Chapter 2 Turbulent Jet and Plumes

2.1 Introduction

2.2 Jets and Plumes

2.3 Environmental parameters

2.4 Buoyant Jet Problem and the Entrainment Hypothesis

2.5 Boundary Effects on Turbulent Buoyant Jets

Objectives:

- Study buoyant jets and plumes, strong man-induced flow patterns used to achieve rapid initial dilutions for water quality control
- Understand the theory of jets and plumes before considering the special type of discharge structure for diluted wastes
- Give the design engineer a firm background in the fundamentals of the theory essential to the prediction of how a given discharge system will perform

2.1 Introduction

•For wastes in a diluted form, a rapidly diluted discharge to the environment is the best means of recycling.

→ Turbulent jets and plumes form an effective mechanism to accomplish rapid initial dilution because they entrain large volumes of ambient fluid and mix it with the discharge fluid.

•The actual discharge structure is essentially the open end of a submerged pipe.

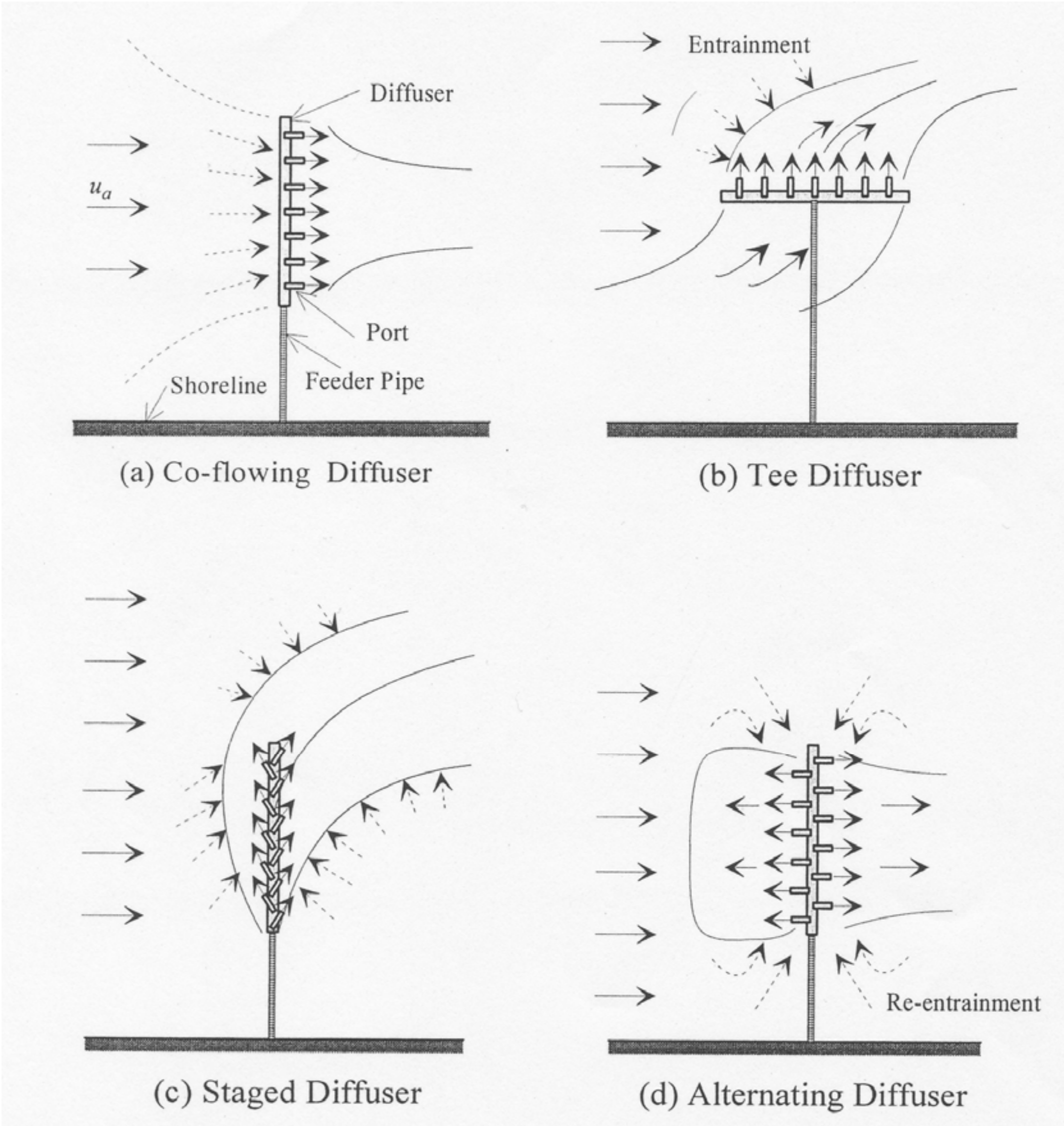
→ design a structure to achieve much higher initial dilution in order to minimize the immediate effects of the discharge on the environment.

→ Submerged multiport diffuser

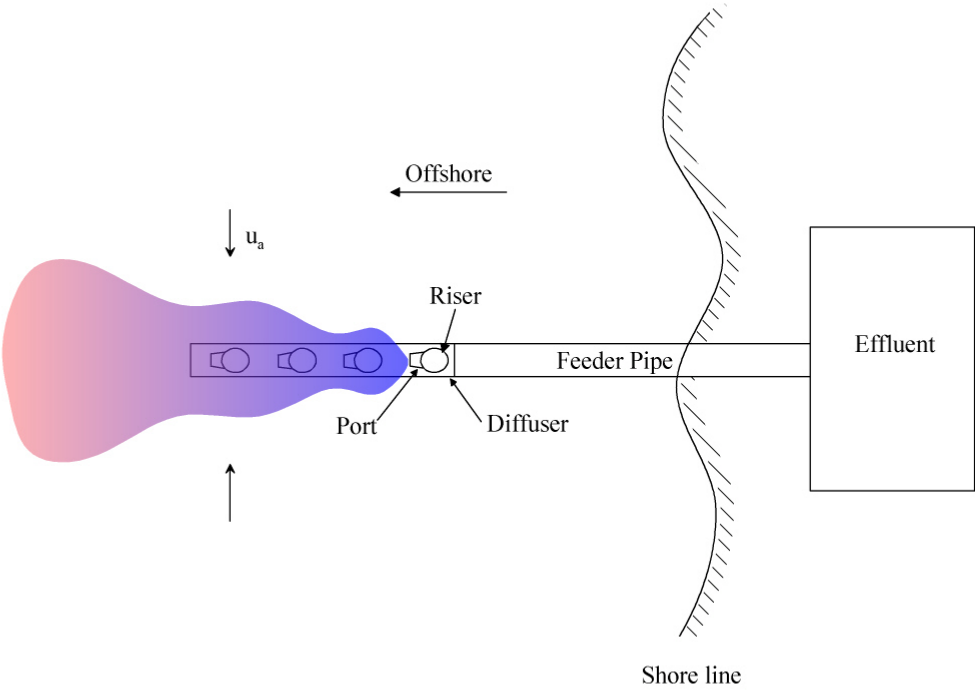
- Uni-directional diffuser:
  - Co-flowing diffuser
  - Tee diffuser
- Staged diffuser
- Alternating diffuser

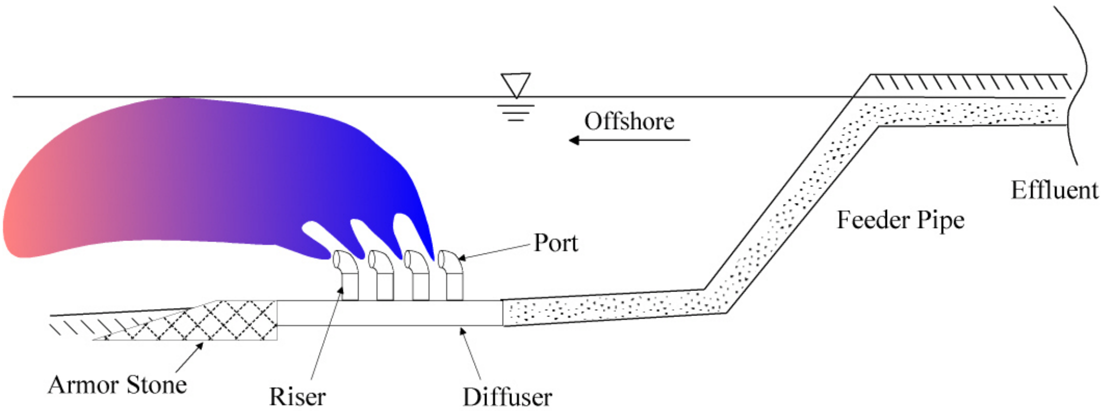
• Three classes of parameters affecting turbulent jet behavior

