[2008][03-1]

Computer aided ship design

Part 1. Curve & Surface

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Advanced Course
Knot Insertion

1) Greville abscissa-’Moving Average’ of the knots
2) de Casteljau Algorithm by Knot Insertion
3) de Boor Algorithm by Knot Insertion

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Bezier function &
Bezier (control) ordinates

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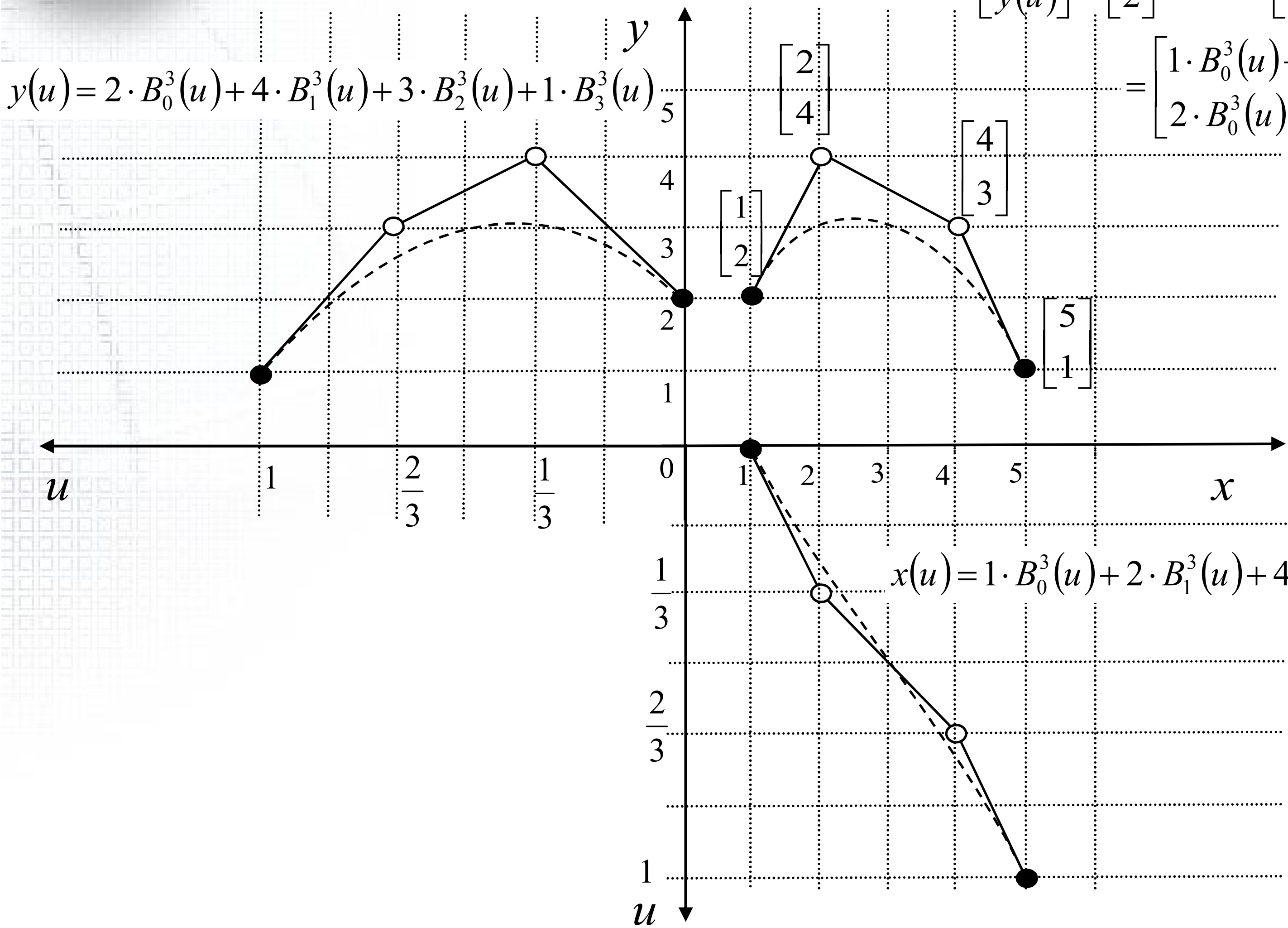
Bezier curves : Bezier (control) points
vs. Bezier function : Bezier (control) ordinates

Bezier function의 다른 이름은 ‘functional Bezier curve’ 또는 ‘non-parametric Bezier curves’라고도 한다.

$$\mathbf{b}(u)=\begin{bmatrix}x(u)\\y(u)\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}B_0^3(u)+\begin{bmatrix}2\\4\end{bmatrix}B_1^3(u)+\begin{bmatrix}4\\3\end{bmatrix}B_2^3(u)+\begin{bmatrix}5\\1\end{bmatrix}B_3^3(u)$$

$$=\begin{bmatrix}1\cdot B_0^3(u)+2\cdot B_1^3(u)+4\cdot B_2^3(u)+5\cdot B_3^3(u)\\2\cdot B_0^3(u)+4\cdot B_1^3(u)+3\cdot B_2^3(u)+1\cdot B_3^3(u)\end{bmatrix}$$



- Bezier curve: x, y, z component(point)를 합쳐서 표현한 것
- Bezier function: Bezier curve를 x, y, z component 별로 표현한 것

Greville abscissae

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Definition of Greville abscissae

- Greville abscissae(ξ_i)
  - 주어진 값(u_i)을 차수(n)만큼 순차적으로 더해서 평균을 낸 것

$$\xi_i = \frac{u_i + \ldots + u_{i+n-1}}{n}$$

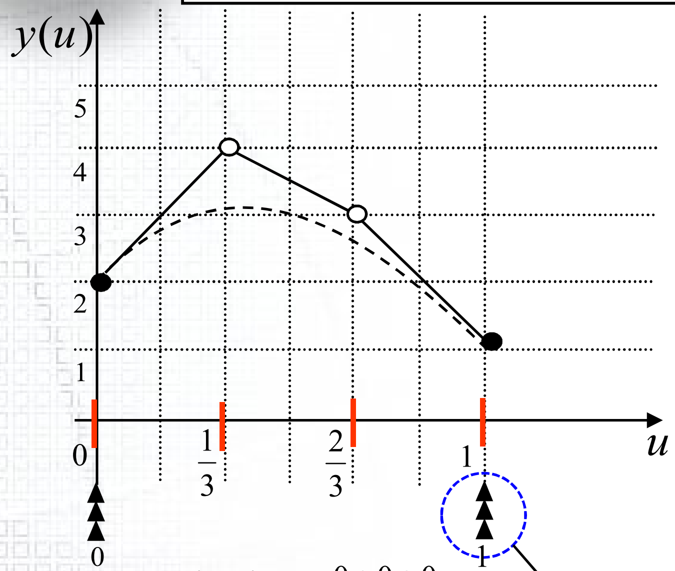
예) n = 3일 경우

$$\xi_0 = \frac{u_0 + u_1 + u_2}{3}, \quad \xi_1 = \frac{u_1 + u_2 + u_3}{3}, \quad \xi_2 = \frac{u_2 + u_3 + u_4}{3}, \quad \cdots$$

Bezier function & Greville abscissae:

The ordinates of the Bezier function are positioned over the Greville abscissae.

Given: Ordinates of Bezier function(2, 4, 3, 1)
Find: Over which Greville abscissae are the ordinates of Bezier function(2, 4, 3, 1) positioned?



$$\xi_0 = \frac{u_0+u_1+u_2}{3} = \frac{0+0+0}{3} = 0$$

$$\xi_1 = \frac{u_1+u_2+u_3}{3} = \frac{0+0+1}{3} = \frac{1}{3}$$

$$\xi_2 = \frac{u_2+u_3+u_4}{3} = \frac{0+1+1}{3} = \frac{2}{3}$$

$$\xi_3 = \frac{u_3+u_4+u_5}{3} = \frac{1+1+1}{3} = 1$$

$y(u)$: Bezier function(y-component of Bezier curve)

$$y(u) = 2 \bullet B_0^3(u) + 4 \bullet B_1^3(u) + 3 \bullet B_2^3(u) + 1 \bullet B_3^3(u)$$

$$\begin{pmatrix} u \\ y(u) \end{pmatrix} = \begin{pmatrix} u \\ 2 \bullet B_0^3(u) + 4 \bullet B_1^3(u) + 3 \bullet B_2^3(u) + 1 \bullet B_3^3(u) \end{pmatrix}$$

$$u = 0 \bullet B_0^3(u) + \frac{1}{3} \bullet B_1^3(u) + \frac{2}{3} \bullet B_2^3(u) + 1 \bullet B_3^3(u)$$

로 표현된다.

Knot 3개가 중첩되면,
control point와 곡선 상의 점이 일치

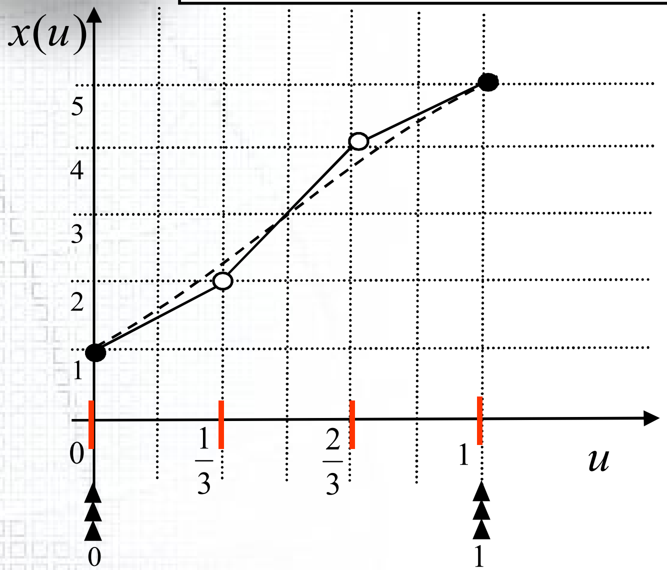
$$\begin{pmatrix} u \\ y(u) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} B_0^3(u) + \begin{pmatrix} \frac{1}{3} \\ 4 \end{pmatrix} B_1^3(u) + \begin{pmatrix} \frac{2}{3} \\ 3 \end{pmatrix} B_2^3(u) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} B_3^3(u)$$

1. Bezier function ordinates는 u= 0, 1/3, 2/3, 1 **상에 위치하였다**(“Linear Precision”).
2. Greville abscissae(u=0,0,0,1,1,1)를 **구해 보면** 0, 1/3, 2/3, 1**이다**.
3. **따라서** Bezier function y- ordinates는 Graville abscissae **상에 위치한다**.

Bezier function & Greville abscissae:

The ordinates of the Bezier function are positioned over the Greville abscissae.

Given: Ordinates of Bezier function(1, 2, 4, 5)
Find: Over which Greville abscissae are the ordinates of Bezier function(1, 2, 4, 5) positioned?



$$\xi_0 = \frac{u_0 + u_1 + u_2}{3} = \frac{0+0+0}{3} = 0$$

$$\xi_2 = \frac{u_2 + u_3 + u_4}{3} = \frac{0+1+1}{3} = \frac{2}{3}$$

$$\xi_1 = \frac{u_1 + u_2 + u_3}{3} = \frac{0+0+1}{3} = \frac{1}{3}$$

$$\xi_3 = \frac{u_3 + u_4 + u_5}{3} = \frac{1+1+1}{3} = 1$$

$x(u)$: Bezier function(x-component of Bezier curve)

$$x(u) = 1 \bullet B_0^3(u) + 2 \bullet B_1^3(u) + 4 \bullet B_2^3(u) + 5 \bullet B_3^3(u)$$

$$\begin{pmatrix} u \\ x(u) \end{pmatrix} = \begin{pmatrix} u \\ 1 \bullet B_0^3(u) + 2 \bullet B_1^3(u) + 4 \bullet B_2^3(u) + 5 \bullet B_3^3(u) \end{pmatrix}$$

$$u = 0 \bullet B_0^3(u) + \frac{1}{3} \bullet B_1^3(u) + \frac{2}{3} \bullet B_2^3(u) + 1 \bullet B_3^3(u)$$

로 표현된다.

$$\begin{pmatrix} u \\ x(u) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} B_0^3(u) + \begin{pmatrix} \frac{1}{3} \\ 2 \end{pmatrix} B_1^3(u) + \begin{pmatrix} \frac{2}{3} \\ 4 \end{pmatrix} B_2^3(u) + \begin{pmatrix} 1 \\ 5 \end{pmatrix} B_3^3(u)$$

1. Bezier x-control ordinates는 0, 1/3, 2/3, 1 **위에 위치한다**(“Linear Precision”).
2. Greville abscissae를 **구해 보면** 0, 1/3, 2/3, 1**이다**.
3. **따라서** Bezier x-control ordinates는 Graville abscissae **위에 위치한다**.

Greville abscissae of B-spline curves - "Moving average of the knots"

- Control polygons of degree n B-Spline functions:

$$\mathbf{d}_i = \begin{bmatrix} \xi_i \\ d_i \end{bmatrix}$$

$$\text{where} \quad \xi_i = \frac{1}{n}(u_i + \ldots + u_{i+n-1}).$$

- These Greville abscissae are the moving average of the knots.
- $\mathbf{d}_i$ are called the control ordinates of the function.

