



16 Heat Equation

16.1 Solution by Fourier Series

- Energy conservation in the control volume

$$\dot{E}_{st} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen}$$

where \dot{E}_{st} : Energy stored in the control volume
 \dot{E}_{in} : Energy entering in the control volume through the surfaces
 \dot{E}_{out} : Energy exiting out of the control volume through the surfaces
 \dot{E}_{gen} : Energy generated inside the control volume

$$\dot{E}_{st} = \frac{\partial}{\partial t}(dm \cdot c_p T) = \rho c_p dV \frac{\partial T}{\partial t} = \rho c_p \frac{\partial T}{\partial t} dx dy dz$$

ρ : density (kg/m³), c_p : heat capacity (J/kg·K)

$$\dot{E}_{in} = \dot{q}_x dy dz + \dot{q}_y dz dx + \dot{q}_z dx dy$$

$$\dot{E}_{out} = \dot{q}_{x+dx} dy dz + \dot{q}_{y+dy} dz dx + \dot{q}_{z+dz} dx dy$$

Fourier's law of heat conduction

$$\dot{q}_x = -k \frac{\partial T}{\partial x}, \quad \dot{q}_y = -k \frac{\partial T}{\partial y}, \quad \dot{q}_z = -k \frac{\partial T}{\partial z}$$

k : thermal conductivity (W/m·K)

By Taylor series expansion, ignoring higher order terms

$$\dot{q}_{x+dx} = \dot{q}_x + \frac{\partial \dot{q}_x}{\partial x} dx \quad \dot{q}_{y+dy} = \dot{q}_y + \frac{\partial \dot{q}_y}{\partial y} dy \quad \dot{q}_{z+dz} = \dot{q}_z + \frac{\partial \dot{q}_z}{\partial z} dz$$

$$\begin{aligned} \dot{E}_{in} - \dot{E}_{out} &= (\dot{q}_x dy dz + \dot{q}_y dz dx + \dot{q}_z dx dy) \\ &- \left(\dot{q}_x dy dz + \frac{\partial \dot{q}_x}{\partial x} dx dy dz + \dot{q}_y dz dx + \frac{\partial \dot{q}_y}{\partial y} dy dz dx + \dot{q}_z dx dy + \frac{\partial \dot{q}_z}{\partial z} dz dx dy \right) \\ &= - \left(\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z} \right) dx dy dz \\ &= \left[\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) \right] dx dy dz \end{aligned}$$

$$\dot{E}_{gen} = \dot{g} dx dy dz \quad \dot{g} : \text{volumetric heat generation (W/m}^3\text{)}$$

$$\rho c_p \frac{\partial T}{\partial t} dx dy dz = \left[\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) \right] dx dy dz + \dot{g} dx dy dz$$

For $k = k(x, y, z)$,

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{g}$$

For $k = \text{constant}$,

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \dot{g}$$

For steady-state,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \dot{g} = 0$$

No heat generation,

$$\begin{aligned} \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} &= 0 \\ \therefore \nabla^2 T &= 0 \end{aligned}$$

- Transient heat conduction equation with no heat generation:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T, \quad \alpha = \frac{k}{c\rho}$$

α : thermal diffusivity (m^2/s)

- One-dimensional heat equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad (1)$$

- Boundary conditions

$$T(0, t) = 0, \quad T(L, t) = 0 \quad \text{for all } t \quad (2)$$

- Initial condition

$$T(x, 0) = f(x) \quad (3)$$

First Step. Two Ordinary Differential Equations

$$T(x, t) = F(x) \cdot G(t) \quad (4)$$

$$F \dot{G} = \alpha F'' G$$

$$\frac{\dot{G}}{\alpha G} = \frac{F''}{F} = -p^2 \quad (5)$$

because $\dot{G}/\alpha G$ depends only on t and F''/F only on x . Both must be equals to a constant $-p^2$.

$$F'' + p^2 F = 0 \quad (6)$$

$$\dot{G} + \alpha p^2 G = 0 \quad (7)$$

Second Step. Satisfying the Boundary conditions

- A general solution of (6) is

$$F(x) = A \cos px + B \sin px \quad (8)$$

- From the boundary conditions (2) it follow that

$$T(0, t) = F(0)G(t) = 0, \quad \text{and} \quad T(L, t) = F(L)G(t) = 0$$

Since $G \equiv 0$, it would give a trivial solution $u \equiv 0$.

