ENGINEERING MATHEMATICS II

010.141

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CHAP. 12
Partial Differential Equations (PDEs)

ABSTRACT OF CHAP. 12

- ***PDEs in Chap. 12 are models of various physical and geometrical problems (the solutions) depend on two or more variables, usually on time t and one or several space variables.***
  - We concentrate on the most important PDEs of applied mathematics, the wave equations governing the vibrating string (Sec. 12.2) and vibrating membrane (Sec. 12.7), the heat equation (12.5), and Laplace equation (Secs. 12.5 and 12.10).
  - We derive these PDEs from physics and consider methods for solving initial and boundary value problems, that is, methods of obtaining solutions satisfying conditions that are given by the physical situation.

CHAP. 12.1
BASIC CONCEPTS

Classes of the PDEs.

EXAMPLES OF PDE

A PDE is an equation involving one or more partial derivatives of a function.

One-D Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Two-D Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

One-D Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Two-D Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Two-D Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Three-D Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Here c is a positive constant, t is time, x, y, z are Cartesian coordinates, and *dimension* is the number of these coordinates in the equation.

Fundamental Theorem

A **solution** of a PDE in some region R of the space of the independent variables is a function that has all the partial derivatives appearing in the PDE in some domain D (definition in Sec. 9.6) containing R, and satisfies the PDE everywhere in R.

In general, the totality of solutions of a PDE is very large. For example, the functions

$$u = x^2 - y^2, \qquad u = e^x \cos y, \qquad u = \sin x \cosh y, \qquad u = \ln (x^2 + y^2) \tag{7}$$

which are entirely different from each other, are solutions of (3) , as you may verify. We shall see later that the unique solution of a PDE corresponding to a given physical problem will be obtained by the use of **additional conditions** arising from the problem. For instance, this may be the condition that the solution u assume given values on the boundary of the region R **("boundary conditions")**. Or, when time t is one of the variables, u (or $u_t = \partial u/\partial t$ or both) may be prescribed at $t = 0$ **("initial conditions")**.

THEOREM 1

Fundamental Theorem on Superposition

If u_1 and u_2 are solutions of a ***homogeneous linear*** *PDE in some region R, then*

$$u = c_1 u_1 + c_2 u_2$$

with any constants c_1 and c_2 is also a solution of that PDE in the region R.

EXAMPLE OF WAVE EQUATION

- Problem, as a solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad u(t,x)$$

$$u(t,x) = \sin 9t \cdot \sin \tfrac{1}{4}x$$

$$\frac{\partial u}{\partial t} = (9 \cos 9t)(\sin \tfrac{1}{4}x)$$

$$\frac{\partial^2 u}{\partial t^2} = -81 \sin 9t \cdot \sin \tfrac{1}{4}x$$

$$\frac{\partial u}{\partial x} = \sin 9t \cdot (\tfrac{1}{4} \cos \tfrac{1}{4}x)$$

$$\frac{\partial^2 u}{\partial x^2} = \sin 9t \cdot (-\tfrac{1}{16} \cos \tfrac{1}{4}x)$$

$-81 \sin 9t \cdot \sin \tfrac{1}{4}x = c^2 (-\tfrac{1}{16}) \sin 9t \cdot \sin \tfrac{1}{4}x$ is identically true,

if $c^2 = (16)(81) \rightarrow c = \pm 36$

EXAMPLE OF HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u = e^{-4t} \cos 3x$$

$$\frac{\partial u}{\partial t} = -4e^{-4t} \cos 3x$$

$$\frac{\partial u}{\partial x} = -3e^{-4t} \sin 3x$$

$$\frac{\partial^2 u}{\partial x^2} = -9e^{-4t} \cos 3x$$

$$-4e^{-4t} \cos 3x = -9e^{-4t} \cos 3x \cdot c^2$$

Then $c^2 = \dfrac{4}{9}$

EXAMPLE OF LAPLACE EQUATION

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -e^x \sin y$$

$$e^x \sin y - e^x \sin y = 0$$

$$0 = 0$$

HOMEWORK IN 12.1

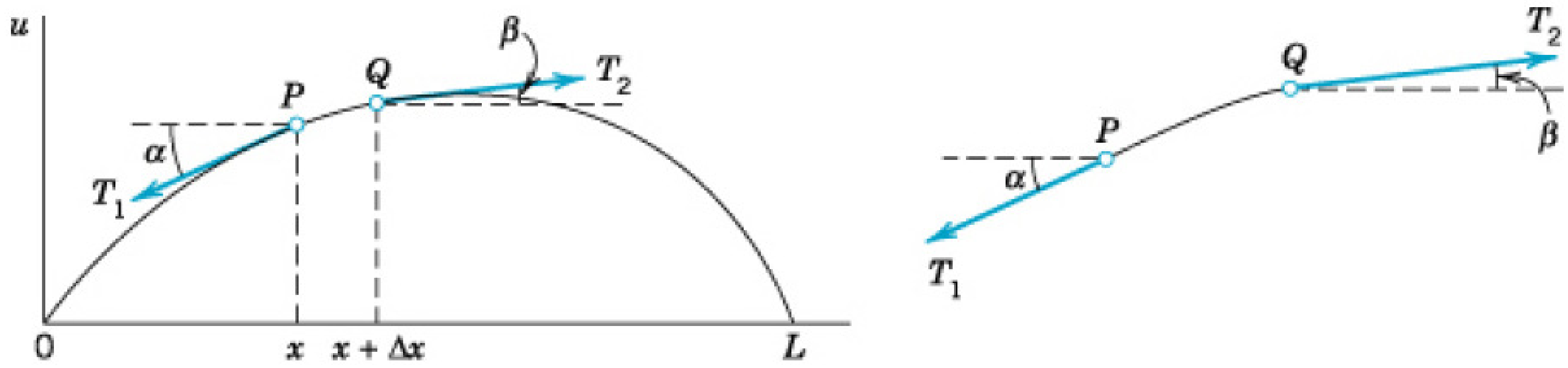
- HW1. Problems 1, 3, 5
- HW2. Problems 14, 15
- HW3. Problems 18, 19
- HW4. Problem 23

CHAP. 12.2
MODELING: VIBRATING STRING, WAVE EQUATION

Modeling the wave equation.

