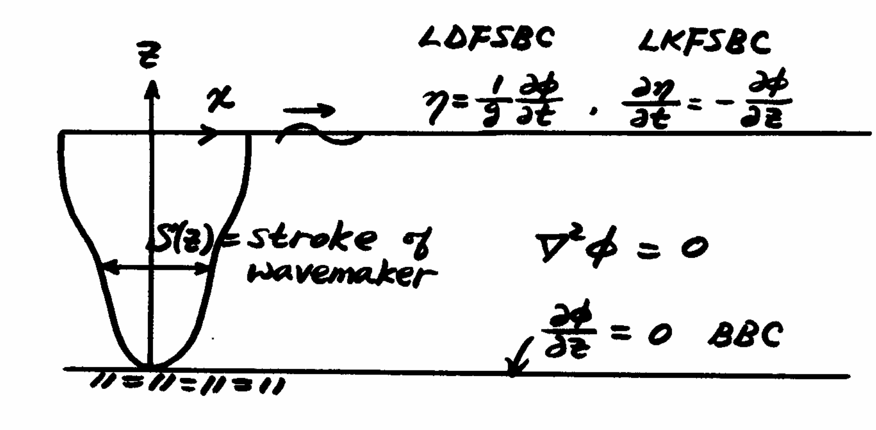
Chapter 6. Wavemaker Theory



LCFSBC(Linearized Combined Free Surface Boundary Condition): Take time derivative of LDFSBC and substitute into LKFSBC.

$$\frac{1}{g}\frac{\partial^2\phi}{\partial t^2} = -\frac{\partial\phi}{\partial z} \qquad \text{on} \quad z = 0$$

Since $\phi \propto e^{i\sigma t}$, $\partial^2\phi/\partial t^2 = -\sigma^2\phi$. Therefore,

$$\frac{\partial\phi}{\partial z} - \frac{\sigma^2}{g}\phi = 0 \qquad \text{on} \quad z = 0 \quad \text{(LCFSBC)}$$

Lateral boundary condition:

(1) Kinematic boundary condition on wavemaker

(2) Radiation condition: Waves outgoing at $x = +\infty$

Paddle displacement is described by

$$x = \frac{S(z)}{2}\sin\sigma t$$

$$F(x,z,t) = x - \frac{S(z)}{2}\sin\sigma t$$

$$\vec{u}\cdot\vec{n} = \frac{-\partial F/\partial t}{|\nabla F|}$$

$$\vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{\hat{i} - \frac{1}{2}\frac{dS}{dz}\sin\sigma t\hat{k}}{|\nabla F|}$$

$$u - \frac{w}{2}\frac{dS}{dz}\sin\sigma t - \frac{S(z)}{2}\sigma\cos\sigma t = 0 \qquad \text{on} \quad x = \frac{S(z)}{2}\sin\sigma t$$

By Taylor series expansion about $x = 0$

$$\left(u - \frac{w}{2}\frac{dS}{dz}\sin\sigma t - \frac{S(z)}{2}\sigma\cos\sigma t\right)_{x=0}$$
$$+\frac{S(z)}{2}\sin\sigma t\frac{\partial}{\partial x}\left(u - \frac{w}{2}\frac{dS}{dz}\sin\sigma t - \frac{S(z)}{2}\sigma\cos\sigma t\right)_{x=0} + \cdots = 0$$

Assume S is small so that $S = O(H)$. Then the linearized wavemaker KBC is

$$u = -\frac{\partial\phi}{\partial x} = \frac{S(z)}{2}\sigma\cos\sigma t \qquad \text{on} \quad x = 0$$

Referring to Table 3.1 of textbook, assume the solution as

$$\begin{aligned}\phi(x,z,t) = {} & A\cosh k(h+z)\sin(kx-\sigma t) + (Bx+C)(Dz+E)\cos\sigma t \\ & + \left(Fe^{k_s x} + Ge^{-k_s x}\right)\left(H\cos k_s z + I\sin k_s z\right)\cos\sigma t\end{aligned}$$

$$\phi(x,z,t) = A\cosh k_p(h+z)\sin(k_p x - \sigma t) + Be^{-k_s x}\left(C\cos k_s z + D\sin k_s z\right)\cos\sigma t$$

