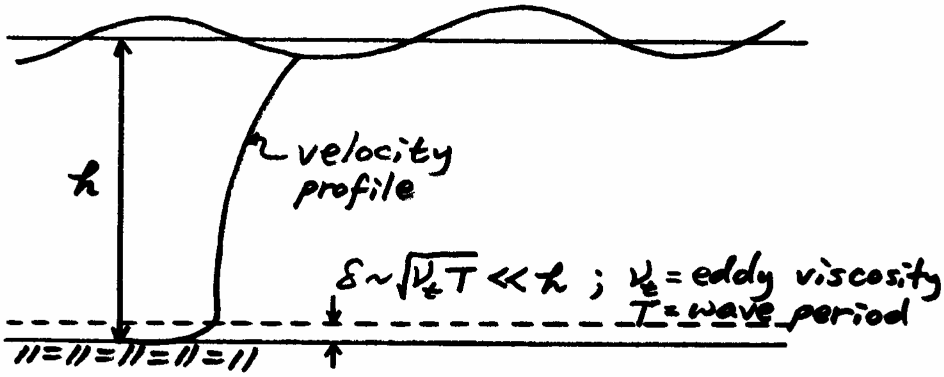
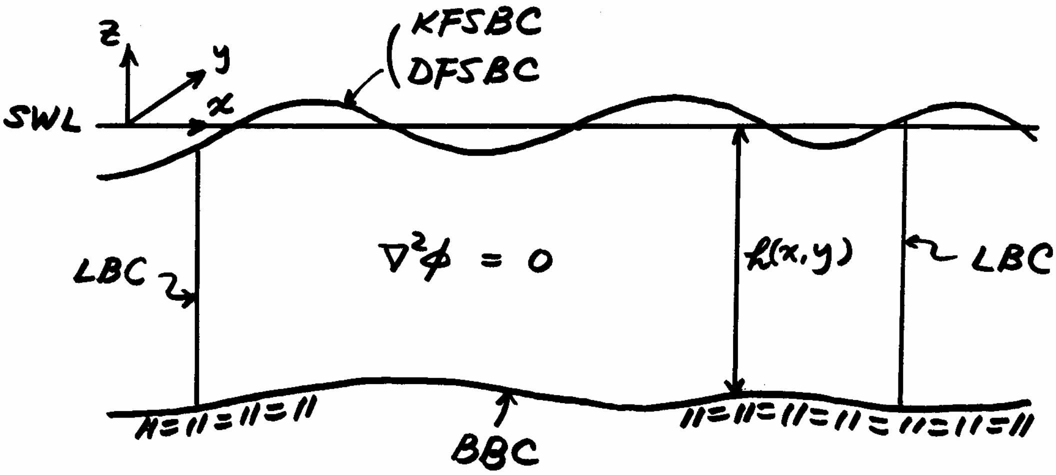
Chapter 3. Small-Amplitude Water Wave Theory

3.1 Introduction



Assuming inviscid, incompressible fluid and irrotational flow, ϕ and Ψ exist, which satisfy $\nabla^2\phi = \nabla^2\Psi = 0$.