VIBRATING STRING AND THE WAVE EQUATION

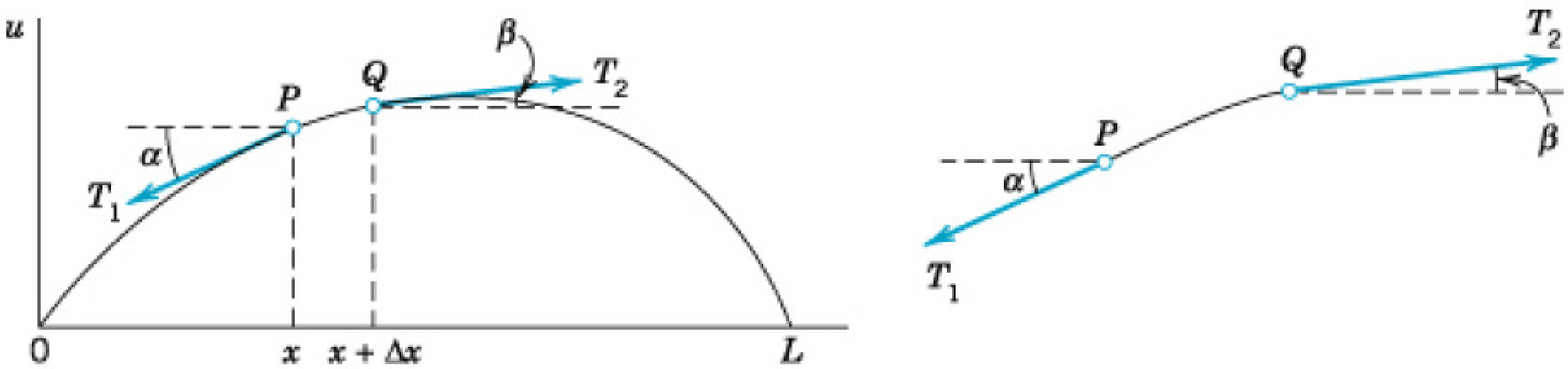
Deflected string at fixed time t



- General assumptions for vibrating string problem:
- mass per unit length is constant; string is perfectly elastic and no resistance to bending.
- tension in string is sufficiently large so that gravitational forces may be neglected.
- string oscillates in a plane; every particle of string moves vertically only; and, deflection and slope at every point are small in terms of absolute value.

DERIVATION OF WAVE EQUATION

Deflected string at fixed time t



T_1, T_2 = tension in string at point P and Q

$T_1 \cos \alpha = T_2 \cos \beta = T$, a constant (as string does not move in horizontal dir.)

Vertical components of tension:

$$-T_1 \sin \alpha \quad \text{and} \quad T_2 \sin \beta$$

DERIVATION OF WAVE EQUATION (Cont.)

Let Δx = length $\overline{\text{PQ}}$, $\rho =$ mass/unit length, and Δx has mass $\rho\Delta x$

Newton's Law: $F = mass \times acceleration$

If u is the vertical position, $\dfrac{\partial^2 u}{\partial t^2} = acceleration$

$$\underline{\underline{T_2 \sin\beta - T_1 \sin\alpha}} = (\rho\Delta x)\frac{\partial^2 u}{\partial t^2}$$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2} \quad (\text{equation } 2)$$

At P(x is distance from origin), $\tan\alpha$ is slope $= \left.\dfrac{\partial u}{\partial x}\right|_x$

Likewise at Q, $\tan\beta = \left.\dfrac{\partial u}{\partial x}\right|_{x+\Delta x}$

DERIVATION OF WAVE EQUATION (Cont.)

Substituting and $\div\ \Delta x :\quad \dfrac{1}{\Delta x}\left[\left.\dfrac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\dfrac{\partial u}{\partial x}\right|_{x} \right] = \dfrac{\rho}{T}\dfrac{\partial^2 u}{\partial t^2}$

as $\Delta x \ \to \ 0$, L.H.S. becomes $\dfrac{\partial^2 u}{\partial x^2}$

Let $c^2 \ = \ \dfrac{T}{\rho}$, *so that* $\dfrac{\partial^2 u}{\partial t^2} \ = \ c^2 \dfrac{\partial^2 u}{\partial x^2}$

This is the 1-D Wave equation

If $T \uparrow$ or $\rho \downarrow \quad c^2 \uparrow$

CHAP. 12.3
SOLUTION BY SEPARATING VARIABLES. USE OF FOURIER SERIES

Solving the PDEs using separating variables and Fourier series and interpreting the solution.

Solution by Separating Variables

one-dimensional wave equation

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho}$$

for the unknown deflection $u(x, t)$ of the string.

Since the string is fastened at the ends $x = 0$ and $x = L$ (see Sec. 12.2), we have the two **boundary conditions**

(2) $$(a) \quad u(0, t) = 0, \qquad (b) \quad u(L, t) = 0 \qquad \text{for all } t.$$

Furthermore, the form of the motion of the string will depend on its *initial deflection* (deflection at time $t = 0$), call it $f(x)$, and on its *initial velocity* (velocity at $t = 0$), call it $g(x)$. We thus have the two **initial conditions**

(3) $$(a) \quad u(x, 0) = f(x), \qquad (b) \quad u_t(x, 0) = g(x) \qquad (0 \leqq x \leqq L)$$

where $u_t = \partial u/\partial t$. We now have to find a solution of the PDE (1) satisfying the conditions (2) and (3) .

Solution by Separating Variables

Step 1. By the **"method of separating variables"** or *product method*, setting $u(x, t) = F(x)G(t)$, we obtain from (1) two ODEs, one for $F(x)$ and the other one for $G(t)$.

Step 2. We determine solutions of these ODEs that satisfy the boundary conditions (2) .

Step 3. Finally, using **Fourier series**, we compose the solutions gained in Step 2 to obtain a solution of (1) satisfying both (2) and (3) , that is, the solution of our model of the vibrating string.

