21 Analyticity

21.1 Limit, Continuity.

limit.

$$\lim_{z\to z_0} f(z) = l.$$

$$\left(\begin{array}{l} |f(z)-l| < \varepsilon. \\ |z-z_0| < \delta. \end{array} \right.$$

continuity : $\lim_{z\to z_0} f(z) = f(z_0)$
a function $f(z)$ is said to be continuous at $z = z_0$

Derivative.

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$\Delta z = z - z_0, z = z_0 + \Delta z$$

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

Example 1. Differentiability, Derivative.

$$f(z) = z^2$$

$$f'(z) = \lim_{\Delta z\to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} (2z+\Delta z) = 2z$$

Differentiation rules.

$$(cf)' = cf', \quad (f+g)' = f'+g', \quad (fg)' = f'g + fg'$$

$$(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$$

$$(z^n)' = nz^{n-1} \text{ (n : integer)}$$

Example 2. $\bar{z}$ not differentiable.

$$f(z) = \bar{z} = x - iy.$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{(\overline{z+\Delta z}) - \Delta z}{\Delta z} = \frac{\Delta\bar{z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \qquad \begin{array}{l} \Delta x \to 0 : -1. \\ \Delta y \to 0 : +1. \end{array}$$

Analytic Functions.

Definition (Analyticity).
A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0. Also, by an analytic we mean a function that is analytic in some domain.

Example 3. Polynomials, rational functions.

polynomials $\quad 1, z, z^2, \cdots$
quotient of two polynomials $g(z)$, $h(z)$

$$f(z) = \frac{g(z)}{h(z)} \text{ : rational function.}$$

21.2 Cauchy-Riemann Equations. Laplace's Equation.

$$w = f(z) = u(x, y) + iv(x, y)$$

Cauchy-Riemann equations.
(1)

$$u_x = v_y, u_y = -v_y$$

f is analytic in a domain D if and only if the first partial derivatives of u, v satisfy the two so-called Cauchy-Riemann equations everywhere in D.

Theorem 1 (Cauchy-Riemann equations).

Let $f(z) = u(x, y) + iu(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations (1). Hence if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (1) at all points of D.

Proof.
(2)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad \left(\begin{array}{l} f(z) = u + iv \\ z = x + iy \end{array} \right.$$

$$\Delta z = \Delta x + i\Delta y.$$

(3)

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

i) path I. $\Delta y \to 0$.

$$f'(z) = \lim_{\Delta x \to 0} \frac{ux+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

(4)

$$f'(z) = u_x + iv_x$$

ii) Path II. $\Delta x \to 0$.

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+y+\Delta y) - v(x,y)}{i\Delta y}$$

(5)

$$f'(z) = -iu_y + v_y$$

(4)=(5) $\quad \therefore u_x = v_y \; v_x = -u_y.$

Theorem 2 (Cauchy-Riemann equations)

If two real-valued continuous functions $u(x, y)$ and $v(x, y)$ of two real variables x and y have continuous derivatives that satisfy the Cauchy-Riemann equations in some domain D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

Example 3. An analytic function of constant absolute value is constant. Show that if $f(z)$ is analytic in a domain D and $|f(z) = k$ =constant in D, then $f(z)$ =const in D.

Solution. $u^2 + v^2 = k^2$.

$$uu_x + vv_x = 0, uu_y + vv_y = 0$$

Cauchy-Riemann equations. $v_x = -u_y$, & $v_y = u_x$

$$u \cdot u_x - v \cdot u_y = 0 \qquad \text{a)}$$

$$u \cdot u_y - v \cdot u_x = 0 \qquad \text{b)}$$

(a)$\cdot u$+(b)$\cdot v$: $(u^2 + v^2) \cdot u_x = 0$.
(a)$\cdot(-v)$+(b)$\cdot u$: $(u^2 + v^2) \cdot u_y = 0$.
i) if $k^2 = u^2 + v^2 = 0$ then $u = v = 0 \to f = 0$.
ii) $u_x = u_y = 0$ by C-R eq. $v_x = v_y = 0$