- jet parameter
- environmental parameters
- geometrical parameters

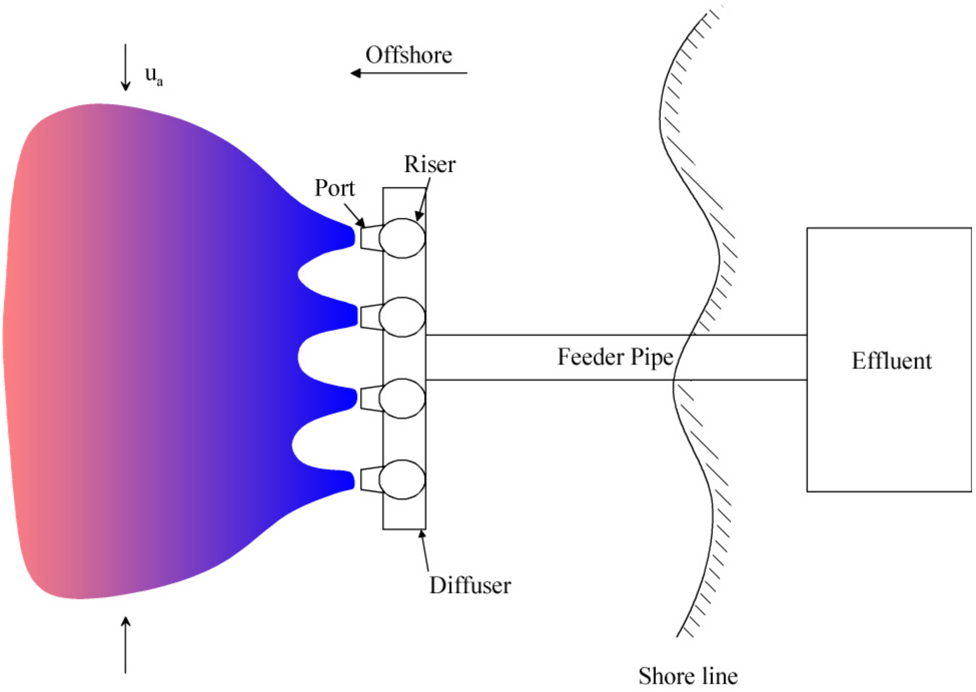


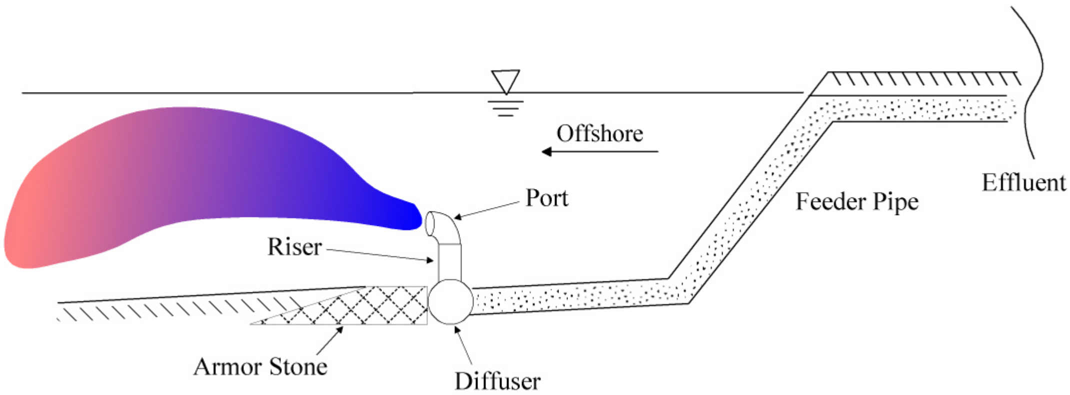
Types of submerged multiport diffuser





The staged diffuser





The Tee diffuser

i) Jet parameter:

- initial jet velocity and turbulence level
- jet mass flux: mass of fluid passing a jet cross section per unit time
- jet momentum flux: streamwise momentum of fluid passing a jet cross section per unit time
- flux of jet material (heat, salinity, contaminant, buoyancy

→ If the tracer concentration is sufficiently low that the density of the jet efflux is essentially equal to the ambient density level, then tracer concentration may not have any effect on the jet dynamics at all.

ii) Environmental parameters:

- ambient currents and turbulent levels
- density stratification

→ begin to influence jet behavior at some distance from the jet orifice

iii) Geometrical parameters:

- jet orifice shape – round/slot
- orientation, attitude (angle), spacing

• Analysis of influence of parameters on jet behavior

- understand how each of the above factors does modify the diluting capability of a jet by considering the effect of each factor in turn
- first study a simple jet emanating from a single round source
- then, extend the arguments to plane jets

• Basic method

First, seek limiting equilibrium asymptotic solution for simple flows

Then, combine these solutions into general description of more complex flows

[Ex] Buoyant jet in a cross flow

~ deduce the qualitative influences of momentum, buoyancy, cross flows on jet trajectory and rate of dilution

2.2 Jets and Plumes

▪Definitions and concepts

1) jet: discharge of fluid from an orifice or slot into a large body of the same or similar fluid

~ driven by the momentum

2) plume: a flow that looks like a jet, but is caused by a potential energy source that provides the fluid with positive or negative buoyancy relative to its surroundings

~ driven by the buoyancy

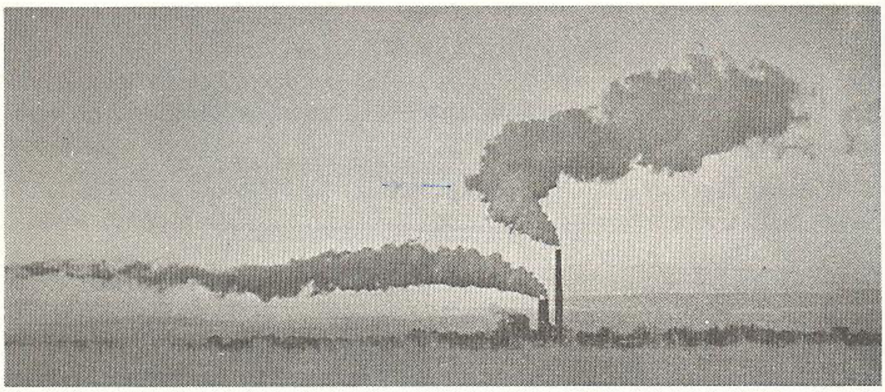


Figure 9.15 Turbulent buoyant jets in a density-stratified shear flow. [Photo by Ralph Turcotte, *Beverly* (Massachusetts) *Times*; see text for details.]

3) buoyant jet

- It is derived from sources of both momentum and buoyancy.
- <u>Initial flow is driven mostly by the momentum</u> of the fluid exiting an orifice.
- All buoyant jets <u>eventually act like plumes</u> given enough flow distance.

- jets and plumes

~ can be either laminar and turbulent flow

~ most flow generated by the discharge will be turbulent

- Jet Reynolds number > 2,000 ~ 4,000 → turbulent jet

$$\mathrm{Re}_j = \frac{DW}{\nu}$$

$D =$ diameter of jet orifice

$W =$ velocity at the jet orifice

▪ Important factors to jet dynamics

(i) Mass flux of jet

= mass of fluid passing a jet cross section per unit time

$$\rho\mu = \int_A \rho w \, dA \qquad (2.1)$$

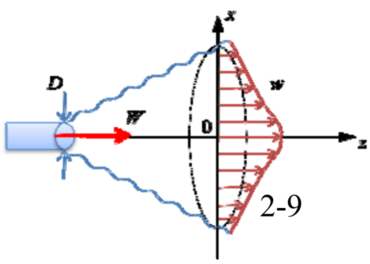
in which $A =$ cross-sectional area of the jet

$w =$ time-averaged jet velocity in the axial direction

$\mu =$ specific mass flux (volume flux=volume/time) of jet, $\mu = \int_A w \, dA$

Initial value of $\mu \Rightarrow Q = \frac{\pi}{4} D^2 W$

in which W = mean outflow velocity



(ii) Momentum flux

= amount of streamwise momentum passing a jet cross section per unit time

$$\rho m = \int_A \rho w^2 \, dA \qquad (2.2)$$

in which m = *specific momentum flux* = $\dfrac{momentum\ flux}{\rho}$

Initial value of $m \Rightarrow M = \dfrac{\pi}{4} D^2 W^2 \quad [L^4 / T^2]$

(iii) Buoyancy flux

= buoyancy or submerged weight of the fluid passing through a cross-section per unit time

$$\rho\beta = \int_A g \, \Delta\rho \, w \, dA \qquad (2.3)$$

in which $\Delta\rho$ = difference in density between the surrounding fluid (ρ_a) and the jet fluid (ρ_d)

$$\Delta\rho = \rho_a - \rho_d$$

g' = effective gravitational acceleration

$$g' = g\frac{\Delta\rho}{\rho} = g\frac{\rho_a - \rho_d}{\rho_d}$$

• Initial value of $\beta \Rightarrow$ B $= g_0{}' \dfrac{\pi}{4} D^2 W = g_0{}' Q \qquad [L^4 / T^3]$