Step 1. Two ODEs from the Wave Equation (1)

(4) $$u(x,t) = F(x)G(t)$$

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \qquad \frac{\partial^2 u}{\partial x^2} = F''G \quad \Rightarrow \quad F\ddot{G} = c^2 F''G .$$

Dividing by c^2FG and simplifying gives

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k .$$

(5) $$F'' - kF = 0$$

(6) $$\ddot{G} - c^2 kG = 0.$$

Solution by Separating Variables

Step 2. Satisfying the Boundary Conditions (2)

$u = FG$ satisfies the boundary conditions (2) ,

$$u(0,t) = F(0)G(t) = 0, \qquad u(L,t) = F(L)G(t) = 0 \qquad \text{for all } t. \tag{7}$$

We first solve (5) . If $G \equiv 0$, then $u = FG \equiv 0$, which is of no interest. Hence $G \not\equiv 0$ and then by (7) ,

$$(a) \quad F(0) = 0, \qquad (b) \quad F(L) = 0. \tag{8}$$

We show that k must be negative. For $k = 0$ the general solution of (5) is $F = ax + b$, and from (8) we obtain $a = b = 0$, so that $F \equiv 0$ and $u = FG \equiv 0$, which is of no interest. For positive $k = \mu^2$ a general solution of (5) is

$$F = Ae^{\mu x} + Be^{-\mu x}$$

and from (8) we obtain $F \equiv 0$ as before (verify!). Hence we are left with the possibility of choosing k negative, say, $k = -p^2$. Then (5) becomes $F'' + p^2F = 0$ and has as a general solution

$$F(x) = A\cos px + B\sin px\ .$$

Solution by Separating Variables

From this and (8) we have

$$F(0) = A = 0 \quad \text{and then} \quad F(L) = B \sin pL = 0.$$

We must take $B \neq 0$ since otherwise $F \equiv 0$. Hence $\sin pL = 0$. Thus

(9) $$pL = n\pi, \quad \text{so that} \quad p = \frac{n\pi}{L} \quad (n \text{ integer}).$$

Setting $B = 1$, we thus obtain infinitely many solutions $F(x) = F_n(x)$, where

(10) $$F_n(x) = \sin \frac{n\pi}{L} x \quad (n = 1, 2, \cdots).$$

We now solve (6) with $k = -p^2 = -(n\pi/L)^2$ resulting from (9), that is,

(11*) $$\ddot{G} + \lambda_n^2 G = 0 \quad \text{where} \quad \lambda_n = cp = \frac{cn\pi}{L}.$$

A general solution is

$$G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t.$$

Hence solutions of (1) satisfying (2) are $u_n(x, t) = F_n(x)G_n(t) = G_n(t)F_n(x)$, written out

(11) $$u_n(x, t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x \quad (n = 1, 2, \cdots).$$

These functions are called the **eigenfunctions**, or *characteristic functions*, and the values $\lambda_n = cn\pi/L$ are called the **eigenvalues**, or *characteristic values*, of the vibrating string. The set $\{\lambda_1, \lambda_2, \cdots\}$ is called the **spectrum**.

Solution by Separating Variables

Discussion of Eigenfunctions. We see that each u_n represents a harmonic motion having the **frequency** $\lambda_n/2\pi = cn/2L$ cycles per unit time. This motion is called the nth **normal mode** of the string. The first normal mode is known as the *fundamental mode* ($n = 1$), and the others are known as *overtones;* musically they give the octave, octave plus fifth, etc. Since in (11)

$$\sin \frac{n\pi x}{L} = 0 \qquad \text{at} \qquad x = \frac{L}{n}, \frac{2L}{n}, \cdots, \frac{n-1}{n}L,$$

the nth normal mode has $n - 1$ **nodes**, that is, points of the string that do not move (in addition to the fixed endpoints); see Fig. 284.

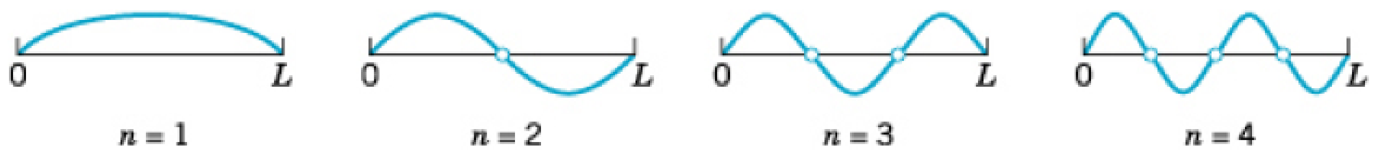


Fig. 284. Normal modes of the vibrating string

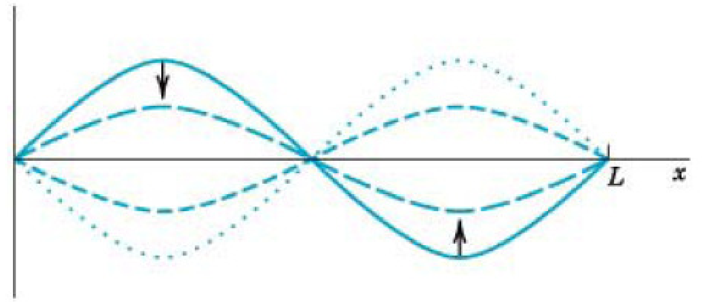


Fig. 285. Second normal mode for various values of t

Solution by Separating Variables

Step 3. Solution of the Entire Problem. Fourier Series

The eigenfunctions (11) satisfy the wave equation (1) and the boundary conditions (2) (string fixed at the ends). A single u_n will generally not satisfy the initial conditions (3) .

(12) $$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x .$$

Satisfying Initial Condition (3a) (Given Initial Displacement). From (12) and (3a) we obtain

(13) $$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x) .$$

(14) $$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, \cdots .$$

Satisfying Initial Condition (3b) (Given Initial Velocity). Similarly, by differentiating (12) with respect to t and using (3b) , we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty} (-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{L}\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x) .$$

(15) $$B_n{}^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, \cdots .$$

Physical Interpretation

Let us consider when the initial velocity g(x) be identically zero.

$$u(t,x) = \sum_{n=1}^{\infty} B_n \cos\lambda_n \sin\frac{n\pi x}{L}, \qquad \lambda_n = \frac{cn\pi}{L}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x+ct)\right\}$$

$$u(t,x) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right]$$

The first term represents a wave that is traveling to the right as *t* increases. The second is a wave that is traveling to the left as *t* increases.

