Chapter 1 Fundamentals of Turbulent Jet

166. Turbulent water jet. Laser-induced fluorescence shows the concentration of jet fluid in the plane of symmetry of an axisymmetric jet of water directed downward into water. The Reynolds number is approximately 2300. The spatial resolution is adequate to resolve the Kolmogorov scale in the downstream half of the photograph. *Dimotakis, Lye & Papantoniou 1981*

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Objectives

- Derive governing equations of turbulent jets
- Derive constancy of momentum flux along jet axis
- Analyze velocity profile of plane and round jets using geometric similarity
- Determine the amount of fluid entrainment into jets

1.1Turbulent Jets

1.1.1 Turbulent Motions

(1) Wall turbulence

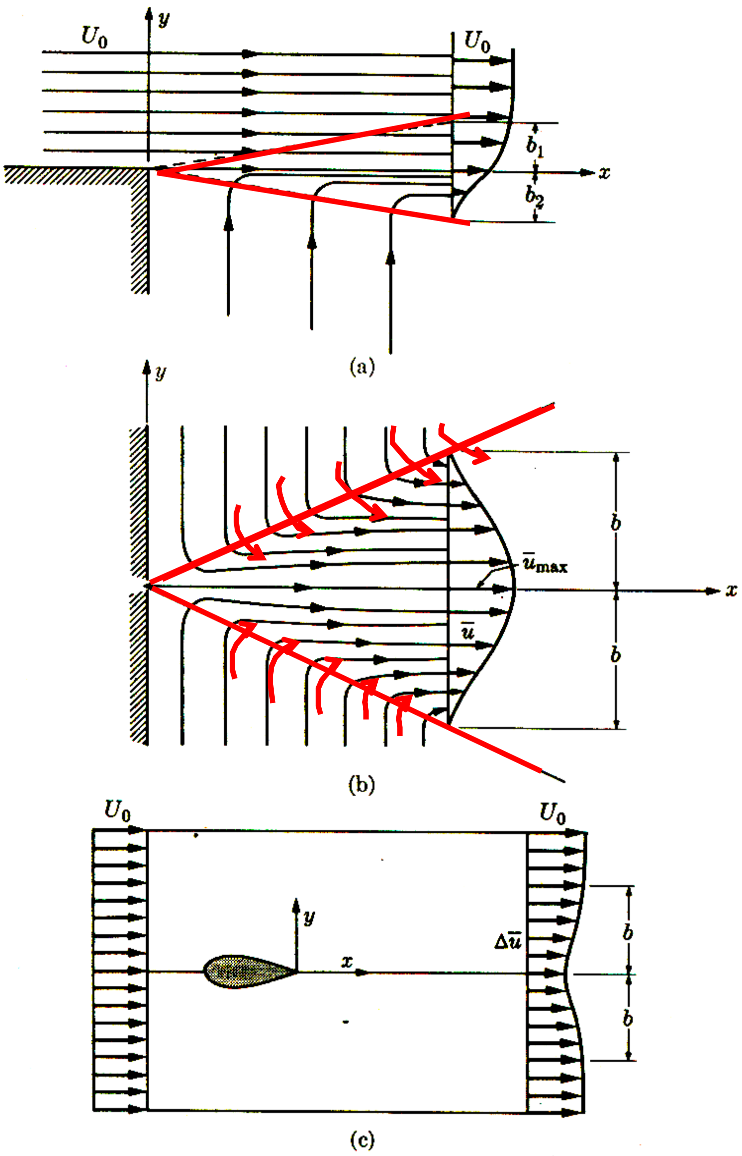
- turbulent motions which are constrained by one or more boundaries
- turbulent generated in velocity gradient caused by the no-slip condition

(2) Free turbulence

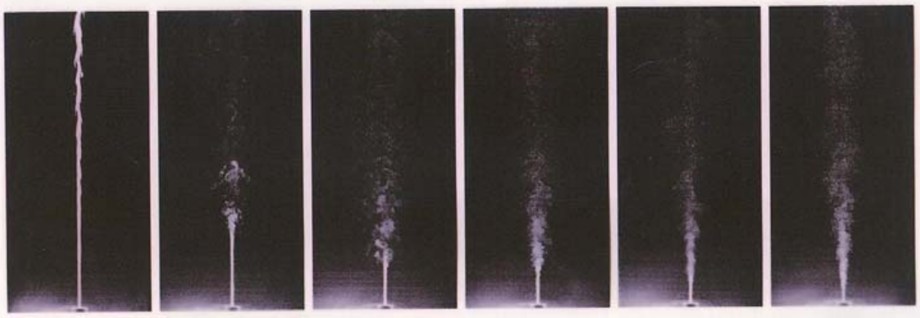
- turbulent motions which are not affected by the presence of solid boundaries
- Example: Fig.16.1
  - shear layer (mixing layer)
  - immersed jet
  - wake of an immersed body
- Velocity (shear) gradients are generated.
- Viscous (molecular) shear stress usually can be neglected in comparison with turbulent eddy stresses throughout the entire flow field.

[Cf] In wall turbulence, due to the damping of turbulent by wall, viscous stresses in the laminar sublayer must be considered.

- In jets and wakes in large bodies of fluid, pressure gradient in the direction of motion is zero.


U_0
y
U_0
b_1
x
b_2
(a)
y
b
$\bar{u}_{max}$
x
$\bar{u}$
b
(b)
U_0
U_0
y
b
$\Delta\bar{u}$
x
b
(c)

PIV measurements of round jets



(R100J, Re =177) (R200J, Re = 437) (R400J, Re =1,305) (R500J, Re = 2,163) (R600J, Re =3,208) (R900J, Re =5,142)

Figs. Evolution of Round Jet with Increase of Reynolds Number (Instantaneous Images)

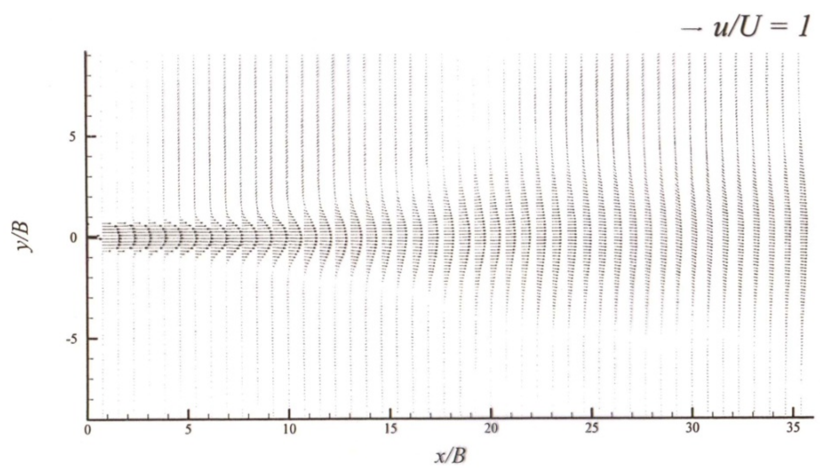


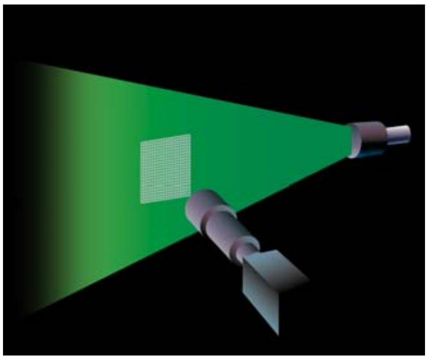
Fig. 4.6(c) Velocity Vector Fields for Case NFJ300 by PIV

