Graphs and Complexity

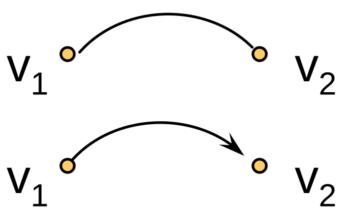
(4541.554 Introduction to Computer-Aided Design)

School of EECS

Seoul National University

Graphs

- Graph: G = (V, E)
  V : vertices v_1, v_2, ...
  E : edges = pairs of distinct vertices (v_i, v_j)
  ordered pairs ---> directed edges

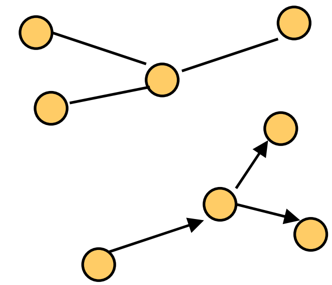


v_1 is adjacent to v_2

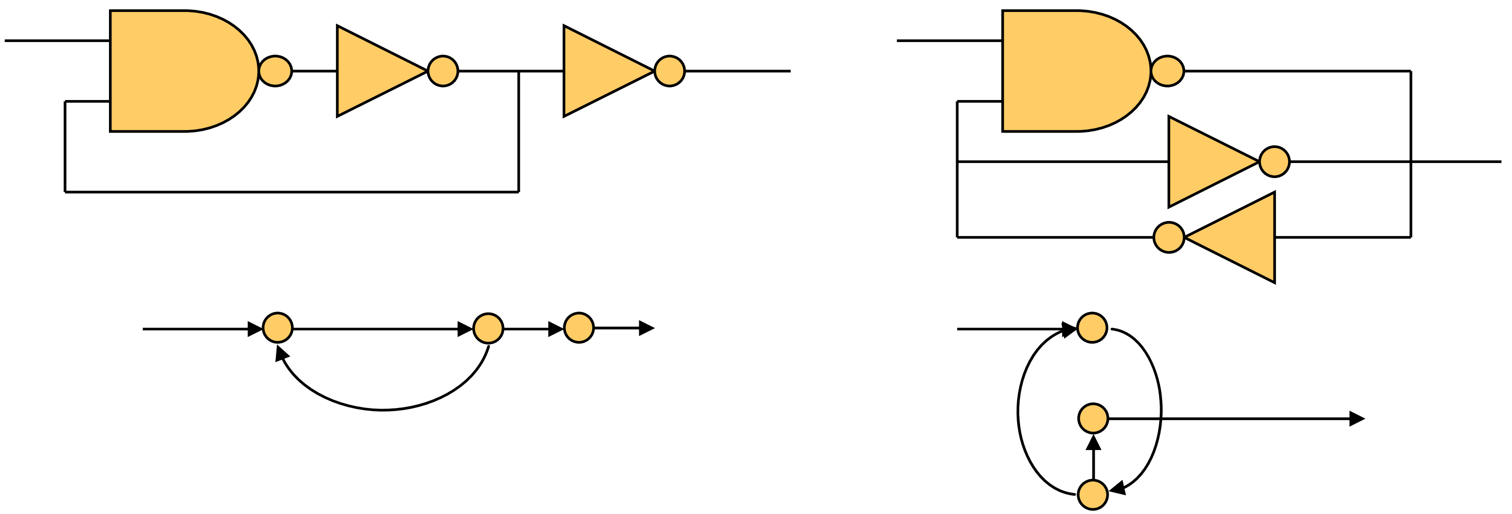
- Walk: sequence of vertices $(v_1, v_2, ..., v_n)$ such that v_i is adjacent to v_{i+1}
- Trail: walk with distinct edges
- Path: trail with distinct vertices
- Circuit: $v_1 = v_n$

Path

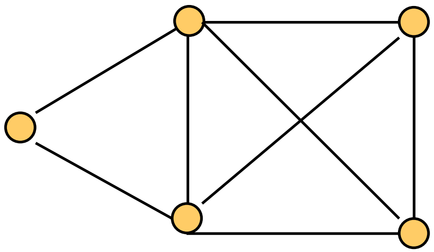
- **Degree: # of edges incident to the vertex**