Knot Insertion

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Knot Insertion

- G. Farin의 지도교수인 W. Boehm에 의해 제안됨
- de Boor 알고리즘, Cox-de Boor 알고리즘 등이 모두 Knot Insertion으로 유도될 수 있음

- Knot Insertion의 전체적인 과정
  - Step 1: 하나의 knot을 삽입한다.
  - Step 2: Knot의 변화에 따라 Greville abscissa가 update 된다.
  - Step 3: Greville abscissa 위에 Bezier(or B-Spline) function ordinate가 위치하므로 update 된 Greville abscissa 위에서의 Bezier(or B-Spline) function ordinate를 각각 구한다.

de Casteljau Algorithm by Knot Insertion

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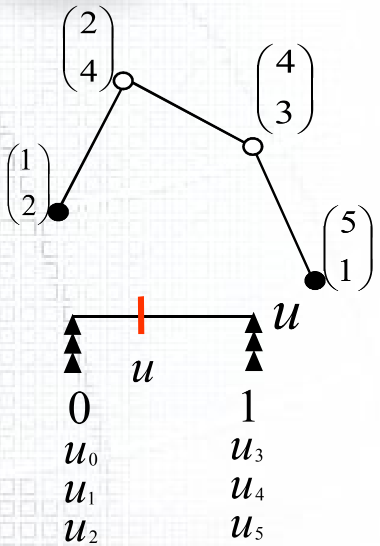
de Casteljau Algorithm by Knot Insertion

:: knot insertion을 왜 하는가?

➔ n번의 knot insertion을 하면 n차 Bezier function의 ordinate를 구할 수 있다.

knot Insertion 1



Greville abscissae updated

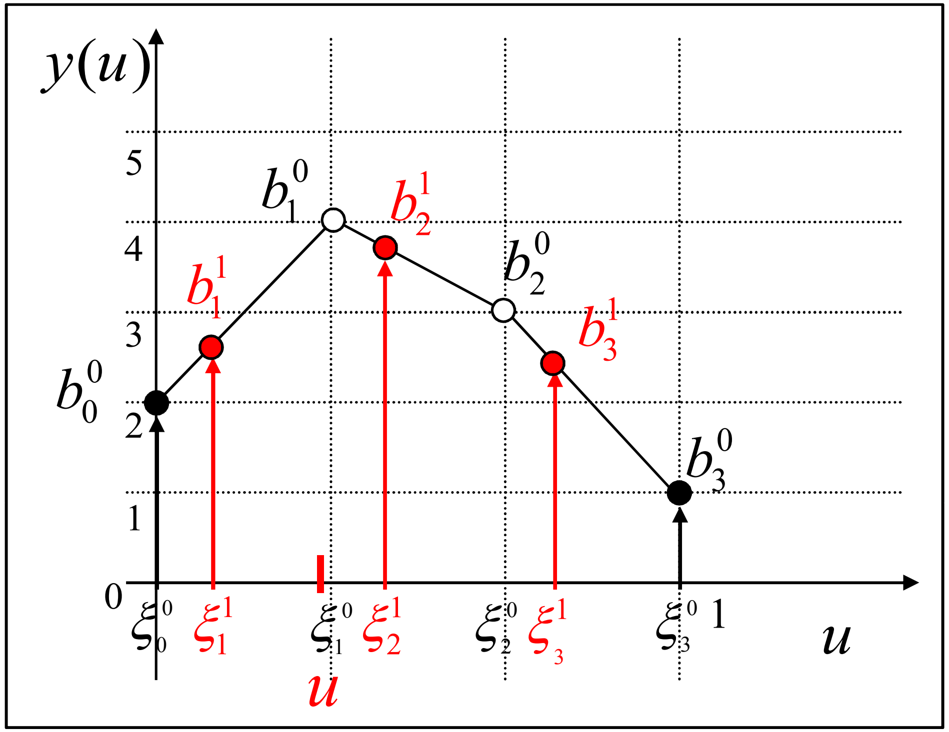
$$\xi_0^1 = \frac{0+0+0}{3} = 0 = \xi_0^0$$

$$\xi_1^1 = \frac{0+0+u}{3} = \frac{u}{3}$$

$$\xi_2^1 = \frac{0+u+1}{3} = \frac{u+1}{3}$$

$$\xi_3^1 = \frac{u+1+1}{3} = \frac{u+2}{3}$$

$$\xi_4^1 = \frac{1+1+1}{3} = 1 = \xi_3^0$$



Given
- Bezier y-control ordinate $b_0^0, b_1^0, b_2^0, b_3^0$
- Knot Insertion에 의해 계산되는 $\xi_0^0, \xi_1^1, \xi_2^1, \xi_3^1, \xi_3^0$

Find
- ξ_i 위에 위치하는 Bezier function ordinates (b_1^1, b_2^1, b_3^1)

두 ordinates의 내분점은 두 ordinates를 잇는 직선 상에 위치함

(예) b_1^1 는 직선 $\overline{b_0^0 b_1^0}$ 사이에 위치

$b_1^1 = ?$

$$\overline{b_0^0 b_1^1} : \overline{b_1^1 b_1^0} = \overline{\xi_0^0 \xi_1^1} : \overline{\xi_1^1 \xi_1^0}$$

$$\overline{\xi_0^0 \xi_1^1} = \frac{u}{3} - 0 = \frac{u}{3}$$

$$\overline{\xi_1^1 \xi_1^0} = \frac{1}{3} - \frac{u}{3} = \frac{1-u}{3}$$

$$\overline{\xi_1^0 \xi_0^0} = \frac{1}{3} - 0 = \frac{1}{3}$$

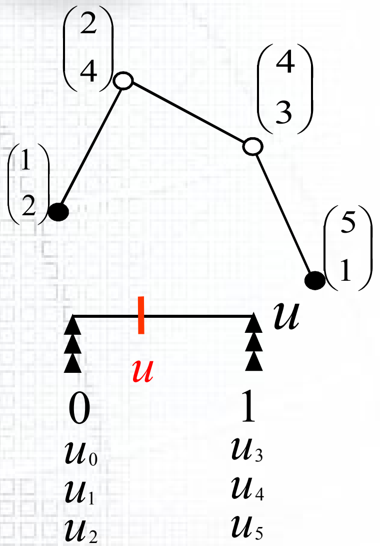
$$b_1^1 = (1-u)b_0^0 + u b_1^0$$

de Casteljau Algorithm by Knot Insertion

:: knot insertion을 왜 하는가?

➔ n번의 knot insertion을 하면 n차 Bezier function의 ordinate를 구할 수 있다.

knot Insertion 1



Greville abscissae updated

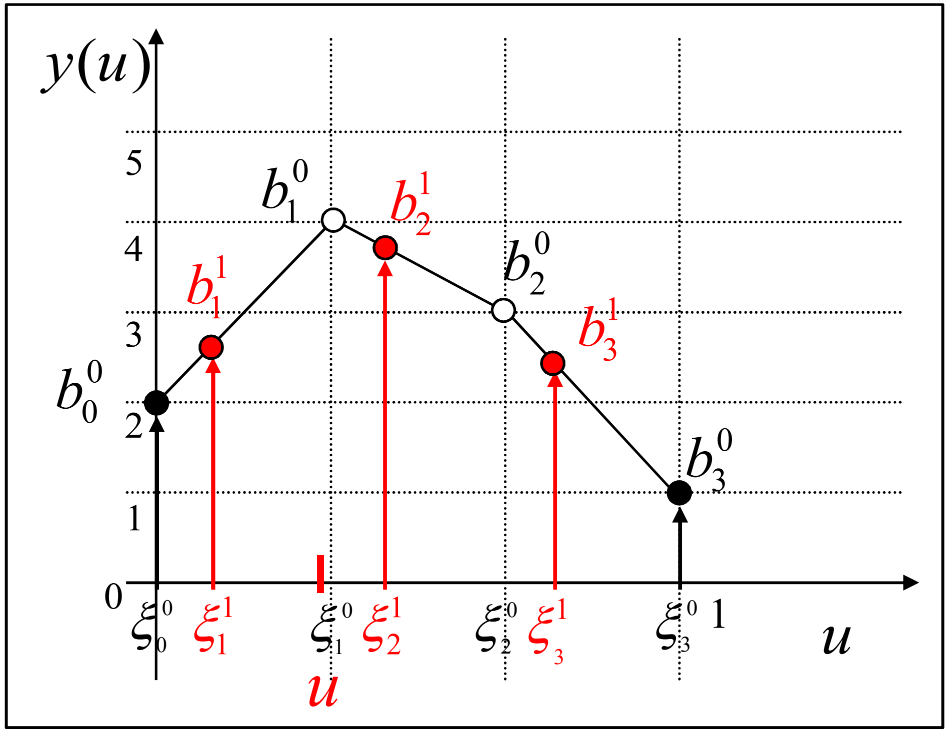
$$\xi_0^1 = \frac{0+0+0}{3} = 0 = \xi_0^0$$

$$\xi_1^1 = \frac{0+0+u}{3} = \frac{u}{3}$$

$$\xi_2^1 = \frac{0+u+1}{3} = \frac{u+1}{3}$$

$$\xi_3^1 = \frac{u+1+1}{3} = \frac{u+2}{3}$$

$$\xi_4^1 = \frac{1+1+1}{3} = 1 = \xi_3^0$$



Given
- Bezier y-control ordinate $b_0^0, b_1^0, b_2^0, b_3^0$
- Knot Insertion에 의해 계산되는 $\xi_0^0, \xi_1^1, \xi_2^1, \xi_3^1, \xi_3^0$

Find
- ξ_i 위에 위치하는 Bezier function ordinates (b_1^1, b_2^1, b_3^1)

두 ordinates의 내분점은 두 ordinates를 잇는 직선 상에 위치함

(예) b_2^1 는 직선 $\overline{b_1^0 b_2^0}$ 사이에 위치

$b_2^1 = ?$

$$\overline{b_1^0 b_2^1} : \overline{b_2^1 b_2^0} = \overline{\xi_1^0 \xi_2^1} : \overline{\xi_2^1 \xi_2^0}$$

$$\begin{cases} \overline{\xi_1^0 \xi_2^1} = \frac{u+1}{3} - \frac{1}{3} = \frac{u}{3} \\ \overline{\xi_2^1 \xi_2^0} = \frac{2}{3} - \frac{u+1}{3} = \frac{1-u}{3} \\ \overline{\xi_2^0 \xi_1^0} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \end{cases}$$

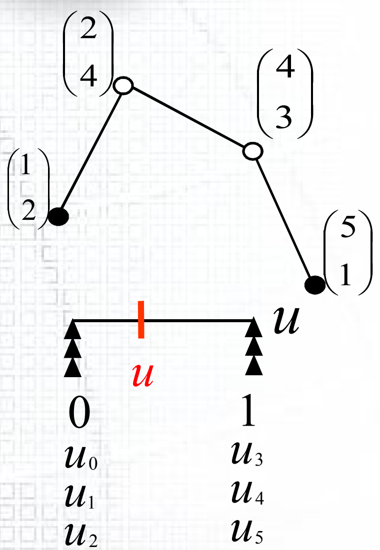
$$b_2^1 = (1-u)b_1^0 + ub_2^0$$

de Casteljau Algorithm by Knot Insertion

:: knot insertion을 왜 하는가?

➔ n번의 knot insertion을 하면 n차 Bezier function의 ordinate를 구할 수 있다.

knot Insertion 1



Greville abscissae updated

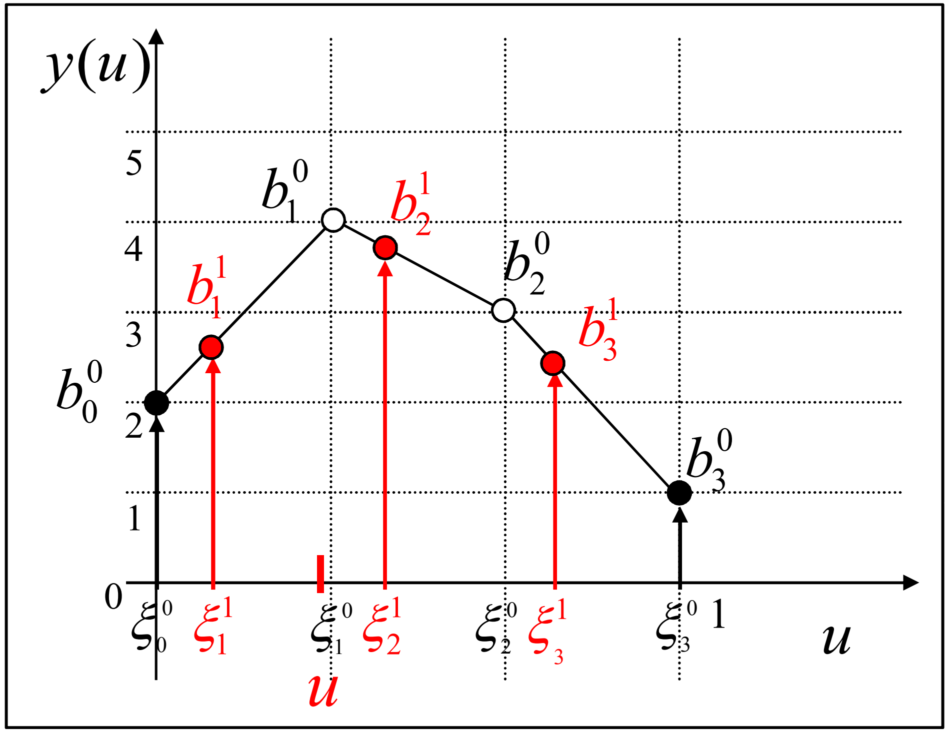
$$\xi_0^1 = \frac{0+0+0}{3} = 0 = \xi_0^0$$

$$\xi_1^1 = \frac{0+0+u}{3} = \frac{u}{3}$$

$$\xi_2^1 = \frac{0+u+1}{3} = \frac{u+1}{3}$$

$$\xi_3^1 = \frac{u+1+1}{3} = \frac{u+2}{3}$$

$$\xi_4^1 = \frac{1+1+1}{3} = 1 = \xi_3^0$$



Given
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- Knot Insertion에 의해 계산되는 $\xi_0^0, \xi_1^1, \xi_2^1, \xi_3^1, \xi_3^0$