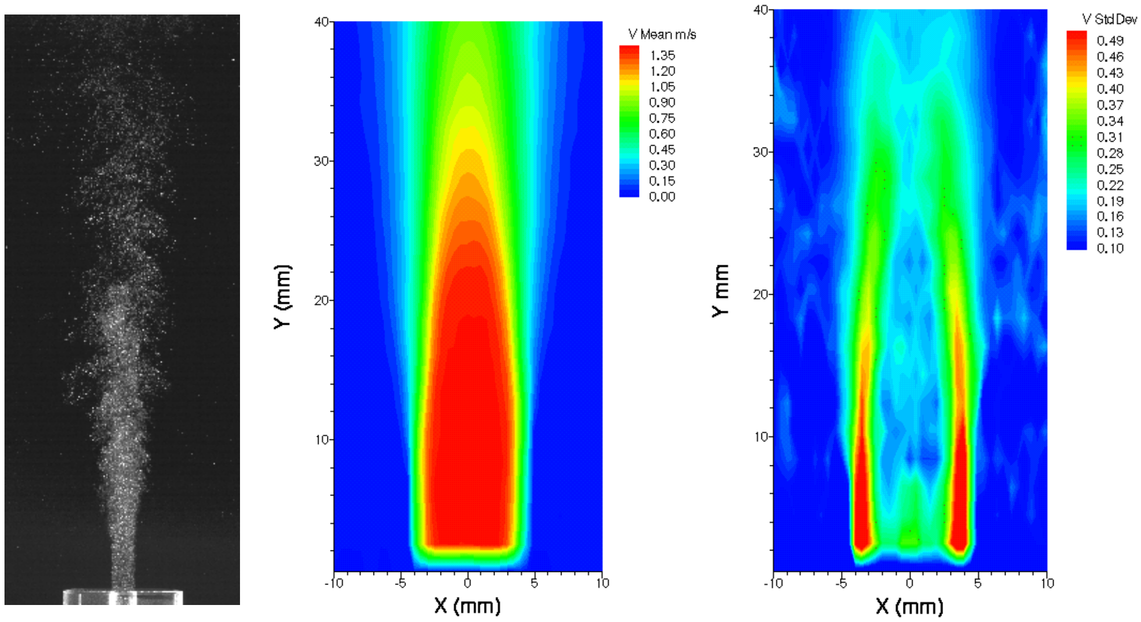
PIV system

Velocity of particle A: $u_x = \dfrac{\Delta x}{\Delta t} \; as \, \Delta t \rightarrow 0$

$$u_y = \frac{\Delta y}{\Delta t} \; as \, \Delta t \rightarrow 0$$

PIV: field measurement





a) Image b)Velocity c)Turbulence Intensity

Jet Characteristics Measured by PIV (Seo et al., 2002)

1.1.2 Equation of Motion for Turbulent Jets

• Common properties of free turbulence and wall turbulence

- width of mixing zone < longitudinal distance, x

$$\rightarrow \frac{\partial u}{\partial y} > \frac{\partial u}{\partial x}$$

~ same assumption as those by Prandtl

→ use Prandtl's 2-D boundary-layer equations for steady 2-D free turbulent flows with zero-pressure gradient and neglecting molecular-viscosity

For instantaneous velocity,

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} \quad \text{- x-com} \qquad \text{(a)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(b)}$$

Decompose velocity and pressure, then take average over time

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{\partial \bar{p}}{\partial x} - \rho\frac{\partial \overline{u'^2}}{\partial x} + \mu\frac{\partial^2 \bar{u}}{\partial y^2} - \rho\frac{\partial \overline{u'v'}}{\partial y} \qquad \text{(c)}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \qquad \text{(d)}$$

$$\overline{u'^2} = \varepsilon\frac{\partial \bar{u}}{\partial x} \quad \rightarrow \quad \frac{\partial \overline{u'^2}}{\partial x} = \frac{\partial}{\partial x}\left(\varepsilon\frac{\partial \bar{u}}{\partial x}\right) \qquad \text{(f)}$$

$$\overline{u'v'} = \varepsilon \frac{\partial \bar{u}}{\partial y} \quad \rightarrow \quad \frac{\partial \overline{u'v'}}{\partial y} = \frac{\partial}{\partial y}\left(\varepsilon \frac{\partial \bar{u}}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\tau}{\rho}\right) \tag{g}$$

where $\varepsilon = \dfrac{\eta}{\rho}$ = kinematic eddy viscosity;

$\tau = \eta \dfrac{\partial \bar{u}}{\partial y}$ = turbulent shear stress

Substituting (g) into (c) gives

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y} \tag{1.1}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{1.2}$$

~ B.C.s are different for plane and axisymmetric jets and wakes.

~ Eq. (1.1) and Eq. (1.2) are equations of motion for the 2-D turbulent free jet with a zero pressure gradient in the axial direction.

[Re] Equations for a turbulent boundary layer

Apply Prandtl's 2-D boundary-layer equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \tag{8.7a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rightarrow u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} = 0 \tag{8.7b}$$

Add Continuity Eq. and Eq. (8.7a)

$$\frac{\partial u}{\partial t} + \underbrace{2u\frac{\partial u}{\partial x}}_{\downarrow\ \frac{\partial u^2}{\partial x}} + \underbrace{\left(v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} \right)}_{\downarrow\ \frac{\partial uv}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \qquad \text{(A)}$$

Substitute velocity decomposition into (A) and average over time

$$\overline{\frac{\partial(\overline{u} + u')}{\partial t}} = \frac{\partial \overline{u}}{dt}$$

$$\overline{\frac{\partial(\overline{u} + u')^2}{\partial x}} = \frac{\partial \overline{u}^2}{dx} + \frac{\partial \overline{u'^2}}{\partial x}$$

$$\overline{\frac{\partial(\overline{u} + u')(\overline{v} + v')}{\partial y}} = \frac{\partial \overline{u}\,\overline{v}}{\partial y} + \frac{\partial \overline{u'v'}}{\partial y}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial x}\overline{(\overline{p} + p')} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}$$

$$\frac{\mu}{\rho}\frac{\partial^2}{\partial y^2}(\overline{u} + u') = \frac{\mu}{\rho}\frac{\partial^2 \overline{u}}{\partial y^2}$$

Thus, (A) becomes

$$\therefore \frac{\partial \overline{u}}{\partial t} + \frac{\partial \overline{u}^2}{\partial x} + \frac{\partial \overline{u}\,\overline{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial \overline{u'^2}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} \qquad \text{(B)}$$

Subtract Continuity eq. from (B)

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial \overline{u'^2}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y}$$

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{\partial \bar{p}}{\partial x} - \rho\frac{\partial \overline{u'^2}}{\partial x} + \mu\frac{\partial^2 \bar{u}}{\partial y^2} - \rho\frac{\partial \overline{u'v'}}{\partial y}$$

→ Equation of motion in x-direction

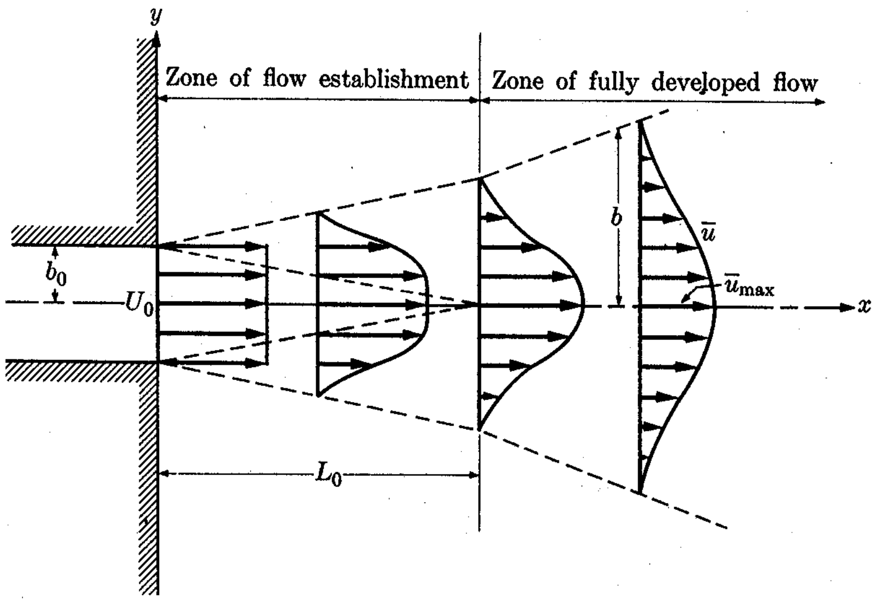
Adopt similar equation as Eq. (8.25) for y-eq.

$$0 = -\frac{\partial}{\partial y}(\bar{p} + \rho\overline{v'^2})$$

Continuity eq.:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

1.2 Plane Jets



• Plane jet:

height of slot $= 2b_0$

length of potential core $= L_0$

$x \le L_0$ $\quad\quad \overline{u_c} = U_0$

$x > L_0$ $\quad\quad \overline{u_c} < U_0$

• Entrainment of surrounding fluid

→ volume rate of flow past any section in the jet increases in the x-direction

1.2.1 Constancy of momentum flux

Derive integral momentum equation

Integrate Eq.(1.1) w.r.t. y

$$\underbrace{\rho\int_{-\infty}^{\infty}\bar{u}\frac{\partial\bar{u}}{\partial x}dy}_{①}+\underbrace{\rho\int_{-\infty}^{\infty}\bar{v}\frac{\partial\bar{u}}{\partial y}dy}_{②}=\underbrace{\int_{-\infty}^{\infty}\frac{\partial\tau}{\partial y}dy}_{③} \tag{1.4}$$

Integral by parts:
$\int uv'\,dx = uv - \int u'v\,dx$

①: $\rho\int_{-\infty}^{\infty}\bar{u}\frac{\partial\bar{u}}{\partial x}dy=\rho\int_{-\infty}^{\infty}\frac{1}{2}\frac{\partial\bar{u}^2}{\partial x}dy=\frac{\rho}{2}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\bar{u}^2dy$