① Case 1: initial buoyancy is generally contained in the discharge

$$B = g\left(\frac{\Delta\rho_0}{\rho}\right)Q = g_0{}'Q = g_0{}'\frac{\pi}{4}D^2W$$

in which $\Delta\rho_0 =$ difference in density between the receiving fluid (ρ_{a_0})

and the fluid(ρ_{d_0}) being discharged

$g_0{}' =$ initial apparent(effective) gravitational acceleration

$$= g\frac{\rho_{a_0} - \rho_{d_0}}{\rho_{d_0}}$$

② Case 2: plume formed by a source of buoyancy, such as a source of heat

→ The buoyancy imparted to the fluid is determined by the heat added.

$$\rho B = \alpha g P / C_P$$

in which $\alpha =$ volume coefficient of thermal expansion

$P =$ heat flux added by the heat sources

$C_P =$ specific heat at constant pressure

• Plane jets

- flow from a slot

- μ, m and β are interpreted as specific fluxes per unit length of slot

- dimension of Q, M and B are reduced in length by one order

2.2.1 The Simple Jet

• simple jet from a round nozzle

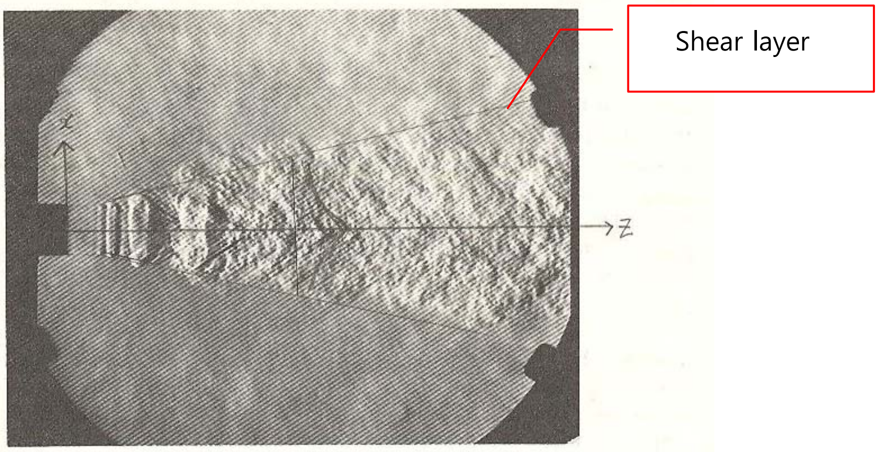


Figure 9.1 Schlieren photograph of a turbulent jet at a Reynolds number of 1.87×10^4. [Courtesy of Steven Crow and Cambridge University Press.]

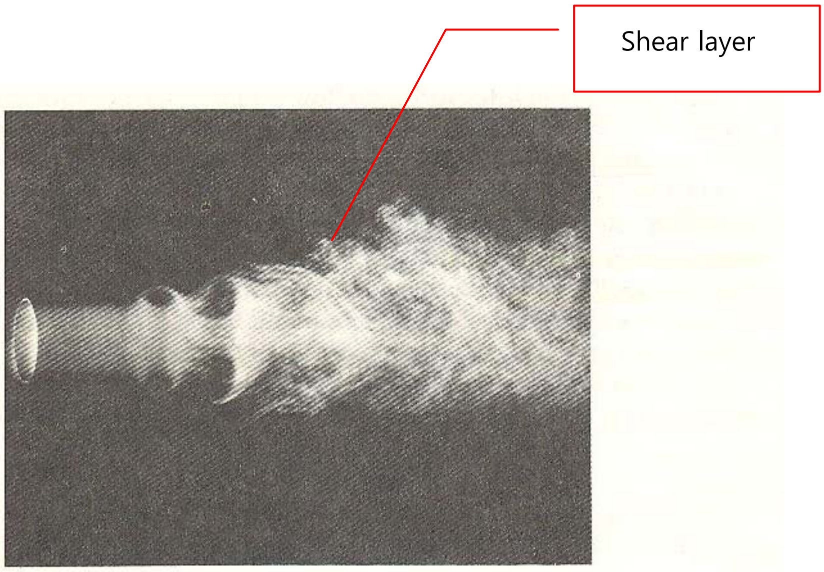


Figure 9.2 Flow visualization of a turbulent jet at a Reynolds number of 10.5×10^4. [Courtesy of Steven Crow and Cambridge University Press.]

- Jet boundary

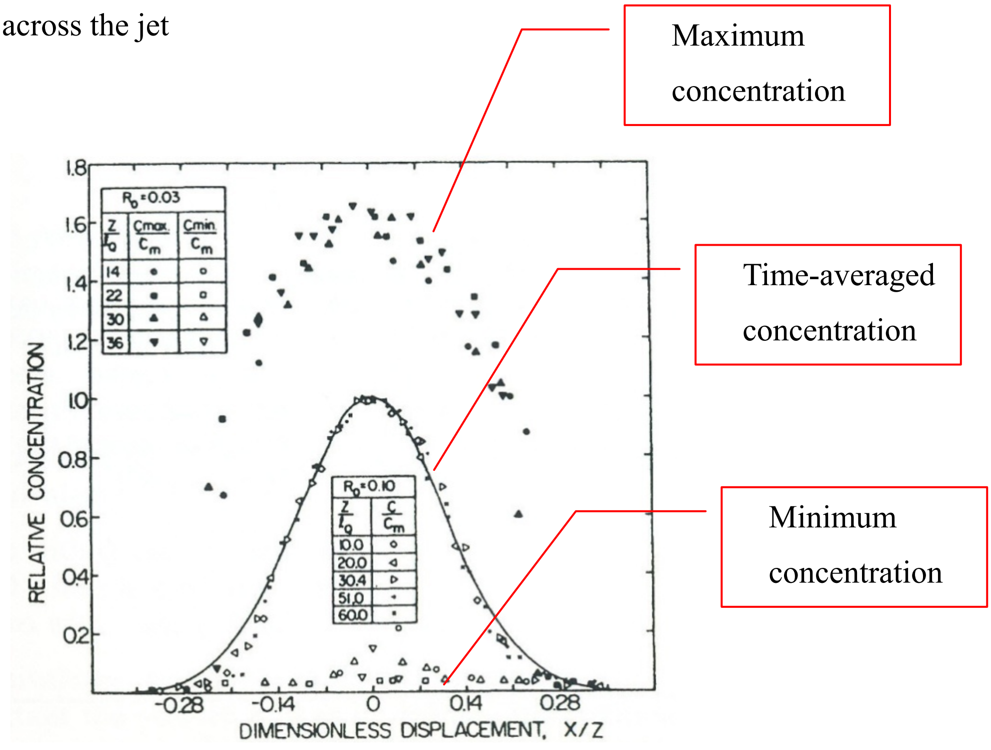
~ a shear layer forms between jet fluid and ambient fluid

→ the shear layer has large cylindrical shaped waves that entrain the ambient fluid in large "gulps" which then break down and mix the two fluids

~ it shows a rapidly fluctuating environment which may be, at times, purely jet fluid, purely ambient fluid or a mixture of both

→ Intermittent nature

~ time-averaged measurement shows an essentially Gaussian distribution of tracer concentration across the jet



$$l_Q = \frac{Q}{M^{0.5}} = \text{ characteristic length scale = volume flux / momentum flux}$$

$$R_0 = \frac{Q^2 B^{2/3}}{M^2} = \text{Richardson number at outlet}$$

(1) Time-averaged concentration and velocity across the jet

→ Gaussian distribution

$$C = C_m \exp\left[-C_T \left(\frac{x}{z} \right)^2 \right] \tag{1}$$

where $C_m =$ centerline concentration;

$z =$ distance along the jet axis;

$x =$ transverse (or radial) distance from the jet axis;

C_T = Gaussian constant

- time-averaged velocity profile,

→ Gaussian distribution

$$w = w_m \exp\left[-C_w \left(\frac{x}{z} \right)^2 \right], \quad z > 6D \quad \text{(ZEF)}$$

[Cf] Fig. 9.4 shows that the turbulence intensity reaches a state of steady decay after $z \geq 10D$ (ZFE: $0 < z \leq 6D \sim 10D$).