Find
- ξ_i 위에 위치하는 Bezier function ordinates
  (b_1^1, b_2^1, b_3^1)

두 ordinates의 내분점은 두 ordinates를 잇는 직선 상에 위치함

(예) b_3^1 는 직선 $\overline{b_2^0 b_3^0}$ 사이에 위치

$b_3^1 = ?$

$$\overline{b_2^0 b_3^1} : \overline{b_3^1 b_3^0} = \overline{\xi_2^0 \xi_3^1} : \overline{\xi_3^1 \xi_3^0}$$

$$\begin{cases} \overline{\xi_2^0 \xi_3^1} = \dfrac{u+2}{3} - \dfrac{2}{3} = \dfrac{u}{3} \\ \overline{\xi_3^1 \xi_3^0} = 1 - \dfrac{u+2}{3} = \dfrac{1-u}{3} \\ \overline{\xi_3^0 \xi_2^0} = 1 - \dfrac{2}{3} = \dfrac{1}{3} \end{cases}$$

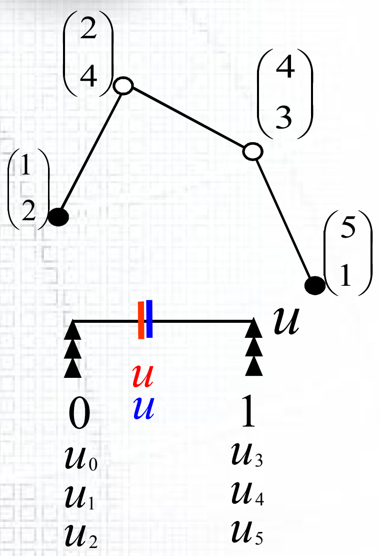
$$b_3^1 = (1-u) b_2^0 + u b_3^0$$

de Casteljau Algorithm by Knot Insertion

:: knot insertion을 왜 하는가?

➔ n번의 knot insertion을 하면 n차 Bezier function의 ordinate를 구할 수 있다.

knot Insertion 2



Greville abscissae updated

$$\xi_0^2 = \frac{0+0+0}{3} = 0 \qquad = \xi_0^0$$

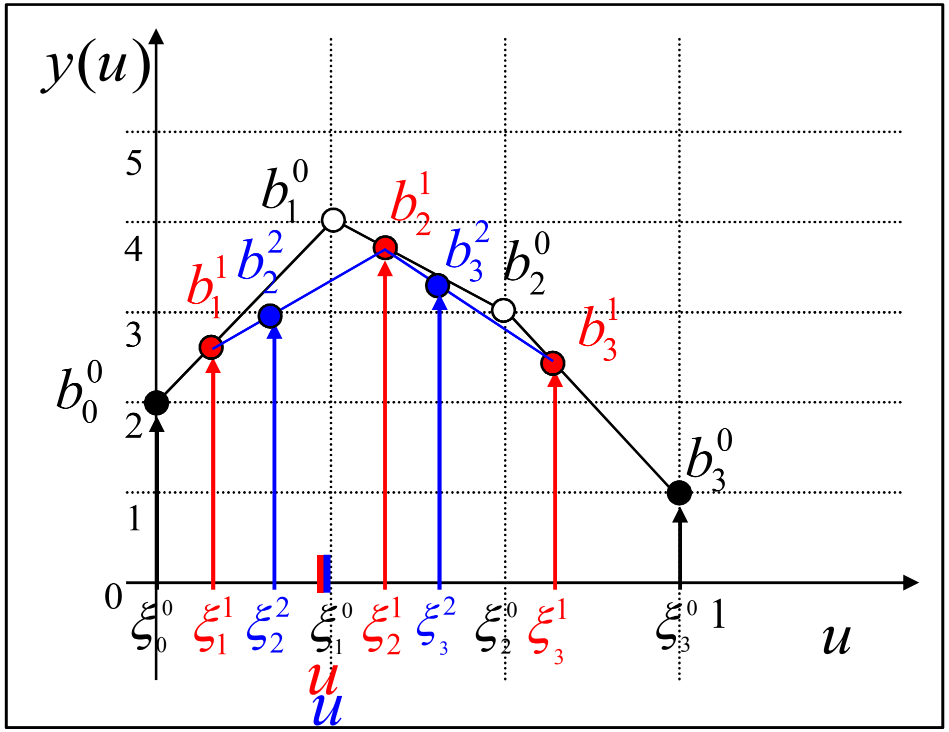
$$\xi_1^2 = \frac{0+0+u}{3} = \frac{u}{3} \qquad = \xi_1^1$$

$$\xi_2^2 = \frac{0+u+u}{3} = \frac{2u}{3}$$

$$\xi_3^2 = \frac{u+u+1}{3} = \frac{2u+1}{3}$$

$$\xi_4^2 = \frac{u+1+1}{3} = \frac{u+2}{3} = \xi_3^1$$

$$\xi_5^2 = \frac{1+1+1}{3} = 1 \qquad = \xi_3^0$$



Given

- Bezier y-control ordinate $b_0^0, b_1^0, b_2^0, b_3^0$
- Knot Insertion에 의해 계산되는 $\xi_0^0, \xi_1^1, \xi_2^2, \xi_3^2, \xi_3^1, \xi_3^0$

Find

- ξ_i 위에 위치하는 Bezier function ordinates (b_2^2, b_3^2)

두 ordinates의 내분점은 두 ordinates를 잇는 직선 상에 위치함

(예) b_2^2 는 직선 $\overline{b_1^1 b_2^1}$ 사이에 위치

$$b_2^2 = ?$$

$$\overline{b_1^1 b_2^2} : \overline{b_2^2 b_2^1} = \overline{\xi_1^1 \xi_2^2} : \overline{\xi_2^2 \xi_2^1}$$

$$\left\{ \begin{array}{l} \overline{\xi_1^1 \xi_2^2} = \dfrac{2u}{3} - \dfrac{u}{3} = \dfrac{u}{3} \\ \overline{\xi_2^2 \xi_2^1} = \dfrac{u+1}{3} - \dfrac{2u}{3} = \dfrac{1-u}{3} \\ \overline{\xi_2^1 \xi_1^1} = \dfrac{u+1}{3} - \dfrac{u}{3} = \dfrac{1}{3} \end{array} \right.$$

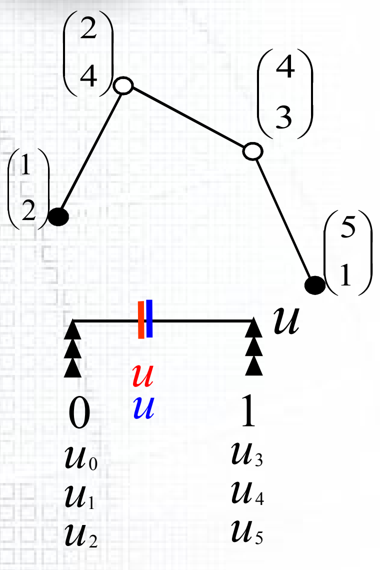
$$b_2^2 = (1-u)b_1^1 + u b_2^1$$

de Casteljau Algorithm by Knot Insertion

:: knot insertion을 왜 하는가?

➔ n번의 knot insertion을 하면 n차 Bezier function의 ordinate를 구할 수 있다.

knot Insertion 2



Greville abscissae updated

$$\xi_0^2 = \frac{0+0+0}{3} = 0 \qquad = \xi_0^0$$

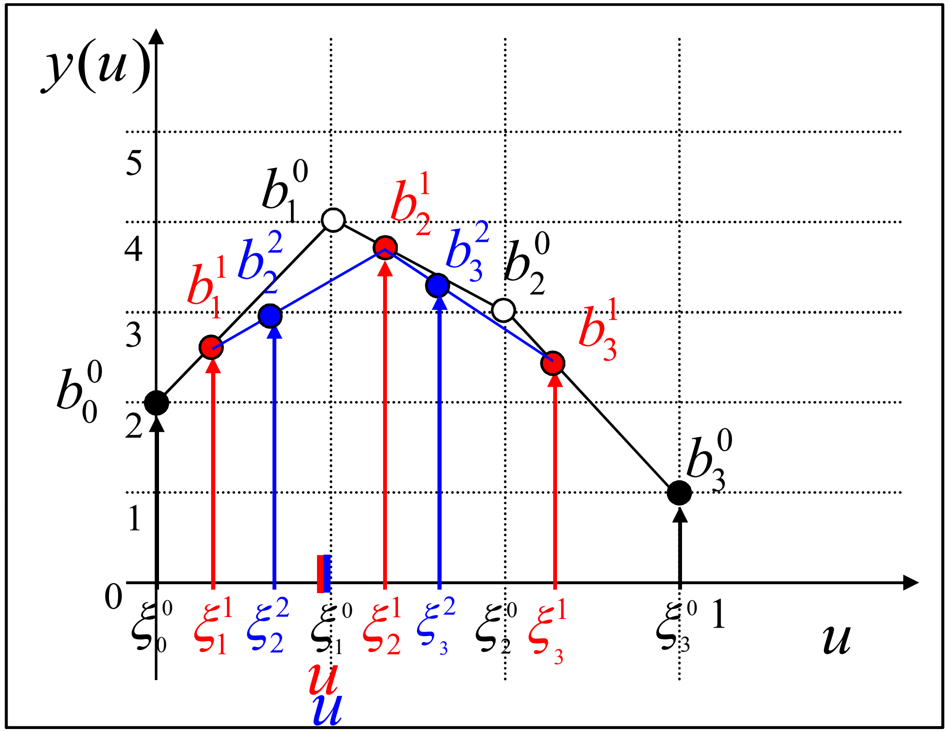
$$\xi_1^2 = \frac{0+0+u}{3} = \frac{u}{3} \qquad = \xi_1^1$$

$$\xi_2^2 = \frac{0+u+u}{3} = \frac{2u}{3}$$

$$\xi_3^2 = \frac{u+u+1}{3} = \frac{2u+1}{3}$$

$$\xi_4^2 = \frac{u+1+1}{3} = \frac{u+2}{3} = \xi_3^1$$

$$\xi_5^2 = \frac{1+1+1}{3} = 1 \qquad = \xi_3^0$$



Given
- Bezier y-control ordinate $b_0^0, b_1^0, b_2^0, b_3^0$
- Knot Insertion에 의해 계산되는 $\xi_0^0, \xi_1^1, \xi_2^2, \xi_3^2, \xi_3^1, \xi_3^0$

Find
- ξ_i 위에 위치하는 Bezier function ordinates (b_2^2, b_3^2)

두 ordinates의 내분점은 두 ordinates를 잇는 직선 상에 위치함

(예) b_3^2 는 직선 $\overline{b_1^1 b_2^1}$ 사이에 위치

$b_3^2 = ?$

$$\overline{b_2^1 b_3^2} : \overline{b_3^2 b_3^1} = \overline{\xi_2^1 \xi_3^2} : \overline{\xi_3^2 \xi_3^1}$$

$$\begin{cases} \overline{\xi_2^1 \xi_3^2} = \dfrac{2u+1}{3} - \dfrac{u+1}{3} = \dfrac{u}{3} \\ \overline{\xi_3^2 \xi_3^1} = \dfrac{u+2}{3} - \dfrac{2u+1}{3} = \dfrac{1-u}{3} \\ \overline{\xi_3^1 \xi_2^1} = \dfrac{u+2}{3} - \dfrac{u+1}{3} = \dfrac{1}{3} \end{cases}$$

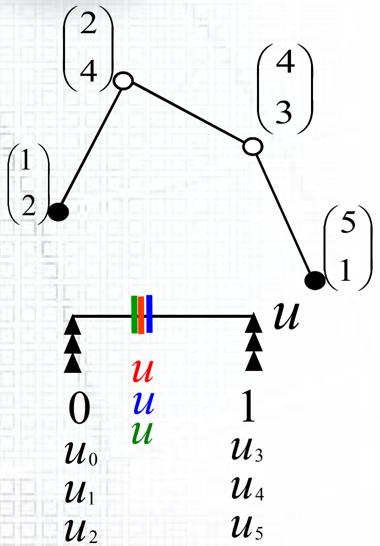
$$b_3^2 = (1-u)b_2^1 + u b_3^1$$

de Casteljau Algorithm by Knot Insertion

:: knot insertion을 왜 하는가?