②: $\rho\int_{-\infty}^{\infty}\bar{v}\frac{\partial\bar{u}}{\partial y}dy=\rho\left\{[\bar{v}\bar{u}]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\bar{u}\frac{\partial\bar{v}}{\partial y}dy\right\}=-\rho\int_{-\infty}^{\infty}\bar{u}\frac{\partial\bar{v}}{\partial y}dy$

$\bar{u}=0\,at\,x=\pm\infty$

$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}=0$

$=-\rho\int_{-\infty}^{\infty}\bar{u}\left(-\frac{\partial\bar{u}}{\partial x}\right)dy=\rho\int_{-\infty}^{\infty}\bar{u}\frac{\partial\bar{u}}{\partial x}dy$ = same as ①

$=\frac{\rho}{2}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\bar{u}^2dy$

③: $\int_{-\infty}^{\infty}\frac{\partial\tau}{\partial y}dy=\tau\Big|_{-\infty}^{\infty}=\eta\frac{\partial\bar{u}}{\partial y}\Big|_{\infty}-\eta\frac{\partial\bar{u}}{\partial y}\Big|_{-\infty}=0$

Eq. (1.4) becomes

$$\rho\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\bar{u}^2dy=0 \tag{1.5}$$

$$\int_{-\infty}^{\infty}\rho\bar{u}^2dy=constant=J \tag{1.6}$$

$\rho\bar{u}$ = momentum per unit volume $\quad (m\bar{u} / vol. = \rho\bar{u})$

$\int \bar{u}dy$ = volume per unit time $\quad (vol./t = Q = \bar{u} \cdot A = \bar{u}dy)$

$\therefore \rho\bar{u}^2 dy$ = total momentum per unit time passing any section of the jet

= momentum flux per unit length of slot

→ Eq. (1.5) states that the flux of momentum of the jet is constant and independent of x

→ There is no change in the longitudinal momentum flux.

• Constant in Eq.(1.6) can be evaluated from momentum influx at $x = 0$

At outlet $\bar{u} = U_0 = const.$ for $-b_0 \le y \le b_0$

J_0 = momentum influx = $\rho U_0 \times U_0 2b_0 = 2\rho U_0{}^2 b_0$

$$\int_{-\infty}^{\infty} \rho\bar{u}^2 dy = 2\rho U_0^2 b_0 \tag{1.7}$$

1.2.2 Velocity profile

In fully developed region of the jet, assume that

$$b \sim x^m$$

$$\overline{u}_{\max} \sim x^{-n}$$

Then, consider order of magnitude of each term in eq. of motion, Eq. (1.1)

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

① $$\overline{u}\frac{\partial \overline{u}}{\partial x} \sim \frac{(\overline{u}_{\max})^2}{x} \sim \frac{x^{-2n}}{x} \sim x^{-2n-1}$$

② $$\overline{v}\frac{\partial \overline{u}}{\partial y} = \left(-\int \frac{\partial \overline{u}}{\partial x} dy\right)\left(\frac{\partial \overline{u}}{\partial y}\right) \sim \left(\frac{\overline{u}_{\max} b}{x}\right)\left(\frac{\overline{u}_{\max}}{b}\right) \sim \frac{(\overline{u}_{\max})^2}{x} \sim \frac{x^{-2n}}{x} \sim x^{-2n-1}$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial x} = 0 \;\rightarrow\; \frac{\partial \overline{v}}{\partial y} = -\frac{\partial \overline{u}}{\partial x} \;\rightarrow\; \overline{v} = \int -\frac{\partial \overline{u}}{\partial x} dy$$

③ $$\frac{\partial}{\partial y}\left(\frac{\tau}{\rho}\right) \sim \frac{(\overline{u}_{\max})^2}{b} \sim x^{-2n-m}$$

$$\frac{\tau}{\rho} \sim (\overline{u}_{\max})^2$$

LHS of (1.1) $\sim x^{-2n-1}$

RHS of (1.1) $\sim x^{-2n-m}$

$\therefore\; m = 1$

$\rightarrow$ $b \sim x^1$ (a)

→ The plane jet expands as a linear function of *x*.

Now consider order of Eq. (1.7)

$$\int_{-\infty}^{\infty} \rho \overline{u}^2 dy = 2\rho U_0^2 b_0$$

$$\rho U_0^2 b_0 \sim x^{-2n} \cdot x^m \sim x^{-2n+m}$$

→ Eq. (1.7) is independent of x only if $-2n + m = 0$

$$\therefore\ n = \frac{m}{2} = \frac{1}{2}$$

$$\boxed{\overline{u}_{\max} \sim x^{-\frac{1}{2}}} \qquad \text{(b)}$$

→ The centerline velocity decreases as $\dfrac{1}{\sqrt{x}}$.

- Jet Reynolds number, Re_i

$$Re_i = \frac{\overline{u}_{\max} b}{\nu} \sim x^{-\frac{1}{2}} x^1 \sim x^{\frac{1}{2}}$$

→ Reynolds number increases as $\sqrt{x}$

1.2.3 Similarity of velocity profiles

We don't know yet about velocity distribution, rate of entrainment, and actual jet dimensions. Beyond some transition distance past end of potential core (L_0), the velocity profiles are similar.

→ transition = 6 ~ 40 ($2b_0$)

• Use semiempirical approaches based on the assumption or geometric similarity of velocity profiles.

$$\frac{\bar{u}}{\bar{u}_{\max}} = f\left(\frac{y}{x}\right) = f(\xi) \tag{1.8}$$

$$\bar{u} = \bar{u}_{\max} f(\xi)$$

in which $\zeta = \frac{y}{x} \rightarrow y = z\xi \rightarrow dy = xd\xi$

• Assume function f as Gaussian curves which is found to be satisfactory from the experiments.

$$\frac{\bar{u}}{\bar{u}_{\max}} = f(\xi) = \exp\left(-\frac{y^2}{2C_1^2 x^2}\right) \tag{1.9}$$