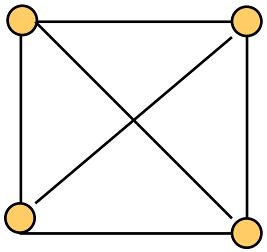
  - **for directed graph: in-degree, out-degree**
  - **sum of degrees of all vertices = 2 x (number of edges)**
    => number of vertices of odd degree is even
- **Isomorphism: one-to-one correspondence of vertices**
  - **Used for LVS**



- **Subgraph: graph formed by a subset of vertices and edges**

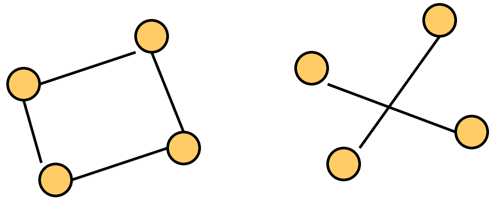


- **Complete graph: for every pair of vertices v_i and v_j, there exists an edge**

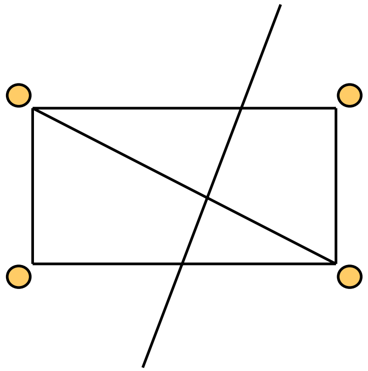


- **Clique: complete subgraph**

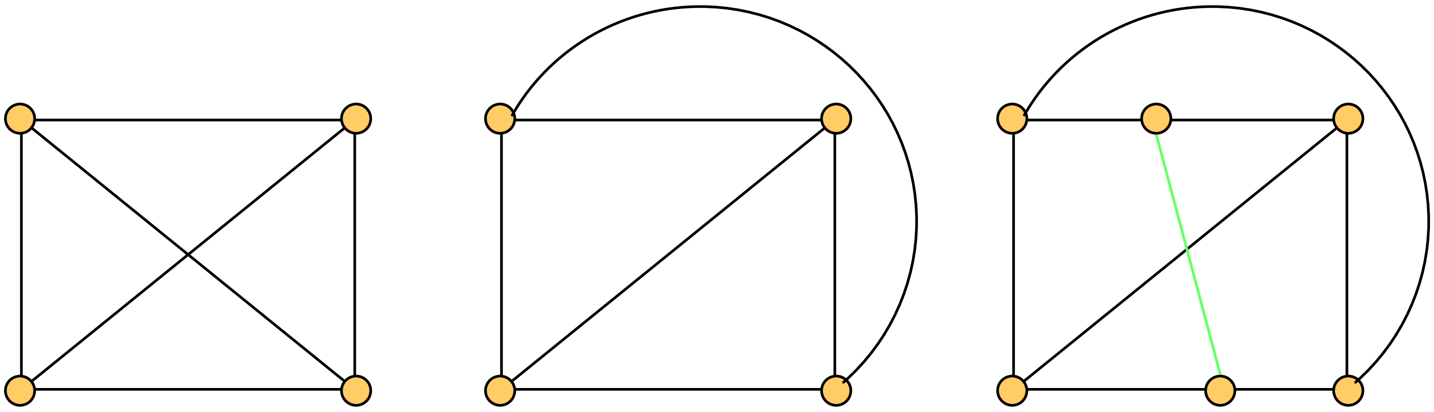
- **Complement of G = (V, E): $\overline{G}$ = (V, $\overline{E}$)**
  - **same set of vertices**
  - **edges between pairs not linked in G**



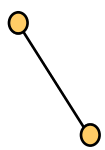
- **Cut-set: set of edges in a connected graph whose removal disconnects the graph, but no proper subset causes disconnection**



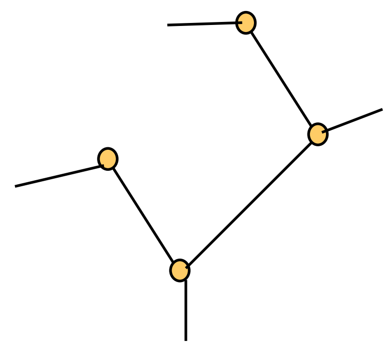
- **Planar graph : can be drawn on a plane without edges crossing**