➔ n번의 knot insertion을 하면 n차 Bezier function의 ordinate를 구할 수 있다.

knot Insertion 3



Greville abscissae updated

$$\xi_0^3 = \frac{0+0+0}{3} = 0 \quad = \xi_0^0$$

$$\xi_1^3 = \frac{0+0+u}{3} = \frac{u}{3} \quad = \xi_1^1$$

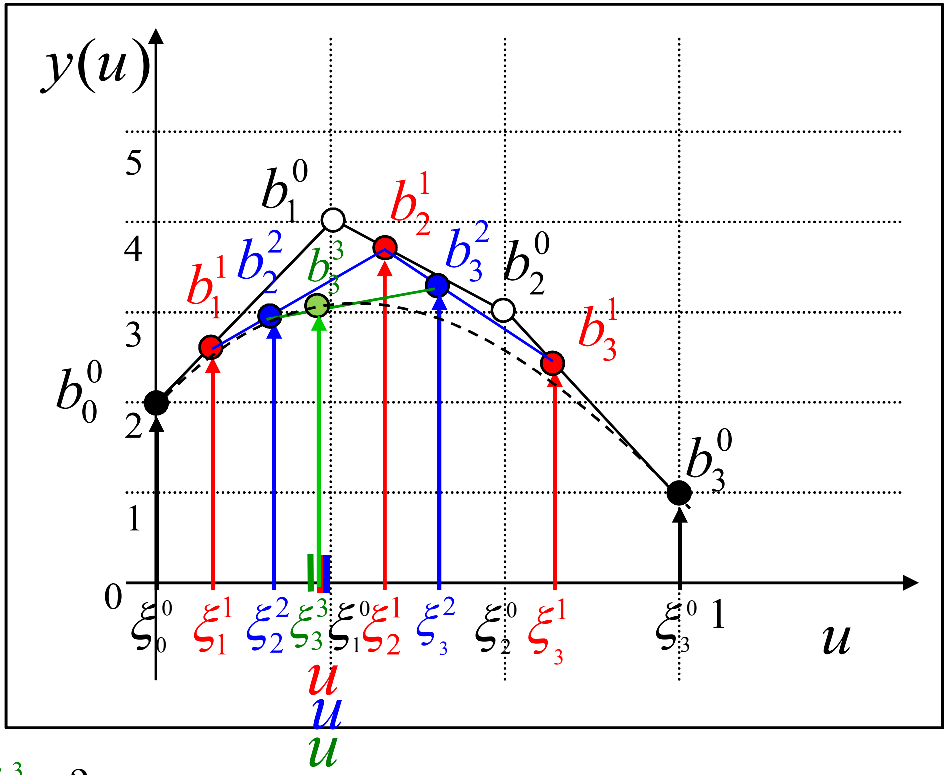
$$\xi_2^3 = \frac{0+u+u}{3} = \frac{2u}{3} \quad = \xi_2^2$$

$$\xi_3^3 = \frac{u+u+u}{3} = u$$

$$\xi_4^3 = \frac{u+u+1}{3} = \frac{2u+1}{3} = \xi_3^2$$

$$\xi_5^3 = \frac{u+1+1}{3} = \frac{u+2}{3} = \xi_3^1$$

$$\xi_6^3 = \frac{1+1+1}{3} = 1 \quad = \xi_3^0$$



Given
- Bezier y-control ordinate $b_0^0, b_1^0, b_2^0, b_3^0$
- Knot Insertion에 의해 계산되는 $\xi_0^0, \xi_1^1, \xi_2^2, \xi_3^3, \xi_3^2, \xi_3^1, \xi_3^0$

Find
- ξ_i 위에 위치하는 Bezier function ordinates
  (b_3^3)

두 ordinates의 내분점은 두 ordinates를 잇는 직선 상에 위치함

(예) b_3^3 는 직선 $\overline{b_2^2 b_3^2}$ 사이에 위치

$$b_3^3 = ?$$

$$\overline{b_2^2 b_3^3} : \overline{b_3^3 b_3^2} = \overline{\xi_2^2 \xi_3^3} : \overline{\xi_3^3 \xi_3^2}$$

$$\overline{\xi_2^2 \xi_3^3} = u - \frac{2u}{3} = \frac{u}{3}$$

$$\overline{\xi_3^3 \xi_3^2} = \frac{2u+1}{3} - u = \frac{1-u}{3}$$

$$\overline{\xi_3^2 \xi_2^2} = \frac{2u+1}{3} - \frac{2u}{3} = \frac{1}{3}$$

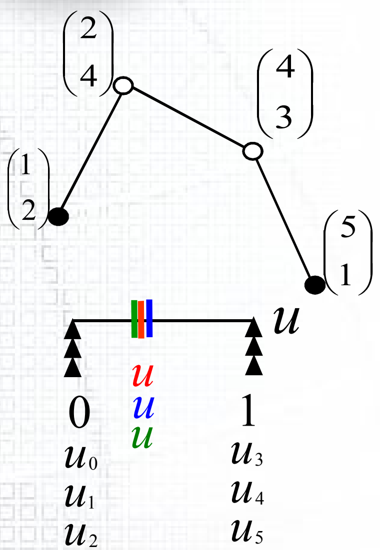
$$b_3^3 = (1-u)b_2^2 + u b_3^2$$

de Casteljau Algorithm by Knot Insertion

:: knot insertion을 왜 하는가?

➔ n번의 knot insertion을 하면 n차 Bezier function의 ordinate를 구할 수 있다.

knot Insertion 3



Greville abscissae updated

$$\xi_0^3=\frac{0+0+0}{3}=0 \qquad =\xi_0^0$$

$$\xi_1^3=\frac{0+0+u}{3}=\frac{u}{3} \qquad =\xi_1^1$$

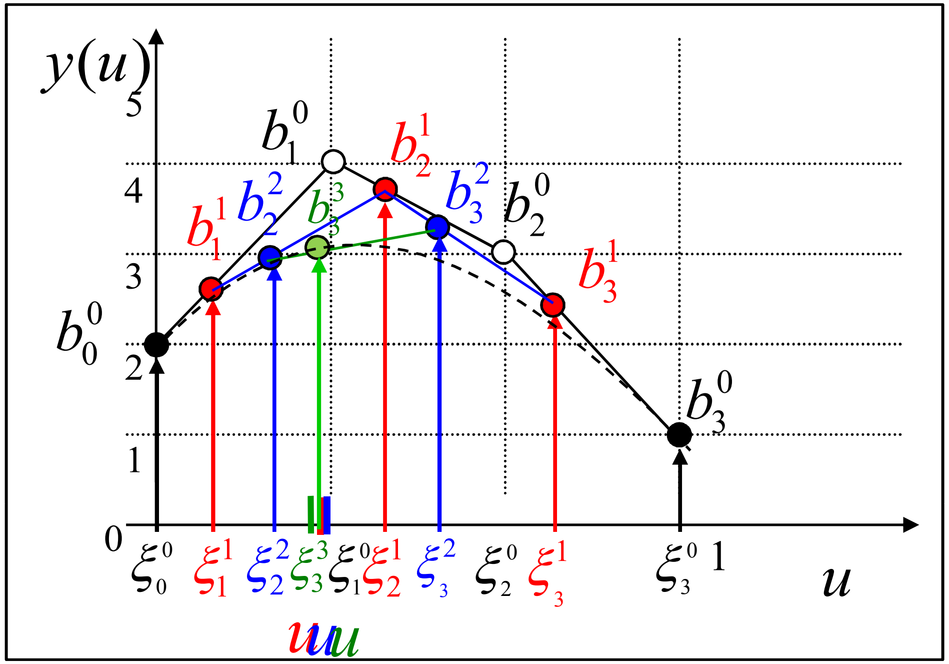
$$\xi_2^3=\frac{0+u+u}{3}=\frac{2u}{3} \qquad =\xi_2^2$$

$$\xi_3^3=\frac{u+u+u}{3}=u$$

$$\xi_4^3=\frac{u+u+1}{3}=\frac{2u+1}{3}=\xi_3^2$$

$$\xi_5^3=\frac{u+1+1}{3}=\frac{u+2}{3}=\xi_3^1$$

$$\xi_6^3=\frac{1+1+1}{3}=1 \qquad =\xi_3^0$$



$$b_1^1=(1-u)b_0^0+ub_1^0 \quad b_2^1=(1-u)b_1^0+ub_2^0 \quad b_3^1=(1-u)b_2^0+ub_3^0$$

$$b_2^2=(1-u)\boxed{b_1^1}+u\boxed{b_2^1} \quad b_3^2=(1-u)\boxed{b_2^1}+u\boxed{b_3^1}$$

$$b_3^3=(1-u)\boxed{b_2^2}+u\boxed{b_3^2}$$

Given
- Bezier y-control ordinate $b_0^0, b_1^0, b_2^0, b_3^0$
- Knot Insertion에 의해 계산되는 $\xi_0^0, \xi_1^1, \xi_2^2, \xi_3^3, \xi_3^2, \xi_3^1, \xi_3^0$

Find
- ξ_i 위에 위치하는 Bezier function ordinates
  (b_3^3)

두 ordinates의 내분점은 두 ordinates를 잇는 직선 상에 위치함

(예) b_3^3 는 직선 $\overline{b_2^2 b_3^2}$ 사이에 위치

Knot Insertion을 3번하여 구한 점은 매개 변수 u에 대한 3차식임
➡ u를 변경시키면 3차 곡선을 얻을 수 있음

$$b_3^3=b_0^0(1-u)^3+b_1^0\,3u(1-u)^2+b_2^0\,3u^2(1-u)+b_3^0u^3$$

Point Evaluation & Subdivision of Bezier Curve

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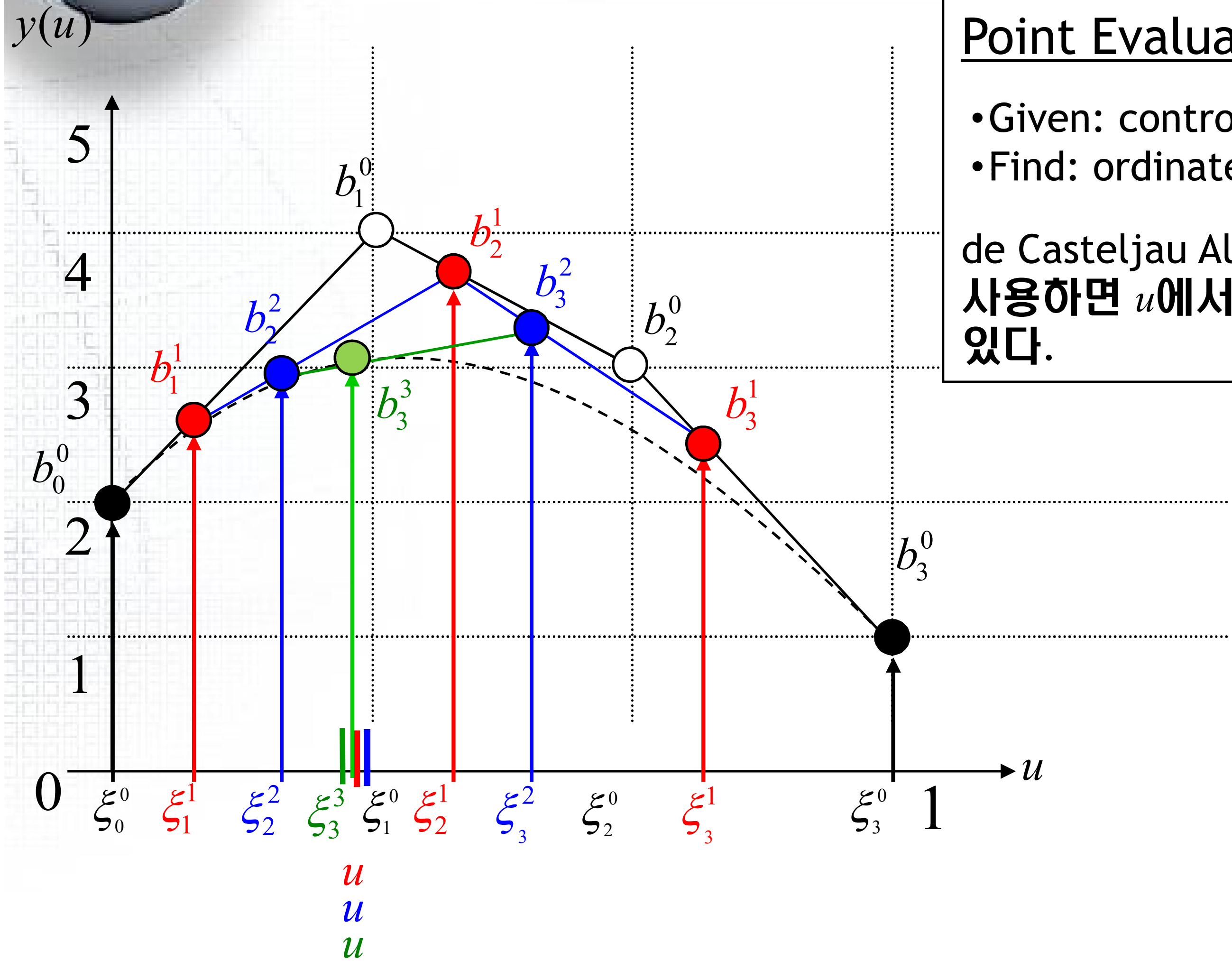
Point Evaluation & Subdivision of Bezier Curve