3.2 Boundary Value Problem



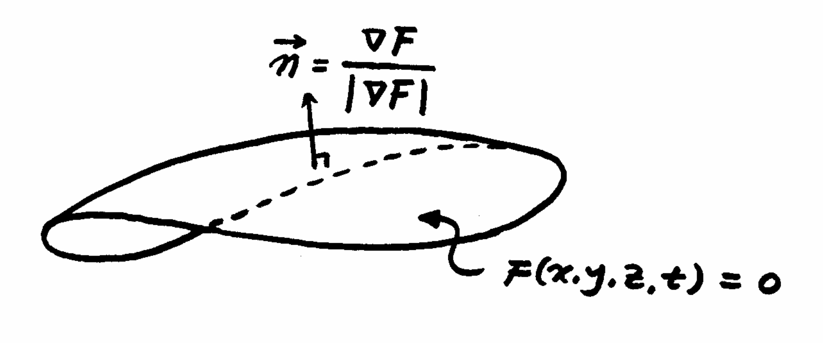
3.3 Boundary Conditions

3.3.1 Kinematic boundary condition

The kinematic boundary condition is a condition that describes the water particle kinematics at a boundary (either fixed or moving). If we sit on a boundary (fixed or moving) and move with the boundary, we do not feel any change of the surface that constitutes the boundary. Mathematically, the rate of change of the surface must be zero, i.e., the total derivative of the surface is zero on the surface. Let the surface of the boundary be represented by $F(x, y, z, t) = 0$. Then

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \quad \text{on} \quad F = 0$$

$$-\frac{\partial F}{\partial t} = \vec{u} \cdot \nabla F = \vec{u} \cdot \vec{n}|\nabla F|; \quad \nabla F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}, \quad |\nabla F| = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}$$



$$\boxed{\vec{u} \cdot \vec{n} = \frac{-\partial F / \partial t}{|\nabla F|} \quad \text{on} \quad F = 0}$$

3.3.2 BBC

Assume a fixed sea bed:

$$z = -h(x, y) \quad \Rightarrow \quad F(x, y, z, t) = z + h(x, y) = 0$$

The kinematic boundary condition on sea bed is

$$\vec{u} \cdot \vec{n} = \frac{-\partial F / \partial t}{|\nabla F|} = 0 \quad \text{on} \quad z = -h(x, y)$$

Recall that $\vec{a} \perp \vec{b}$ if $\vec{a} \cdot \vec{b} = 0$. Therefore, $\vec{u} \cdot \vec{n} = 0$ means that $\vec{u}$ is parallel to the sea bed surface, i.e. no flow perpendicular to the bed.

Using

$$\vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{\frac{\partial h}{\partial x}\vec{i} + \frac{\partial h}{\partial y}\vec{j} + \vec{k}}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}}$$

we have

$$\vec{u} \cdot \vec{n} = \frac{u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}} = 0$$

$$w = -u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y} \quad \text{on} \quad z = -h(x, y)$$

$$-\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial h}{\partial y} \quad \text{on} \quad z = -h(x, y)$$

On a horizontal bottom, $\partial h / \partial x = \partial h / \partial y = 0$, so that $\partial \phi / \partial z = 0$ on $z = -h$.

3.3.3 KFSBC

On water surface,

$$z = \eta(x, y, t) \quad \Rightarrow \quad F(x, y, z, t) = z - \eta(x, y, t) = 0$$

The kinematic boundary condition on free surface is

$$\vec{u} \cdot \vec{n} = \frac{-\partial F / \partial t}{|\nabla F|} = \frac{\partial \eta / \partial t}{\sqrt{(\partial \eta / \partial x)^2 + (\partial \eta / \partial y)^2 + 1}} \quad \text{on} \quad z = \eta(x, y, t)$$

Using

$$\vec{u} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{-\frac{\partial \eta}{\partial x}\vec{i} - \frac{\partial \eta}{\partial y}\vec{j} + \vec{k}}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + 1}}$$

we have

$$-u\frac{\partial \eta}{\partial x} - v\frac{\partial \eta}{\partial y} + w = \frac{\partial \eta}{\partial t}$$

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \quad \text{on} \quad z = \eta(x, y, t)$$

$$-\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} \quad \text{on} \quad z = \eta(x, y, t)$$