(2) Self-similarity in ZEF (Self-preserving region)

- Time-averaged velocity and tracer distributions can be expressed in terms of maximum value (measured at the jet centerline) and a measure of the width.

$$w = w_{\mathrm{m}} f\left(\frac{x}{b_w}\right) \tag{2.8}$$

$$C = C_{\mathrm{m}} f_1\left(\frac{x}{b_T}\right)$$

where b_w, b_T are the values of x at which w or C reduces to some specified fraction of w_m, C_m, e.g. 0.50 or 0.37 (e^{-1}).

b_w, $b_T \rightarrow$ nominal jet boundary (Ch. 1)

• Gaussian form

$$C = C_m \exp\left[-\left(\frac{x}{b_T}\right)^2\right] \tag{2.9}$$

where b_T is value of x at which C takes the value of $0.37 C_m$

• Width parameters, k_T, k_w

1) Comparison of Eqs. (1) and (2.9)

$$b_T = \sqrt{\frac{1}{C_T}}\, z$$

2) Albertson et al.

$$C = C_m \exp\left[-\frac{1}{2}\left(\frac{x}{c_T z} \right)^2 \right]$$

$$\rightarrow b_T = \sqrt{2}\, c_T z$$

3) Assume that

$$b_T = k_T z$$

Then, we have Gaussian distribution as

$$C = C_m \exp\left[-\left(\frac{x}{k_T z} \right)^2 \right] = C_m \exp\left[-\frac{1}{k_T{}^2}\left(\frac{x}{z} \right)^2 \right]$$

• Characteristic length scale

→ $l_Q =$ ratio of volume flux to momentum flux

i) Round Jet

$$l_Q = \frac{Q}{M^{1/2}} = \frac{AW}{\left(AW^2\right)^{1/2}} = \sqrt{A} \tag{2.10}$$

where $A =$ initial cross sectional area of the jet at jet orifice $= \frac{\pi}{4} D^2$

$$\therefore \; l_Q = \frac{\sqrt{\pi}}{2} D \quad \sim \quad 0.886D$$

ii) Planar Jet (Slot jet, 2-D Jet)

$$l_Q = \frac{Q^2}{M} = \frac{(DW)^2}{DW^2} = D \ . \ . \ . \text{ Slot height}$$

(3) Asymptotic solution by dimensional analysis

i) Centerline velocity

$$w_m = \phi(Q, M, z)$$

Apply Buckingham π theorem

→ Repeating variables M, Q

$$\phi_1(w_m, Q, M, z) = 0$$

$$\phi_2\left(w_m \frac{Q}{M}, \frac{z}{Q / M^{1/2}}\right) = 0$$

$$\therefore \ w_m \frac{Q}{M} = \phi_3\left(\frac{z}{l_Q}\right) \qquad (2.11)$$

Asymptotic cases

i) $z \le l_Q$, $\quad w_m \ \rightarrow \ \frac{M}{Q}$ - initial value at the outlet

$$w_m = \frac{M}{Q} = W$$

ii) $z \gg l_Q$, $\quad Q \ \rightarrow \ 0 \quad or \quad M \ \rightarrow \ \infty$

→ The further we are from the jet orifice the less important the volume flow is in defining the solution and the more important the momentum flux becomes.

→ All properties of the jet are defined solely in term of z, and M.

$$\therefore \; w_m \frac{Q}{M} \rightarrow a_1 \left(\frac{l_Q}{z} \right)^1 \qquad (2.12)$$

where a_1 = empirical costant

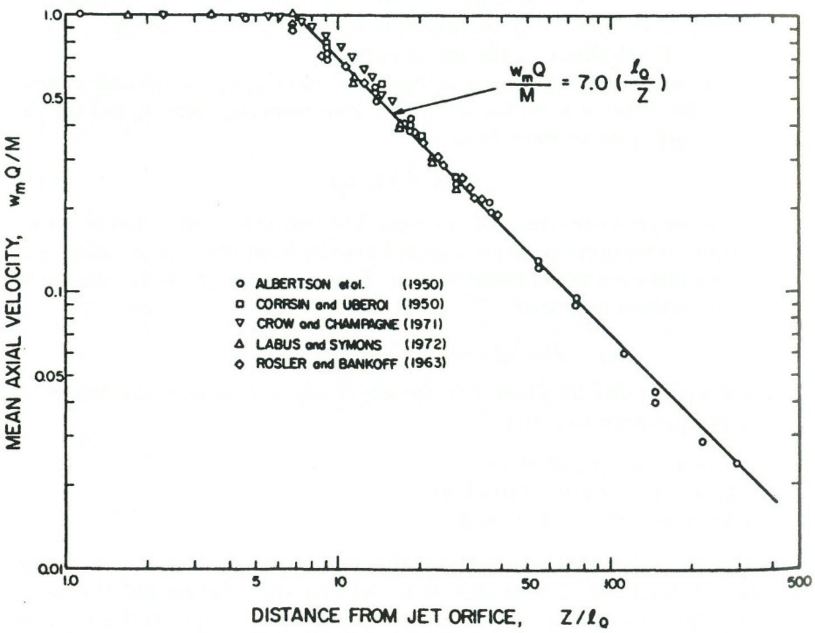


Fig. 2.5 Decay of peak time-averaged velocity on the axis of a round turbulent jet

From data fit (Fig. 2.5)

$$w_m \frac{Q}{M} = 7.0 \frac{l_Q}{z} \qquad (2.12a)$$

$$\frac{w_m}{W} = 7.0 \frac{l_Q}{z} = 6.2 \frac{D}{z}$$

[Cf] Ch.1 (Daily & Harleman, 1966)

$$\frac{w_m}{W} = 6.4 \frac{d_0}{z} \qquad (1.28)$$

ii) Width parameter

Dimensional analysis gives

$$\frac{b}{l_Q} = f\left(\frac{z}{l_Q}\right) \qquad (A)$$

when $z \gg l_Q$, $\quad Q \to 0 \quad or \quad M \to \infty$

→ f must be such as to have Q vanish from the relationship.

$$\therefore \frac{b}{l_Q} = a_2 \left(\frac{z}{l_Q}\right)^1 \quad \to \quad \boxed{b = a_2 z}$$

Table 2.1 Width Parameters for Turbulent Round Jets

Investigator	b_w/z	b_T/z
Albertson *et al.* (1950)	0.114	—
Becker *et al.* (1967)	—	0.127
Corrsin (1943)	0.100	0.132
Corrsin and Uberoi (1950)	0.114	0.140
	0.130	0.156
Forstall and Gaylord (1955)	0.107	0.115
Hinze and van der Hegge Zijnen (1949)	0.102	0.115
Keagy and Weller (1949)	0.099	0.107
	0.106	0.126
Kizer (1963)	0.099	0.125
Rosenweig *et al.* (1961)	0.108	0.120
Ruden (1933)	0.103	0.124
Sunavala *et al.* (1957)	—	0.141
Uberoi and Garby (1967)	0.090	0.101
	0.101	0.114
Wilson and Danckwerts (1964)	0.120	0.156
	0.114	0.138
Mean values	0.107 ±0.003	0.127 ±0.004

- Average values of width parameters

$$\frac{b_w}{z} = 0.107 \quad (w = 0.37\, w_m) \tag{B}$$

$$\frac{b_T}{z} = 0.127 \quad (c = 0.37\, c_m) \tag{C}$$

$$\frac{b_T}{b_w} = 1.19 \tag{D}$$

→ Mean concentration profile is wider than the mean velocity profile.

[Re] Velocity profile

$$w = w_m \exp\left[-k_w \left(\frac{x}{z} \right)^2 \right] \tag{1}$$

$$b_w = 0.107\, z \tag{2}$$

Combine (1) and (2)

$$w = w_m \exp\left[-k_w (0.107)^2 \left(\frac{x}{b_w} \right)^2 \right] \tag{3}$$

If we substitute $k_w = 87.3$ into (3), then we get

$$w = w_m \exp\left[-\left(\frac{x}{b_w} \right)^2 \right]$$

iii) Volume flux

Dimensional analysis gives

$$\frac{\mu}{Q} = f\left(\frac{z}{l_Q}\right) \tag{2.13}$$

i) $z/l_Q \to 0, \quad f(z/l_Q) \to 1$

$$\mu = Q$$

ii) $z/l_Q \to \infty, \quad Q \to 0$

$$\therefore \quad \frac{\mu}{Q} = c_j\left(\frac{z}{l_Q}\right)^1$$

Substituting (2.10) yields

$$\frac{\mu}{Q} = c_j \frac{z}{Q/\sqrt{M}}$$

For round jet,

$$\frac{\mu}{Q} = c_j\left(\frac{z}{l_Q}\right), \quad z \gg l_Q \tag{2.14}$$

The value of c_j can be found by using self-similar velocity profile.

$$\mu = \int_A w \, dA$$

$$= \int_A w_m \exp\left[-\left(\frac{x}{b_w}\right)^2\right] dA$$

$$= \int_0^\infty w_m \exp\left[-\left(\frac{x}{b_w}\right)^2\right] 2\pi x \, dx$$

$$= -\pi w_m b_w^{\ 2} \int_0^\infty \left(-\frac{2x}{b_w^{\ 2}}\right) \exp\left(-\frac{x^2}{b_w^{\ 2}}\right) dx$$

$$= -\pi w_m b_w^{\ 2} \left[e^{-\frac{x^2}{b_w^{\ 2}}} \right]_0^\infty = \pi w_m b_w^{\ 2} \qquad (2.16)$$

Substitute (2.12a) & (B) into (2.16)

$$\mu = \pi \left[7.0 \frac{M}{Q} \frac{l_Q}{z} \right] (0.107 z)^2$$

w_m

b_w

$$\therefore \ \frac{\mu}{Q} = 0.25 \frac{M}{Q} \frac{l_Q}{Q} z = 0.25 \frac{M}{Q^2} l_Q \, z$$