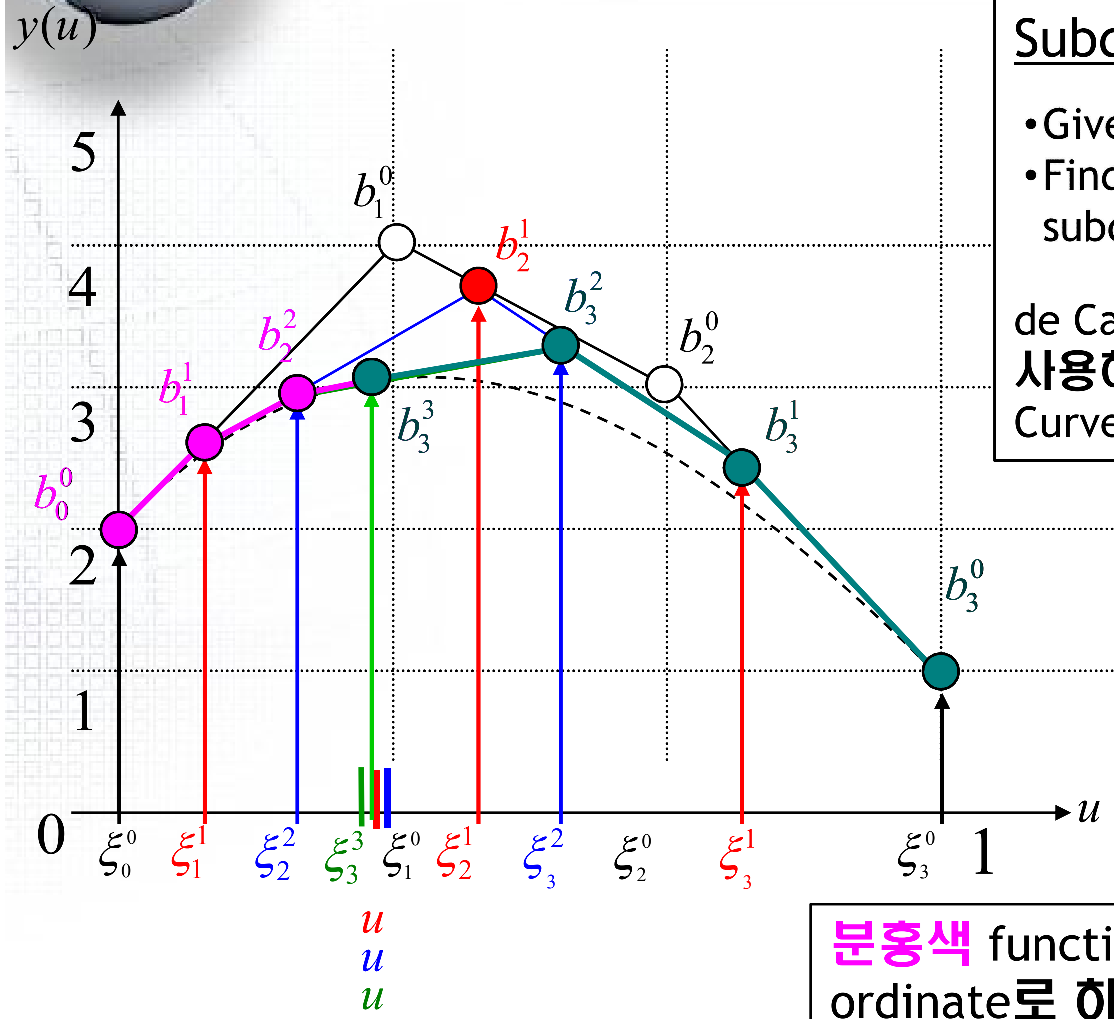


Point Evaluation

- Given: control ordinate of Bezier Curve
- Find: ordinate at parameter u

de Casteljau Algorithm(혹은 knot insertion)을 사용하면 u에서의 control ordinate를 구할 수 있다.

Point Evaluation & Subdivision of Bezier Curve



Subdivision

- Given: control ordinate of Bezier Curve
- Find: control ordinate of two Bezier Curve subdivided at parameter u

de Casteljau Algorithm(혹은 knot insertion)을 사용하면 u를 기준으로 나누어지는 두 Bezier Curve의 control ordinate를 구할 수 있다.

분홍색 function: $b_0^0,\ b_1^1,\ b_2^2,\ b_3^3$ 를 y-control ordinate로 하는 Bezier Function

청록색 function: $b_3^3,\ b_3^2,\ b_3^1,\ b_3^0$ 를 y-control ordinate로 하는 Bezier Function

de Boor algorithm

1) de Boor algorithm by Knot Insertion
2) de Boor algorithm
3) Relationship between de Boor algorithm & B-spline curves

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de Boor Algorithm by Knot Insertion

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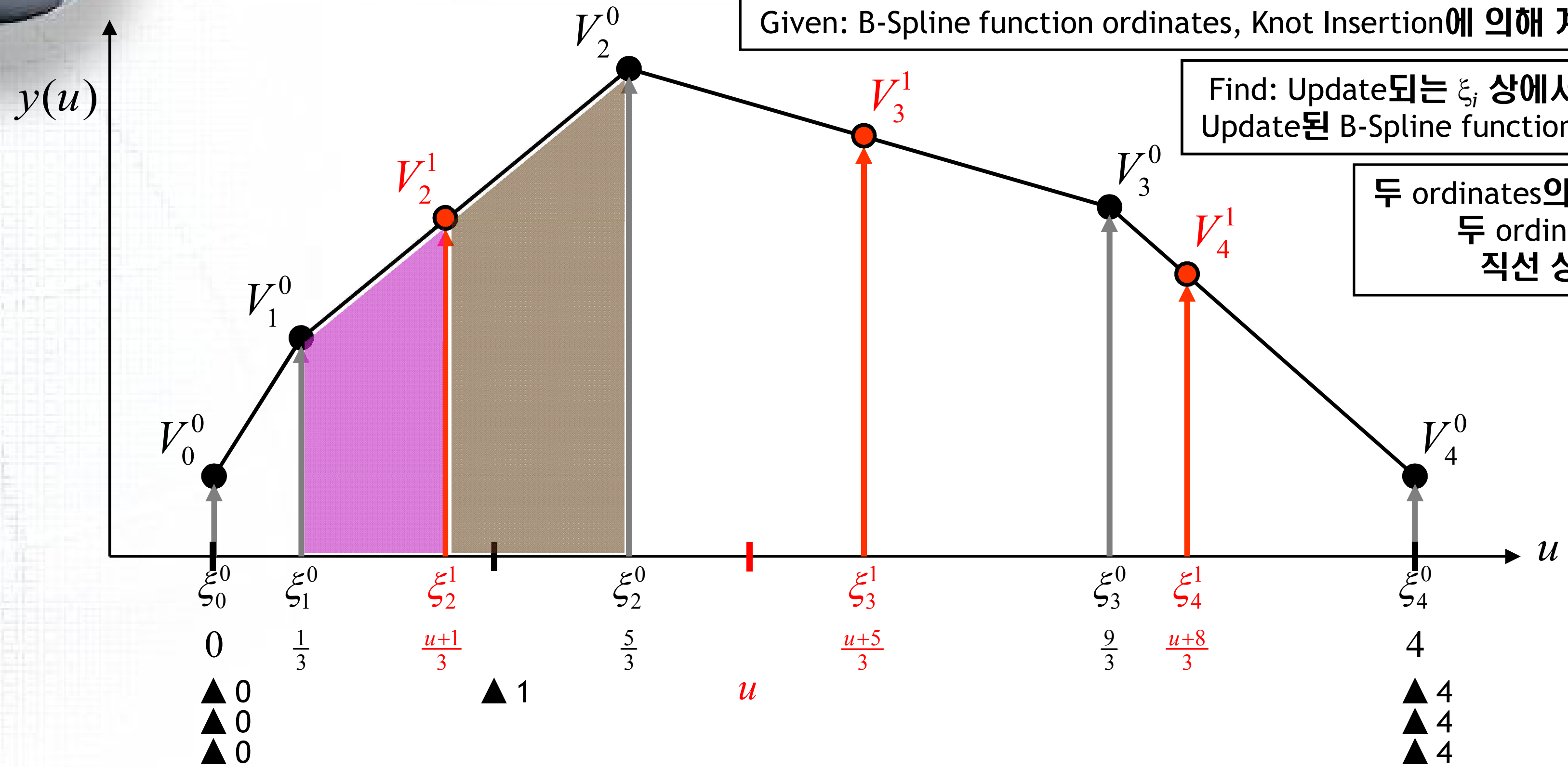
de Boor Algorithm by Knot Insertion(1)

- n-times knot Insertion ➡ Evaluation of n-th B-Spline function

Given: B-Spline function ordinates, Knot Insertion에 의해 계산되는 ξ_i

Find: Update되는 ξ_i 상에서 위치하는 Update된 B-Spline function ordinates

두 ordinates의 내분점은 두 ordinate를 잇는 직선 상에 위치함



$$\xi_0^1 = \frac{0+0+0}{3} = 0 = \xi_0^0$$

$$\xi_1^1 = \frac{0+0+1}{3} = \frac{1}{3} = \xi_1^0$$

$$\xi_2^1 = \frac{0+1+u}{3} = \frac{u+1}{3}$$

$$\xi_3^1 = \frac{1+u+4}{3} = \frac{u+5}{3}$$

$$\xi_4^1 = \frac{u+4+4}{3} = \frac{u+8}{3}$$

$$\xi_5^1 = \frac{4+4+4}{3} = 4 = \xi_4^0$$

$$V_2^1 = ?$$

$$\overline{V_1^0 V_2^1} : \overline{V_2^1 V_2^0} = \overline{\xi_1^0 \xi_2^1} : \overline{\xi_2^1 \xi_2^0}$$

$$\begin{cases} \overline{\xi_1^0 \xi_2^1} = \frac{u+1}{3} - \frac{1}{3} = \frac{u}{3} \\ \overline{\xi_2^1 \xi_2^0} = \frac{5}{3} - \frac{u+1}{3} = \frac{4-u}{3} \\ \overline{\xi_1^0 \xi_2^0} = \frac{5}{3} - \frac{1}{3} = \frac{4}{3} \end{cases}$$

$$V_2^1 = \frac{4-u}{4} V_1^0 + \frac{u}{4} V_2^0$$

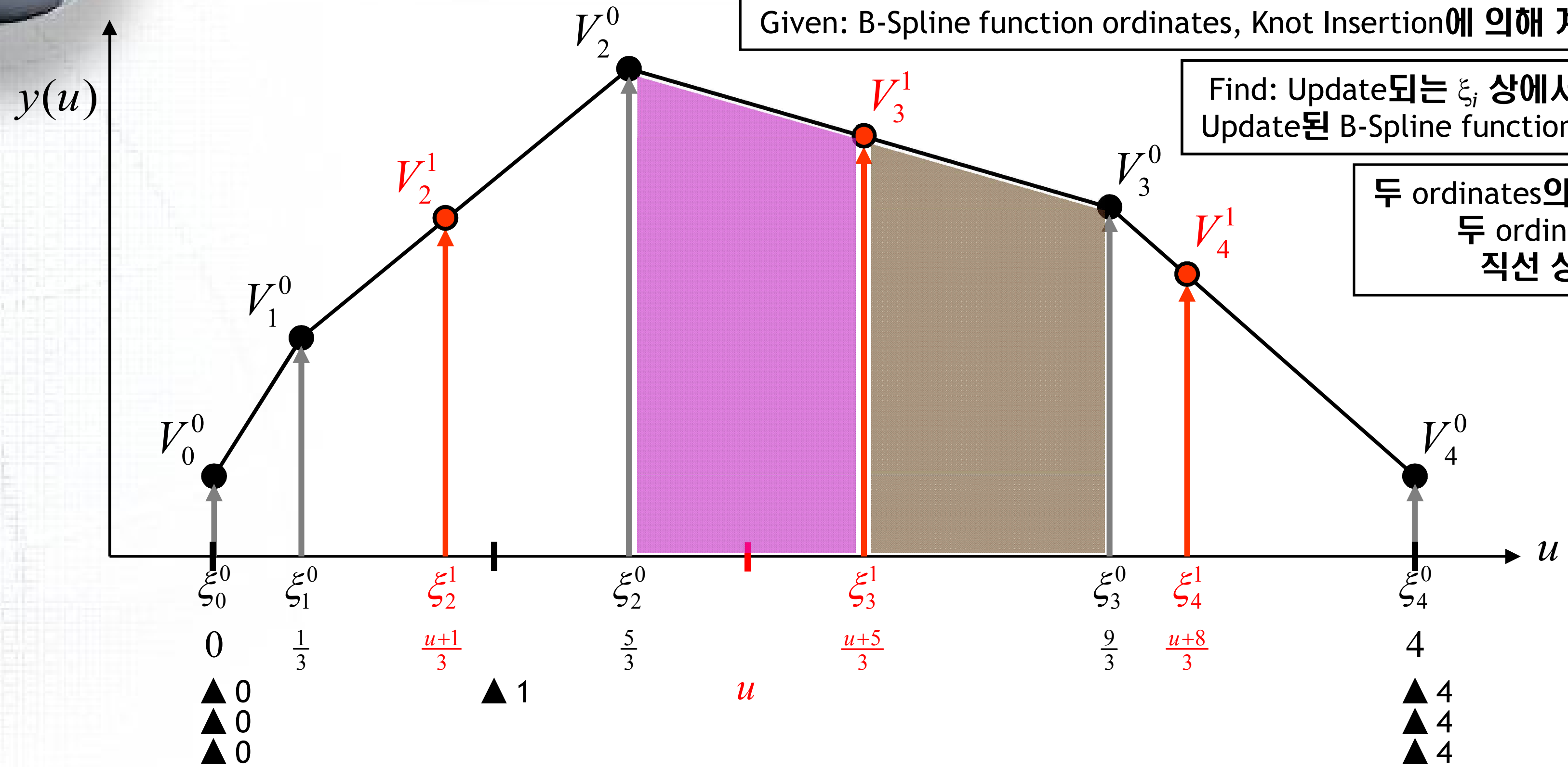
de Boor Algorithm by Knot Insertion(1)