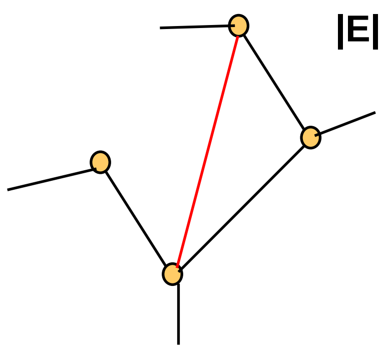
– **Theorem (Euler's formula)**
  - **Given a connected planar graph**
  - **|R| = |E| - |V| + 2**
    where R is the set of regions including the unbounded region
  - **Proof by induction starting with |E| = 1**



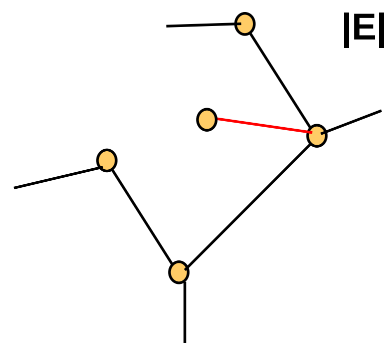
|E|=1



|E|=n-1

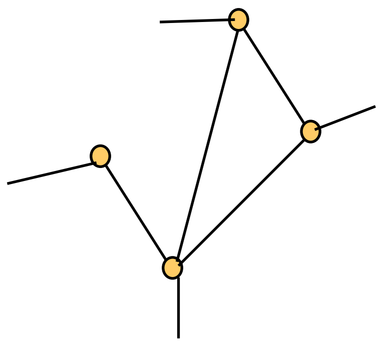


|E|=n



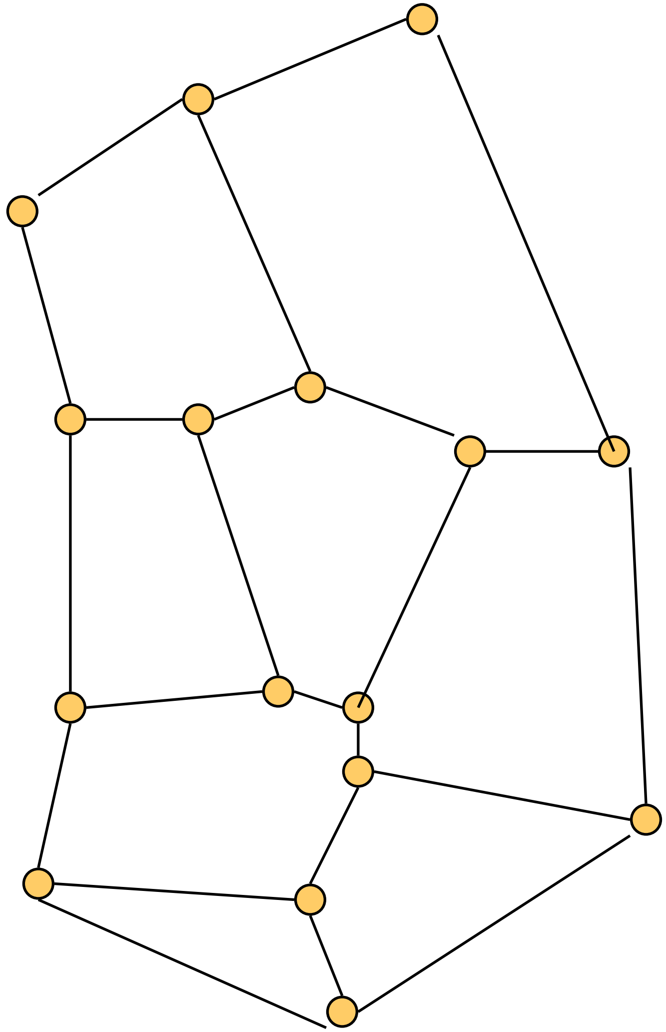
|E|=n

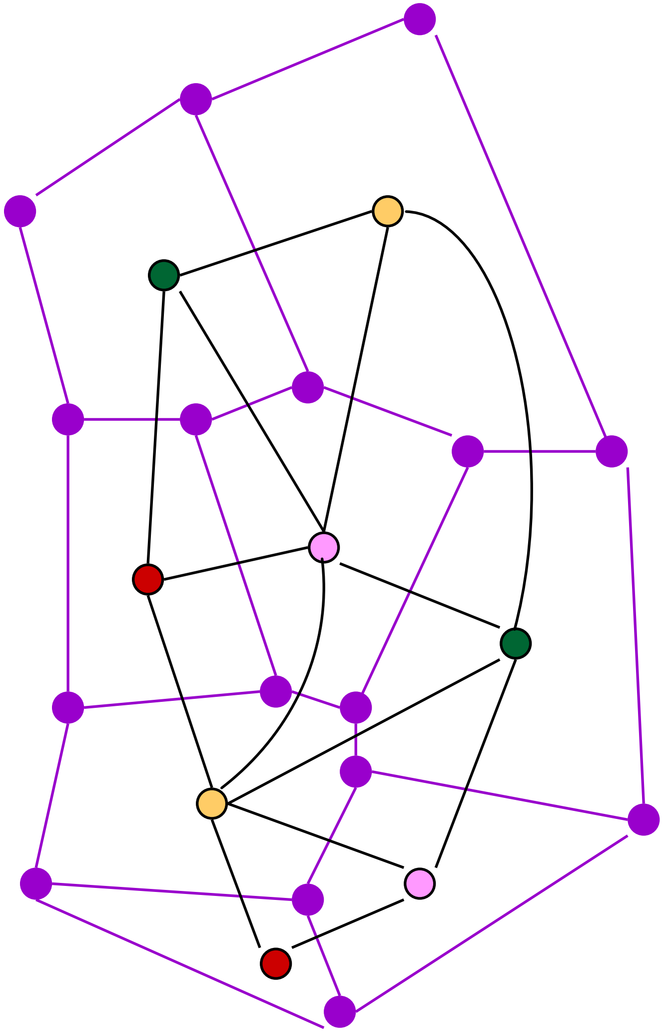
- Corollary
  - $|E| \leq 3|V| - 6$ for $|E|>1$
  - Proof