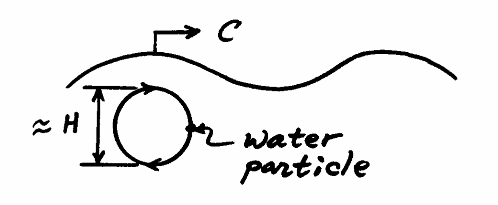
Small-amplitude wave theory assumes

$L \sim O(h)$

H / L = wave steepness << 1

$\eta \sim O(H)$

$t \sim O(T)$
$x \sim O(L)$
$z \sim O(h) \sim O(L)$
$u = \dfrac{\partial \phi}{\partial x} \sim O\left(\dfrac{H}{T}\right)$
$\phi \sim O\left(\dfrac{HL}{T}\right)$



$$-\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} \quad \text{on} \quad z = \eta(x,y,t)$$

$$\left(\frac{H}{T}\right) \quad \left(\frac{H}{T}\right) \quad \left(\frac{H}{T}\frac{H}{L}\right) \quad \left(\frac{H}{T}\frac{H}{L}\right)$$

$$(1) \quad (1) \quad \left(\frac{H}{L}\right) \quad \left(\frac{H}{L}\right) << 1$$

$$\therefore \; -\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{on} \quad z = \eta(x,y,t)$$

Taylor series expansion about $z = 0$ gives

$$\left(-\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t}\right)_{z=\eta} = \left(-\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t}\right)_{z=0} + \eta \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t}\right)_{z=0} + \frac{\eta^2}{2}\frac{\partial^2}{\partial z^2}\left(-\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t}\right)_{z=0} + \cdots = 0$$

$$\left(\frac{H}{T}\right) \left(\frac{H}{T}\right) \qquad \left(\frac{H^2}{LT}\right) \left(\frac{H^2}{LT}\right)$$

$$-\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{on} \quad z = 0 \quad \text{(LKFSBC)}$$

3.3.4 DFSBC

The Bernoulli equation on free surface is

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p_\eta}{\rho} + g\eta = C(t) \quad \text{on} \quad z = \eta(x, y, t)$$

Using $p_\eta = 0$ (gauge pressure),

$$\begin{array}{cccc}
-\dfrac{\partial \phi}{\partial t} & + \dfrac{1}{2}\left(u^2 + v^2 + w^2\right) & + g\eta & = C(t) \quad \text{on} \quad z = \eta(x, y, t) \\
\left(\dfrac{HL}{T^2}\right) & \left(\dfrac{H^2}{T^2}\right) & (gH) & \\
(1) & \left(\dfrac{H}{L}\right) << 1 & \left(g\dfrac{T^2}{L}\right) \sim O(1) &
\end{array}$$

$$\therefore \; -\frac{\partial \phi}{\partial t} + g\eta = C(t) \quad \text{on} \quad z = \eta(x, y, t)$$

Taylor series expansion about $z = 0$ gives

$$\begin{array}{ccccc}
\left(-\dfrac{\partial \phi}{\partial t} + g\eta\right)_{z=\eta} = & \left(-\dfrac{\partial \phi}{\partial t} + g\eta\right)_{z=0} & + \eta \dfrac{\partial}{\partial z}\left(-\dfrac{\partial \phi}{\partial t} + g\eta\right)_{z=0} & + \cdots = C(t) \\
 & \left(\dfrac{HL}{T^2}\right) \; (gH) & \left(\dfrac{H^2}{T^2}\right) \; \left(g\dfrac{H^2}{L}\right) & \\
 & (1) \; \left(g\dfrac{T^2}{L}\right) & \left(\dfrac{H}{L}\right) \; \left(g\dfrac{T^2}{L}\dfrac{H}{L}\right) &
\end{array}$$

$$-\frac{\partial \phi}{\partial t} + g\eta = C(t) \quad \text{on} \quad z = 0 \quad \text{(LDFSBC)}$$

3.3.5 LBC

Assuming 2-D ($x - z$) periodic wave,

$$\phi(x,t) = \phi(x+L,t)$$
$$\phi(x,t) = \phi(x,t+T)$$

3.3.6 Summary of 2-D periodic wave boundary value problem

GE	$\nabla^2\phi = 0$	$0 \le x \le L, \quad -h \le z \le \eta$
BBC	$-\dfrac{\partial\phi}{\partial z} = 0$	on $z = -h$
LKFSBC	$-\dfrac{\partial\phi}{\partial z} = \dfrac{\partial\eta}{\partial t}$	on $z = 0$
LDFSBC	$-\dfrac{\partial\phi}{\partial t} + g\eta = C(t)$	on $z = 0$
LBC	$\phi(x,t) = \phi(x+L,t)$ $\phi(x,t) = \phi(x,t+T)$	

Note: The Laplace equation is linear, so the superposition of solutions is valid.