- n-times knot Insertion ➡ Evaluation of n-th B-Spline function

Given: B-Spline function ordinates, Knot Insertion에 의해 계산되는 ξ_i

Find: Update되는 ξ_i 상에서 위치하는 Update된 B-Spline function ordinates

두 ordinates의 내분점은 두 ordinate를 잇는 직선 상에 위치함



$$\xi_0^1 = \frac{0+0+0}{3} = 0 = \xi_0^0$$

$$\xi_1^1 = \frac{0+0+1}{3} = \frac{1}{3} = \xi_1^0$$

$$\xi_2^1 = \frac{0+1+u}{3} = \frac{u+1}{3}$$

$$\xi_3^1 = \frac{1+u+4}{3} = \frac{u+5}{3}$$

$$\xi_4^1 = \frac{u+4+4}{3} = \frac{u+8}{3}$$

$$\xi_5^1 = \frac{4+4+4}{3} = 4 = \xi_4^0$$

$$V_3^1 = ?$$

$$\overline{V_2^0 V_3^1} : \overline{V_3^1 V_3^0} = \overline{\xi_2^0 \xi_3^1} : \overline{\xi_3^1 \xi_3^0}$$

$$\left\{ \begin{aligned} \overline{\xi_2^0 \xi_3^1} &= \frac{u+5}{3} - \frac{5}{3} = \frac{u}{3} \\ \overline{\xi_3^1 \xi_3^0} &= \frac{9}{3} - \frac{u+5}{3} = \frac{4-u}{3} \\ \overline{\xi_2^0 \xi_3^0} &= \frac{9}{3} - \frac{5}{3} = \frac{4}{3} \end{aligned} \right.$$

$$V_3^1 = \frac{4-u}{4} V_2^0 + \frac{u}{4} V_3^0$$

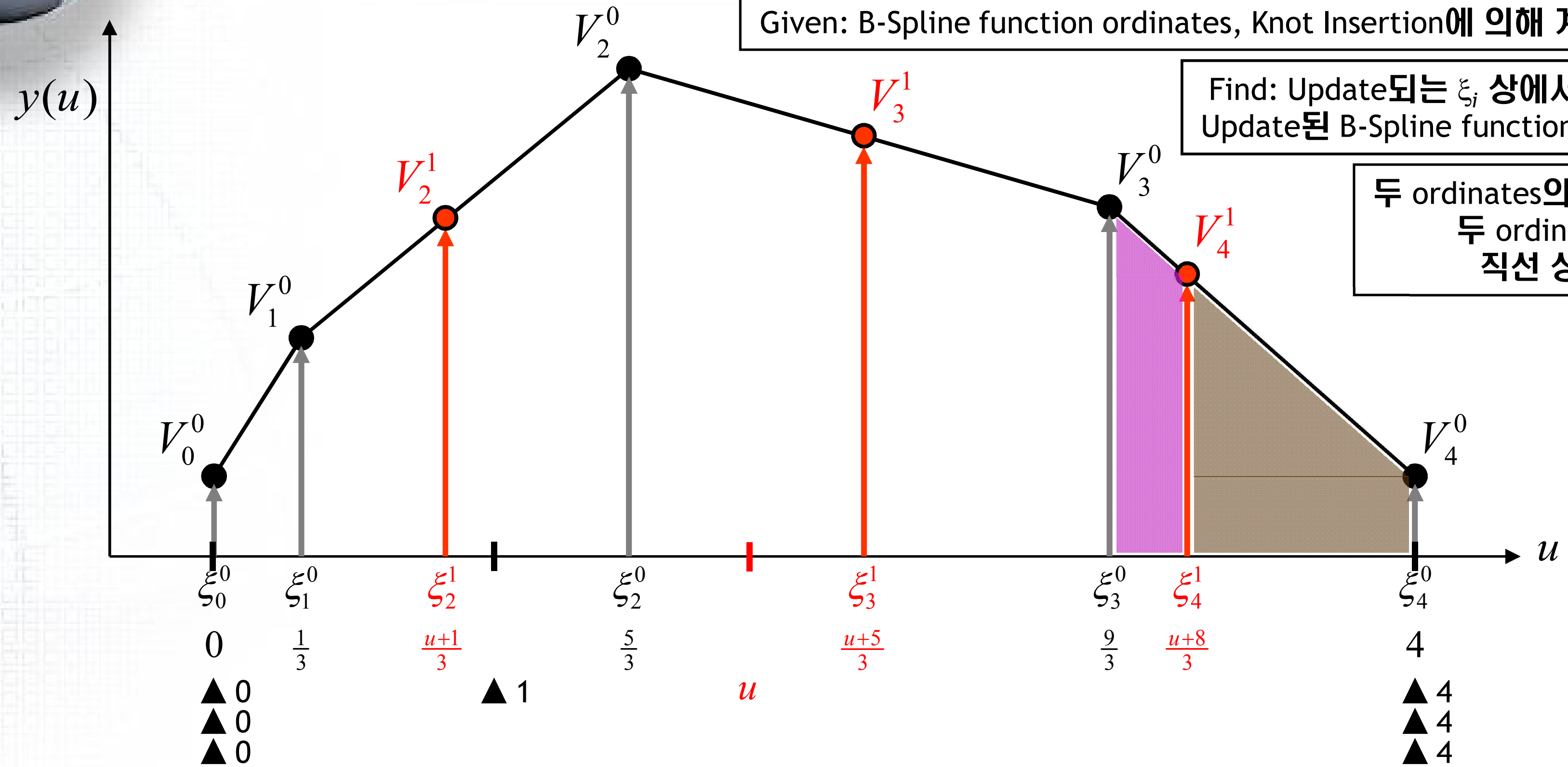
de Boor Algorithm by Knot Insertion(1)

- n-times knot Insertion ➡ Evaluation of n-th B-Spline function

Given: B-Spline function ordinates, Knot Insertion에 의해 계산되는 ξ_i

Find: Update되는 ξ_i 상에서 위치하는 Update된 B-Spline function ordinates

두 ordinates의 내분점은 두 ordinate를 잇는 직선 상에 위치함



$$\xi_0^1 = \frac{0+0+0}{3} = 0 = \xi_0^0$$

$$\xi_1^1 = \frac{0+0+1}{3} = \frac{1}{3} = \xi_1^0$$

$$\xi_2^1 = \frac{0+1+u}{3} = \frac{u+1}{3}$$

$$\xi_3^1 = \frac{1+u+4}{3} = \frac{u+5}{3}$$

$$\xi_4^1 = \frac{u+4+4}{3} = \frac{u+8}{3}$$

$$\xi_5^1 = \frac{4+4+4}{3} = 4 = \xi_4^0$$

$$V_4^1 = ?$$

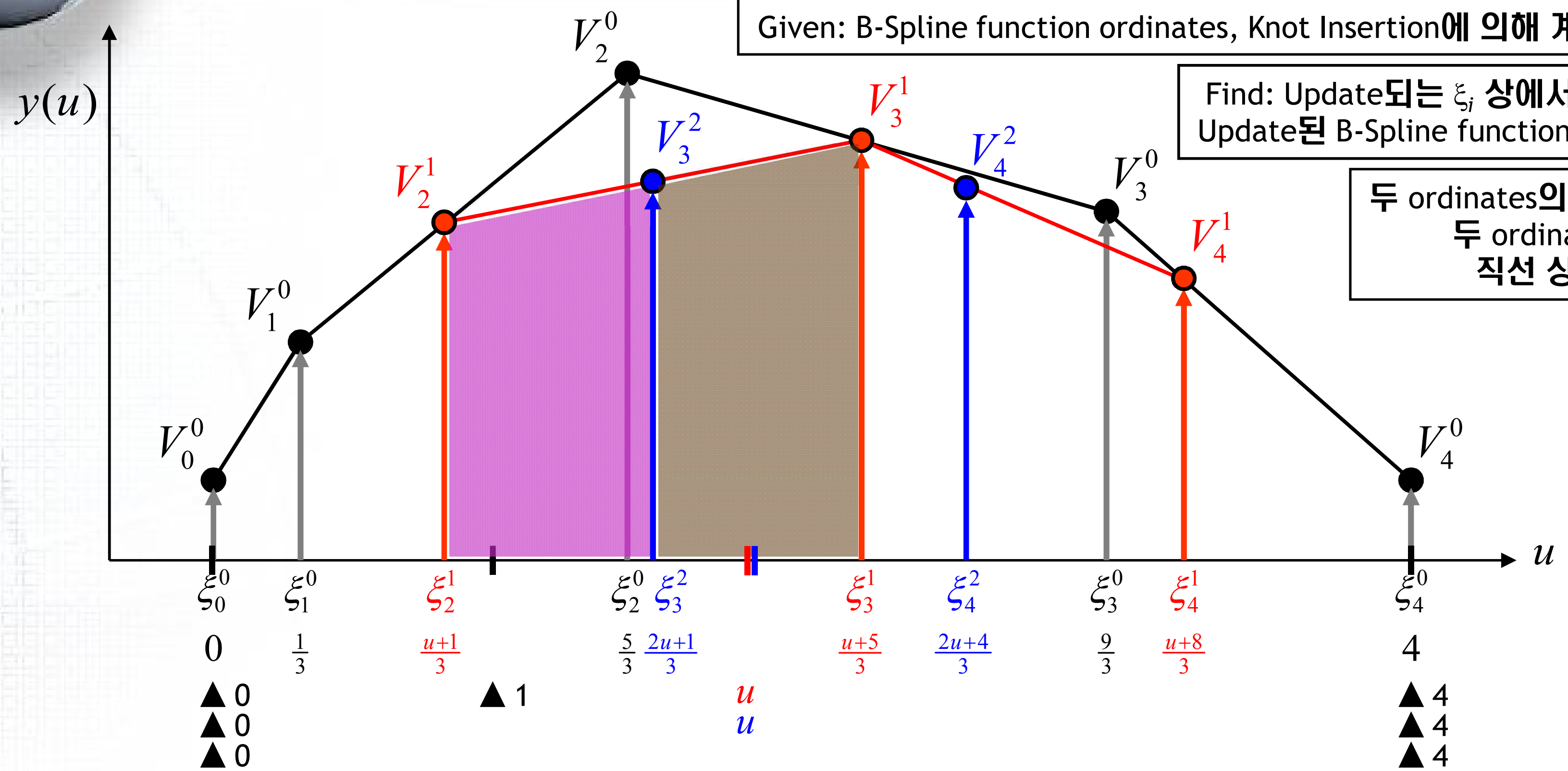
$$\overline{V_3^0 V_4^1} : \overline{V_4^1 V_4^0} = \overline{\xi_3^0 \xi_4^1} : \overline{\xi_4^1 \xi_4^0}$$

$$\left\{ \begin{aligned} \overline{\xi_3^0 \xi_4^1} &= \frac{u+8}{3} - \frac{9}{3} = \frac{u-1}{3} \\ \overline{\xi_4^1 \xi_4^0} &= 4 - \frac{u+8}{3} = \frac{4-u}{3} \\ \overline{\xi_3^0 \xi_4^0} &= 4 - \frac{9}{3} = \frac{3}{3} \end{aligned} \right.$$

$$V_4^1 = \frac{4-u}{3} V_3^0 + \frac{u-1}{3} V_4^0$$

de Boor Algorithm by Knot Insertion(2)