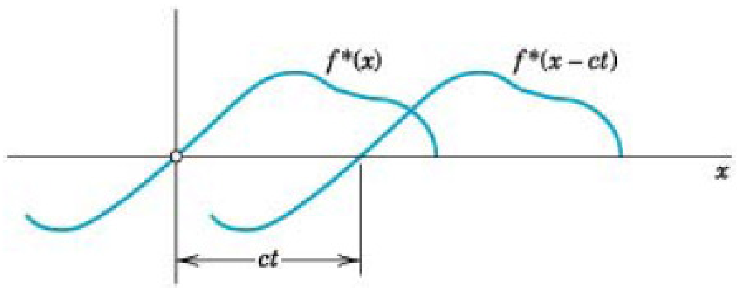


Fig. 287. Interpretation of (17)

EXAMPLE 1

EXAMPLE 1 Vibrating String if the Initial Deflection is Triangular

Find the solution of the wave equation (1) corresponding to the triangular initial deflection

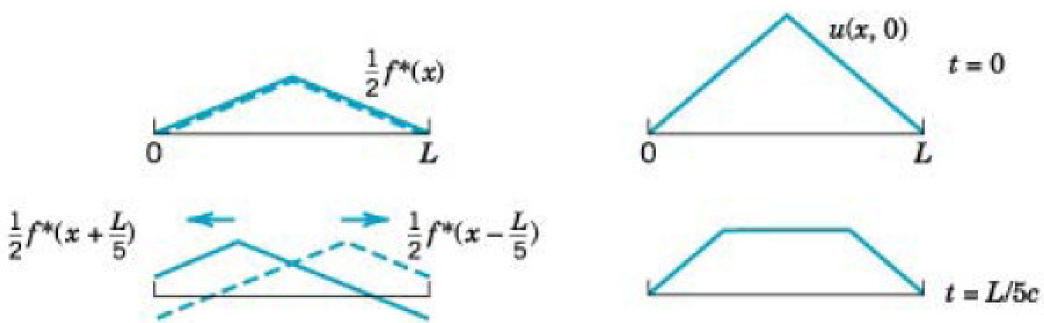
$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if } 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L-x) & \text{if } \dfrac{L}{2} < x < L \end{cases}$$

and initial velocity zero. (Figure 288 shows $f(x) = u(x, 0)$ at the top.)

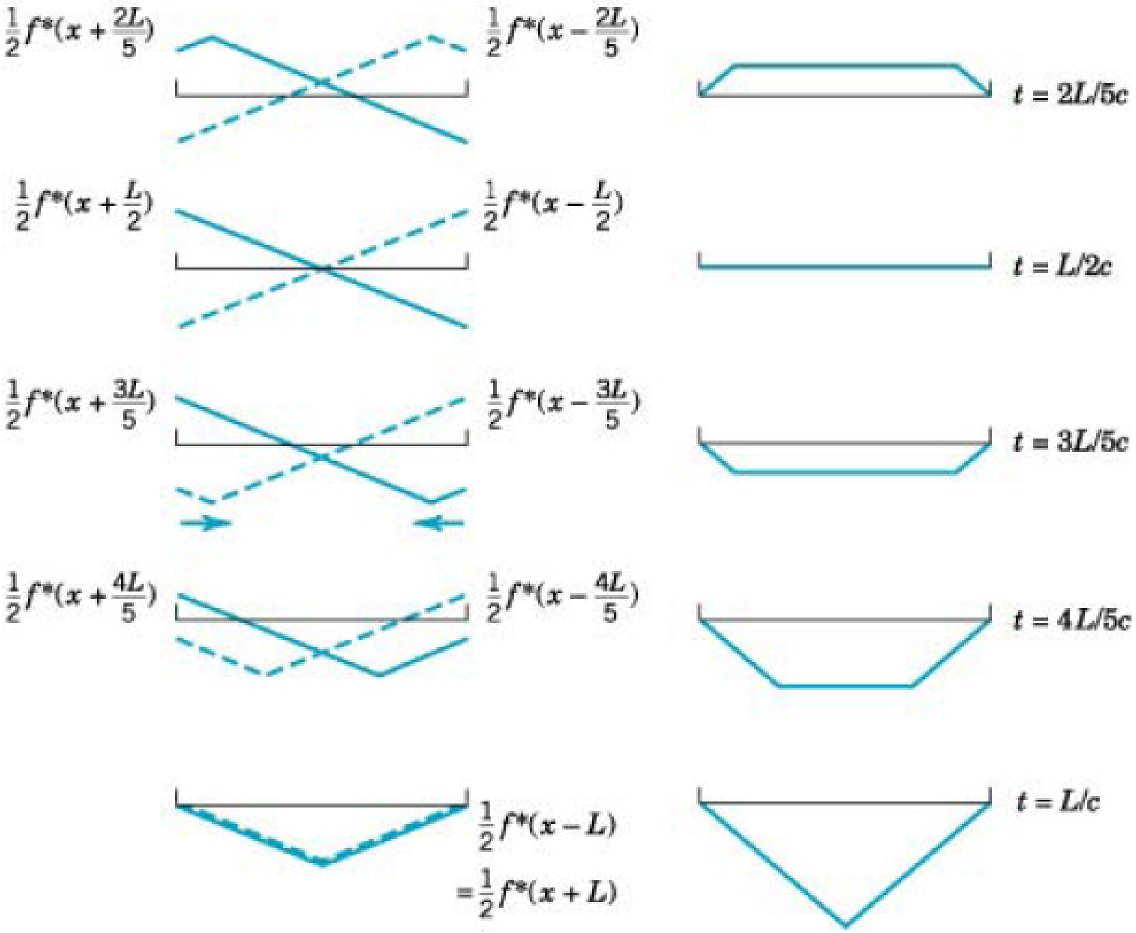
Solution:

Since $g(x) \equiv 0$, we have $B_n^* = 0$ in (12) , and from Example 4 in Sec. 11.3 we see that the B_n are given by (5) , Sec. 11.3. Thus (12) takes the form

$$u(x,t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t - \frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi c}{L}t + - \cdots\right].$$



EXAMPLE 1



$\frac{1}{2}f^*(x+\frac{2L}{5})$
$\frac{1}{2}f^*(x-\frac{2L}{5})$
$t = 2L/5c$
$\frac{1}{2}f^*(x+\frac{L}{2})$
$\frac{1}{2}f^*(x-\frac{L}{2})$
$t = L/2c$
$\frac{1}{2}f^*(x+\frac{3L}{5})$
$\frac{1}{2}f^*(x-\frac{3L}{5})$
$t = 3L/5c$
$\frac{1}{2}f^*(x+\frac{4L}{5})$
$\frac{1}{2}f^*(x-\frac{4L}{5})$
$t = 4L/5c$
$\frac{1}{2}f^*(x-L)$
$=\frac{1}{2}f^*(x+L)$
$t = L/c$

HOMEWORK IN 12.3

- HW1. Problem 1
- HW2. Problems 11, 12
- HW3. Problems 15, 16

CHAP. 12.5
HEAT EQUATION: SOLUTION BY FOURIER SERIES

Also called the diffusion equation.