    - Degree of a region = # of edges on the boundary
    - Degree of each region ≥ 3
    - Sum of degrees of all regions $\geq 3|R|$
    - Sum of degrees of all regions = $2|E|$
    - $2|E| \geq 3|R| = 3(|E| - |V| + 2)$
    - $|E| \leq 3|V| - 6$



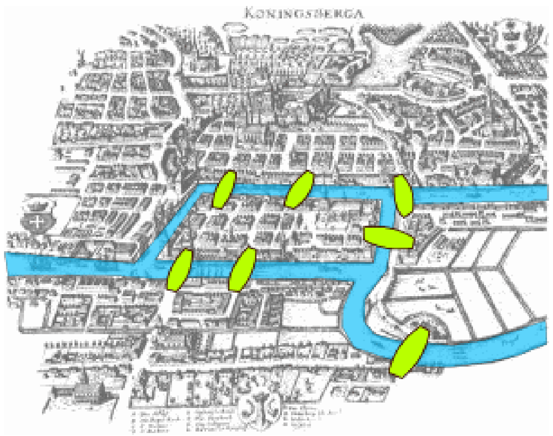
- Degrees of vertices
  - If each vertex has degree ≥ 6, then
    $|E| = (\text{sum of degrees})/2 \geq 6/2\ |V| > 3|V| - 6$
    and the graph cannot be planar.
  - A planar graph has a vertex of degree at most 5.