- n-times knot Insertion ➡ Evaluation of n-th B-Spline function



Given: B-Spline function ordinates, Knot Insertion에 의해 계산되는 ξ_i

Find: Update되는 ξ_i 상에서 위치하는 Update된 B-Spline function ordinates

두 ordinates의 내분점은 두 ordinate를 잇는 직선 상에 위치함

$$\xi_0^2 = \frac{0+0+0}{3} = 0 = \xi_0^0$$

$$\xi_1^2 = \frac{0+0+1}{3} = \frac{1}{3} = \xi_1^0$$

$$\xi_2^2 = \frac{0+1+u}{3} = \frac{u+1}{3} = \xi_2^1$$

$$\xi_3^2 = \frac{1+u+u}{3} = \frac{2u+1}{3}$$

$$\xi_4^2 = \frac{u+u+4}{3} = \frac{2u+4}{3}$$

$$\xi_5^2 = \frac{u+4+4}{3} = \frac{u+8}{3} = \xi_4^1$$

$$\xi_6^2 = \frac{4+4+4}{3} = 4 = \xi_4^0$$

$V_3^2 = ?$

$$\overline{V_2^1V_3^2} : \overline{V_3^2V_3^1} = \overline{\xi_2^1\xi_3^2} : \overline{\xi_3^2\xi_3^1}$$

$$\begin{cases} \overline{\xi_2^1\xi_3^2} = \frac{2u+1}{3} - \frac{u+1}{3} = \frac{u}{3} \\ \overline{\xi_3^2\xi_3^1} = \frac{u+5}{3} - \frac{2u+1}{3} = \frac{4-u}{3} \\ \overline{\xi_2^1\xi_3^1} = \frac{u+5}{3} - \frac{u+1}{3} = \frac{4}{3} \end{cases}$$

$$V_3^2 = \frac{4-u}{4}V_2^1 + \frac{u}{4}V_3^1$$

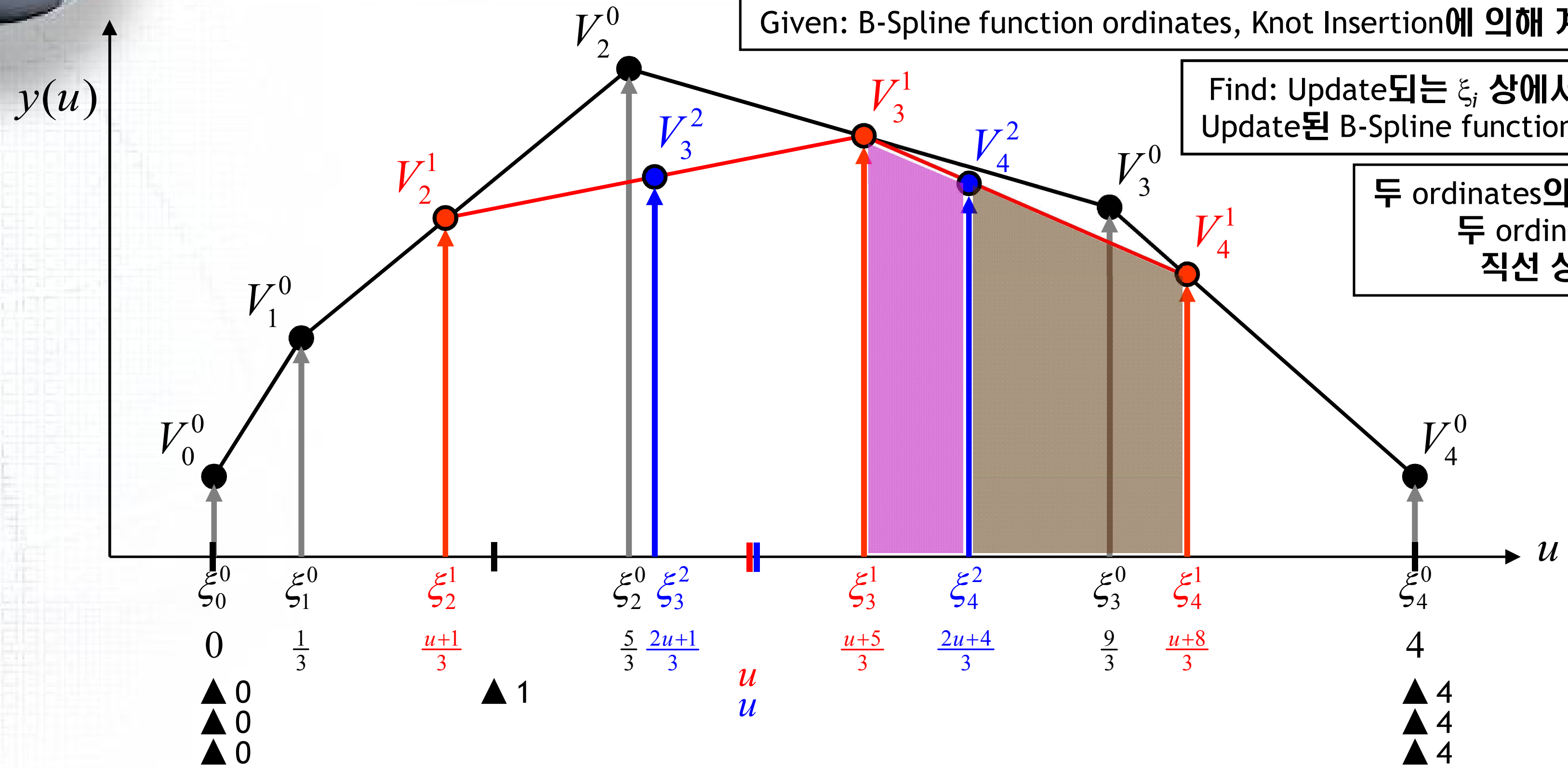
de Boor Algorithm by Knot Insertion(2)

- n-times knot Insertion ➡ Evaluation of n-th B-Spline function

Given: B-Spline function ordinates, Knot Insertion에 의해 계산되는 ξ_i

Find: Update되는 ξ_i 상에서 위치하는 Update된 B-Spline function ordinates

두 ordinates의 내분점은 두 ordinate를 잇는 직선 상에 위치함



$$\xi_0^2 = \frac{0+0+0}{3} = 0 = \xi_0^0$$

$$\xi_1^2 = \frac{0+0+1}{3} = \frac{1}{3} = \xi_1^0$$

$$\xi_2^2 = \frac{0+1+u}{3} = \frac{u+1}{3} = \xi_2^1$$

$$\xi_3^2 = \frac{1+u+u}{3} = \frac{2u+1}{3}$$

$$\xi_4^2 = \frac{u+u+4}{3} = \frac{2u+4}{3}$$

$$\xi_5^2 = \frac{u+4+4}{3} = \frac{u+8}{3} = \xi_4^1$$

$$\xi_6^2 = \frac{4+4+4}{3} = 4 = \xi_4^0$$

$V_4^2 = ?$

$$\overline{V_3^1 V_4^2} : \overline{V_4^2 V_4^1} = \overline{\xi_3^1 \xi_4^2} : \overline{\xi_4^2 \xi_4^1}$$

$$\begin{cases} \overline{\xi_3^1 \xi_4^2} = \dfrac{2u+4}{3} - \dfrac{u+5}{3} = \dfrac{u-1}{3} \\ \overline{\xi_4^2 \xi_4^1} = \dfrac{u+8}{3} - \dfrac{2u+4}{3} = \dfrac{4-u}{3} \\ \overline{\xi_3^1 \xi_4^1} = \dfrac{u+8}{3} - \dfrac{u+5}{3} = \dfrac{3}{3} \end{cases}$$

$$V_4^2 = \frac{4-u}{3} V_3^1 + \frac{u-1}{4} V_4^1$$

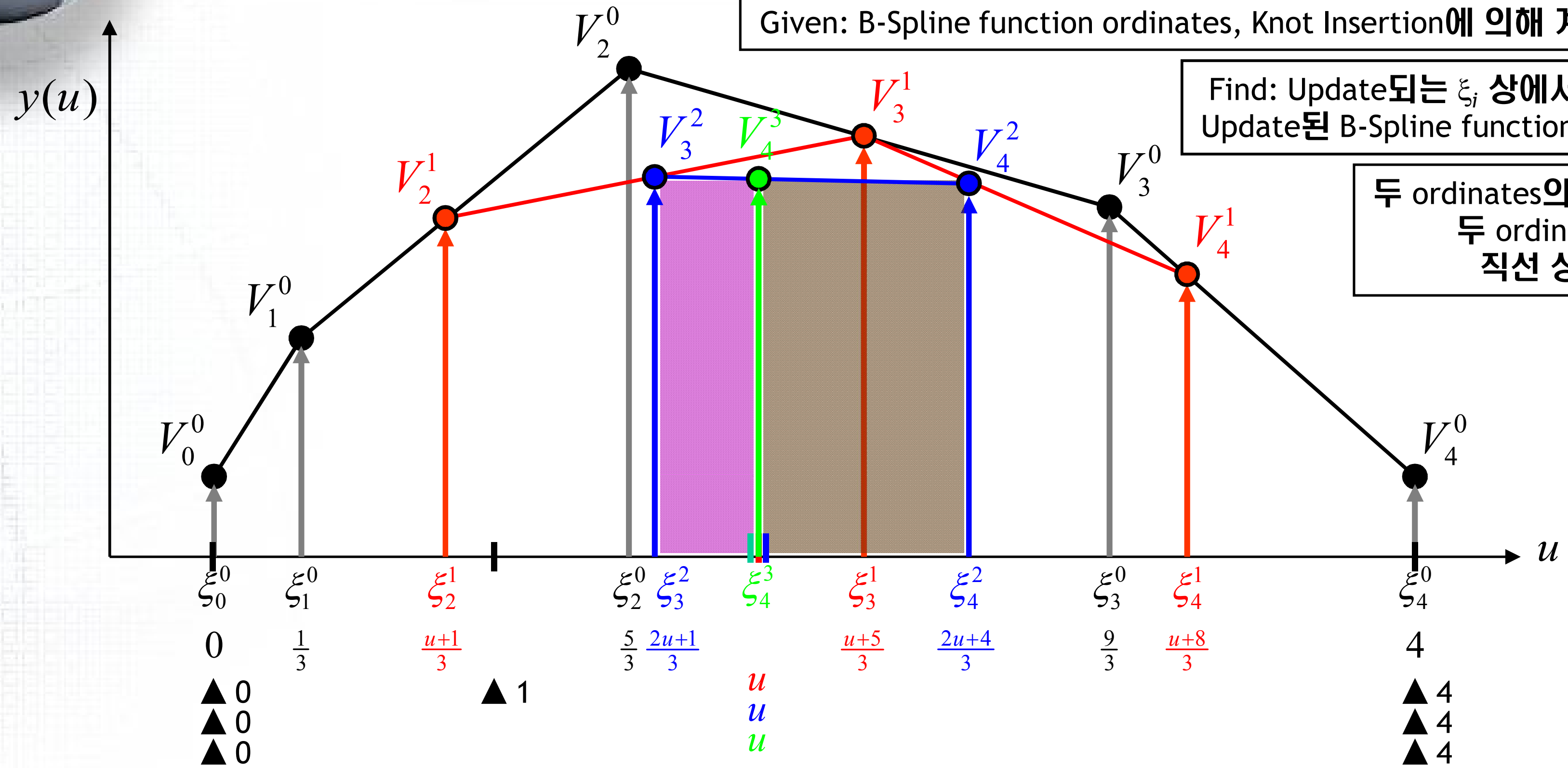
de Boor Algorithm by Knot Insertion(3)

- n-times knot Insertion ➡ Evaluation of n-th B-Spline function

Given: B-Spline function ordinates, Knot Insertion에 의해 계산되는 ξ_i

Find: Update되는 ξ_i 상에서 위치하는 Update된 B-Spline function ordinates

두 ordinates의 내분점은 두 ordinate를 잇는 직선 상에 위치함



$$\xi_0^3 = \frac{0+0+0}{3} = 0 = \xi_0^0$$

$$\xi_1^3 = \frac{0+0+1}{3} = \frac{1}{3} = \xi_1^0$$

$$\xi_2^2 = \frac{0+1+u}{3} = \frac{u+1}{3} = \xi_2^1$$

$$\xi_3^3 = \frac{1+u+u}{3} = \frac{2u+1}{3} = \xi_3^2$$

$$\xi_4^3 = \frac{u+u+u}{3} = u$$

$$\xi_5^3 = \frac{u+u+4}{3} = \frac{2u+4}{3} = \xi_4^2$$

$$\xi_6^3 = \frac{u+4+4}{3} = \frac{u+8}{3} = \xi_4^1$$

$$\xi_7^3 = \frac{4+4+4}{3} = 4 = \xi_4^0$$

$V_4^3 = ?$

$$\overline{V_3^2 V_4^3} : \overline{V_4^3 V_4^2} = \overline{\xi_3^2 \xi_4^3} : \overline{\xi_4^3 \xi_4^2}$$

$$\begin{cases} \overline{\xi_3^2 \xi_4^3} = u - \dfrac{2u+1}{3} = \dfrac{u-1}{3} \\ \overline{\xi_4^3 \xi_4^2} = \dfrac{2u+4}{3} - u = \dfrac{4-u}{3} \\ \overline{\xi_3^2 \xi_4^2} = \dfrac{2u+4}{3} - \dfrac{2u+1}{3} = \dfrac{3}{3} \end{cases}$$

$$V_4^3 = \frac{4-u}{3} V_3^2 + \frac{u-1}{3} V_4^2$$

de Boor algorithm by Knot Insertion(4)

- n-times knot Insertion ➡ Evaluation of n-th B-Spline function

$$V_2^1 = \frac{(4-u)}{4}V_1^0 + \frac{u}{4}V_2^0$$

$$V_3^1 = \frac{(4-u)}{4}V_2^0 + \frac{u}{4}V_3^0$$

$$V_4^1 = \frac{(4-u)}{3}V_3^0 + \frac{(u-1)}{3}V_4^0$$

$$V_3^2 = \frac{(4-u)}{4}V_2^1 + \frac{u}{4}V_3^1$$

$$V_4^2 = \frac{(4-u)}{3}V_3^1 + \frac{(u-1)}{4}V_4^1$$

$$V_4^3 = \frac{(4-u)}{3}V_3^2 + \frac{(u-1)}{3}V_4^2$$

이상과 같이 de-Boor Algorithm을
기하학으로 풀어 보니
V_4^3 는 아래와 같이 됨
여기서 ( )부분은 u의 3차식이며
이것은 Cox de-Boor의
basis function과 일치함

$$V_4^3 = \frac{(4-u)}{3}\left(\frac{(4-u)}{4}V_2^1 + \frac{u}{4}V_3^1\right) + \frac{(u-1)}{3}\left(\frac{(4-u)}{3}V_3^1 + \frac{(u-1)}{3}V_4^1\right)$$