in which C_1 = spreading coefficient = const. to be determined experimentally

[Cf] Top Hat distribution

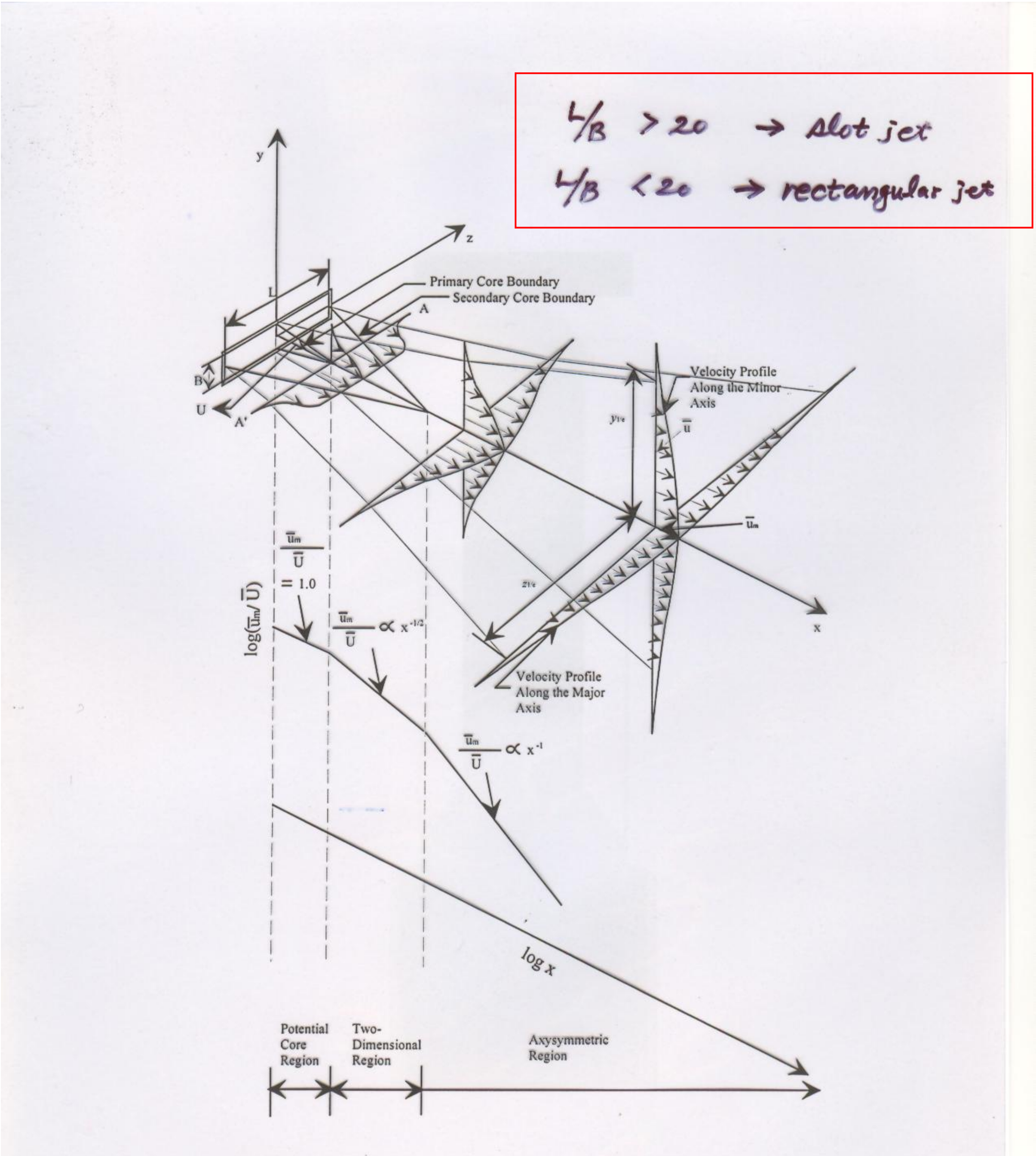


Fig. 1. Schematic Diagram of Three-Dimensional Rectangular Jet

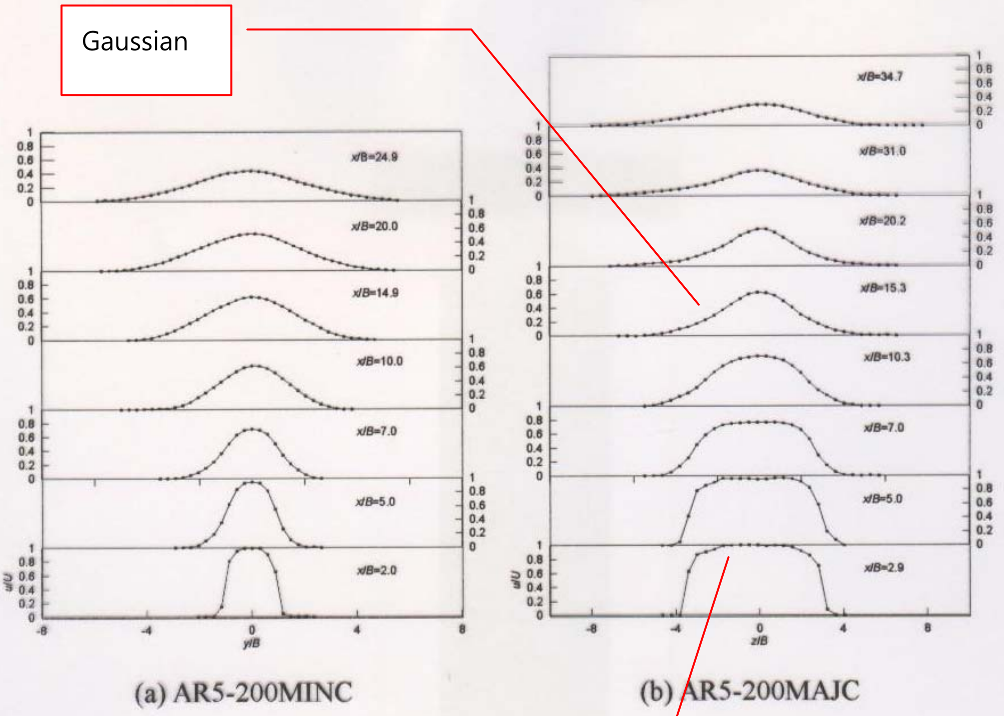
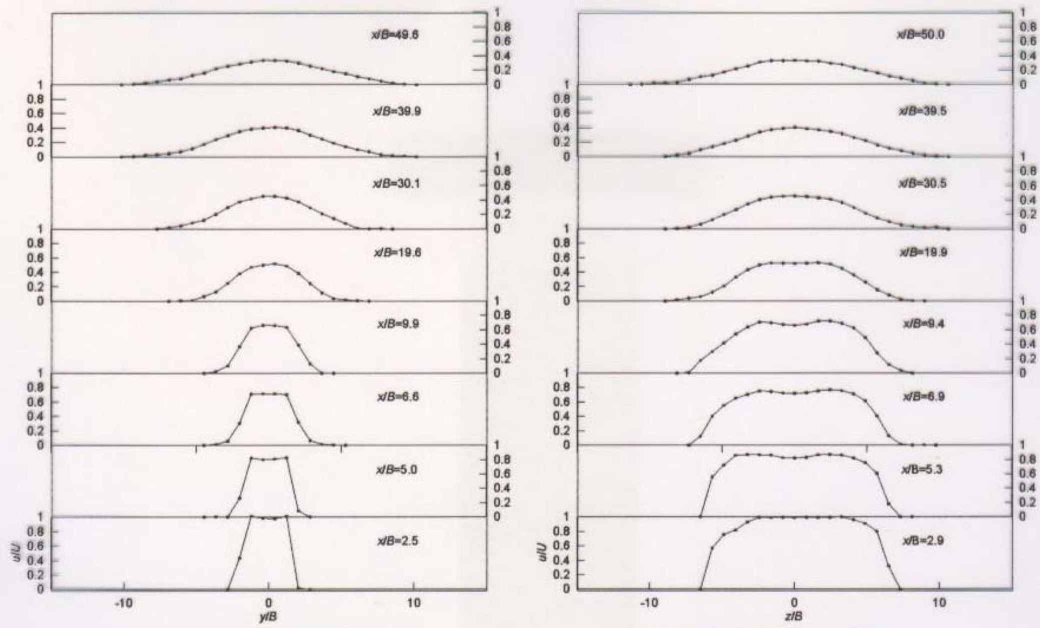
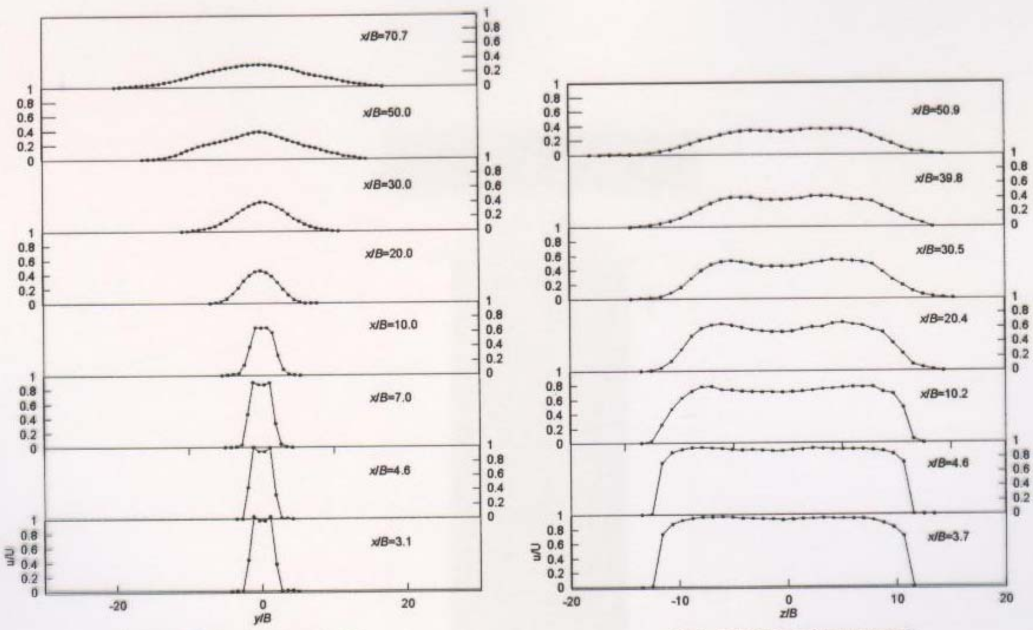


Fig. 4. Lateral Velocity Distribution Measured in Center Sections of Minor and Major Axes for AR = 5



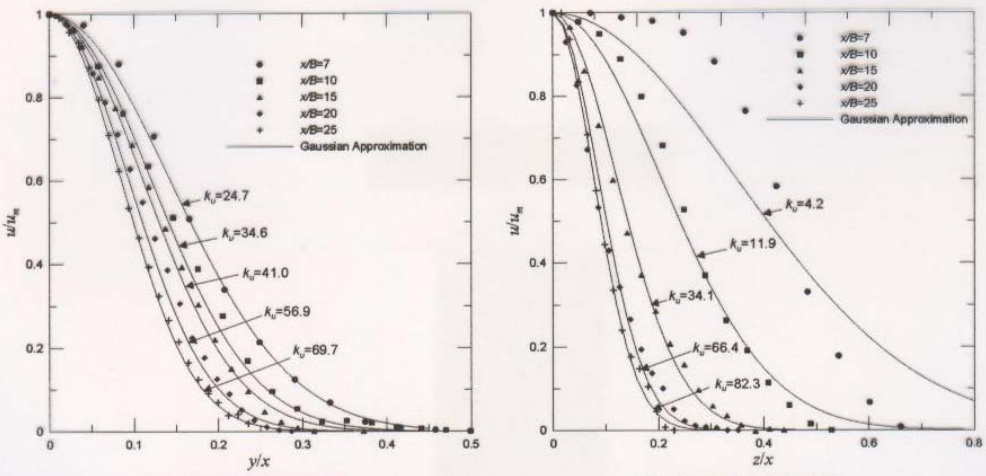
(a) AR10-250MINC (b) AR10-250MAJC

Fig. 5. Lateral Velocity Distribution Measured in Center Sections of Minor and Major Axes for AR = 10