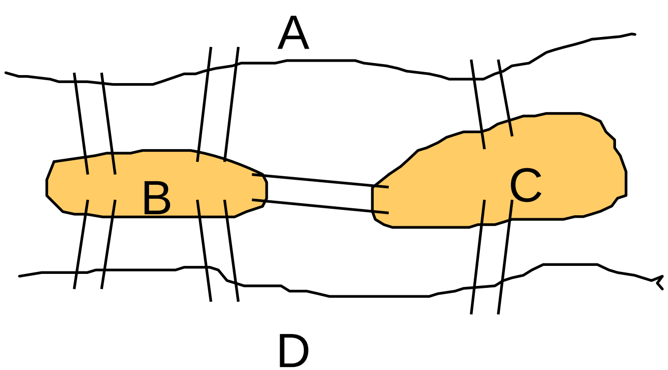
- **Dual graph and vertex coloring**

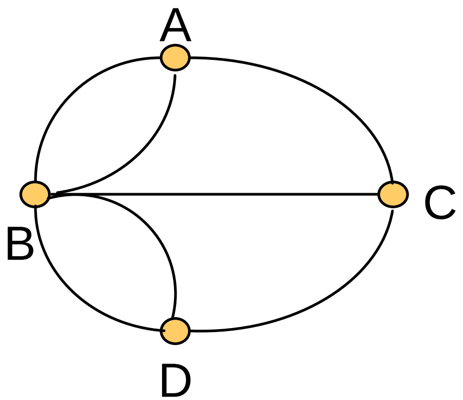


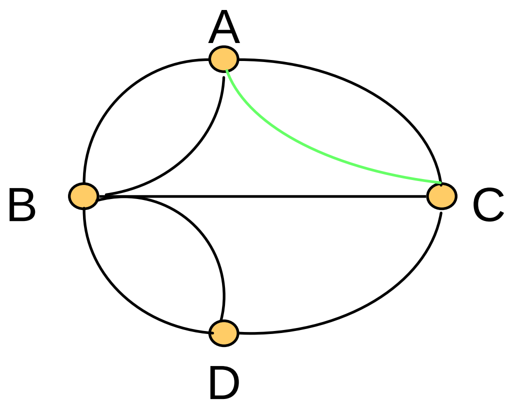


- **Euler circuit, Euler trail**
  - **Traverse all edges once**
  - **Euler circuit: the degree of each vertex must be even**
  - **Euler trail: no more than 2 vertices must have odd degree**
  - **Königsberg (Kaliningrad) bridges**

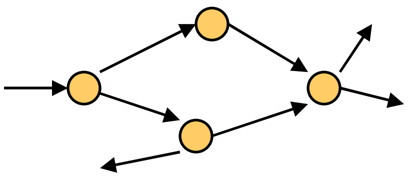




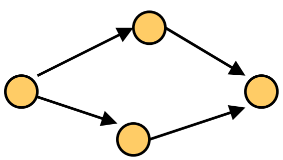




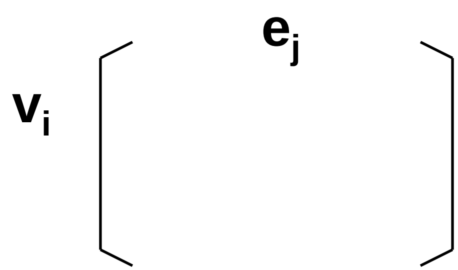
- **Hamilton circuit/path**
  - **Visit each vertex once**
  - **Exactly two edges are incident at each vertex.**
- **Traveling Salesman Problem**
  - **Find a minimum cost Hamilton circuit in a complete graph**
- **DAG: directed acyclic graph**
  - **set of vertices: strict partial order**
  - **(direct) successor (or descendant)**
  - **(direct) predecessor (or ancestor)**



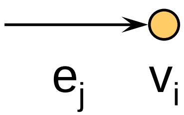
- **(Strict) partial order: relation R that is**
  - **reflexive: $(x, x) \in R$ (irreflexive : $(x, x) \notin R$)**
  - **antisymmetric: $(x, y) \in R$ and $(y, x) \in R \Rightarrow x=y$ (asymmetric: $(x, y) \in R \Rightarrow (y, x) \notin R$)**
  - **transitive: $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$**
- **Polar DAG: any v is reachable from source, and sink is reachable from any v**



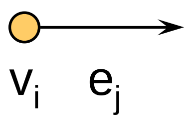
- **Incidence matrix :**
  - **undirected : (i, j) = 1 if e_j is incident to v_i else 0**



  - **directed : (i, j) = 1 if**

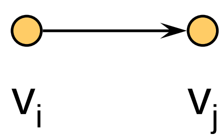


= -1 if



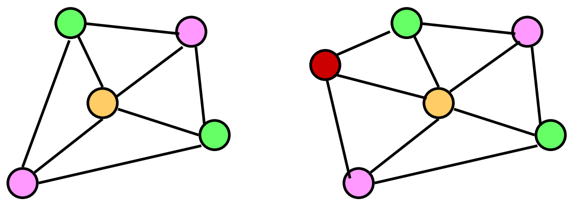
= 0 otherwise

- **Adjacency matrix :**
  - **(i, j) = 1 if v_i is adjacent to v_j**



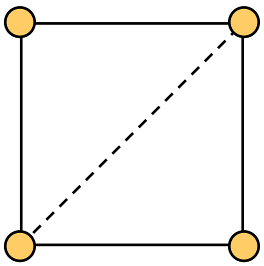
- **Clique number: cardinality of the largest clique**
- **Clique partition: disjoint**
- **Clique cover: possibly overlapping**
- **Clique cover number: cardinality of a minimum clique cover**
- **Independent set: no two vertices in the set are adjacent**
- **independence number: cardinality of the largest independent set**
- **Coloring: partition of vertices into independent sets**
- **Chromatic number: minimum number of colors needed**