3.4 Solution to Boundary Value Problem

Using separation of variables, the velocity potential can be expressed as

$$\phi(x,z,t) = \mathrm{X}(x)\mathrm{Z}(z)\mathrm{T}(t)$$

Assume

$$\mathrm{T}(t) = \sin\sigma t; \quad \sigma = \frac{2\pi}{T}$$

where σ = wave angular frequency. The above equation satisfies the periodicity condition in time, that is

$$\mathrm{T}(t+T) = \sin(\sigma t + \sigma T) = \sin\sigma t\cos\sigma T + \cos\sigma t\sin\sigma T = \sin\sigma t$$

Now

$$\phi(x,z,t) = \mathrm{X}(x)\mathrm{Z}(z)\sin\sigma t$$

Substitution into the Laplace equation gives

$$\frac{d^2\mathrm{X}}{dx^2}\mathrm{Z}\sin\sigma t + \mathrm{X}\frac{d^2\mathrm{Z}}{dz^2}\sin\sigma t = 0$$
$$\frac{1}{\mathrm{X}}\frac{d^2\mathrm{X}}{dx^2} + \frac{1}{\mathrm{Z}}\frac{d^2\mathrm{Z}}{dz^2} = 0$$

The first term is a function of x alone, whereas the second term is a function of z alone. Therefore, the two terms should be constants with opposite signs, that is,

$$\frac{1}{\mathrm{X}}\frac{d^2\mathrm{X}}{dx^2} = -k^2$$
$$\frac{1}{\mathrm{Z}}\frac{d^2\mathrm{Z}}{dz^2} = k^2$$

The possible solutions that satisfy these ODE's are given in Table 3.1 of textbook.

Among these, the solution which satisfies the periodicity condition in x is

$$\phi(x,z,t) = (A\cos kx + B\sin kx)\left(Ce^{kz} + De^{-kz}\right)\sin\sigma t$$

The periodicity condition in x gives

$$\begin{aligned} A\cos kx + B\sin kx &= A\cos k(x+L) + B\sin k(x+L) \\ &= A(\cos kx\cos kL - \sin kx\sin kL) + B(\sin kx\cos kL + \cos kx\sin kL) \end{aligned}$$

For the above relation to be satisfied, the followings are needed:

$$\cos kL = 1 \quad \text{and} \quad \sin kL = 0$$

which give the wave number,

$$k = \frac{2\pi}{L}$$

Recalling that the superposition of solutions is valid for the Laplace equation, keep only

$$\phi(x,z,t) = A\cos kx\left(Ce^{kz} + De^{-kz}\right)\sin\sigma t$$

Using BBC,

$$\begin{aligned} &-\frac{\partial\phi}{\partial z} = A\cos kx\left(kCe^{kz} - kDe^{-kz}\right)\sin\sigma t = 0 \quad \text{on} \quad z = -h \\ &Ce^{-kh} - De^{kh} = 0 \\ &C = De^{2kh} \end{aligned}$$

Therefore,

$$
\begin{aligned}
\phi(x,z,t) &= A\cos kx\left(De^{2kh}e^{kz} + De^{-kz}\right)\sin\sigma t \\
&= ADe^{kh}\cos kx\left(e^{k(h+z)} + e^{-k(h+z)}\right)\sin\sigma t \\
&= G\cos kx\cosh k(h+z)\sin\sigma t
\end{aligned}
$$