$$F(0) = F(L) = 0$$

$$F(0) = A = 0$$

$$F(L) = B \sin pL = 0 \Rightarrow \sin pL = 0$$

$$p = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

- Setting $B = 1$,

$$F_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots$$

- From (7)

$$\dot{G}_n + \lambda_n^2 G_n = 0 \quad \text{where} \quad \lambda_n = \alpha p_n = \sqrt{\alpha} \frac{n\pi}{L}.$$

- General solution

$$G_n(t) = B_n e^{-\alpha p_n^2 t} \quad n = 1, 2, \dots$$

$$T_n(x, t) = F_n(x)G_n(t) = B_n \sin \frac{n\pi x}{L} e^{-\alpha p_n^2 t} \quad (n = 1, 2, \dots) \quad (9)$$

- Eigenfunctions: $B_n \sin \frac{n\pi x}{L} e^{-\alpha p_n^2 t}$

- Eigenvalue: $\lambda_n = \sqrt{\alpha} \cdot p_n = \sqrt{\alpha} \frac{n\pi}{L}$

Third Step. Solution of the Entire Problem

$$T(x, t) = \sum_{n=1}^{\infty} T_n(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\alpha p_n^2 t}, \quad \left(p_n = \frac{n\pi}{L}\right) \quad (10)$$

- From (3)

$$T(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} dx = f(x)$$

$$B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi x}{L} dx \quad (n = 1, 2, \dots) \quad (11)$$

Example 1. Sinusoidal initial temperature

$\rho=8.92 \text{ g/cm}^3$, $c=0.092 \text{ cal/(g } ^\circ\text{C)}$, $k=0.95 \text{ cal/(cm sec } ^\circ\text{C)}$, $L=80 \text{ cm}$.

Solution.

$$T(x, 0) = f(x) = 100 \cdot \sin \frac{\pi x}{80} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

By orthogonality of the trigonometric functions,

$$B_1 = 100, \quad B_2 = B_3 = \dots = 0$$

$$p_1 = \frac{\pi}{L}, \quad \alpha = \frac{k}{\rho c} = \frac{0.95}{8.92 \times 0.092} = 1.158 \text{ cm}^2/\text{sec}$$

$$T(x, t) = 100 \cdot \sin \frac{\pi x}{80} \cdot e^{-1.158 \frac{\pi^2}{L^2} t}$$

At $x = L/2 = 40$ cm,

$$100 \cdot e^{-1.158 \frac{\pi^2}{80^2} t} = 50$$
$$t = \frac{\ln 0.5}{-1.158 \frac{\pi^2}{80^2}} = 388 \text{ seconds}$$

Example 2. Speed of decay

Solution.

$$T(x, 0) = f(x) = 100 \cdot \sin \frac{3\pi x}{80}$$

$$\therefore T(x, t) = 100 \cdot \sin \frac{3\pi x}{80} e^{-1.158 \cdot \frac{3^2 \pi^2}{80^2} t}$$

At $x = 40/3$ cm,

$$50 = 100 \cdot \sin \frac{3\pi x}{80} e^{-10.422 \frac{\pi^2}{80^2} t}$$

$$t = \frac{\ln 0.5}{-10.422 \frac{\pi^2}{80^2}} \approx 43 \text{ seconds}$$

Example 3. 'Triangular' initial temperature in a bar

$$T(x, 0) = f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ L - x & \text{if } L/2 < x < L \end{cases}$$

Solution.

From (11)

$$\begin{aligned}
B_n &= \frac{2}{L} \left[\int_0^{L/2} x \sin \frac{n\pi x}{L} dx + \int_{L/2}^L (L-x) \sin \frac{n\pi x}{L} dx \right] \\
&= \frac{2}{L} \left[-\frac{L}{n\pi} x \cos \frac{n\pi x}{L} \Big|_0^{L/2} + \frac{L}{n\pi} \int_0^{L/2} \cos \frac{n\pi x}{L} dx \right] \\
&+ \frac{2}{L} \left[-\frac{L}{n\pi} (L-x) \cos \frac{n\pi x}{L} \Big|_{L/2}^L - \frac{L}{n\pi} \int_{L/2}^L \cos \frac{n\pi x}{L} dx \right] \\
&= \frac{2}{L} \left[-\frac{L^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi x}{L} \Big|_0^{L/2} + \frac{L^2}{2n\pi} \cos \frac{n\pi}{2} - \frac{L^2}{n^2\pi^2} \sin \frac{n\pi x}{L} \Big|_{L/2}^L \right] \\
&= \frac{2L}{n^2\pi^2} \left(\sin \frac{n\pi}{2} - \sin n\pi + \sin \frac{n\pi}{2} \right) = \frac{4L}{n^2\pi^2} \sin \frac{n\pi}{2}
\end{aligned}$$

If n is even, $B_n = 0$.