- **clique number ≤ chromatic number**
  (vertices in a clique ⇒ different colors)

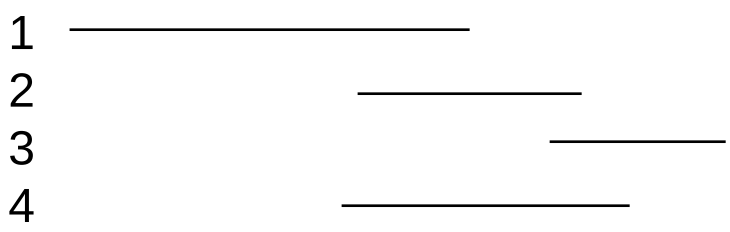


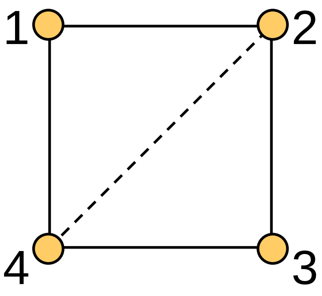
- **independence number ≤ clique cover number**
  (vertices in an independent set ⇒ different cliques)

- **Chordal (triangulated) graph: every cycle with more than 3 edges possesses a chord**



- **Interval graph: subclass of chordal graph**





- **Tree: root vertex + unique path from the root to each vertex**
- **parent, child, sibling (same parent)**
- **ancestor, descendant**
- **A tree with n vertices has n-1 edges.**
- **Leaf: vertex with no children**

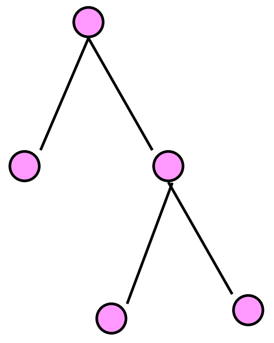
  Internal vertex: non-leaf vertex
- **m-ary tree: each non-leaf vertex has m children**

  m=2: binary tree

  i = # of internal vertices => (mi+1) vertices in total

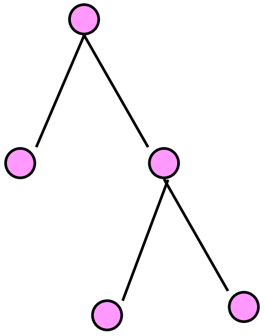
  l = # of leaves => l + i vertices in total