Applying LDFSBC,

$$
\begin{aligned}
&-\frac{\partial\phi}{\partial t} + g\eta = C(t) \quad \text{on} \quad z=0 \\
&-\sigma G\cos kx\cosh kh\cos\sigma t + g\eta = C(t) \\
&\eta = \frac{\sigma G\cosh kh}{g}\cos kx\cos\sigma t + \frac{C(t)}{g}
\end{aligned}
$$

The spatial mean of η must be zero. Thus, $C(t)$ must be zero. Expressing η as

$$
\eta = \frac{H}{2}\cos kx\cos\sigma t
$$

we have

$$
G = \frac{gH}{2\sigma\cosh kh}
$$

Finally,

$$
\phi(x,z,t) = \frac{gH}{2\sigma}\frac{\cosh(h+z)}{\cosh kh}\cos kx\sin\sigma t
$$

Applying LKFSBC,

$$
\begin{aligned}
&-\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} \quad \text{on} \quad z=0 \\
&-\frac{gkH}{2\sigma}\frac{\sinh kh}{\cosh kh}\cos kx\sin\sigma t = -\frac{\sigma H}{2}\cos kx\sin\sigma t \\
&\frac{gkH}{2\sigma}\tanh kh = \frac{\sigma H}{2}
\end{aligned}
$$

$$\therefore\ \sigma^2 = gk\tanh kh \quad \text{(dispersion relationship)}$$

The dispersion relationship gives relation among h, σ and k or the relation among h, T and L. The dispersion relationship can be solved for k by Newton-Raphson method for given σ and h. Approximate solutions are also available:

Eckart (1951):

$$\sigma^2 = gk\sqrt{\tanh\frac{\sigma^2 h}{g}}$$

Hunt (1979):

$$(kh)^2 = y^2 + \frac{y}{1+\sum_{n=1}^{6} d_n y^n}; \qquad y = \frac{\sigma^2 h}{g}, \quad d_1 \sim d_6 \text{ given in p. 72 of textbook}$$

The dispersion relationship gives

$$\left(\frac{2\pi}{T}\right)^2 = g\frac{2\pi}{L}\tanh kh$$

$$L = \frac{gT^2}{2\pi}\tanh kh$$

In deepwater ($kh > \pi$), $\tanh kh \cong 1$. Therefore, the deepwater wavelength is

$$L_0 = \frac{gT^2}{2\pi}$$

Then

$$L = L_0 \tanh kh$$

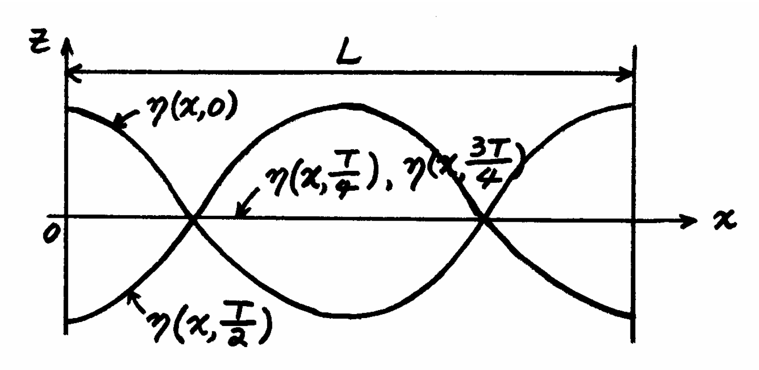
$$C = \frac{L}{T} = \frac{L_0}{T} \tanh kh = C_0 \tanh kh$$

$$C_0 = \frac{L_0}{T} = \frac{gT}{2\pi}$$

In summary,

$$\phi(x,z,t) = \frac{Hg}{2\sigma}\frac{\cosh(h+z)}{\cosh kh}\cos kx \sin\sigma t$$
$$\eta(x,t) = \frac{H}{2}\cos kx \cos\sigma t$$

which represents a standing wave.