$\frac{1}{l_Q^{\ 2}}$

$$= 0.25 \frac{l_Q}{l_Q^{\ 2}} z = 0.25 \frac{z}{l_Q}$$

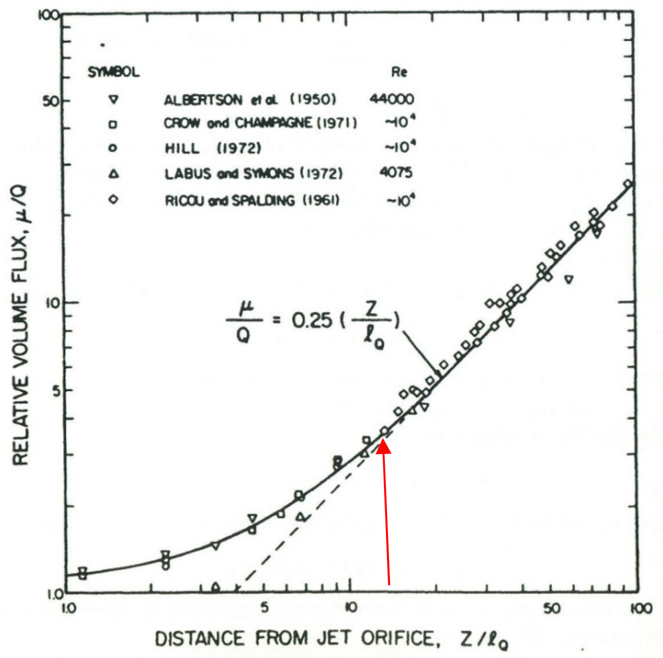
$$\frac{\mu}{Q} = 0.25 \frac{z}{l_Q}, \quad z \gg l_Q \qquad (2.18)$$

[Cf] $\frac{\mu}{Q} = 0.25 \frac{z}{l_Q} = 0.25 \frac{z}{0.88 D} = 0.28 \frac{z}{D}$

→ the same as Eq. (1.33)

Comparison with experimental data: Fig. 2.6

$$\frac{\mu}{Q} = \begin{cases} f(z/l_Q), & z/l_Q \le 15 \\ 0.25\dfrac{z}{l_Q}, & \dfrac{z}{l_Q} > 15 \end{cases}$$



iv) Dilution of a tracer

~ We are mostly interested in the dilution of a tracer material discharged in the jet.

C_m = <u>centerline</u> concentration measured on the jet axis

$$C_m \sim \frac{1}{z}$$

Similar to w_m (Eq. 2.12)

Suppose that Y is the rate of supply of tracer mass to the jet

Then, Y = mass flux (mass/time)

$$Y = QC_0 \tag{2.19}$$

where C_0 = initial mass concentration of tracer; Q = initial volume flux

For $z \gg l_Q$, $M^{1/2}$ is the only jet parameter with time involved

$$\frac{C_m}{Y} = a_2 (M^{1/2} z)^{-1}$$

(M/L^3)/(M/t) = t/L^3

[(L^{4}t^{-2})$^{0.5}$ L]$^{-1}$ = t/L^3

$$\frac{C_m}{QC_0} = a_2 \frac{1}{M^{1/2} z}$$

$$l_Q = \frac{Q}{M^{1/2}}$$

$$\frac{C_m}{C_0} = a_2 \frac{l_Q}{z}, \qquad z \gg l_Q \tag{2.21}$$

By experiments by Chen and Rodi (1976): $a_2 = 5.64$

$$\frac{C_m}{C_0} = 5.64 \frac{l_Q}{z} \tag{2.21a}$$

- Centerline dilution, S_m

The centerline dilution can be derived from (2.21a).

$$S_m = \frac{C_0}{C_m} = 0.18 \frac{z}{l_Q} \tag{2.21b}$$

- Mean dilution, S_{av}

$$S_{av} = \frac{C_0}{C_{av}}$$

where C_{av} = average concentration

By the way, we can define the average concentration for a jet in the following way

$$Y = \mu C_{av} = QC_0 \tag{2.22}$$

Therefore, rearranging (2.22) gives

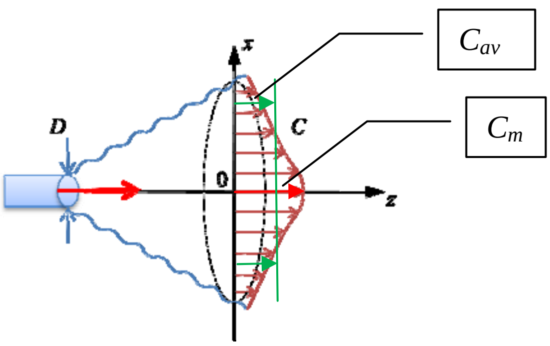
$$S_{av} = \frac{C_0}{C_{av}} = \frac{\mu}{Q} \tag{2.22a}$$

Combine (2.22a) and (2.18)

$$S_{av} = \frac{\mu}{Q} = 0.25\frac{z}{l_Q} \tag{2.22b}$$

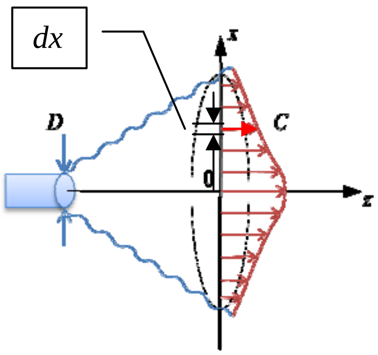
Dividing (2.22b) by (2.21b) yields

$$\frac{S_{av}}{S_m} = \frac{C_m}{C_{av}} \cong 1.4$$



- Rate of mass flux

$$Y = \mu C_{av} = \int_{jet} 2\pi\, x\, w\, C\, dx + \underline{turbulent\ \ transport} \tag{2.23}$$



Assume Gaussian forms for w and C, and integrate (2.23)

$$\mu C_{av} = \pi\, w_m\, C_m \left(\frac{b_w^{\ 2}\, b_T^{\ 2}}{b_w^{\ 2} + b_T^{\ 2}} \right) + turbulent\ \ transport$$

$$\therefore\ turbulent\ transport = Y - \pi\, w_m\, C_m \left(\frac{b_w^{\ 2}\, b_T^{\ 2}}{b_w^{\ 2} + b_T^{\ 2}} \right)$$

$$\therefore\ \frac{turbulent\ \ flux\ \ of\ \ tracer\ \ mass}{total\ \ flux\ \ of\ \ tracer\ \ mass}$$

$Y = QC_0$

$$= 1 - \frac{\pi w_m C_m}{QC_0} \left(\frac{b_w^{\ 2}\, b_T^{\ 2}}{b_w^{\ 2} + b_T^{\ 2}} \right) = 0.17 \pm 0.12 \tag{2.25}$$

→ Thus, the turbulent flux is not zero.

• Summary: Table 2.2

$$\frac{\mu}{Q} = 0.25\frac{z}{l_Q}, \text{ round jet}$$

$$\frac{\mu}{q} = 0.5\left(\frac{z}{l_Q}\right)^{1/2}, \text{ plane jet}$$

$$l_Q = \frac{q}{\sqrt{m}}, \text{ plane jet}$$

Table 2.2 Summary of Properties of Turbulent Jet

Parameter	Round jet	Plane jet
Initial volume flow rate Q	Dimensions L^3T^{-1}	Dimensions L^2T^{-1}
Initial specific momentum flux M	Dimensions L^4T^{-2}	Dimensions L^3T^{-2}
Characteristic length scale l_Q	$\frac{Q}{M^{1/2}}$	$\frac{Q^2}{M}$
Maximum time-averaged velocity w_m	$w_m\frac{Q}{M} = (7.0 \pm 0.1) l_Q/z$	$w_m\frac{Q}{M} = (2.41 \pm 0.04)\left(\frac{l_Q}{z}\right)^{1/2}$
Maximum time-averaged tracer concentration C_m	$\frac{C_m}{C_0} = (5.6 \pm 0.1)\left(\frac{l_Q}{z}\right)$	$\frac{C_m}{C_0} = (2.38 \pm 0.04)\left(\frac{l_Q}{z}\right)^{1/2}$
Mean dilution μ/Q	$\frac{\mu}{Q} = (0.25 \pm 0.01)\left(\frac{z}{l_Q}\right)$	$\frac{\mu}{Q} = (0.50 \pm 0.02)\left(\frac{z}{l_Q}\right)^{1/2}$
Velocity scale of half-width b_w/z	0.107 ± 0.003	0.116 ± 0.002
Concentration scale of half-width b_T/z	0.127 ± 0.004	0.157 ± 0.003
Ratio C_m/C_{av}	1.4 ± 0.1	1.2 ± 0.1