(a) AR20-150MINC (b) AR20-200MAJC

Fig. 6. Lateral Velocity Distribution Measured in Center Sections of Minor and Major Axes for AR = 20



(a) AR5-200MINC (b) AR5-200MAJC

Fig. 8. Lateral Velocity Profiles on Center Sections of Major and Minor Axes (AR = 5)

Combine Eq. (1.7) and Eq. (1.8)

$$2\rho U_0^2 b_0 = \rho(\overline{u})_{\max}^2 \int_{-\infty}^{\infty} f^2(\xi)dy = \rho(\overline{u})_{\max}^2 \int_{-\infty}^{\infty} f^2(\xi)d\xi\, x$$

Let $I_2 = \int_{-\infty}^{\infty} f^2(\xi)d\xi$

Then, the ratio of the centerline velocity to the initial jet velocity can be given as

$$2U_0^2 b_0 = (\overline{u})_{\max}^2 x I_2 \tag{1.11}$$

$$\frac{\overline{u}_{\max}}{U_0} = \sqrt{\frac{2b_0}{xI_2}} \tag{1.12}$$

→ $\overline{u}_{\max} \sim \dfrac{1}{\sqrt{x}}$

→ the same result as Eq. (b)

- Length of potential core, L_0

$$\frac{\overline{u}_{\max}}{U_0} = 1 \quad \text{at} \quad x = L_0$$

$$\therefore 1 = \sqrt{\frac{2b_0}{L_0 I_2}}$$

$$L_0 = \frac{2b_0}{I_2} \tag{1.13}$$

1.2.4 Entrainment Hypothesis

The total discharge per unit width of the jet, $x > L_0$ is given by integrating the local velocity across a section of the jet,

$$Q = \int_{-\infty}^{\infty} \bar{u} dy = \bar{u}_{\max} \int_{-\infty}^{\infty} f(\xi) dy = \bar{u}_{\max} x \int_{-\infty}^{\infty} f(\xi) d\xi \qquad (1.14)$$

where Q = volume flux

The initial discharge per unit width, Q_0 is given

$$Q_0 = 2b_0 U_0 \qquad (1.15)$$

Divide Eq. (1.14) by Eq. (1.15) to obtain the ratio of the total rate of flow to the initial discharge

$$\frac{Q}{Q_0} = \frac{\bar{u}_{\max}}{U_0} \frac{x}{2b_0} \underbrace{\int_{-\infty}^{\infty} f(\xi) d\xi}_{I_1}$$

$$= \frac{\bar{u}_{\max}}{U_0} \frac{x}{2b_0} I_1 = \sqrt{\frac{2b_0}{xI_2}} \frac{x}{2b_0} I_1 = \sqrt{\frac{xI_1^2}{2b_0 I_2}}$$

$$\boxed{\frac{Q}{Q_0} = \sqrt{\frac{I_1^2}{2b_0 I_2}} \sqrt{x}} \qquad (1.16)$$

→ $Q = \sqrt{x}$

where $\frac{Q}{Q_0}$ = volume dilution

• Experimental results by Albertson et al. (1950)

$\longrightarrow \quad C_1 = 0.109; \quad I_1 = 0.272; \quad I_2 = 0.192$

Then, we have

$$\bar{u}_{\max} / U_0 = 2.28\sqrt{2b_0 / x} = 3.22\sqrt{b_0 / x}, \quad x > L_0 \tag{1.17}$$

$$L_0 = 10.4 b_0 \tag{1.18}$$

$$Q / Q_0 = 0.62\sqrt{x / 2b_0}, \qquad x > L_0 \tag{1.19}$$

- Gaussian velocity distribution

Eq. (1.9):

$$\frac{\bar{u}}{\bar{u}_{\max}} = \exp\left(-\frac{1}{2C_1^2} \frac{y^2}{x^2} \right)$$

$$= \exp\left\{ -\frac{1}{2(0.109)^2} \left(\frac{y}{x} \right)^2 \right\} = \exp\left\{ -42.08 \left(\frac{y}{x} \right)^2 \right\}$$

[Cf] Abramovich (1963)

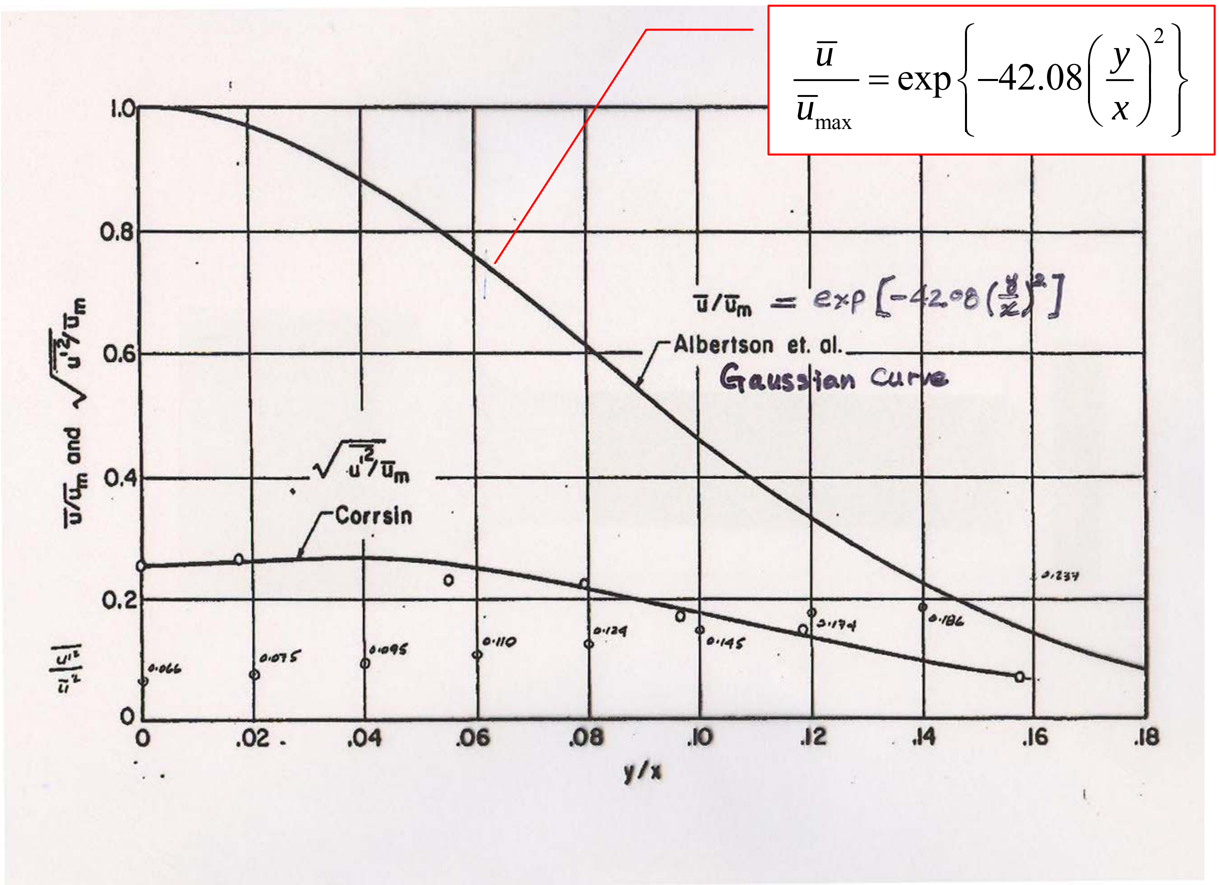
$$\frac{\bar{u}_{\max}}{U_0} = 3.78\sqrt{b_0 / x}$$

Zijnen (1958)

$$\frac{\bar{u}_{\max}}{U_0} = 3.12 \sim 3.52\sqrt{b_0 / x}$$

Newman (1961)

$$\frac{\bar{u}_{\max}}{U_0} = 3.39\sqrt{b_0 / x}$$



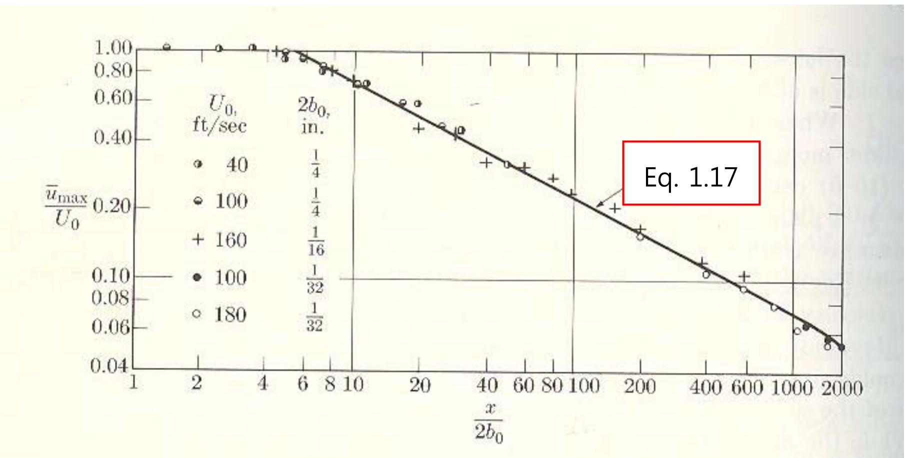


FIG. 16-3. Distribution of centerline velocity in flow from slot [1].