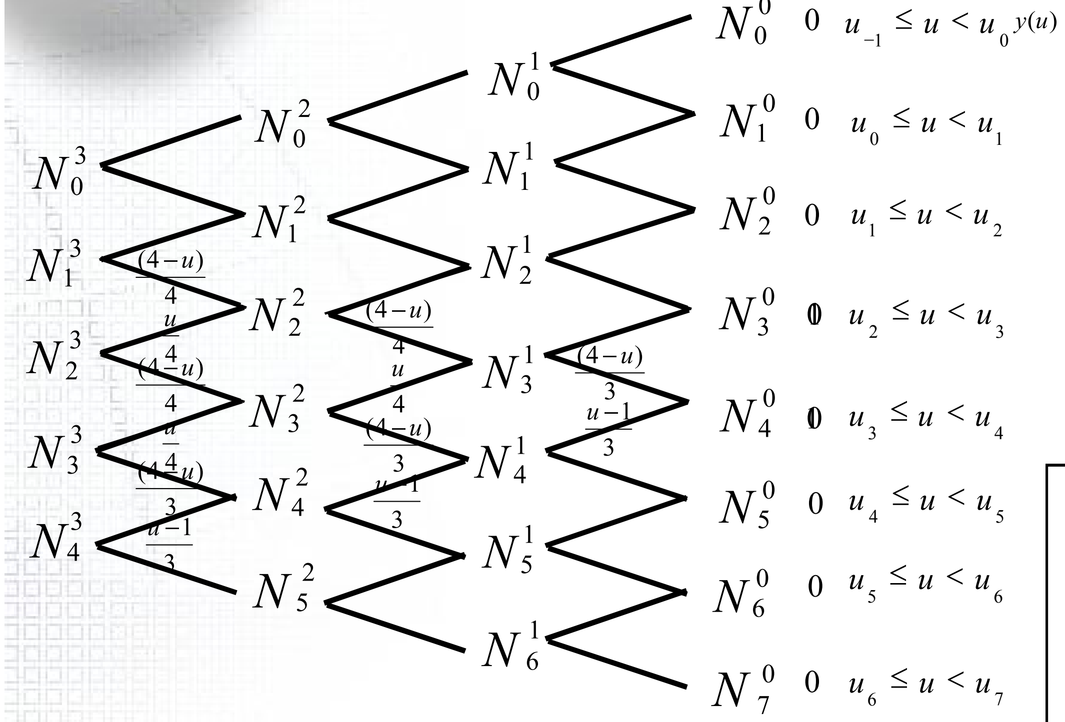
$$= \frac{(4-u)^2}{3\cdot 4}V_2^1 + \frac{(4-u)u}{3\cdot 4}V_3^1 + \frac{(u-1)(4-u)}{3\cdot 3}V_3^1 + \frac{(u-1)^2}{3\cdot 3}V_4^1$$

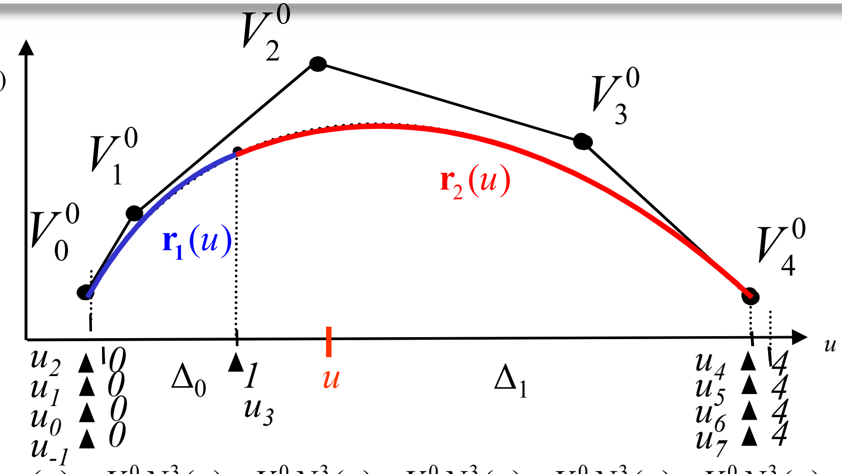
$$= \frac{(4-u)^2}{3\cdot 4}V_2^1 + \left(\frac{(4-u)u}{3\cdot 4} + \frac{(u-1)(4-u)}{3^2}\right)V_3^1 + \frac{(u-1)^2}{3^2}V_4^1$$

$$V_4^3 = \frac{(4-u)^2}{3\cdot 4}\left(\frac{(4-u)}{4}V_1^0 + \frac{u}{4}V_2^0\right) + \left(\frac{(4-u)u}{12} + \frac{(u-1)(4-u)}{9}\right)\left(\frac{(4-u)}{4}V_2^0 + \frac{u}{4}V_3^0\right) + \frac{(u-1)^2}{9}\left(\frac{(4-u)}{3}V_3^0 + \frac{(u-1)}{3}V_4^0\right)$$

$$V_4^3 = \frac{(4-u)^3}{48}V_1^0 + (4-u)^2\left(\frac{u}{24} + \frac{(u-1)}{36}\right)V_2^0 + (4-u)\left(\frac{u^2}{48} + \frac{u(u-1)}{36} + \frac{(u-1)^2}{27}\right)V_3^0 + \frac{(u-1)^3}{27}V_4^0$$

Cox-de Boor Algorithm: Evaluation of Cox-de Boor basis function





$$\mathbf{r}(u) = V_0^0 N_0^3(u) + V_1^0 N_1^3(u) + V_2^0 N_2^3(u) + V_3^0 N_3^3(u) + V_4^0 N_4^3(u)$$

Cox-de Boor Algorithm

$$N_i^n(u) = \frac{u - u_{i-1}}{u_{i+n-1} - u_{i-1}} N_i^{n-1}(u) + \frac{u_{i+n} - u}{u_{i+n} - u_i} N_{i+1}^{n-1}(u)$$

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_{i-1} \le u < u_i \\ 0 & \text{else} \end{cases}$$

$$\mathbf{r}_2(u) = \frac{(4-u)^3}{48} V_1^0 + (4-u)^2 \left(\frac{u}{24} + \frac{u-1}{36} \right) V_2^0 + (4-u) \left(\frac{u^2}{48} + \frac{u(u-1)}{36} + \frac{(u-1)^2}{27} \right) V_3^0 + \frac{(u-1)^3}{27} V_4^0$$

$$V_4^3 = \frac{(4-u)^3}{48} V_1^0 + (4-u)^2 \left(\frac{u}{24} + \frac{(u-1)}{36} \right) V_2^0 + (4-u) \left(\frac{u^2}{48} + \frac{u(u-1)}{36} + \frac{(u-1)^2}{27} \right) V_3^0 + \frac{(u-1)^3}{27} V_4^0$$

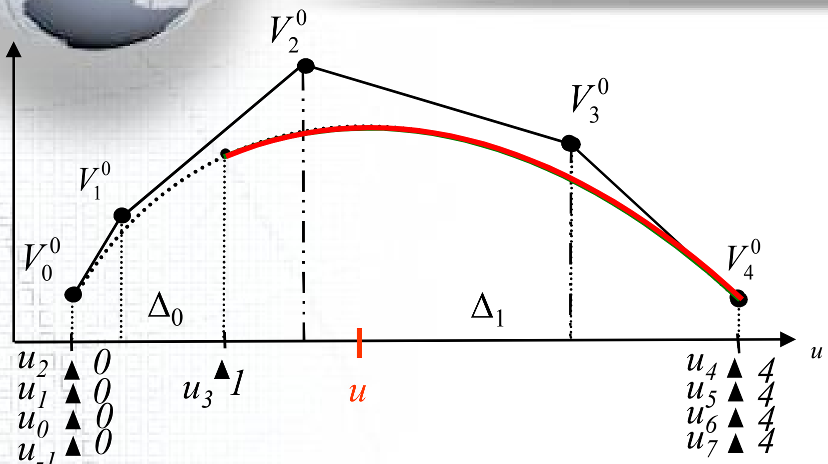
de Boor algorithm으로 구한 결과와 Cox- de Boor algorithm으로 구한 결과가 같다.

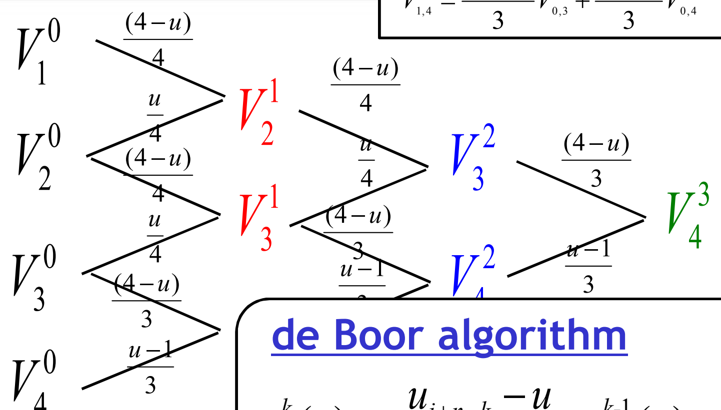
de Boor Algorithm & Cox-de Boor Algorithm

$$V^y_{1,2}=\frac{(4-u)}{4}V^y_{0,1}+\frac{u}{4}V^y_{0,2} \qquad V^y_{2,3}=\frac{(4-u)}{4}V^y_{1,2}+\frac{u}{4}V^y_{1,3}$$

$$V^y_{1,3}=\frac{(4-u)}{4}V^y_{0,2}+\frac{u}{4}V^y_{0,3} \qquad V^y_{2,4}=\frac{(4-u)}{3}V^y_{1,3}+\frac{(u-1)}{3}V^y_{1,4}$$

$$V^y_{1,4}=\frac{(4-u)}{3}V^y_{0,3}+\frac{(u-1)}{3}V^y_{0,4} \qquad V^y_{3,4}=\frac{(4-u)}{3}V^y_{2,3}+\frac{(u-1)}{3}V^y_{2,4}$$



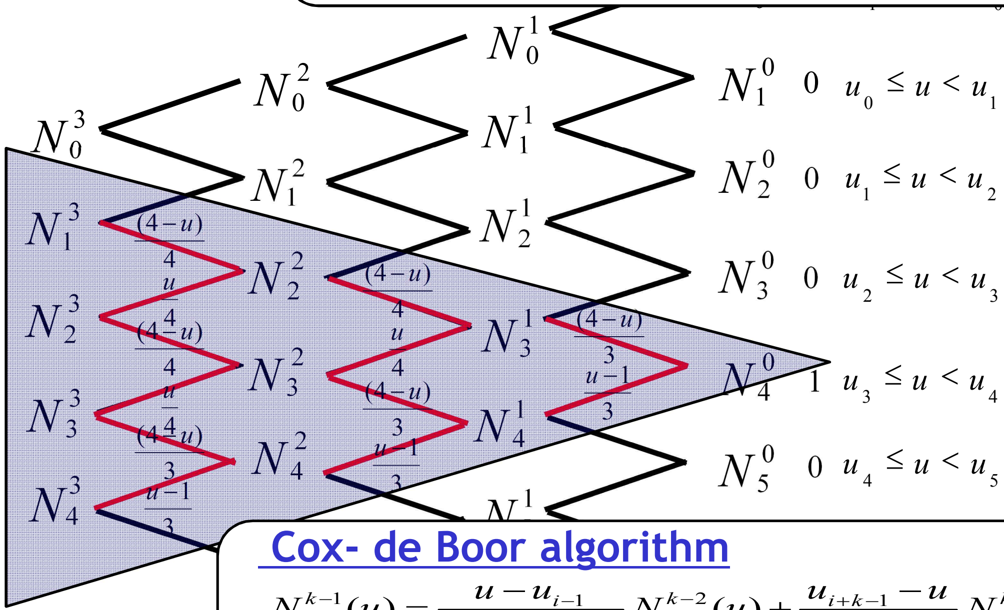


$$V_4^3=\frac{(4-u)^3}{48}V_1^0+(4-u)^2\left(\frac{u}{24}+\frac{(u-1)}{36}\right)V_2^0+$$

$$(4-u)\left(\frac{u^2}{48}+\frac{u(u-1)}{36}+\frac{(u-1)^2}{27}\right)V_3^0+\frac{(u-1)^3}{27}V_4^0$$

de Boor algorithm

$$\mathrm{v}_i^k(u)=\frac{u_{i+n-k}-u}{u_{i+n-k}-u_{i-1}}\mathrm{v}_{i-1}^{k-1}(u)+\frac{u-u_{i-1}}{u_{i+n-k}-u_{i-1}}\mathrm{v}_i^{k-1}(u)$$



매개 변수 u의 위치에서
동일한 3차식을 얻을 수 있음
➡ de Boor Algorithm과
Cox de-Boor Algorithm은 동일한 것

$$\mathbf{r}_2(u)=\frac{(4-u)^3}{48}V_1^0+(4-u)^2\left(\frac{u}{24}+\frac{u-1}{36}\right)V_2^0+$$

$$(4-u)\left(\frac{u^2}{48}+\frac{u(u-1)}{36}+\frac{(u-1)^2}{27}\right)V_3^0+\frac{(u-1)^3}{27}V_4^0$$

Cox- de Boor algorithm

$$N_i^{k-1}(u)=\frac{u-u_{i-1}}{u_{i+k-2}-u_{i-1}}N_i^{k-2}(u)+\frac{u_{i+k-1}-u}{u_{i+k-1}-u_i}N_{i+1}^{k-2}(u)$$

$$N_i^0(u)=\begin{cases}1 & \text{if } u_{i-1}\le u<u_i\\ 0 & \text{else}\end{cases}$$