Otherwise n is odd,

$$B_n = \frac{4L}{n^2\pi^2} \quad (n = 1, 5, 9, \dots) \quad \text{and} \quad B_n = -\frac{4L}{n^2\pi^2} \quad (n = 3, 7, 11, \dots)$$

$$T(x, t) = \frac{4L}{\pi^2} \left[\sin \frac{\pi x}{L} \exp \left\{ -\alpha \left(\frac{\pi}{L} \right)^2 t \right\} - \frac{1}{9} \sin \frac{3\pi x}{L} \exp \left\{ -\alpha \left(\frac{3\pi}{L} \right)^2 t \right\} + \dots \right]$$

Example 4. Bar with insulated ends. Eigenvalue 0

- Governing equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} = 0$$

- Boundary conditions

$$T_x(0, t) = 0, \quad T_x(L, t) = 0$$

- Initial condition

$$T(x, 0) = f(x)$$

- Separation of variables

$$T(x, t) = F(x) \cdot G(t)$$

$$\dot{G}F = \alpha GF''$$

$$\frac{\dot{G}}{\alpha G} = \frac{F''}{F} = -p^2$$

$$F'' + p^2 F = 0$$

$$\dot{G} + \alpha p^2 G$$

$$F(x) = A \cos px + B \sin px$$

$$T_x(0, t) = F'(0)G(t) = 0, \quad T_x(L, t) = F'(L)G(t) = 0$$

$$\therefore F'(0) = F'(L) = 0$$

$$F'(x) = -Ap \sin px + Bp \cos px$$

$$F'(0) = Bp = 0, \quad \Rightarrow \quad B = 0$$

$$F'(L) = -Ap \sin pL = 0, \quad \Rightarrow \quad \sin pL = 0$$

$$\therefore p = p_n = \frac{n\pi}{L} \quad (n = 0, 1, 2, \dots)$$

$$F_n(x) = \cos \frac{n\pi x}{L} \quad (n = 0, 1, 2, \dots)$$

$$G_n(t) = A_n e^{-\alpha \left(\frac{n\pi}{L}\right)^2 t}$$

- Eigenfunctions

$$T_n(x, t) = F_n(x)G_n(t) = A_n \cos \frac{n\pi x}{L} e^{-\alpha \left(\frac{n\pi}{L}\right)^2 t} \quad (n = 0, 1, 2, \dots) \quad (12)$$

- Eigenvalues

$$\lambda_n = \sqrt{\alpha} \frac{n\pi}{L}$$

$$T(x, t) = \sum_{n=0}^{\infty} T_n(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-\alpha \left(\frac{n\pi}{L}\right)^2 t} \quad (13)$$

- With the initial condition

$$T(x, 0) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x)$$

$$A_0 = \frac{1}{L} \int_0^L f(x) dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad (n = 0, 1, 2, \dots) \quad (14)$$

Example 5. 'Triangular' initial temperature in a bar with insulated ends

$$T(x, 0) = f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ L - x & \text{if } L/2 < x < L \end{cases}$$

Solution.

$$\begin{aligned}
A_0 &= \frac{1}{L} \int_0^L f(x) dx = \frac{1}{L} \cdot \frac{1}{2} L \cdot \frac{L}{2} = \frac{L}{4} \\
A_n &= \frac{2}{L} \left[\int_0^{L/2} x \cos \frac{n\pi x}{L} dx + \int_{L/2}^L (L-x) \cos \frac{n\pi x}{L} dx \right] \\
&= \frac{2}{L} \left[\frac{L}{n\pi} x \sin \frac{n\pi x}{L} \Big|_0^{L/2} - \frac{L}{n\pi} \int_0^{L/2} \sin \frac{n\pi x}{L} dx \right] \\
&\quad + \frac{2}{L} \left[\frac{L}{n\pi} (L-x) \sin \frac{n\pi x}{L} \Big|_{L/2}^L + \frac{L}{n\pi} \int_{L/2}^L \sin \frac{n\pi x}{L} dx \right] \\
&= \frac{2}{L} \left[\frac{L^2}{2n\pi} \sin \frac{n\pi}{2} + \left(\frac{L}{n\pi} \right)^2 \cos \frac{n\pi}{L} \Big|_0^{L/2} - \frac{L^2}{2n\pi} \sin \frac{n\pi}{2} - \left(\frac{L}{n\pi} \right)^2 \cos \frac{n\pi}{L} \Big|_{L/2}^L \right] \\
&= \frac{2L}{n^2\pi^2} \left(\cos \frac{n\pi}{2} - 1 + \cos \frac{n\pi}{2} - \cos n\pi \right) \\
&\quad \therefore A_n = \frac{2L}{n^2\pi^2} \left(2 \cos \frac{n\pi}{2} - \cos n\pi - 1 \right)
\end{aligned}$$