1.2.5 Theoretical Solution for Equation of Motion for 2-D Boundary Layer Flow

Solve equations of motion and continuity

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho}\frac{\partial \tau}{\partial y} \qquad (1)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \qquad (2)$$

Given: 2 equations

Unknown: u, v, τ

→ We need 1 more eq. for τ

1) Tollmien solution (1926)

→ use Prandtl's mixing length formula

$$\tau=\rho l^2\left(\frac{\partial u}{\partial y}\right)^2$$

$$l\propto b \;\; \rightarrow \;\; l=\beta b \;\; \rightarrow \;\; l=\beta C_2 x$$

2) Goertler solution (1942)

→ use Prandtl's 2nd eq.

$$\tau=\rho\varepsilon\frac{\partial u}{\partial y}$$

$$u/u_m=1-\tanh^2(\sigma y/x)$$

$$u/u_m=\frac{1}{\sigma}\left(\frac{\sigma y}{x}-\frac{\sigma y}{x}\tanh^2\frac{\sigma y}{x}-0.5\tanh\frac{\sigma y}{x}\right)$$

• Abramovich, p.64

$\varphi = \frac{y}{ax}$; $a = 0.09$

$$\frac{u}{u_0} = F'(\varphi) = 0.0176e^{-\varphi} + 0.6623e^{\frac{\varphi}{2}} \cos\left(\frac{\sqrt{3}}{2}\varphi\right) + $$
$$+ 0.228e^{\frac{\varphi}{2}} \sin\left(\frac{\sqrt{3}}{2}\varphi\right). \qquad (2.60)$$

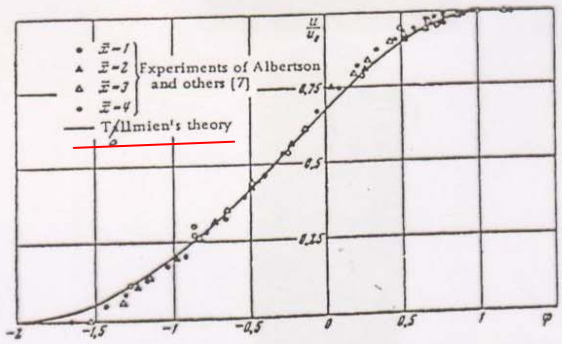


Fig. 2.2. Velocity profile in the boundary layer of a submerged jet ($\bar{x} = x/b_0$).

using Prandtl's Mixing Length Hypothesis.

with $\ell = cx$

$c = \sqrt{\frac{a^3}{2}}$

$\tau_{xy} = \rho \ell^2 \frac{\partial u}{\partial y}\left|\frac{\partial u}{\partial y}\right|$

• Schlichting, p499 (1955)

p747 (1979)

$$u = \frac{\sqrt{3}}{2}\sqrt{\frac{K\sigma}{x}}\,(1 - \tanh^2\eta),$$
$$v = \frac{\sqrt{3}}{4}\sqrt{\frac{K}{x\sigma}}\left\{2\eta\,(1 - \tanh^2\eta) - \tanh\eta\right\},$$
$$\eta = \sigma\frac{y}{x}. \qquad (23.45)$$

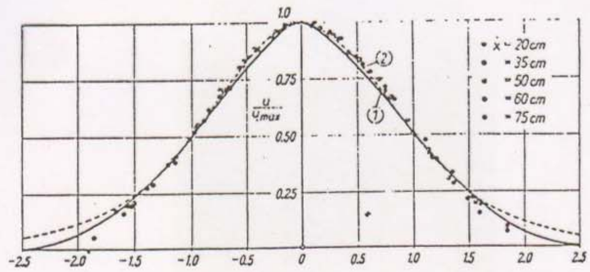
If $\eta = 0$ ($u = u_{max}$)

$v = 0$.

Görtler

$\sigma = 7.67$

$K = \frac{J}{\rho}$



$\eta = \frac{y}{y\left(\frac{u_m}{2}\right)}$

using Prandtl's second hypothesis

$\varepsilon = \kappa_1 b\,\bar{u}_{max}$

Fig. 23.7 Velocity distribution in a two-dimensional, turbulent jet. Measurements due to Foerthmann [13]. Theory: curve (1) due to Tollmien; curve (2) from eqn. (23.45)

32*

• hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}(e^{x} - e^{-x})}{\frac{1}{2}(e^{x} + e^{-x})}$$

1.3 Axially Symmetric Jets

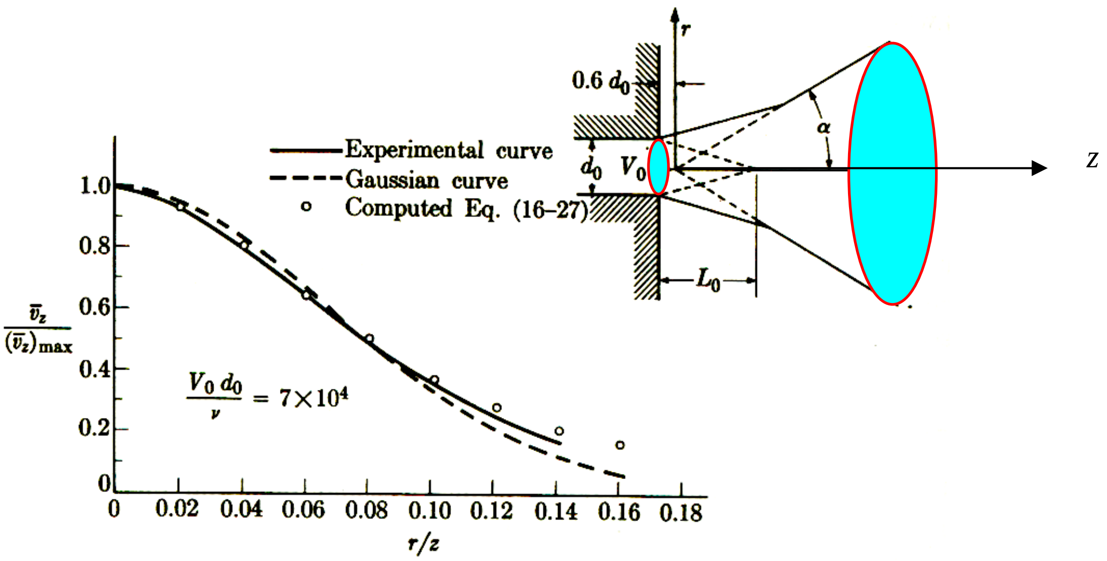
• Axially symmetric jets

~ round jet issuing from a circular hole, pipe, nozzle

~ symmetrical about longitudinal axis of the jet

axial - z, v_z

radial - r, v_r



1.3.1 Derivation of Equation of Motion

Employ the same boundary-layer approximation as in the plane jets in cylindrical coordinates

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right\}+\rho g_z$$

steady

Axisymmetrical jet

pressure gradient = 0

$\frac{\partial^2 v_z}{\partial r^2} > \frac{\partial^2 v_z}{\partial z^2}$

$$\rho v_r \frac{\partial v_z}{\partial r} + \rho v_z \frac{\partial v_z}{\partial z} = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right] \tag{1.20}$$

Substitute $\tau = \mu \dfrac{\partial v_z}{\partial r}$

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = \frac{1}{\rho r} \frac{\partial r \tau}{\partial r} \tag{1.21}$$

Continuity eq. (From (6-30)) becomes

$$\frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z}(v_z) = 0 \tag{1.22}$$

Assume half width and centerline velocity as a power function of *z* as in the plane-jet analysis

$$d \sim z^m$$

$$(\overline{v}_z)_{\max} \sim z^{-n}$$

Evaluate the order of magnitude of the terms in the eq. of motion

$$m = 1$$

$$d \sim z^1 \tag{A}$$

Evaluate the order of magnitude of the terms in the eq. of constant-momentum flux

$$2\pi\rho \int_0^\infty (\overline{v}_z)^2 r dr = \rho V_0^2 \left(\frac{\pi d_0^2}{4} \right) \tag{1.23}$$

$(\rho V_0) \times (V_0 A_0)$

Therefore, the order of magnitude of the terms gives

$$(\overline{v}_z)^2_{max} d^2 \sim z^{-2n+2m} \sim z^0$$

flux is independent of z

$$\therefore \; -2n + 2m = 0$$

$$\therefore \; n = m = 1$$

Finally we get

$d \sim z$ → Jet boundary increases linearly with z.