HEAT EQUATION

From prior work the heat equation is:

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u, \qquad c^2 = \frac{k}{\sigma\rho}$$

Fig. 291. Bar under consideration

In one dimension (laterally insulated):

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Some boundary conditions at each end:

$$u(0, t) = u(L, t) = 0,$$

for all t

Initial Condition:

$$u(x, 0) = f(x),$$

specified $\quad 0 \leq x \leq L$

HEAT EQUATION

***Step 1*. Two ODEs from the Heat Equation (1) .** Substitution of a product $u(x, t) = F(x)G(t)$ into (1) gives $F\dot{G} = c^2F'' G$ with $\dot{G} = dG/dt$ and $F'' = d^2F/dx^2$. To separate the variables, we divide by c^2FG, obtaining

(4) $$\frac{\dot{G}}{c^2G} = \frac{F''}{F}.$$

The left side depends only on t and the right side only on x, so that both sides must equal a constant k (as in Sec. 12.3). You may show that for $k = 0$ or $k > 0$ the only solution $u = FG$ satisfying (2) is $u \equiv 0$. For negative $k = -p^2$ we have from (4)

$$\frac{\dot{G}}{c^2G} = \frac{F''}{F} = -p^2.$$

Multiplication by the denominators gives immediately the two ODEs

(5) $$F'' + p^2F = 0$$

and

(6) $$\dot{G} + c^2p^2G = 0.$$

***Step 2*. Satisfying the Boundary Conditions (2) .** We first solve (5) . A general solution is

(7) $$F(x) = A\cos px + B\sin px.$$

From the boundary conditions (2) it follows that

$$u(0, t) = F(0)G(t) = 0 \qquad \text{and} \qquad u(L, t) = F(L)G(t) = 0.$$

HEAT EQUATION

Setting $B = 1$, we thus obtain the following solutions of (5) satisfying (2) :

$$F_n(x) = \sin\frac{n\pi x}{L}, \qquad \text{n} = 1, 2, \cdots .$$

(As in Sec. 12.3, we need not consider *negative* integral values of n.)

All this was literally the same as in Sec. 12.3. From now on it differs since (6) differs from (6) in Sec. 12.3. We now solve (6) . For $p = n\pi/L$, as just obtained, (6) becomes

$$\dot{G} + \lambda_n^2 G = 0 \qquad \text{where} \qquad \lambda_n = \frac{cn\pi}{L} .$$

It has the general solution

$$G_n(t) = B_n e^{-\lambda_n^2 t}, \qquad n = 1, 2, \cdots$$

where B_n is a constant. Hence the functions

(8) $$u_n(x, t) = F_n(x)G_n(t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t} \qquad (n = 1, 2, \cdots)$$

are solutions of the heat equation (1) , satisfying (2) . These are the **eigenfunctions** of the problem, corresponding to the **eigenvalues** $\lambda_n = cn\pi/L$.

***Step 3*. Solution of the Entire Problem. Fourier series.** So far we have solutions (8) satisfying the boundary conditions (2) . To obtain a solution that also satisfies the initial condition (3) , we consider a series of these eigenfunctions,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n{}^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right). \tag{9}$$

From this and (3) we have

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x).$$

Hence for (9) to satisfy (3) , the B_n's must be the coefficients of the **Fourier sine series**, as given by (4) in Sec. 11.3; thus

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx \qquad (n = 1, 2, \cdots). \tag{10}$$

The solution of our problem can be established, assuming that $f(x)$ is piecewise continuous (see Sec. 6.1) on the interval $0 \leqq x \leqq L$ and has one-sided derivatives (see Sec. 11.1) at all interior points of that interval; that is, under these assumptions the series (9) with coefficients (10) is the solution of our physical problem. A proof requires knowledge of uniform convergence and will be given at a later occasion (Probs. 19, 20 in Problem Set 15.5).

Because of the exponential factor, all the terms in (9) approach zero as t approaches infinity. The rate of decay increases with n.

EXAMPLE 1

EXAMPLE 1 Sinusoidal Initial Temperature

Find the temperature $u(x, t)$ in a laterally insulated copper bar 80 cm long if the initial temperature is 100 sin $(\pi x/80)$ °C and the ends are kept at 0°C. How long will it take for the maximum temperature in the bar to drop to 50°C? First guess, then calculate. *Physical data for copper:* density 8.92 gm/cm^3, specific heat 0.092 cal/(gm °C), thermal conductivity 0.95 cal/(cm sec °C).

Solution:

The initial condition gives

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{80} = f(x) = 100 \sin\frac{\pi x}{80}.$$

Hence, by inspection or from (9) we get $B_1 = 100$, $B_2 = B_3 = \cdots = 0$. In (9) we need $\lambda_1{}^2 = c^2\pi^2/L^2$, where $c^2 = K/(\sigma\rho) = 0.95/(0.092 \cdot 8.92) = 1.158$ [cm^2/sec]. Hence we obtain

$$\lambda_1^2 = 1.158 \cdot 9.870 / 80^2 = 0.001785 [\sec^{-1}].$$

The solution (9) is

$$u(x, t) = 100 \sin\frac{\pi x}{80} e^{-0.001785t}.$$

Also, $100e^{-0.001785t} = 50$ when $t = (\ln 0.5)/(-0.001785) = 388$ [sec] ≈ 6.5 [min]. Does your guess, or at least its order of magnitude, agree with this result?