- If n is odd, $A_n = 0$.
- If $n = 2, 6, 10, \dots$,

$$A_n = -\frac{8L}{n^2\pi^2}$$

- If $n = 4, 8, 12, \dots$, $A_n = 0$.

$$T(x, t) = \frac{L}{4} - \frac{8L}{\pi^2} \left\{ \frac{1}{2^2} \cos \frac{2\pi x}{L} \exp \left[-\alpha \left(\frac{2\pi}{L} \right)^2 t \right] + \frac{1}{6^2} \cos \frac{6\pi x}{L} \exp \left[-\alpha \left(\frac{6\pi}{L} \right)^2 t \right] + \dots \right\}$$

Steady-State Two-Dimensional Heat Flow

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

- If the heat flow is steady, then $\partial T / \partial t = 0$.

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{15}$$

- Boundary value problem

Dirichlet problem if T is prescribed on C ,

Neumann problem if the normal derivative $T_n = \partial T / \partial n$ is prescribed on C , **Mixed problem** if T is prescribed on a portion of C and T_n on the rest of C .

Dirichlet Problem in a Rectangle R

- Governing equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

- Boundary conditions

$$\begin{aligned} T(0, y) &= 0, & T(a, y) &= 0 \\ T(x, 0) &= 0, & T(x, b) &= f(x) \end{aligned}$$

- Separation of variables

$$\begin{aligned} T(x, y) &= F(x) \cdot G(y) \\ \frac{1}{F} \cdot \frac{d^2 F}{dx^2} &= -\frac{1}{G} \cdot \frac{d^2 G}{dy^2} = -k = -p^2 \\ \frac{d^2 F}{dx^2} + p^2 F &= 0 \end{aligned}$$

- Boundary conditions

$$\begin{aligned} T(0, y) = F(0)G(y) &= 0, & \Rightarrow & F(0) = 0 \\ T(a, y) = F(a)G(y) &= 0, & \Rightarrow & F(a) = 0 \end{aligned}$$

$$F(x) = A \cos px + B \sin px$$

$$F(0) = A = 0 \quad \text{and} \quad F(a) = B \sin pa = 0$$

$$\therefore p_n = \frac{n\pi}{a} \quad (n = 1, 2, \dots)$$

$$F(x) = F_n(x) = \sin \frac{n\pi x}{a} \quad (n = 1, 2, \dots) \quad (16)$$

$$\frac{d^2 G_n}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G_n = 0$$

$$G_n(y) = A_n \sinh \frac{n\pi y}{a} + B_n \cosh \frac{n\pi y}{a}$$

- Boundary condition

$$T_n(x, 0) = 0 = F_n(x)G_n(0) = 0 \quad \Rightarrow \quad G_n(0) = 0$$

$$G_n(0) = B_n = 0$$

$$G_n(y) = A_n \sinh \frac{n\pi y}{a}$$

- Eigenfunctions

$$T_n(x, y) = F_n(x) \cdot G_n(y) = A_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \quad (17)$$

- Last boundary condition to obtain A_n

$$T(x, b) = f(x) \quad (18)$$

$$T(x, y) = \sum_{n=1}^{\infty} T_n(x, y)$$

$$T(x, b) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}$$

- By using the orthogonality of the trigonometric functions,

$$A_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

$$T(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \quad (19)$$

where

$$A_n = \frac{2}{a \sinh(n\pi b/a)} \int_0^a f(x) \sin \frac{n\pi x}{a} dx \quad (20)$$

16.2 Solution by Fourier Integrals and Transforms

- Specifically, we shall find solutions of the heat equation in a bar that extends to infinity on both sides (and is laterally insulated, as before).