The first term satisfies BBC. For the second term to satisfy BBC, we need

$$-\left.\frac{\partial\phi}{\partial z}\right|_{z=-h} = 0 = -Be^{-k_s x}\left(-k_s C\sin(-k_s h) + k_s D\cos(-k_s h)\right)$$

$$-C\sin(-k_s h) + D\cos(-k_s h) = 0$$

$$D = -C \tan k_s h$$

$$C \cos k_s z + D \sin k_s z = C \cos k_s z - C \tan k_s h \sin k_s z$$

$$= \frac{C}{\cos k_s h}\left(\cos k_s z \cos k_s h - \sin k_s z \sin k_s h\right) = E \cos k_s (h+z)$$

$$\therefore\ \phi(x,z,t) = A \cosh k_p (h+z) \sin(k_p x - \sigma t) + B e^{-k_s x} \cos k_s (h+z) \cos \sigma t$$

Applying LCFSBC,

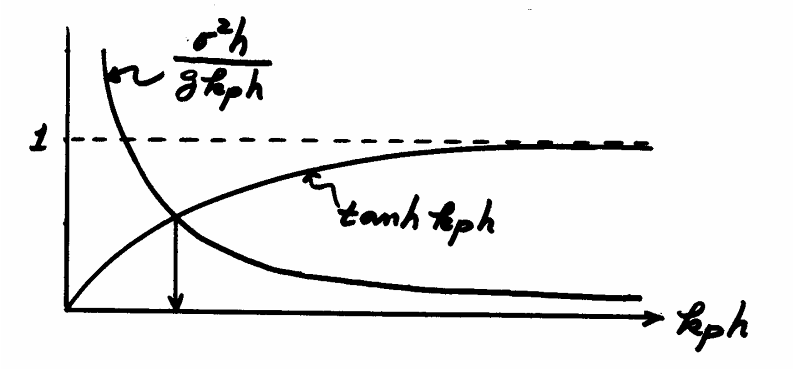
$$\frac{\partial \phi}{\partial z} - \frac{\sigma^2}{g}\phi = 0 \qquad \text{on} \quad z = 0$$

$$A k_p \sinh k_p h \sin(k_p x - \sigma t) - B k_s e^{-k_s x} \sin k_s h \cos \sigma t$$
$$- \frac{\sigma^2}{g}\left\{A \cosh k_p h \sin(k_p x - \sigma t) + B e^{-k_s x} \cos k_s h \cos \sigma t\right\} = 0$$

$$\therefore\ k_p \sinh k_p h - \frac{\sigma^2}{g}\cosh k_p h = 0 \quad \Rightarrow \quad \sigma^2 = g k_p \tanh k_p h$$

$$- k_s \sin k_s h - \frac{\sigma^2}{g}\cos k_s h = 0 \quad \Rightarrow \quad \sigma^2 = -g k_s \tan k_s h$$

$$\frac{\sigma^2 h}{g k_p h} = \tanh k_p h$$



$$\frac{\sigma^2 h}{g k_s h} = -\tan k_s h$$

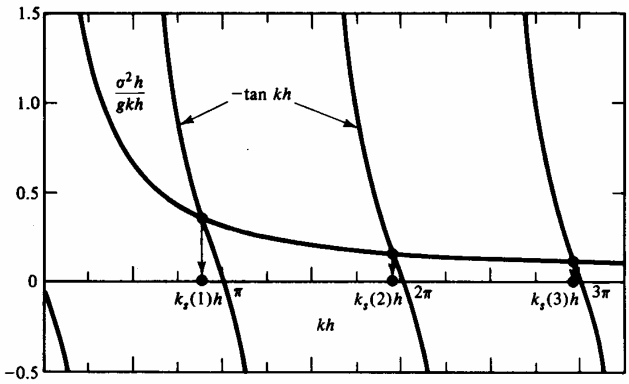


Figure 6.3 Graphical representation of the dispersion relationship for the standing wave modes, showing three of the infinite numbers of roots, $k_s(n)$. Here, $\sigma^2 h/g = 1.0$.