EXAMPLE 2

EXAMPLE 2 Speed of Decay

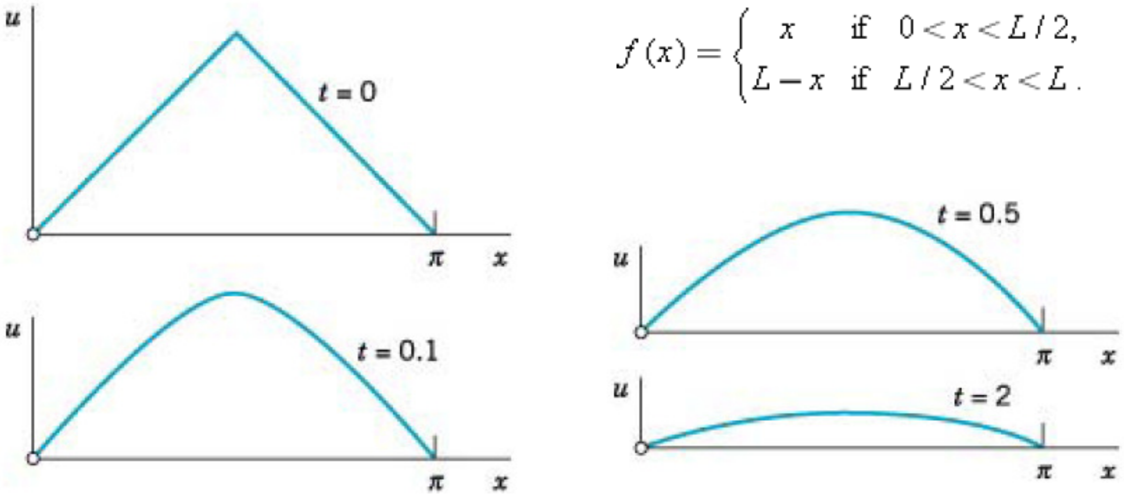
Solve the problem in Example 1 when the initial temperature is 100 sin ($3\pi x/80$) °C and the other data are as before.

$$u(x, t) = 100 \sin \frac{3\pi x}{80} e^{-0.01607t}.$$

Hence the maximum temperature drops to 50°C in $t = (\ln 0.5)/(-0.01607) \approx 43$ [seconds], which is much faster (9 times as fast as in Example 1; why?).

Had we chosen a bigger n, the decay would have been still faster, and in a sum or series of such terms, each term has its own rate of decay, and terms with large n are practically 0 after a very short time. Our

$$f(x) = \begin{cases} x & \text{if } 0 < x < L/2, \\ L - x & \text{if } L/2 < x < L. \end{cases}$$



EXAMPLE 3

EXAMPLE 3 "Triangular" Initial Temperature in a Bar

Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature 0, assuming that the initial temperature is

$$f(x) = \begin{cases} x & \text{if } 0 < x < L/2, \\ L - x & \text{if } L/2 < x < L. \end{cases}$$

(The uppermost part of Fig. 292 shows this function for the special $L = \pi$.)

Solution:

$$B_n = \frac{2}{L}\left(\int_0^{L/2} x \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} (L-x)\sin\frac{n\pi x}{L}\,dx\right).$$

Integration gives $B_n = 0$ if n is even,

$$B_n = \frac{4L}{n^2\pi^2} \quad (n = 1, 5, 9, \cdots) \quad \text{and} \quad B_n = -\frac{4L}{n^2\pi^2} \quad (n = 3, 7, 11, \cdots).$$

(see also Example 4 in Sec. 11.3 with $k = L/2$). Hence the solution is

$$u(x,t) = \frac{4L}{\pi^2}\left[\sin\frac{\pi x}{L}\exp\left[-\left(\frac{c\pi}{L}\right)^2 t\right] - \frac{1}{9}\sin\frac{3\pi x}{L}\exp\left[-\left(\frac{3c\pi}{L}\right)^2 t\right] + - \cdots\right].$$

Figure 292 shows that the temperature decreases with increasing t, because of the heat loss due to the cooling of the ends.

EXAMPLE 4

EXAMPLE 4 Bar with Insulated Ends.

Find a solution formula of (1) , (3) with (2) replaced by the condition that both ends of the bar are insulated.

Solution:

(2*) $$u_x(0,t)=0, \qquad u_x(L,t)=0 \qquad \text{for all } t.$$

Since $u(x, t) = F(x)G(t)$, this gives $u_x(0, t) = F'(0)G(t) = 0$ and $u_x(L, t) = F'(L)G(t) = 0$. Differentiating (7) , we have $F'(x) = -Ap \sin px + Bp \cos px$, so that

$$F'(0)=Bp=0 \qquad \text{and then} \qquad F'(L)=-Ap\sin pL=0.$$

The second of these conditions gives $p = p_n = n\pi/L$, ($n = 0, 1, 2, \cdots$). From this and (7) with $A = 1$ and $B = 0$ we get $F_n(x) = \cos(n\pi x/L)$, ($n = 0, 1, 2, \cdots$). With G_n as before, this yields the eigenfunctions

(11) $$u_n(x,t)=F_n(x)G_n(t)=A_n\cos\frac{n\pi x}{L}e^{-\lambda_n^2 t} \qquad (n=0,1,\cdots)$$

corresponding to the eigenvalues $\lambda_n = cn\pi/L$. The latter are as before, but we now have the additional

HOMEWORK IN 12.5

- HW1. Problems 5,7

CHAP. 12.6
HEAT EQUATION: SOLUTION BY FOURIER INTEGRALS AND TRANSFORMS

Extension to bars of infinite length.

INTRODUCTION

Our discussion of the heat equation

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

in the last section extends to bars of infinite length, which are good models of very long bars or wires (such as a wire of length, say, 300 ft). Then the role of Fourier series in the solution process will be taken by **Fourier integrals** (Sec. 11.7).

we do not have boundary conditions, but only the **initial condition**

(2) $$u(x, 0) = f(x) \qquad (-\infty < x < \infty)$$

(3) $$F'' + p^2 F = 0$$ [see (5), Sec. 12.5]

and

(4) $$\dot{G} + c^2 p^2 G = 0$$ [see (6), Sec. 12.5]

Solutions are

$$F(x) = A\cos px + B\sin px \qquad \text{and} \qquad G(t) = e^{-c^2 p^2 t},$$

respectively, where A and B are any constants. Hence a solution of (1) is

(5) $$u(x, t; p) = FG = (A\cos px + B\sin px)e^{-c^2 p^2 t}.$$

Here we had to choose the separation constant k negative, $k = -p^2$, because positive values of k would lead to an increasing exponential function in (5), which has no physical meaning.