Width parameter

$l_Q = \frac{q}{\sqrt{m}}$

$w_m \frac{q}{m}$

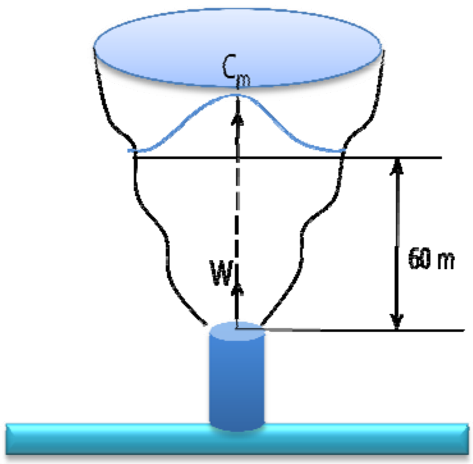
$\frac{\mu}{q}$

[Example 2.1]

Turbulent round jet:

$$Q = 1\ m^3/s,\ C_0 = 1\ kg/m^3 = 10^3\ mg/l = 10^3\ ppm,\ W = 3\ m/s$$

Find : $\overline{w}_m$, $\overline{C}_m$, $\dfrac{\mu}{Q}$ at $z = 60\ m$



[Sol]

$$Q = A_0 W$$

$$\rightarrow A_0 = \frac{Q}{W} = \frac{1}{3}\ m^2 \rightarrow D = 0.651\, m$$

$$l_Q = \frac{Q}{\sqrt{M}} = \frac{1}{\sqrt{A_0 W^2}} = \frac{1}{\sqrt{\frac{1}{3}(3)^2}} = \frac{1}{\sqrt{3}} = 0.577\ m$$

$$M = A_0 W^2 = \frac{1}{3}(3)^2 = 3\ m^4/s^2$$

$$\left.\frac{z}{l_Q}\right|_{z=60\,m} = \frac{60}{0.577} = \underline{103.9}$$

(2.12): $w_m \dfrac{Q}{M} = 7.0 \dfrac{l_Q}{z}$

$$\therefore w_m = 7.0 \frac{l_Q}{z}\frac{M}{Q} = 7.0\left(\frac{1}{103.9}\right)\left(\frac{3}{1}\right) = 0.20 \ m/s$$

(2.21) : $\dfrac{C_m}{C_0} = 5.6 \dfrac{l_Q}{z}$

$$\therefore C_m = 5.6\,(\frac{1}{103.9})(10^3) = 53.9 \ \ ppm$$

$$C_{av} = 53.9 / 1.4 = 38.5 \ \ ppm$$

(2.18): $\dfrac{\mu}{Q} = 0.25 \dfrac{z}{l_Q} = 0.25\,(103.9) = 26.0$

Minimum dilution, S_m

$$S_m = \frac{C_o}{C_m} = \frac{1000}{53.9} = 18.6$$

Bulk (average) dilution, S_{av}

i) $S_{av} = \dfrac{C_0}{C_{av}} = \dfrac{1000}{38.5} = 26.0$

$$S_{av} = \frac{C_0}{C_{av}} = 1 \Big/ \left\{\frac{C_m}{C_0}\frac{C_{av}}{C_m}\right\} = 1 \Big/ \left\{5.6 \frac{l_Q}{z}\right\}\left(\frac{1}{1.4}\right) = 0.25 \frac{z}{l_Q}$$

ii) $\dfrac{\mu}{Q} = 0.25 \dfrac{z}{l_Q} = 0.25\,(103.9) = 26.0$

2.2.2 The Simple Plume

▪ Mechanics of pure plume

The pure plume is easier to analyze than the pure jet because in the pure plume there is no initial volume flux, or initial momentum flux.

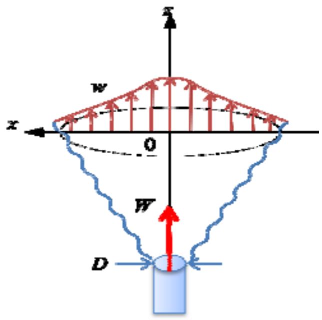
~ smoke plume above a fire

~ all flow variables for a plume must be function only of B (buoyancy flux), z, and v (viscosity).

▪ **Time-averaged vertical velocity for round plume**

$$w_m = f(B, z, v)$$

$$\phi(w_m, B, z, v) = 0$$



Buckingham π theorem

Select repeating variables as B, z

$$\phi_1\left(w_m\left(\frac{z}{B}\right)^{1/3}, \frac{B^{1/3} z^{2/3}}{v} \right) = 0$$

$$\therefore\ w_m\left(\frac{z}{B}\right)^{1/3} = \phi_2\left(\frac{B^{1/3} z^{2/3}}{v}\right) \tag{2.27}$$

$$\text{L.H.S.} = w_m\left(\frac{z}{B}\right)^{1/3} = \frac{L}{t}\left(\frac{L}{g' Q}\right)^{1/3} = \frac{L}{t}\left(\frac{L}{\frac{L}{t^2}\frac{L^3}{t}}\right)^{1/3} = \frac{L}{t}\frac{t}{L} = 1$$

$$\text{R.H.S.} = \frac{B^{1/3} z^{2/3}}{\nu} = \frac{(g'Q)^{1/3} z^{2/3}}{\nu}$$

$$= \frac{[(L/t^2)(L^3/t)]^{1/3} L^{2/3}}{\nu} = \frac{(L/t)L}{\nu} = \frac{V\,l}{\nu} = \text{Re}$$

→ Re is usually large enough so that the flow is turbulent and the effect of viscosity becomes essentially absent. $\left(z \gg \frac{\nu^{2/3}}{B^{1/2}},\ \text{Re} \gg 1 \right)$

→ Then, Eq. (2.27) becomes

$$w_m \left(\frac{z}{B} \right)^{1/3} \approx const \qquad \rightarrow \qquad w_m \left(\frac{z}{B} \right)^{1/3} = b_1$$

Experiments by Rouse et al. (1952); $b_1 = 4.7$

$$w_m \left(\frac{z}{B} \right)^{1/3} = 4.7$$

$$w_m = 4.7 \left(\frac{B}{z} \right)^{1/3} \qquad (2.28)$$

[Cf] For pure jet, $\frac{w_m}{W} = 7.0 \frac{l_Q}{z}$

- **Change of momentum by buoyancy**

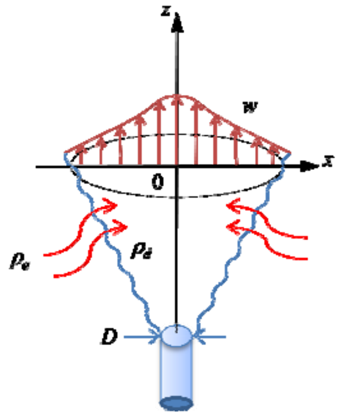
The fluid within the plume is less dense (or more) than its surroundings

→ Force of gravity acts to change the momentum in the flow.

→ Flux of momentum increases along the axis of the plumes.

[Cf] For the pure jet, momentum flux is constant along the axis of the jet.

$$\rho m = \int_A \rho w^2 \, dA = \text{constant}$$



Consider the momentum flux, $m = \phi(B, z)$

The dimensional analysis gives

$$\frac{m}{B^{2/3} z^{4/3}} = const. = b_2$$

Experiments by Rouse et al. (1952); $b_2 = 0.35$

$$m = 0.35\, B^{2/3} z^{4/3} \tag{2.29}$$

$$\rightarrow m \propto z^{4/3}$$

- **Volume flux**

In similar fashion, μ can be written as

$$\frac{\mu}{B^{1/3} z^{5/3}} = b_3$$

$$\mu = 0.15\, B^{1/3} z^{5/3} \tag{2.30}$$

$$\rightarrow \mu \propto z^{5/3}$$

Eliminate B from Eq. (2.29) & Eq. (2.30)

Eq.(2.29): $B^{1/3} = \dfrac{m^{1/2}}{\sqrt{b_2 z^{2/3}}}$ (1)

Substitute (1) into Eq. (2.30)

$$\mu = b_3 \frac{m^{1/2}}{\sqrt{b_2 z^{2/3}}} z^{5/3} = \frac{b_3}{\sqrt{b_2}} m^{1/2} z$$

$$\mu = c_p m^{1/2} z \qquad (2.31)$$

$m \sim z^{4/3}$

$$c_p = \frac{b_3}{\sqrt{b_2}} = \frac{0.15}{\sqrt{0.35}} = 0.254 \,, \qquad c_p = spreading\ \ coeff.\ \ for\ \ plume$$

$$\mu = 0.25\, m^{1/2} z \qquad (2.31a)$$

[Cf] For pure jet, $\mu = 0.25\, M^{1/2} z$ (2.18a)

Eq. (2.18): $\dfrac{\mu}{Q} = 0.25 \dfrac{z}{l_Q} \rightarrow \mu = 0.25\, Q \dfrac{z}{Q / M^{1/2}} = 0.25\, M^{1/2} z$

• Comparison of (2.31a) and (2.18a)

→ Volume flux equation are the same for plume and jet, except local momentum flux (m) must be used for the plume.