We have infinite number of roots, $k_s(n),\ n = 1,2,\cdots$. Let us examine the first term:

$$\begin{aligned} e^{-k_s(1)x} \le e^{-\frac{\pi/2}{h}x} &= e^{-\pi/2} = 0.208 \quad \text{for} \quad x = h \\ &= e^{-\pi} = 0.043 \quad \text{for} \quad x = 2h \\ &= e^{-3\pi/2} = 0.009 \quad \text{for} \quad x = 3h \end{aligned}$$

This term decays exponentially with the distance from the wavemaker, x. The other terms decay more rapidly with x. These terms are called evanescent modes. Now the velocity potential becomes

$$\phi(x,z,t) = \underbrace{A_p \cosh k_p(h+z)\sin(k_p x - \sigma t)}_{\text{progressive wave}} + \underbrace{\sum_{n=1}^{\infty} C_n e^{-k_s(n)x}\cos[k_s(n)(h+z)]\cos\sigma t}_{\text{evanescent waves decaying exponentially with } x}$$

A_p and $C_n\,(n = 1 \text{ to } \infty)$ should be determined from WBC:

$$-\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = -A_p k_p \cosh k_p(h+z)\cos\sigma t + \sum_{n=1}^{\infty} C_n k_s(n)\cos[k_s(n)(h+z)]\cos\sigma t = \frac{S(z)}{2}\sigma\cos\sigma t$$

Examining variation over depth,

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0$$

$$\frac{dZ}{dz} = 0 \qquad \text{on} \quad z = -h$$

$$\frac{dZ}{dz} - \frac{\sigma^2}{g} Z = 0 \qquad \text{on} \quad z = 0$$

This is the Sturm-Liouville problem, which gives the orthogonality of eigenfunctions, or

$$\int_{-h}^{0} Z_m Z_n dz = 0 \qquad \text{if} \quad m \neq n$$

In our problem, the eigenfunctions are $\cosh k_p(h+z)$ and $\cos[k_s(n)(h+z)]$, $n = 1, 2, \cdots$. To find A_p, multiply $\cosh k_p(h+z)$ on both side of WBC and integrate over depth:

$$-\int_{-h}^{0} A_p k_p \cosh^2 k_p(h+z)dz + \sum_{n=1}^{\infty}\int_{-h}^{0} C_n k_s(n)\cos[k_s(n)(h+z)]\cosh k_p(h+z)dz$$
$$= \int_{-h}^{0} \frac{S(z)}{2}\sigma\cosh k_p(h+z)dz$$

$$\therefore \; A_p = \frac{-\int_{-h}^{0} \frac{S(z)}{2}\sigma\cosh k_p(h+z)dz}{k_p \int_{-h}^{0} \cosh^2 k_p(h+z)dz}$$

Similarly,

$$C_n = \frac{\int_{-h}^{0} \frac{S(z)}{2} \sigma \cos[k_s(n)(h+z)]dz}{k_s(n)\int_{-h}^{0} \cos^2[k_s(n)(h+z)]dz}$$