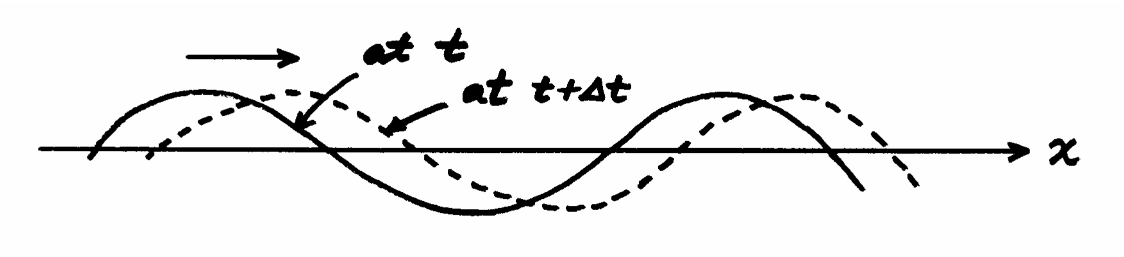
Another standing wave associated with the $\sin kx$ term gives

$$\phi(x,z,t) = -\frac{Hg}{2\sigma}\frac{\cosh(h+z)}{\cosh kh}\sin kx \cos\sigma t$$
$$\eta(x,t) = \frac{H}{2}\sin kx \sin\sigma t$$

Superposition of the two standing waves gives a progressive wave:

$$\begin{aligned}\phi(x,z,t) &= \frac{Hg}{2\sigma}\frac{\cosh(h+z)}{\cosh kh}(\cos kx \sin\sigma t - \sin kx \cos\sigma t)\\ &= -\frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx-\sigma t)\end{aligned}$$
$$\begin{aligned}\eta(x,t) &= \frac{H}{2}(\cos kx \cos\sigma t + \sin kx \sin\sigma t)\\ &= \frac{H}{2}\cos(kx-\sigma t)\end{aligned}$$

Progressive wave propagating to positive x -direction:



For progressive wave propagating to negative x -direction:

$$\sin(kx-\sigma t) \quad \Rightarrow \quad \sin(kx+\sigma t)$$

For the definitions of deepwater, shallow water, and intermediate depth water, see Fig. 3.12 of textbook. Also see Table 3.2 for asymptotic forms of hyperbolic functions.

Dispersion relationship in shallow water:

$$\sigma^2 = gk\tanh kh \cong gk^2 h$$
$$C^2 = \left(\frac{\sigma}{k}\right)^2 = gh$$
$$C = \sqrt{gh}$$

The shallow water wave is non-dispersive, that is, all the waves with different frequencies propagate at the same speed.

Dispersion relationship in deepwater:

$$\sigma^2 = gk\tanh kh \cong gk$$
$$L_0 = \frac{2\pi}{k} = \frac{2\pi}{\frac{\sigma^2}{g}} = \frac{2\pi g}{\left(\frac{2\pi}{T}\right)^2} = \frac{gT^2}{2\pi}$$
$$C_0 = \frac{L_0}{T} = \frac{gT}{2\pi}$$

3.5 Waves with Uniform Current U_0

Assume: 1) Current direction and wave propagation direction is collinear, and 2) Current is uniform vertically as well as horizontally.