$(\overline{v}_z)_{max} \sim \dfrac{1}{z}$ → Centerline velocity deceases inversely with z.

[Cf] For plane jet: $m = 1, \quad n = \dfrac{1}{2}$

$$b \sim z^1$$

$$\overline{u}_{max} \sim \frac{1}{\sqrt{z}}$$

[Re] Spreading coefficient

For plane jet, $b = kz$

For round jet, $d = kz$

•Value of k

	Plane jet	Round jet
Velocity profile	0.116	0.107
Concentration profile	0.157	0.127

• Eddy viscosity

$$\eta = \rho l^2 \left| \frac{d\bar{u}}{dy} \right|$$

$$\eta \sim d^2 \frac{(\bar{v}_z)_{\max}}{d} \sim d\ (\bar{v}_z)_{\max} \sim z^1 \cdot z^{-1} \sim z^0 = const.$$

→ Eddy viscosity is constant throughout the mixing region of the jet.

Eq. of motion Eq. (1.20) for turbulent flow becomes

$$\rho \bar{v}_r \frac{\partial \bar{v}_z}{\partial r} + \rho \bar{v}_z \frac{\partial \bar{v}_z}{\partial z} = \eta \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{v}_z}{\partial r} \right) \right] \tag{1.20}$$

Divide (A) by ρ

$$\bar{v}_r \frac{\partial \bar{v}_z}{\partial r} + \bar{v}_z \frac{\partial \bar{v}_z}{\partial z} = \frac{\varepsilon}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{v}_z}{\partial r} \right) \tag{1.24}$$

1.3.2 Solution for axially symmetric jet

Solve eq. of motion, Eq. (1.24) and continuity eq., Eq. (1.22)

Assume geometrically similar velocity profiles

$$\frac{\bar{v}_z}{(\bar{v}_z)_{\max}} = f'(\xi')$$

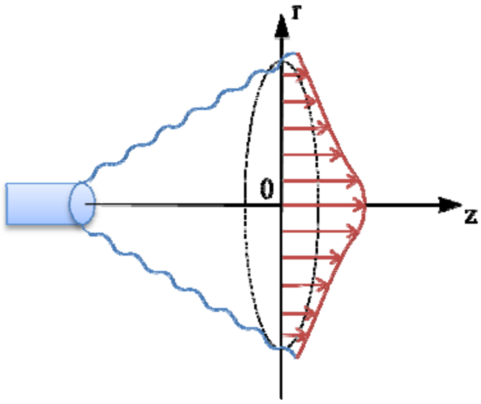
where $\xi' = \dfrac{r}{z}$

∞

Use following boundary conditions

$$r = \infty: \quad \bar{v}_z = 0$$

$$r = 0: \quad \bar{v}_r = 0\ ; \quad \frac{\partial \bar{v}_r}{\partial r} = 0$$



Solution can be obtained by integration of Eq. (1.22) and Eq. (1.24).

[Re] Hinze (1987), pp. 520 ~ 527

$$\frac{\bar{v}_z}{(\bar{v}_z)_{\max}} = \frac{1}{\left[1 + (\bar{v}_z)_{\max} \dfrac{r^2}{8\varepsilon z}\right]^2} \tag{1.25}$$

Substitute experimental data for ε, (1.26) into (1.25)

$$\varepsilon = 0.00196 z (\bar{v}_z)_{\max} \tag{1.26}$$

$$\therefore \frac{\bar{v}_z}{(\bar{v}_z)_{max}} = \frac{1}{\left[1+\dfrac{r^2}{0.016z^2}\right]^2}, \qquad z > L_0$$

$$\frac{\bar{v}_z}{(\bar{v}_z)_{max}} = \frac{1}{\left[1+62.5\left(\dfrac{r}{z}\right)^2\right]^2} \tag{1.27}$$

→ Exact solution

[Cf] For slot jet, $\dfrac{\bar{u}_{max}}{U_0} = 2.28\sqrt{\dfrac{2b_0}{x}}$ (B)

Constancy of momentum flux

Substitute Eq. (1.27) into Eq. (1.23)

$$\frac{(\bar{v}_z)_{max}}{V_0} = 6.4\frac{d_0}{z} \tag{1.28}$$

in which z = distance from geometrical origin of similarity

• Length of potential core, L_0

At $z = L_0$, $(\bar{v}_z)_{max} = V_0$

$$\therefore 1 = 6.4\frac{d_0}{L_0}$$

$$L_0 = 6.4d_0$$

From the actual origin $L_0 = 6.4d_0 + 0.6d_0 = 7.0d_0$

[Re] Detailed derivation of Eq. (1.28)

Eq.(1.23): $2\pi\rho\int_0^\infty (\overline{v}_z)^2 rdr = const. = \rho V_0^2\left(\frac{\pi d_0^2}{4}\right)$

$$\int_0^\infty (\overline{v}_z)^2 rdr = \frac{1}{8}V_0^2 d_0^2 \quad \text{(A)}$$

Substitute Eq.(1.27) into (A)

L.H.S. $\int_0^\infty \frac{\left\{(\overline{v}_z)_{\max}\right\}^2}{\left\{1+\frac{1}{0.016z^2}r^2\right\}^4} rdr$

Set $r^2 = X \quad \rightarrow \quad dr = \frac{dX}{2r}$

$$a = \frac{1}{0.016z^2}$$

Then, L.H.S = $(\overline{v}_z)_{\max}^2 \int_0^\infty \frac{1}{(1+aX)^4}\frac{dX}{2}$

$$= \frac{(\overline{v}_z)_{\max}^2}{2}\left[-\frac{1}{3}\frac{1}{(1+aX)^3}\frac{1}{a}\right]_0^\infty$$

$$= \frac{(\overline{v}_z)_{\max}^2}{2}\left[0+\frac{1}{3a}\right] = \frac{1}{6}(0.016z^2)(\overline{v}_z)_{\max}^2$$

$$= 0.0027\left\{z(\overline{v}_z)_{\max}\right\}^2$$

L.H.S. = R.H.S.; $0.0027z^2(\overline{v}_z)^2_{\max} = \frac{1}{8}V_0^2 d_0^2$

$$\frac{(\overline{v}_z)^2_{\max}}{V_0} = 6.8\frac{d_0^2}{z} \tag{1.28}$$

Combine Eq. (1.26) and Eq. (1.28)

$$\varepsilon = 0.00196z\left(V_0 6.4\frac{d_0}{z}\right) = 0.013V_0 d_0 \rightarrow \text{constant} \tag{1.29}$$

$$\frac{\varepsilon}{v} = \frac{0.013V_0 d_0}{v} = 0.013Re_0 \tag{1.30}$$

[Re] Gaussian curve

$$\frac{\bar{v}_z}{(\bar{v}_z)_{max}} = \exp\left[-c\left(\frac{r}{z}\right)^2\right]$$

→ Comparison of exact solution, Eq. (1.27) and Gaussian curve → Fig. 1.4

• Gaussian curve for axisymmetrical jet

	Constant C	
Reichardt (1951)	48	
Hinze (1959)	108	
Schlichting (1979)	72	
Papanicolaou and List (1988)	80	$z/d_0 < 50$
	93	$z/d_0 > 50$
Yu et al. (1998)	78	

[Example] Air jet

$$V_0 = 100 \text{ fps}; \quad d_0 = 0.1 \text{ ft}; \quad \nu = 1.6\times10^{-4}\,\text{ft}^2/\text{sec} \quad \text{(Table 1-6)}$$

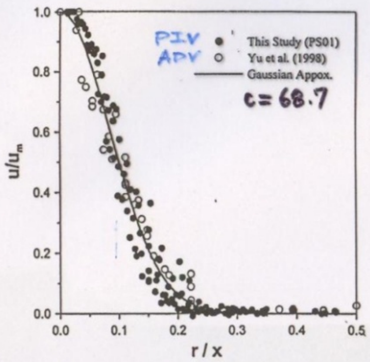
$$\therefore \quad Re_0 = \frac{100(0.1)}{1.6\times10^{-4}} = 62{,}500$$

$$\frac{\varepsilon}{\nu} = 0.013(62{,}500) = 813 \sim 10^3$$

→ ε (turbulent eddy viscosity) is 10^3 times larger than molecular viscosity

For laminar flow,

$$\frac{\varepsilon}{\nu} = 1 \quad \rightarrow \quad Re_0 = \frac{1}{0.013} \approx 80$$