USE OF FOURIER INTEGRALS

Any series of functions (5) , found in the usual manner by taking p as multiples of a fixed number, would lead to a function that is periodic in x when $t = 0$. However, since $f(x)$ in (2) is not assumed to be periodic, it is natural to use **Fourier integrals** instead of Fourier series. Also, A and B in (5) are arbitrary and we may regard them as functions of p, writing $A = A(p)$ and $B = B(p)$. Now, since the heat equation (1) is linear and homogeneous, the function

(6) $$u(x,t) = \int_0^\infty u(x,t;p)\,dp = \int_0^\infty [A(p)\cos px + B(p)\sin px]\,e^{-c^2p^2t}\,dp$$

Determination of $A(p)$ and $B(p)$ from the Initial Condition. From (6) and (2) we get

(7) $$u(x,0) = \int_0^\infty [A(p)\cos px + B(p)\sin px]\,dp = f(x)\,.$$

This gives $A(p)$ and $B(p)$ in terms of $f(x)$; indeed, from (4) in Sec. 11.7 we have

(8) $$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv\,dv, \qquad B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin pv\,dv\,.$$

According to (1*) , Sec. 11.9, our Fourier integral (7) with these $A(p)$ and $B(p)$ can be written

$$u(x,0) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(px - pv)\,dv\right]dp\,.$$

Similarly, (6) in this section becomes

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(px - pv)\,e^{-c^2p^2t}\,dv\right]dp\,.$$

USE OF FOURIER INTEGRALS

Assuming that we may reverse the order of integration, we obtain

(9) $$u(x,t)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{0}^{\infty} e^{-c^2p^2t}\cos(px-pv)\,dp\right]dv\,.$$

Using the inner integral and variable transformation $\displaystyle\int_0^\infty e^{-s^2}\cos 2bs\,ds=\frac{\sqrt{\pi}}{2}e^{-b^2}$

(12) $$u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2cz\sqrt{t})e^{-z^2}dz\,.$$

EXAMPLE1

EXAMPLE 1 Temperature in an Infinite Bar

Find the temperature in the infinite bar if the initial temperature is (Fig. 295)

$$f(x) = \begin{cases} U_0 = const & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

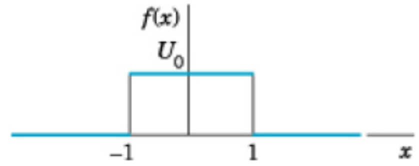


Fig. 295. Initial temperature in Example 1

Solution:

From (11) we have

$$u(x,t) = \frac{U_0}{2c\sqrt{\pi t}} \int_{-1}^{1} \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\} dv .$$

If we introduce the above variable of integration z, then the integration over v from -1 to 1 corresponds to the integration over z from $(-1-x)/(2c\sqrt{t})$ to $(1-x)/(2c\sqrt{t})$, and

(13) $$u(x,t) = \frac{U_0}{\sqrt{\pi}} \int_{-(1+x)/(2c\sqrt{t})}^{(1-x)/(2c\sqrt{t})} e^{-z^2} dz \qquad (t > 0) .$$

We mention that this integral is not an elementary function, but can be expressed in terms of the error function, whose values have been tabulated. (Table A4 in App. 5 contains a few values; larger tables are listed in Ref. [GR1] in App. 1. See also CAS Project 10, p. 568.) Figure 296 shows $u(x, t)$ for $U_0 = 100°$ C, $c^2 = 1$ cm^2/sec, and several values of t.

EXAMPLE1

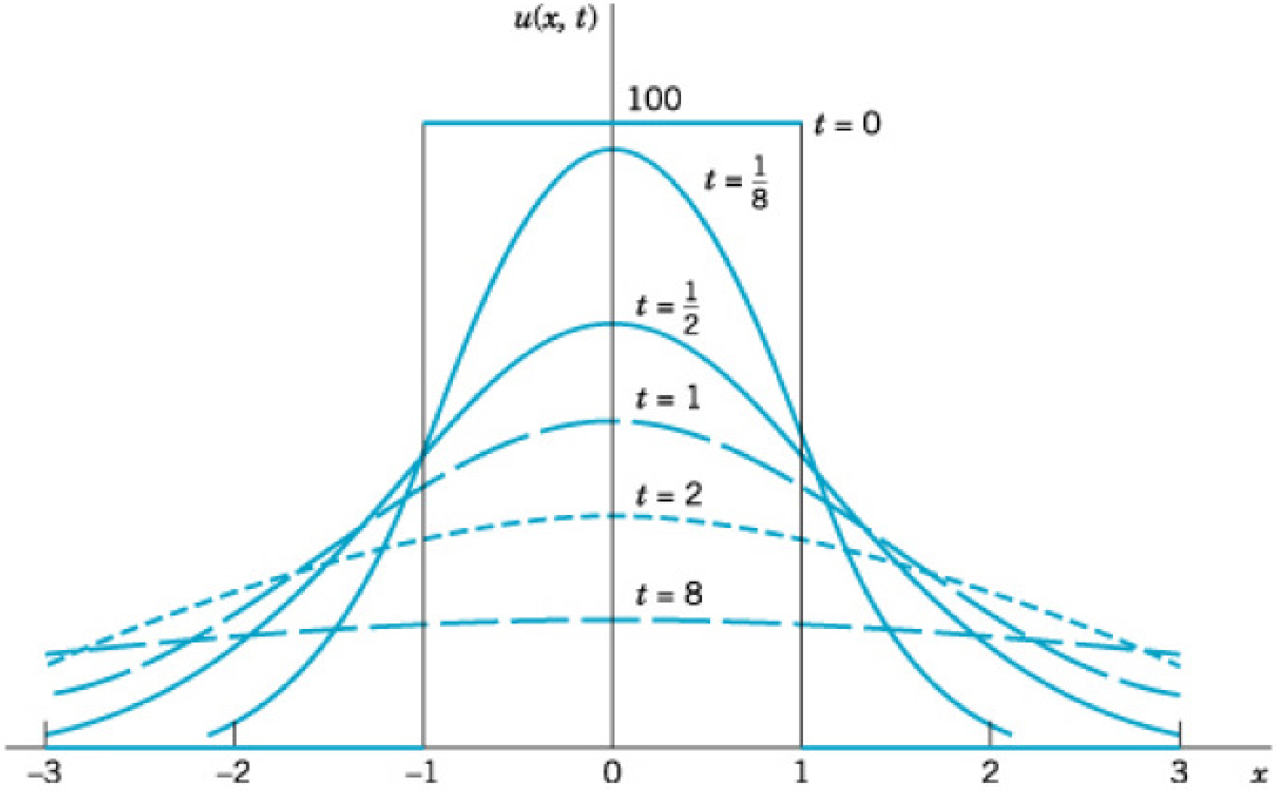


Fig. 296. Solution $u(x, t)$ in Example 1 for $U_0 = 100°C$, $c^2 = 1$ cm^2/sec, and several values of t

HOMEWORK IN 12.6

- HW1. Problems 3,5

CHAP. 12.7
MODELING: MEMBRANE, TWO-DIMENSIONAL WAVE EQUATION

Extension to bars of infinite length.