In the absence of waves,

$$U_0 = -\frac{\partial \phi}{\partial x} \quad \Rightarrow \quad \phi = -U_0 x$$

For wave with current,

$$\phi = -U_0 x + A \cosh k(h+z) \cos(kx - \sigma t)$$

which satisfies the periodicity condition and BBC. Then,

$$u = -\frac{\partial \phi}{\partial x} = U_0 + kA \cosh k(h+z) \sin(kx - \sigma t)$$

$$u^2 = U_0^2 + 2kAU_0 \cosh k(h+z) \sin(kx - \sigma t) + k^2 A^2 \cosh^2 k(h+z) \sin^2 (kx - \sigma t)$$

Neglecting small nonlinear terms,

$$u^2 \cong U_0^2 + 2kAU_0 \cosh k(h+z) \sin(kx - \sigma t)$$

DFSBC is

$$\left[-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + g\eta \right]_{z=\eta}$$
$$= \left[-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + g\eta \right]_{z=0} + \eta \frac{\partial}{\partial z} \left[-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + g\eta \right]_{z=0} = C(t)$$

Note that w^2 is small compared to the linear terms but u^2 is not. Then DFSBC gives

$$-\sigma A\cosh kh\sin(kx-\sigma t)+\frac{1}{2}\left(U_0^2+2kAU_0\cosh kh\sin(kx-\sigma t)\right)+g\eta=C(t)$$

or

$$\eta(x,t)=-\frac{U_0^2}{2g}+\frac{A\sigma}{g}\left(1-\frac{kU_0}{\sigma}\right)\cosh kh\sin(kx-\sigma t)+C(t)$$

Since

$$\bar{\eta}=\frac{1}{T}\int_0^T\eta dt=0=-\frac{U_0^2}{2g}+\frac{1}{T}\int_0^T C(t)dt$$

we have

$$C(t)=\text{constant}=\frac{U_0^2}{2g}$$

Now

$$\eta(x,t)=\frac{A\sigma}{g}\left(1-\frac{kU_0}{\sigma}\right)\cosh kh\sin(kx-\sigma t)=\frac{H}{2}\sin(kx-\sigma t)$$

which gives

$$A=\frac{H}{2}\frac{g}{\sigma\left(1-\frac{kU_0}{\sigma}\right)\cosh kh}=\frac{gH}{2\sigma\left(1-\frac{U_0}{C}\right)\cosh kh}$$

Therefore,

$$\phi(x,z,t) = -U_0 x + \frac{gH}{2\sigma\left(1-\frac{U_0}{C}\right)} \frac{\cosh k(h+z)}{\cosh kh} \cos(kx-\sigma t)$$

Applying KFSBC,

$$\left[\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z}\right]_{z=\eta} = \left[\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z}\right]_{z=0} + \eta \frac{\partial}{\partial z}\left[\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z}\right]_{z=0} + \cdots = 0$$

$$\frac{\partial \eta}{\partial t} + U_0 \frac{\partial \eta}{\partial x} = -\frac{\partial \phi}{\partial z} \quad \text{on} \quad z=0$$

$$-\sigma \frac{H}{2}\cos(kx-\sigma t) + U_0 k \frac{H}{2}\cos(kx-\sigma t) = -\frac{kgH}{2\sigma\left(1-\frac{U_0}{C}\right)} \tanh kh \cos(kx-\sigma t)$$

$$-\sigma + U_0 k = -\frac{gk}{\sigma\left(1-\frac{U_0}{C}\right)} \tanh kh$$

$$-1 + U_0 \frac{k}{\sigma} = -\frac{gk}{\sigma^2\left(1-\frac{U_0}{C}\right)} \tanh kh$$

$$-1 + \frac{U_0}{C} = -\frac{gk}{\sigma^2\left(1-\frac{U_0}{C}\right)} \tanh kh$$

$$\sigma^2 = \frac{gk \tanh kh}{\left(1-\frac{U_0}{C}\right)^2}$$

$$\sqrt{\sigma^2\left(1-\frac{U_0}{C}\right)^2} = \sqrt{gk \tanh kh}$$

$$\sigma\left(1-\frac{U_0}{C}\right) = \sqrt{gk \tanh kh}$$

Finally,

$$\sigma = U_0 k + \sqrt{gk \tanh kh}$$

The second term on RHS indicates the angular frequency in no current, that is the angular frequency in moving frame of reference, while the total indicates the angular frequency in stationary frame of reference.