  => l = (m-1)i + 1

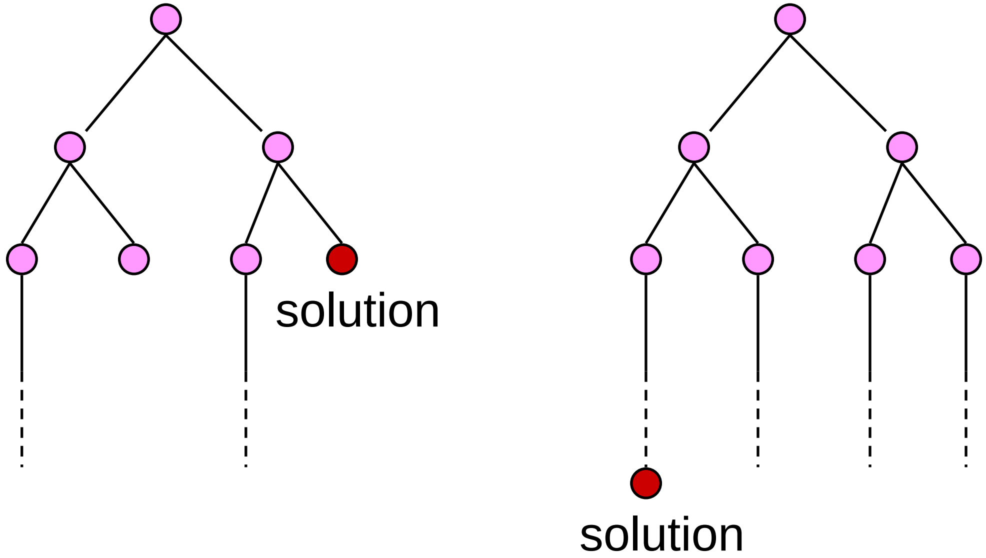


- **Height: length (# of edges) of the longest path from the root**
- **Level of a node: length of the path from the root to the node**
- **Balanced: all leaves are at levels h and h-1 (h=height)**
- **m-ary tree: at most m^h leaves => $h \geq \lceil \log_m l \rceil$**

  balanced => $h = \lceil \log_m l \rceil$

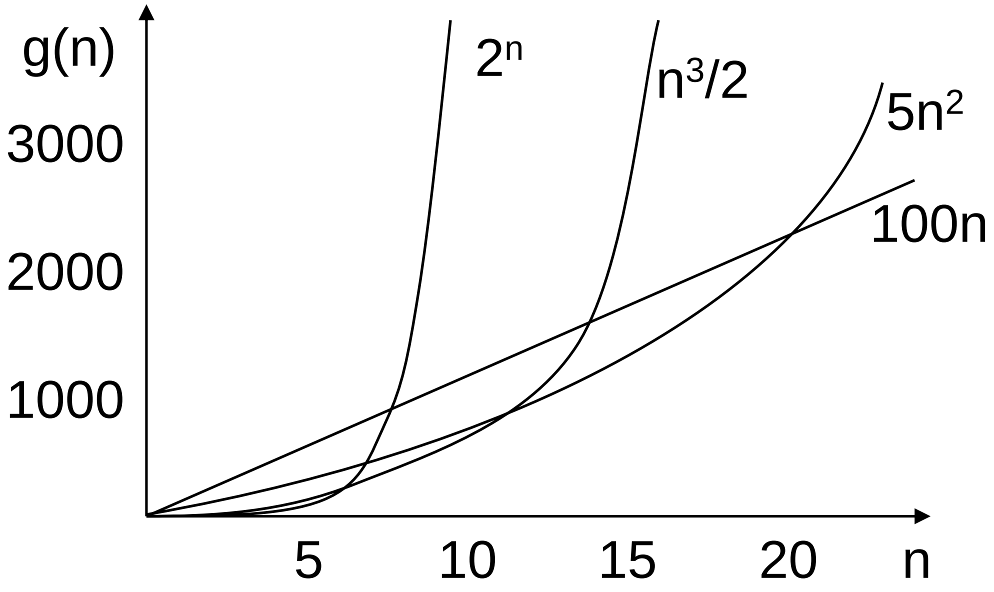


- **Tree enumeration**
  - **Depth-first search (backtracking, branch and bound)**
  - **Breadth-first search**
  - **Balanced binary tree with height=n**
    - **About $2x2^n$ nodes**
    - **Worst case traversal visits all nodes**
    - **Runtime=$c2^n$**



Computational Complexity

- **Notation**
  - **$f(n) = O(g(n))$**
    if there exist $c>0$ and $n_0>0$ such that $f(n) \leq cg(n)$ for $n \geq n_0$
  - **$f(n) = \Omega(g(n))$**
    if there exist $c>0$ and $n_0>0$ such that $f(n) \geq cg(n)$ for $n \geq n_0$
  - **$f(n) = \Theta(g(n))$**
    if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$



Function	Approximate Values		
n	10	100	1000
nlogn	33	664	9966
n^3	1,000	1,000,000	10^9
10^6n^8	10^{14}	10^{22}	10^{30}
2^n	1024	$1.27x10^{30}$	$1.05x10^{301}$
n^{logn}	2099	$1.93x10^{13}$	$7.89x10^{29}$
n!	3,628,800	10^{158}	$4x10^{2567}$

Function	Size of Instance Solved in One Day	Size of Instance Solved in a Computer 10 Times Faster
n	10^{12}	10^{13}
$n \log n$	0.948×10^{11}	0.87×10^{12}
n^2	10^6	3.16×10^6
n^3	10^4	2.15×10^4
$10^8 n^4$	10	18
2^n	40	43
10^n	12	13
$n^{\log n}$	79	95
$n!$	14	15