For jet, $\mu \sim z^1$ (∵ momentum is conserved)

For plume, $\mu \sim z^{5/3}$ (∵ momentum flux is constantly increasing)

- **Plume Richardson Number**

Eliminate z from Eq. (2.29) and Eq. (2.30)

Eq. (2.29): $$z^{5/3} = \left(\frac{m}{b_2 B^{2/3}} \right)^{5/4} \qquad (2)$$

Substitute (2) into Eq. (2.30)

$$\therefore \quad \mu = b_3 B^{1/3} \left(\frac{m}{b_2 B^{2/3}} \right)^{5/4} = \frac{0.15}{(0.35)^{5/4}} \frac{m^{5/4}}{B^{1/2}} = 0.557 \frac{m^{5/4}}{B^{1/2}}$$

Plume Richardson No.

$$R_p = \frac{\mu B^{1/2}}{m^{5/4}}$$

$$R_p = b_3 b_2^{-5/4} = 0.557 \text{ for round plume}$$

$$= 0.735 \text{ for plane plume}$$

[Cf] Richardson No. for buoyant jet

$$R = \frac{\mu^2 \beta^{2/3}}{m^2}$$

$$R_0 = \frac{Q^2 B^{2/3}}{M^2}$$

- **Spreading (growth) coefficient**

Assume a Gaussian profile

$$w = w_m \exp\left\{ -\left(\frac{z}{b_w} \right)^2 \right\}$$

Integrate to get μ and m

$$\mu = \frac{1}{\rho} \int \rho w \, dA \qquad \text{(a)}$$

$$m = \frac{1}{\rho} \int \rho w^2 \, dA \qquad \text{(b)}$$

Eq. (2.31): $\mu = c_p m^{1/2} z$

$$\frac{\mu}{m^{1/2}} = c_p z \qquad \text{(c)}$$

Substitute results of integration of (a), (b) into (c)

$$\sqrt{2\pi}\, b_w = c_p z \quad \rightarrow \quad b_w = \frac{c_p}{\sqrt{2\pi}} z = \frac{0.254}{\sqrt{2\pi}} z = 0.101 z$$

c_p, c_j = spreading (growth) coefficient = 0.25 for both plume and jet

[Cf] For jet, $b_w = 0.107 z$

- **Tracer concentration**

In a buoyancy-driven discharge (plume), the rate of decrease of the maximum (centerline) tracer concentration can be deduced in the same way as for a jet.

Suppose that

Y = equivalent mass flux of tracer = QC_o

$$\left[\frac{C_m}{Y}\right] = \frac{[M\,L^{-3}]}{[M\,t^{-1}]} = \left[\frac{t}{L^3}\right]$$

Dimensional analysis gives

$$\frac{C_m}{Y} = f_5\,(B, z)$$

$$\frac{C_m}{Y} B^{1/3} z^{5/3} = const. = b_4 \qquad (2.34)$$

Experiments by Chen & Rodi (1976); $b_4 = 9.1$

$$C_m = 9.1\,Y\,B^{-1/3} z^{-5/3} = 9.1\,Q\,C_0\,B^{-1/3} z^{-5/3}$$

• Characteristic length scale

If the source of buoyancy is derived from a volume flow, so that B is defined as

$$B = g\frac{\Delta\rho_0}{\rho}Q = g_0{}'\,Q,$$

Then there is a length scale defined as

$$l_B = \frac{Q^{3/5}}{B^{1/5}}$$

$$l_B = \frac{Q^{3/5}}{B^{1/5}} = \frac{Q^{3/5}}{(g_0{}'\,Q)^{1/5}} = \frac{(L^3/t)^{2/5}}{(L/t^2)^{1/5}} = \frac{L^{6/5}/t^{2/5}}{L^{1/5}/t^{2/5}} = L$$

→ distance from the flow source at which buoyancy influence the flow

[Cf] For pure jet,

$$l_Q = \frac{Q}{M^{1/2}}$$

• Summary → Table 2.3

-The <u>round plume</u> results are based on experiments by Rouse et al. (1952).

→ no consideration of turbulent fluxes

→ needs adjustment based on better understanding

-The <u>plane plume</u> results are based on experiments by Kotsovinos (1975).

→ Turbulent transport of tracer in the axial direction is not negligible.

→ flux of turbulent transport ~ 35%

[Cf] For simple jet, flux of turbulent transport ~ 17%

Table 2.3 Summary of Plume Properties

Parameter	Round plume	Plane plume
Initial buoyancy flux B	Dimensions L^4T^{-3}	Dimensions L^3T^{-3}
Maximum time-averaged velocity w_m	$w_m = (4.7 \pm 0.2)B^{1/3}z^{-1/3}$	$w_m = 1.66B^{1/3}$
Maximum time-averaged tracer concentration C_m	$C_m = (9.1 \pm 0.5)YB^{-1/3}z^{-5/3}$	$C_m = 2.38YB^{-1/3}z^{-1}$
Volume flux μ	$\mu = (0.15 \pm 0.015)B^{1/3}z^{5/3}$	$\mu = 0.34B^{1/3}z$
Velocity scale of half-width b_w/z	0.100 ± 0.005	0.116 ± 0.002
Concentration scale of half-width b_T/z	0.120 ± 0.005	0.157 ± 0.003
Ratio C_m/C_{av}	1.4 ± 0.2	0.81 ± 0.1

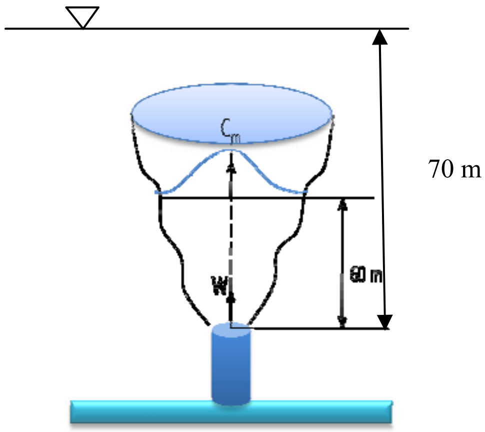
[Example 2.2]

A freshwater discharge (plume) into the sea:

$$Q = 1 \text{ m}^3/\text{s}, \quad C_o = 1 \text{ kg/m}^3, \quad d = 70 \text{ m}, \quad z = 60 \text{ m}$$

Plume: $T_d = 17.8\ ^oC$, fresh water, $C_0 = 10^3$ ppm

Sea: $T_a = 11.1\ ^oC$, salinity = 32.5 ‰ (per thousand) $\rightarrow$ unstratified ambient



[Re] Density of seawater (App. A, p. 443)

Density is a function of temperature, salinity, and pressure.

Density is usually expressed in σ - units.

$$\sigma_t = (\rho - 1)\,1000$$

$$\therefore \rho = 1 + \frac{\sigma_t}{1000} \qquad \text{(A1-1)}$$

where ρ = density (gm/cc)

[Re] 1 gm/cc = 10^6 mg/l

Empirical equations have been developed to calculate σ_t as function of temperature *T* and salinity *S*.

Fig. A.1 can be used to obtain an approximate density over the whole range of temperatures and salinities.

→ The dependence of σ_t on salinity is <u>nearly linear</u> while the dependence on temperature is much more <u>nonlinear</u>.

For a small range of *S*, use linear interpolation (approximation)

$$\sigma_t'(T,S) = \sigma_t(T,S_0) + \frac{\partial \sigma_t}{\partial S}(T,S_0)[S - S_0] \qquad \text{(A1-2)}$$

where S_0 = reference salinity (34 ‰)

Correction is needed.

$$\sigma_t(T,S) = \sigma_t'(T,S) + \Delta\sigma_t \qquad \text{(A1-3)}$$

Procedure:

(1) Use Table A.2 to obtain

$$\sigma_t(T,34), \quad \frac{\partial \sigma_t}{\partial S}(T,34)$$

(2) Calculate $\sigma_t'(T,S)$ by (A1-2)

(3) Use Fig. A.2 to find $\Delta\sigma_t$

(4) Calculate $\sigma_t(T,S)$ by (A1-3)

(5) Calculate ρ by (A1-1)

[Sol]

(i) Plume water: $T_d = 17.8\ ^oC$ ← fresh water

From Table A.1: $\sigma_t = -1.343$

$$\therefore\ \rho_d = 1 + \frac{-1.343}{1000} = 0.9986 \text{ gm/cc} = 998.6 \text{ kg/m}^3$$

(ii) Seawater: $T_a = 11.1\ ^oC$, S = 32.5 ‰

Table A.2: $\sigma_t\,(T, 34) = 25.9992$; $\frac{\partial \sigma_t}{\partial S}(T, 34) = 0.77667$

Eq. (A1-2): $\sigma_t'\left(T, S\right) = 25.9992 + 0.77667\,[32.5 - 34] = 24.8342$