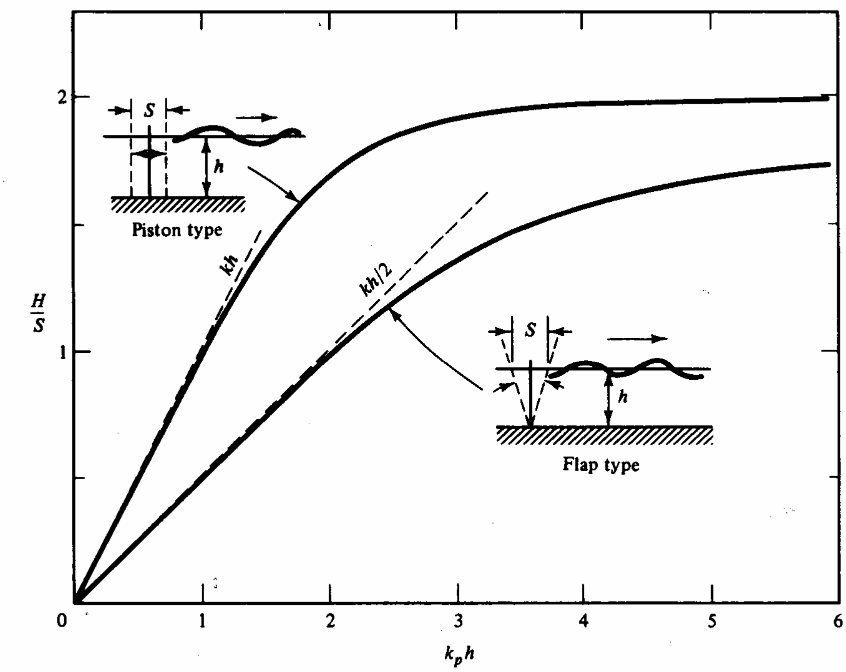


Figure 6.2 Plane wavemaker theory. Wave height to stroke ratios versus relative depths. Piston and flap type wavemaker motions.

For piston-type wavemaker,

$$S(z) = S$$

and for flap type wavemaker,

$$S(z) = S\left(1 + \frac{z}{h}\right)$$

where S = stroke at SWL.

Far from the wavemaker, where the evanescent modes are negligible,

$$\eta = \frac{H}{2}\cos(k_p x - \sigma t) = \frac{1}{g}\frac{\partial \phi}{\partial t}\bigg|_{z=0} = -\frac{\sigma}{g} A_p \cosh k_p h \cos(k_p x - \sigma t)$$

$$H = -2\frac{\sigma}{g} A_p \cosh k_p h$$

See H/S versus $k_p h$ in Figure 6.2 of textbook. See also Eqs. (6.25) and (6.26).

For small $k_p h$,

$$\frac{H}{S} = \begin{cases} k_p h & \text{for piston wavemaker} \\ \dfrac{k_p h}{2} & \text{for flap wavemaker} \end{cases}$$

Mean power (averaged over one wave period) required by the wavemaker is calculated by the energy flux away from the wavemaker:

$$P = EC_g = \frac{1}{T}\int_t^{t+T}\int_{-h}^{0} p_d u dz$$

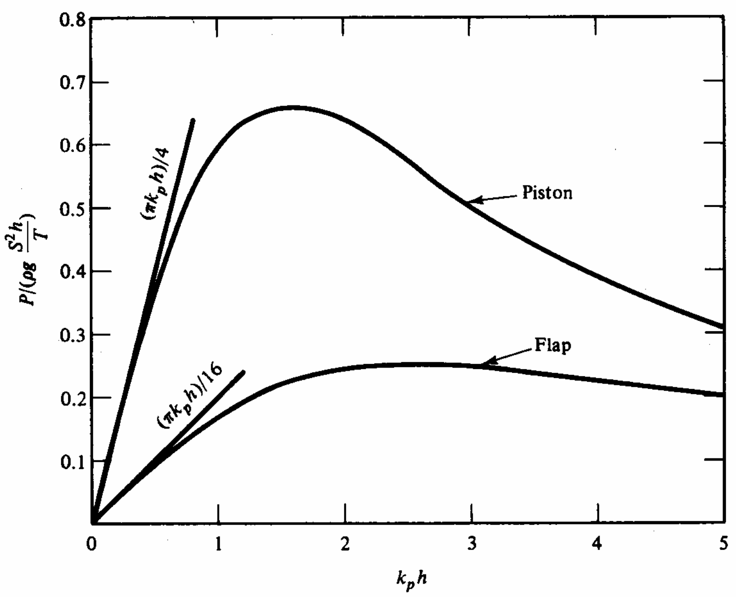
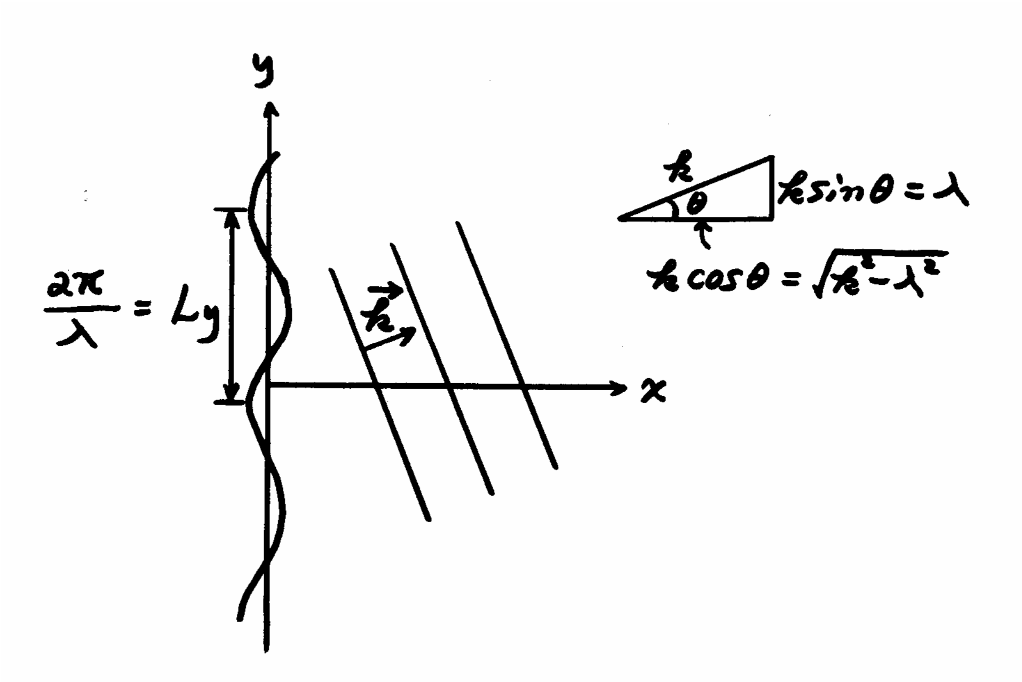