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad (21)$$

- No Boundary Conditions, only the initial condition.

$$T(x, 0) = f(x) \quad (-\infty < x < \infty) \quad (22)$$

- Substituting $T(x, t) = F(x)G(t)$ into (1),

$$F'' + p^2 F = 0 \quad (23)$$

$$\dot{G} + \alpha p^2 G = 0 \quad (24)$$

- Solutions are

$$F(x) = A \cos px + B \sin px \quad \text{and} \quad G(t) = e^{-\alpha p^2 t}$$

- A solution of (1) is

$$T(x, t; p) = FG = (A \cos px + B \sin px) e^{-\alpha p^2 t}. \quad (25)$$

Use of Fourier Integrals

- Since $f(x)$ in (2) is not assumed to be periodic, it is natural to use Fourier integrals instead of Fourier series.

$$T(x, t) = \int_0^{\infty} T(x, t; p) dp = \int_0^{\infty} [A(p) \cos px + B(p) \sin px] e^{-\alpha p^2 t} dp \quad (26)$$

Determination of $A(p)$ and $B(p)$ from the Initial Condition.

$$T(x, 0) = \int_0^{\infty} [A(p) \cos px + B(p) \sin px] dp = f(x) \quad (27)$$

$$A(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos pv dv, \quad B(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin pv dv. \quad (28)$$

- Fourier integral (7) with $A(p)$ and $B(p)$

$$T(x, 0) = \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} (f(v) \cos pv \cos px + f(v) \sin pv \sin px) dv \right] dp$$

$$T(x, 0) = \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(px - pv) dv \right] dp$$

- Similarly, (6) in this section becomes.

$$T(x, t) = \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(px - pv) e^{-\alpha p^2 t} dv \right] dp$$

$$T(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \left[\int_0^{\infty} e^{-\alpha p^2 t} \cos(px - pv) dp \right] dv \quad (29)$$

- Evaluating the inner integral by the formula

$$\int_0^{\infty} e^{-s^2} \cos 2bs ds = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad (30)$$

- Choose $p = s/\sqrt{\alpha t}$ as a new variable of integration and set

$$b = \frac{(x - v)}{2\sqrt{\alpha t}}$$

$$2bs = (x - v)p \quad \text{and} \quad ds = \sqrt{\alpha t} dp$$

$$\begin{aligned} \int_0^{\infty} e^{-\alpha p^2 t} \cos(px - pv) dp &= \frac{1}{\sqrt{\alpha t}} \int_0^{\infty} e^{-s^2} \cos 2bs ds = \frac{\sqrt{\pi}}{2\sqrt{\alpha t}} e^{-b^2} \\ &= \frac{1}{2} \left(\frac{\pi}{\alpha t} \right)^{1/2} \exp \left\{ -\frac{(x - v)^2}{4\alpha t} \right\} \end{aligned}$$

- By inserting this result into (9),

$$T(x, t) = \frac{1}{2\sqrt{\pi\alpha t}} \int_{-\infty}^{\infty} f(v) \exp \left\{ -\frac{(x - v)^2}{4\alpha t} \right\} dv \quad (31)$$

- Taking $z = (v - x)/2\sqrt{\alpha t}$,

$$v = x + 2z\sqrt{\alpha t} \quad \implies \quad dv = 2\sqrt{\alpha t} dz$$

$$T(x, t) = \frac{2\sqrt{\alpha t}}{2\sqrt{\pi\alpha t}} \int_{-\infty}^{\infty} f(x + 2z\sqrt{\alpha t}) e^{-z^2} dz$$

$$T(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2z\sqrt{\alpha t}) e^{-z^2} dz. \quad (32)$$

Example 1. Temperature an infinite bar

$$f(x) = \begin{cases} T_0 = \text{const} & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Solution.

- From (11)

$$T(x, t) = \frac{T_0}{2\sqrt{\pi\alpha t}} \int_{-1}^1 \exp\left\{-\frac{(x-v)^2}{4\alpha t}\right\} dv$$

- Introducing $z = (v-x)/2\sqrt{\alpha t}$,

$$T(x, t) = \frac{T_0}{\sqrt{\pi}} \int_{(-1-x)/2\sqrt{\alpha t}}^{(1-x)/2\sqrt{\alpha t}} e^{-z^2} dz \quad (33)$$

Use of Fourier Transforms

Example 2. Temperature in the infinite bar in Example 1.

$$f(x) = \begin{cases} T_0 = \text{const} & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Solution.