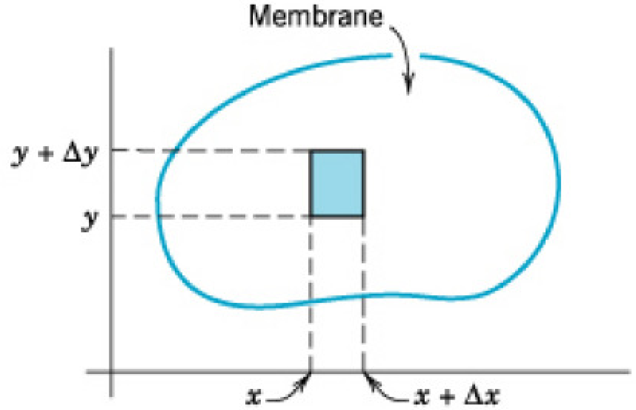
VIBRATING MEMBRANE AND THE TWO-DIMENSIONAL WAVE EQUATION

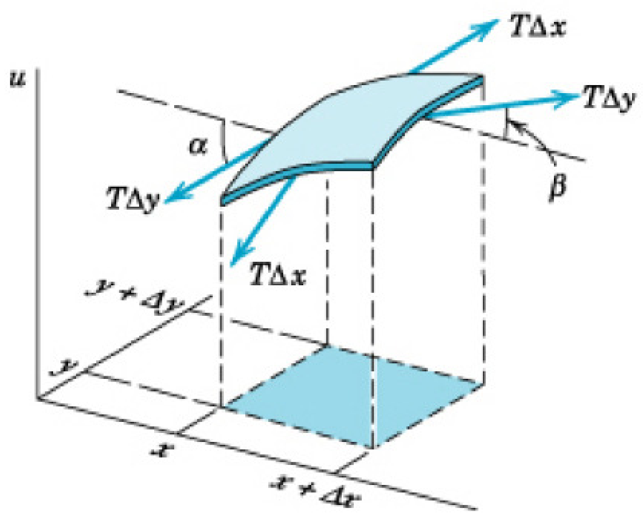
Three Assumptions:

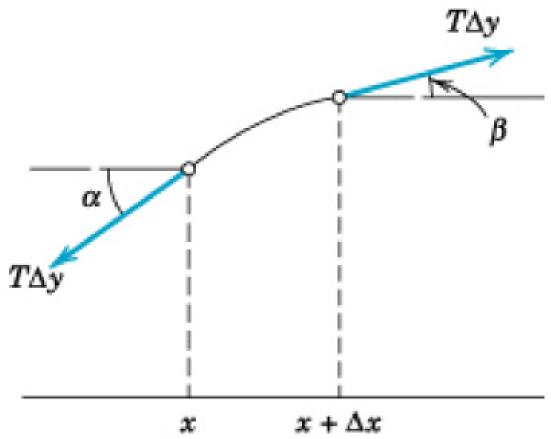
- The mass of the membrane is constant, the membrane is perfectly flexible, and offers no resistance to bending;
- The membrane is stretched and then fixed along its entire boundary in the x-y plane. Tension per unit length (T) which is caused by stretching is the same at all points and in the plane and does not change during the motion;
- The deflection of the membrane $u(x,y,t)$ during vibratory motion is small compared to the size of the membrane, and all angles of inclination are small

GEOMETRY OF VIBRATING "DRUM"

Vibrating Membrane







NOW FOR NEWTON'S LAW

$$T\Delta y(\sin\beta - \sin\alpha) \approx T\Delta y(\tan\beta - \tan\alpha)$$
$$= T\Delta y[u_x(x+\Delta x, y_1) - u_x(x, y_2)]$$

$$(\rho\Delta x\Delta y)\frac{\partial^2 u}{\partial t^2} = T\Delta y\left[u_x(x+\Delta x, y_1) - u_x(x, y_2)\right] + T\Delta x\left[u_y(x_1, y+\Delta y) - u_y(x_2, y)\right]$$

Divide by $\rho\Delta x\Delta y$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[\frac{u_x(x+\Delta x, y_1) - u_x(x, y_2)}{\Delta x} + \frac{u_y(x_1, y+\Delta y) - u_y(x_2, y)}{\Delta y}\right]$$

Take limit as $\Delta x \to 0, \Delta y \to 0$

$$\frac{T}{\rho} = c^2$$

This is the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

WAVE EQUATION

- In Laplacian form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where

$$c^2 = T/\rho$$

  - Some boundary conditions: u = 0 along all edges of the boundary.
  - Initial conditions could be the initial position and the initial velocity of the membrane
  - As before, the solution will be broken into separate functions of x,y, and t.
  - Subscripts will indicate variable for which derivatives are taken.

SOLUTION OF 2-D WAVE EQUATION

Let

$$u(x, y, t) = F(x, y)G(t)$$

substitute into wave equation:

$$F\ddot{G} = c^2\left(F_{xx}G + F_{yy}G\right);$$

divide by c^2FG, to get:

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}\left(F_{xx} + F_{yy}\right) = -\nu^2$$

This gives two equations, one in time and one in space. For time,

$$\ddot{G} + \lambda^2 G = 0$$

where $\lambda = c\nu$, and, what is called the amplitude function:

$$F_{xx} + F_{yy} + \nu^2 F = 0$$

also known as the Helmholtz equation.

SEPARATION OF THE HELMHOLTZ EQUATIONS

Let $F(x, y)=H(x)Q(y)$

and, substituting into the Helmholtz:

$$\frac{d^2H}{dx^2}Q = -\left(H\frac{d^2Q}{dy^2} + \nu^2 HQ\right)$$

Here the variables may be separated by dividing by HQ:

$$\frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right) = -k^2$$

Note: $p^2 = \nu^2 - k^2$

As usual, set each side equal to a constant, $-k^2$. This leads to two ordinary linear differential equations:

$$\frac{d^2H}{dx^2} + k^2H = 0, \qquad \frac{d^2Q}{dy^2} + p^2Q = 0$$