{r}(u)}{du^2} = 3(3-1)\sum_{i=0}^{1}(\mathbf{b}_{i+2} - 2\mathbf{b}_{i+1} + \mathbf{b}_i)B_i^{1}(u)$$

$u = 1$일 때

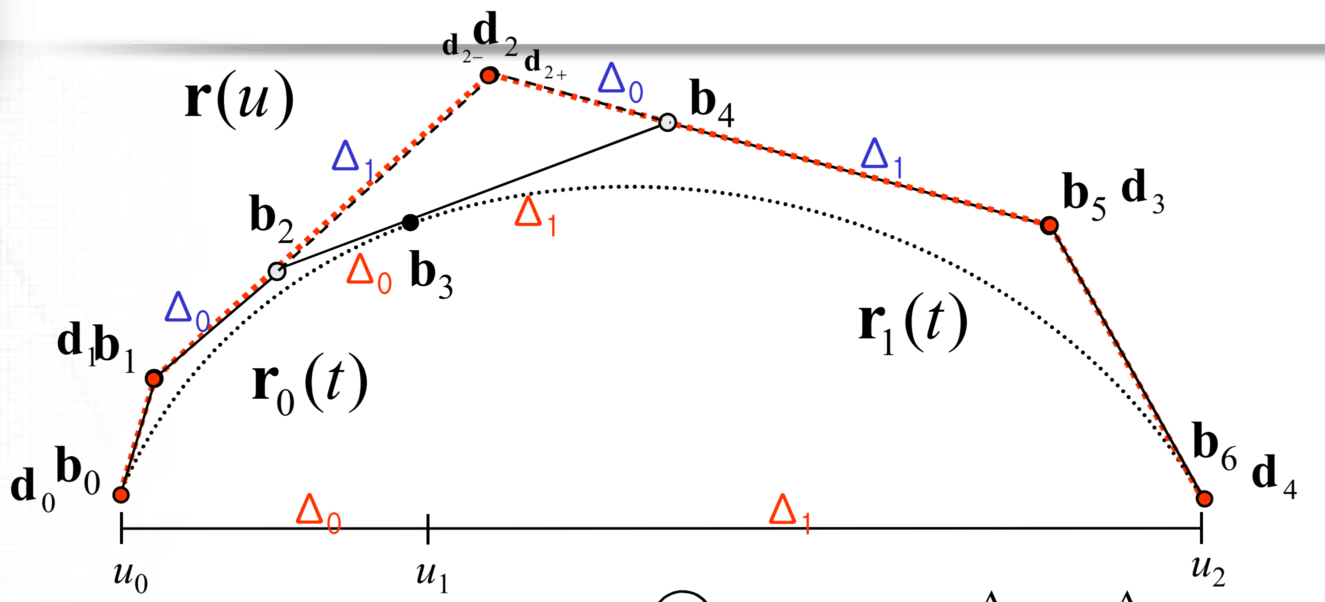
$$\frac{d^2\mathbf{r}(1)}{du^2} = 3(3-1)(\mathbf{b}_3 - 2\mathbf{b}_2 + \mathbf{b}_1)$$

$$\frac{d^2\mathbf{r}(u(t))}{du^2} = \frac{1}{(\Delta)^2}\frac{d^2\mathbf{r}(t)}{dt^2} \quad (\Delta = u_1 - u_0)$$

$u = u_1$ 일 때

$$\frac{d^2\mathbf{r}(u_1)}{du^2} = \frac{1}{(\Delta)^2}\frac{d^2\mathbf{r}(1)}{dt^2} = \frac{1}{(\Delta)^2}3(3-1)(\mathbf{b}_3 - 2\mathbf{b}_2 + \mathbf{b}_1)$$

2.3.4.4. C^2 continuity condition of composite curves



① 연결점 $\mathbf{b}_3$에서 C^2조건

$$\frac{d^2\mathbf{r}(u_{1-})}{du^2} = \frac{1}{(\Delta_0)^2}\frac{d^2\mathbf{r}_0(1)}{dt^2} = \frac{1}{(\Delta_0)^2}3(3-1)(\mathbf{b}_3-2\mathbf{b}_2+\mathbf{b}_1)$$

$$\frac{d^2\mathbf{r}(u_{1+})}{du^2} = \frac{1}{(\Delta_1)^2}\frac{d^2\mathbf{r}_1(0)}{dt^2} = \frac{1}{(\Delta_1)^2}3(3-1)(\mathbf{b}_5-2\mathbf{b}_4+\mathbf{b}_3)$$

② $\frac{d^2\mathbf{r}(u_{1-})}{du^2} = \frac{d^2\mathbf{r}(u_{1+})}{du^2}$ 이어야 하므로

$\frac{6}{(\Delta_0)^2}(\mathbf{b}_3-2\mathbf{b}_2+\mathbf{b}_1) = \frac{6}{(\Delta_0)^2}(\mathbf{b}_5-2\mathbf{b}_4+\mathbf{b}_3)$이다.

③ 그리고 C^1조건$\left(\mathbf{b}_3 = \frac{\Delta_1}{\Delta}\mathbf{b}_2 + \frac{\Delta_0}{\Delta}\mathbf{b}_4\right)$을 대입하여

정리하면

$$\Rightarrow -\frac{\Delta_1}{\Delta_0}\mathbf{b}_1 + \frac{\Delta}{\Delta_0}\mathbf{b}_2 = \frac{\Delta}{\Delta_1}\mathbf{b}_4 - \frac{\Delta_0}{\Delta_1}\mathbf{b}_5$$

④ 좌변을 $\mathbf{d}_{2-} = -\frac{\Delta_1}{\Delta_0}\mathbf{b}_1 + \frac{\Delta}{\Delta_0}\mathbf{b}_2$ 라 하면

$$\mathbf{b}_2 = \frac{\Delta_1}{\Delta}\mathbf{b}_1 + \frac{\Delta_0}{\Delta}\mathbf{d}_{2-}$$

⑤ 우변을 $\mathbf{d}_{2+} = \frac{\Delta}{\Delta_1}\mathbf{b}_4 - \frac{\Delta_0}{\Delta_1}\mathbf{b}_5$ 라 하면

$$\mathbf{b}_4 = \frac{\Delta_1}{\Delta}\mathbf{d}_{2+} + \frac{\Delta_1}{\Delta}\mathbf{b}_5$$

⑥ 즉,$(\mathbf{d}_{2-} = \mathbf{d}_{2+} = \mathbf{d}_2)$인 점이 존재하면 C^2 조건을 만족한다.

⑦ 연결점에서 C^2조건

$$-\frac{\Delta_1}{\Delta_0}\mathbf{b}_1 + \frac{\Delta}{\Delta_0}\mathbf{b}_2 = \frac{\Delta}{\Delta_1}\mathbf{b}_4 - \frac{\Delta_0}{\Delta_1}\mathbf{b}_5$$

$$ratio(\mathbf{b}_1,\mathbf{b}_2,d_2) = ratio(d_2,\mathbf{b}_4,\mathbf{b}_5) = \frac{\Delta_0}{\Delta_1}$$