- Let $\hat{T} = F(T)$ denote the Fourier transform of T , regarded as a function of x .

$$F(T_t) = \alpha F(T_{xx}) = \alpha(-\omega^2)F(T) = -\alpha\omega^2\hat{T}$$

$$F(T_t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T_t e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} T e^{-i\omega x} dx = \frac{\partial \hat{T}}{\partial t}$$

$$\therefore \frac{\partial \hat{T}}{\partial t} = -\alpha\omega^2\hat{T}$$

$$\hat{T}(\omega, t) = C(\omega)e^{-\alpha\omega^2 t}$$

- Initial condition

$$\hat{T}(\omega, 0) = C(\omega) = \hat{f}(\omega) = F(f)$$

$$\hat{T}(\omega, t) = \hat{f}(\omega)e^{-\alpha\omega^2 t}$$

$$T(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-\alpha\omega^2 t} e^{i\omega x} d\omega \quad (34)$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv$$

$$T(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) \left[\int_{-\infty}^{\infty} e^{-\alpha\omega^2 t} e^{i\omega(x-v)} d\omega \right] dv$$

$$e^{-\alpha\omega^2 t} e^{i\omega(x-v)} = e^{-\alpha\omega^2 t} \cos(\omega x - \omega v) + i e^{-\alpha\omega^2 t} \sin(\omega x - \omega v)$$

- Sine is an odd function.

$$i \int_{-\infty}^{\infty} e^{-\alpha\omega^2 t} \sin \omega(x - v) d\omega = 0$$

$$T(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \left[\int_0^{\infty} e^{-\alpha\omega^2 t} \cos \omega(x - v) d\omega \right] dv$$

Example 3. Solution in Example 1 by the method of convolution

Solution.

- The beginning is as in Example 2 and leads to (14).

$$T(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-\alpha\omega^2 t} e^{i\omega x} dx \quad (35)$$

- Revoking the convolution integral in Fourier transform

$$T(x, t) = (f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) \hat{g}(\omega) e^{i\omega x} d\omega \quad (36)$$

where

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\alpha\omega^2 t} \quad (37)$$

- By the definition of convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(p) g(x - p) dp \quad (38)$$

$$F(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$$

with $\alpha t = 1/4a$ or $a = 1/4\alpha t$, using (17)

$$F(e^{-x^2/4\alpha t}) = \sqrt{2\alpha t} e^{-\alpha\omega^2 t} = \sqrt{2\alpha t} \sqrt{2\pi} \hat{g}(\omega)$$

- Hence \hat{g} has the inverse

$$\frac{1}{\sqrt{2\alpha t} \sqrt{2\pi}} e^{-x^2/4\alpha t}$$

- Replacing x with $x - p$ and substituting this into (18),

$$T(x, t) = (f * g)(x) = \frac{1}{2\sqrt{\pi\alpha t}} \int_{-\infty}^{\infty} f(p) \exp \left\{ -\frac{(x-p)^2}{4\alpha t} \right\} dp. \quad (39)$$

Example 4. Fourier sine transform applied to the heat equation.

Solution.

- A laterally insulated bar extends from $x = 0$ to infinity.
- Initial condition: $T(x, 0) = f(x)$.
- Boundary condition: $T(0, t) = 0 \Rightarrow f(0) = 0$.

$$F_s(T_t) = \frac{\partial \hat{T}_s}{\partial t} = \alpha F_s(T_{xx}) = -\alpha\omega^2 F_s(T) = -\alpha\omega^2 \hat{T}_s(\omega, t)$$

$$\hat{T}_s(\omega, t) = C(\omega)e^{-\alpha\omega^2 t}$$

$$T(x, 0) = f(x) \Rightarrow \hat{T}_s(\omega, 0) = \hat{f}_s(\omega) = C(\omega)$$

$$\hat{T}_s(\omega, t) = \hat{f}_s(\omega)e^{-\alpha\omega^2 t}$$

- Taking the inverse Fourier sine transform and substituting

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(p) \sin \omega p \, dp$$

$$T(x, t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(p) \sin \omega p e^{-\alpha\omega^2 t} \sin \omega x \, dp d\omega \quad (40)$$