Figure 6.4 Dimensionless mean power as a function of water depth for piston and flap wavemakers.

3-D Wavemaker



"Snake-type" wavemaker

The boundary value problem is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad \text{on} \quad z = -h$$

$$\frac{\partial\phi}{\partial z} - \frac{\sigma^2}{g}\phi = 0 \qquad \text{on} \quad z = 0$$

$$-\frac{\partial\phi}{\partial x} = \frac{S(z)}{2}\sigma\cos(\lambda y - \sigma t) \qquad \text{on} \quad x = 0$$

Assume $\phi(x, y, z, t) = X(x)Z(z)Y(y, t)$. z -problem is the same as 2-D wavemaker problem. Thus,

$$Z(z) = A\cosh k(h+z) + \sum_{n=1}^{\infty} C_n \cos k_s(n)(h+z)$$

with $\sigma^2 = gk\tanh kh$ and $\sigma^2 = -gk_s \tan k_s h$. Periodicity condition in y -direction gives

$$Y(y,t) = C\sin(\lambda y - \sigma t) + D\cos(\lambda y - \sigma t)$$

Substituting these into the Laplace equation,

$$\frac{1}{X}\frac{d^2 X}{dx^2} \pm k^2 - \lambda^2 = 0$$

where $+$ is for progressive wave, and $-$ is for evanescent waves. The solutions are

$$X = E\cos\sqrt{k^2 - \lambda^2}\,x + F\sin\sqrt{k^2 - \lambda^2}\,x$$

$$X_s = Ee^{\sqrt{k_s^2+\lambda^2}x} + Fe^{-\sqrt{k_s^2+\lambda^2}x}$$

Now the velocity potential is given by

$$\phi = A\cosh k(h+z)\sin(\sqrt{k^2-\lambda^2}\,x+\lambda y-\sigma t)$$
$$+\sum_{n=1}^{\infty} C_n \cos[k_s(n)(h+z)]e^{-\sqrt{k_s^2+\lambda^2}\,x}\cos(\lambda y-\sigma t)$$