For round jet;

$$\frac{u}{u_m} = \exp\left[-C\left(\frac{r}{x}\right)^2\right]$$

그림 10 유속 횡분포의 비교($x/d_p = 6$)

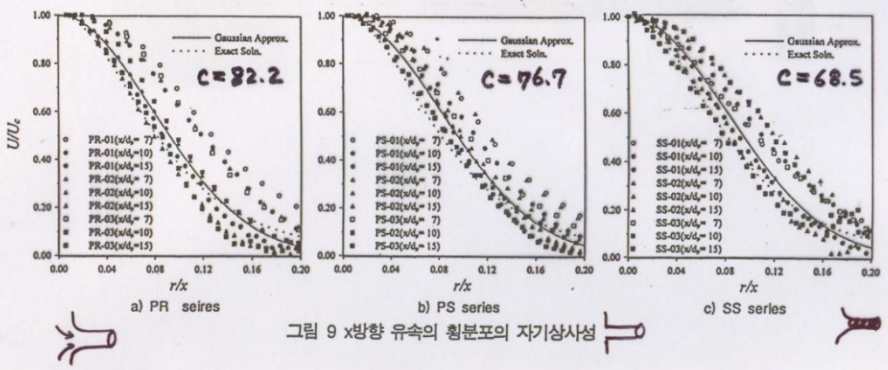


그림 9 x방향 유속의 횡분포의 자기상사성

1.3.3 Lateral Spread of Jets

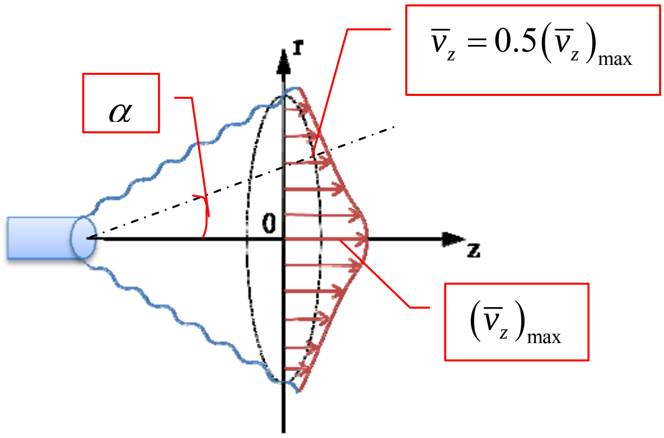
For both plane and round jets, the lateral spread of of jets is linear.

$$b \sim z^1; \; d \sim z^1$$

For line (jet boundary) along which $\frac{\bar{v}_z}{(\bar{v}_z)_{\max}} = 0.5$ (or $\frac{1}{e} = 0.37$)

$\alpha = 6.5^\circ$ for plane jet (A1)

$\alpha = 5^\circ$ for round jet (A2)



[Re] Gaussian profile

$$\frac{\bar{v}_z}{(\bar{v}_z)_{\max}} = 0.5 = \exp\left[-k\left(\frac{r}{z}\right)^2\right]$$

$$-k\left(\frac{r}{z}\right)^2 = \ln(0.5) \;\rightarrow\; \frac{r}{z} = \left(-\frac{\ln(0.5)}{k}\right)^{1/2}$$

Then, use trigonometric function

$$\frac{r}{z} = \tan\alpha \quad \rightarrow \; \alpha = \tan^{-1}\frac{r}{z}$$

•Turbulent nature of jet flow

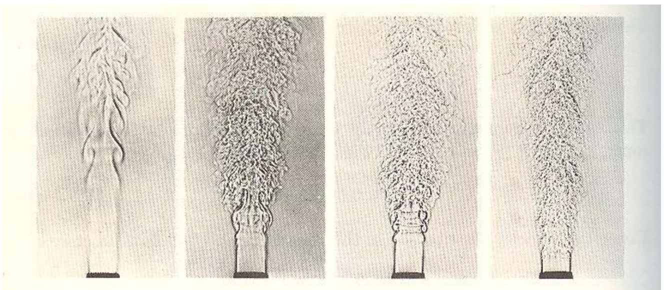


FIG. 16-5. Development of a turbulent jet [4].

→ The precise jet boundaries cannot be defined due to turbulent nature of the flow.

→ Actual jet limits are statistically determined.

→ use intermittency factor Ω

$$\Omega = \frac{\text{time during which the flow is turbulent}}{\text{total elapsed time of measurement}}$$

$\Omega = 1$ for fully turbulent region → center part of the jet

$\Omega = 0$ for nonturbulent region → edge of the jet

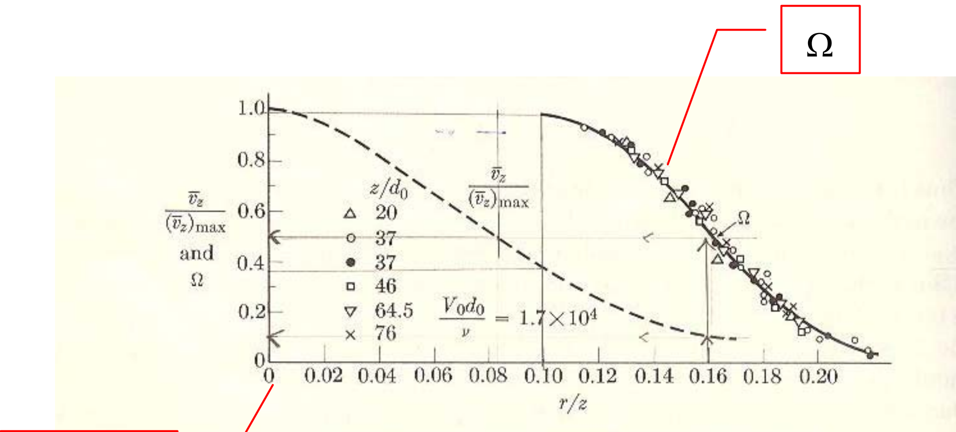


FIG. 16–6. Radial distribution of intermittency factor in a round jet [5].

Radial distribution of intermittency factor

i) At $r/z = 0.16$ → $\bar{v}_z = 0.1(\bar{v}_z)_{max}$; $\Omega = 0.5$

ii) At $r/z = 0.10$ → $v_z = 0.37 v_{z\,max}$; $\Omega = 1.0$

1.3.4 Fluid Entrainment

The amount of fluid entrained by the round jet is determined by integrating the velocity profiles in the zone of established flow.

The volume rate of flow is given as

$$Q = \int_0^\infty \bar{v}_z \, 2\pi r dr = \int_0^\infty \frac{(\bar{v}_z)_{\max}}{\left[1 + (\bar{v}_z)_{\max} \dfrac{r^2}{8\varepsilon z}\right]^2} 2\pi r dr \qquad \text{(a)}$$

Let $X = r^2, \quad a = (\bar{v}_z)_{\max}, \quad b = \dfrac{(\bar{v}_z)_{\max}}{8\varepsilon z}$

$$dx = 2rdr \quad \rightarrow \quad dr = \frac{1}{2r} dx$$

Then, (a) becomes

$$Q = 2\pi a \int_0^\infty \frac{1}{(1+bX)^2} \frac{dX}{2} = \pi a \left[\frac{-1}{(1+bX)^2} \frac{1}{b} \right]_0^\infty = 0 + \pi a \left[\frac{1}{b} \right] = \frac{a}{b} \pi$$

$$Q = 8\pi\varepsilon z \qquad (1.31)$$

Since initial flow rate, Q_0 is given as

$$Q_0 = \left(\pi \frac{d_0^2}{4} \right) V_0$$

The ratio is given as

$$\frac{Q}{Q_0} = \frac{8\pi\varepsilon z}{\pi \frac{d_0^2}{4} V_0} = \frac{32(0.013 V_0 d_0)_z}{d_0^2 V_0} = \frac{0.42z}{d_0}$$

$$\frac{Q}{Q_0} = 0.42 \frac{z}{d_0} \tag{1.32}$$

[Cf] A similar calculation based on the Gaussian curve gives

$$\frac{Q}{Q_0} = 0.28 \frac{z}{d_0} \tag{1.33}$$

→ This is Because velocities are small near the edge of the jet by the Gaussian curve as shown in Fig.1.4.

[Cf] For slot jet, $\frac{Q}{Q_0} = 0.62 \sqrt{\frac{x}{2b_0}}$

Homework # 1-1

Due: 2 weeks from today

1. For both plane and round jets,
   a) Plot Q vs x (z)
   b) Plot $\overline{u}_{\max}$ vs x

2. Prove Eq. (A1) & (A2) using Gaussian solution with k = 48.08 for plane jet and k = 72.0 for round jet.

3. For axially symmetric jet, derive Eq. 1.33.