$$u = v = \text{const} \to f = \text{const}$$

Polar form. $z = r(\cos\theta + i\sin\theta)$

$$f(z) = u(r,\theta) + iv(r,\theta)$$
$$x = r\cos\theta, y = r\sin\theta. \to r = \sqrt{x^2+y^2}, \theta = \arctan\frac{y}{x}$$
$$r_x = \frac{x}{(x^2+y^2)^1/2} = \frac{x}{r}, \qquad r_y = \frac{y}{(x^2+y^2)^1/2} = \frac{y}{r}$$
$$\theta_x = \frac{1}{1+(y/x)^2}(-\frac{y}{x^2}) = -\frac{y}{x^2+y^2} = -\frac{y}{r^2}$$
$$\theta_y = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$
$$u_x = u_r\cdot r_x + u_\theta\cdot\theta_x = \frac{x}{r}\cdot u_r - \frac{y}{r^2}u_\theta.$$
$$u_y = v_r\cdot r_y + v_\theta\cdot\theta_y = \frac{y}{r}\cdot v_r + \frac{x}{r^2}v_\theta.$$

Since $u_x = v_y$ $\qquad rx\cdot u_r - yu_\theta = ryv_r + xv_\theta \cdots$ a)

$$u_y = u_r\cdot r_y + u_\theta\cdot\theta_y = \frac{y}{r}u_r + \frac{x}{r^2}u_\theta.$$
$$-v_x = -v_r\cdot r_x - v_\theta\cdot\theta_x = -\frac{x}{r}\cdot v_r + \frac{y}{r^2}v_\theta.$$

Since $u_y = -v_x$ $\qquad r_yu_r + xu_\theta = -rxv_r + yv_\theta \cdots$ b)
a)$\times x+$ b)$\times y$ $\qquad u_r\cdot r\cdot(x^2+y^2) = v_\theta(x^2+y^2)$

$$\therefore u_r = \frac{1}{r}v_\theta$$

a)$\times y-$b)$\times x$: $-(y^2+x^2)u_\theta = r(y^2+x^2)v_r$

$$\therefore v_r = -\frac{1}{r}u_\theta.$$

(7) $u_r = \frac{1}{r}v_\theta, v_r = -\frac{1}{r}u_\theta \quad (r>0)$

21.3 Laplace's Equation. Harmonic Functions. $\Rightarrow$ (Solution of Laplace's Equation)

Theorem 3 (Laplace's equation)
If $f(z) = u(x,y)+iv(x,y)$ is analytic in a domain D, then u and v satisfy Laplace's equation.
(8) $\nabla^2 u = u_{xx} + u_{yy} = 0.$
and
(9) $\nabla^2 v = v_{xx} + v_{yy} = 0.$
respectively, in D and have continuous second partial derivatives in D.

Proof.

i) $\left.\begin{array}{l} u_x = v_y \Rightarrow u_{xx} = v_{yx} \\ u_y = -v_x \Rightarrow u_{yy} = -vyx \end{array}\right) \Rightarrow u_{xx} + u_{yy} = 0$

ii) $\left.\begin{array}{l} u_x = v_y \Rightarrow u_{xy} = v_{yy} \\ u_y = -v_x \Rightarrow u_{yx} = -vxx \end{array}\right) \Rightarrow v_{xx} + v_{yy} = 0$

If two harmonic functions u and v satisfy the Cauchy-Riemann equations in a domain D, they are the real and imaginary parts of an analytic function f in D.
Then v is said to be a conjugate harmonic function of u in D.

Example 4. How to find a conjugate harmonic function by the Cauchy-Riemann equations.

$$u = x^2 - y^2 - y$$

sol) $u_x = 2x, u_{xx} = 2, u_y = -2y - 1, u_{yy} = -2.$

$$u_{xx} + u_{yy} = 2 - 2 = 0 \qquad \therefore \text{harmonic function.}$$

By C-R. $v_y = u_x = 2x, v_x = -u_y = 2y + 1.$

$$v = 2xy + h(x) \Rightarrow v_x = 2y + dh/dx.$$

$$\therefore dh/dx = 1 \Rightarrow h(x) = x + c$$

$$v = 2xy + x + c.$$