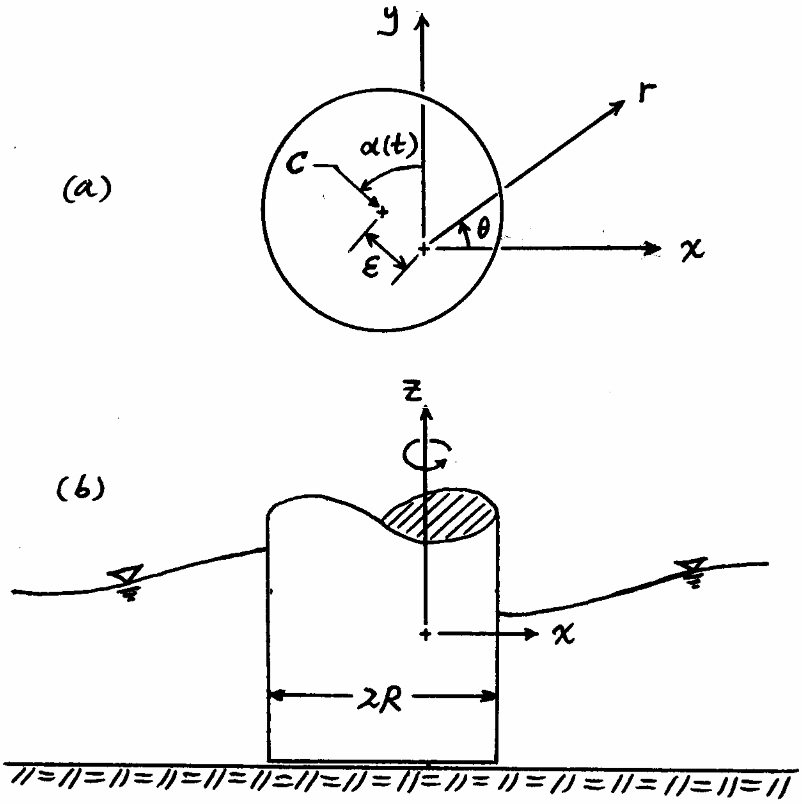
WBC gives

$$-\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = -A\sqrt{k^2-\lambda^2}\cosh k(h+z)\cos(\lambda y-\sigma t)$$
$$+\sum_{n=1}^{\infty} C_n\sqrt{k_s^2+\lambda^2}\cos[k_s(h+z)]\cos(\lambda y-\sigma t)$$
$$=\frac{S(z)}{2}\sigma\cos(\lambda y-\sigma t)$$

Use orthogonality to determine A and C_n.

<u>Cylindrical wavemaker</u>

Spiral wavemaker (Type III in text)



Boundary value problem:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$-\frac{\partial\phi}{\partial z} = 0 \qquad \text{on} \quad z = -h$$

$$\frac{\partial\phi}{\partial z} - \frac{\sigma^2}{g}\phi = 0 \qquad \text{on} \quad z = 0$$

KBC on cylinder wall:

$$r = R + \varepsilon\sin(\theta - \sigma t)$$
$$F(r,\theta,t) = r - R - \varepsilon\sin(\theta - \sigma t) = 0$$
$$\frac{DF}{Dt} = \sigma\varepsilon\cos(\theta - \sigma t) + u_r - \frac{u_\theta}{r^2}\varepsilon\cos(\theta - \sigma t) = 0 \qquad \text{on} \quad F = 0$$

Neglecting the second order term,

$$u_r = -\frac{\partial\phi}{\partial r} = -\sigma\varepsilon\cos(\theta - \sigma t) \qquad \text{on} \quad F = 0$$

Taylor series expansion about $r = R$ gives

$$\left[u_r + \sigma\varepsilon\cos(\theta - \sigma t)\right]_{r=R+\varepsilon\sin(\theta-\sigma t)}$$
$$= \left[u_r + \sigma\varepsilon\cos(\theta - \sigma t)\right]_{r=R} + \varepsilon\sin(\theta - \sigma t)\frac{\partial}{\partial r}\left[u_r + \sigma\varepsilon\cos(\theta - \sigma t)\right]_{r=R} + \cdots = 0$$
$$\therefore\ -\frac{\partial\phi}{\partial r} = -\sigma\varepsilon\cos(\theta - \sigma t) \qquad \text{on} \quad r = R$$

Using separation of variable, the solution satisfying the radiation condition (outgoing waves as $r \to \infty$) is

$$\phi(r,\theta,z,t) = A_p H_1^{(1)}(k_p r)\cosh k_p(h+z)e^{i(\theta-\sigma t)} + \sum_{n=1}^{\infty} C_n K_1(k_n r)\cos k_n(h+z)e^{i(\theta-\sigma t)}$$

A_p and C_n can be determined from orthogonality condition by integrating over z.

Plunger wavemaker (See Figure 6.8 of textbook.)