Fig. A.2: $\Delta\,\sigma_t = 0.0007$

Eq. (A1-3): $\sigma_t(T, S) = 24.8342 + 0.0007 = 24.835$

Eq. (A1-1): $\rho_a = 1 + \frac{24.835}{1000} = 1.0248 \text{ g/cc} = 1024.8 \text{ kg/m}^3$

$$\Delta\,\rho_0 = \rho_{a_0} - \rho_d = 26.2 \text{ kg/m}^3$$

$$g_0' = g\frac{\Delta\,\rho_0}{\rho_d} = 9.8\frac{26.2}{998.6} = 0.257 \text{ m/s}^2$$

$$B = g_0'\,Q = 0.257\,(1) = 0.257 \text{ m}^4/\text{s}^3$$

•Mass flux of tracer,

$$Y = QC_0 = 1(1) = 1 \text{ kg/s}$$

•Use (2.34) to calculate centerline concentration

$$C_m = 9.1 Y B^{-1/3} z^{-5/3}$$

$$C_m = 9.1\,(1)\,(0.257)^{-1/3}\,(60)^{-5/3} = 0.016 \text{ kg/m}^3 = 16 \text{ ppm}$$

[Cf] For momentum jet, $C_m = 54$ ppm

•Use (2.30) to calculate volume flux

$$\mu = 0.15\, B^{1/3} z^{5/3}$$

$$\mu = 0.15\,(0.257)^{1/3}(60)^{5/3} = 87.7 \text{ m}^3\text{/s}$$

•Mean dilution,

$$\frac{\mu}{Q} = \frac{87.7}{1} = 87.7$$

[Cf] For momentum jet: $\frac{\mu}{Q} = 26$

$$C_{av} = \frac{C_0}{\mu / Q} \rightarrow C_{av} = \frac{1000}{87.7} = 11.4 \text{ ppm}$$

• Minimum dilution

$$\frac{C_0}{C_{av}} = \frac{1000}{16} = 62.5$$

$$\rightarrow \frac{C_m}{C_{av}} = \frac{16}{11.4} = 1.4$$

Homework Assignment #2-1

Due: 1 week from today

1. Compare mean dilution of round jet and plane jet using same data given in Example 2.1.

$Q = 1 \text{ m}^3/\text{s}, \quad C_0 = 1 \text{ kg/m}^3, \quad A = \frac{1}{3} \text{ m}^2$ for single round jet

For plane jets, $L = 3.3 \text{ m}$

a) Plot S_{av} vs z

b) Change L, and plot S_{av} vs z

2. Compare mean dilution of single round plume and plane plume using same data given in Example 2.2.

$Q = 1 \text{ m}^3/\text{s}, \quad C_0 = 1 \text{ kg/m}^3, \quad A = \frac{1}{3} \text{ m}^2$ for both round and plane jets

$L = 3.3 \text{ m}, \quad \rho_a = 1024.8 \text{ kg/m}^3, \quad \rho_d = 998.6 \text{ kg/m}^3$

a) Plot S_{av} vs z

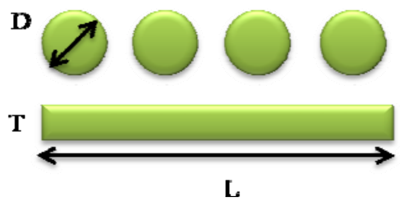
b) Change L, and plot S_{av} vs z

- **Comparison of Mean dilution**

For given discharge, if discharge area is the same for multiple round and plane plumes,

$$\mu = 0.15\,B^{1/3} z^{5/3} \quad \text{, round plume}$$

$$\mu' = 0.34\,B^{1/3} z^{1}, \quad \text{plane plume}$$



1) Multiple round plume: <u>no merge</u> → analyze as a sinlge plume

$$A_t = A_0 \times n = \frac{\pi}{4} D^2 \times n \;,\; Q = A_0 W = \frac{Q_t}{n} = \frac{A_t W}{n} \;,\; B = g' Q$$

$$S_{av} = \frac{\mu}{Q} = 0.15 \frac{B^{1/3} z^{5/3}}{Q} = 0.15 \frac{(g' Q)^{1/3} z^{5/3}}{Q} = 0.15 \frac{g'^{1/3} z^{5/3}}{Q^{2/3}}$$

2) Plane plume: analyze as a whole slot

$$Q_t = A_t W = L T W$$

$$q = \frac{Q_t}{L} = T W \;,\; B = g' q$$

$$\mu' L = 0.34 (g' q)^{1/3} L z = 0.34\, g'^{1/3} (q L)^{1/3} z L^{2/3} = 0.34\, g'^{1/3} Q_t^{1/3} z L^{2/3}$$

$$S_{av} = \frac{\mu' L}{Q_t} = 0.34 \frac{g'^{1/3} Q_t^{1/3} z L^{2/3}}{Q_t} = 0.34 \frac{g'^{1/3} z L^{2/3}}{Q_t^{2/3}} = 0.34 \frac{g'^{1/3} z L^{2/3}}{(nQ)^{2/3}}$$

•Point where the dilution of the round plume equals the dilution of the plane plume is

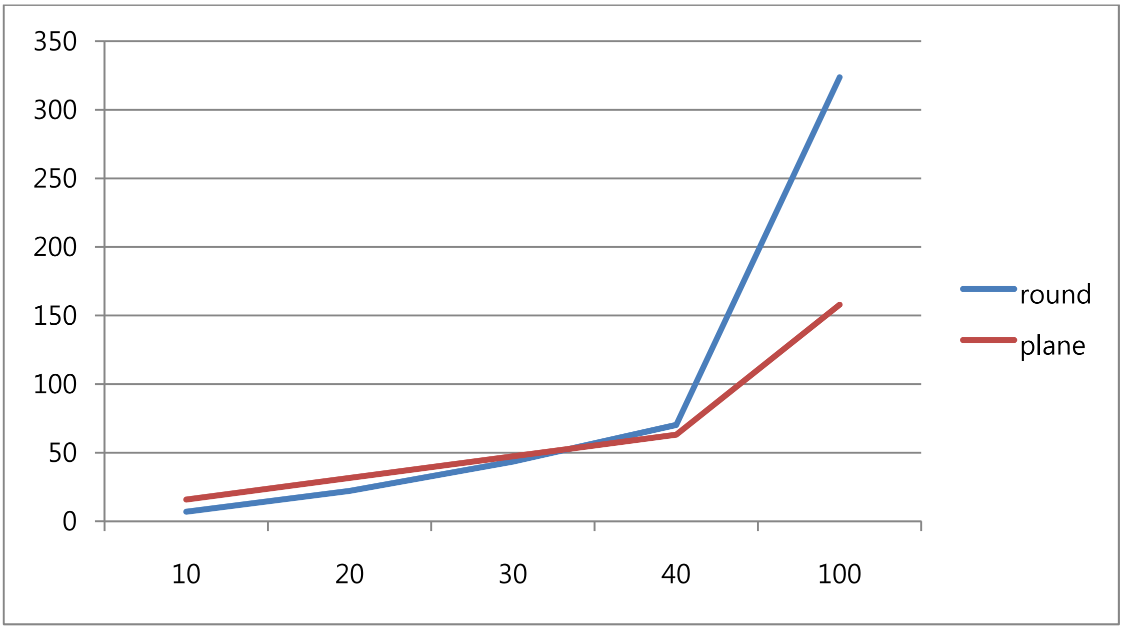
$$\rightarrow 0.15\, z^{5/3} = 0.34\, z\, (L/n)^{2/3}$$

$$\therefore \quad z = 3.41\frac{L}{n} \quad \rightarrow \quad \boxed{\frac{z}{L} = \frac{3.41}{n}}$$

(i) n =10; L=100

$$\frac{z}{100} = \frac{3.41}{10} \rightarrow z = 34.1m$$

z	$S_{av}\dfrac{Q^{2/3}}{g'^{1/3}}$	
	round, $0.15\,z^{5/3}$	plane, $0.34\,\dfrac{z\,L^{2/3}}{n^{2/3}}$
10	6.97	15.79
20	22.17	31.58
30	43.50	47.37
40	70.26	63.16
100	323.71	157.9



- **Comparison of Entrainment Perimeter (Capacity)**

1) Multiple round plume (no merge)

b = half width ≈ 0.1 z

$$E.P. = 2\pi b \times n = 2\pi\,(0.1\,z) \times n = 0.628\,z \times n$$

$$E.P./L = 0.628\,n\,(z/L)$$

b= 0.116 z

2) Plane plume

$$E.P. = 2(L+b) = 2(L+0.116\,z) = 2L + 0.232\,z$$

$$E.P./L = 2 + 0.232\,(z/L)$$

- Point where the entrainment perimeter is the same is

$$\rightarrow\ 0.628\,z/L = 2 + 0.232\,z/L$$

$$\therefore\ z/L = \frac{2}{(0.628\,n - 0.232)}$$

(i) n=10

z/L	$E.A._{round}/L$	$E.A._{plane}/L$
0.1	0.628	2.023
0.2	1.256	2.046
0.3	1.884	2.070
0.4	2.512	2.093
0.5	3.140	2.116

	Entrainment	Dilution
n=10;	$z/L = \dfrac{2}{(0.628\times 10-0.232)} = 0.331$	$z/L = 3.41\dfrac{L}{10} = 0.341\,L$
n=20;	$z/L = \dfrac{2}{(0.628\times 20-0.232)} = 0.162$	$z/L = 3.41\dfrac{L}{20} = 0.171\,L$

• Merging Height

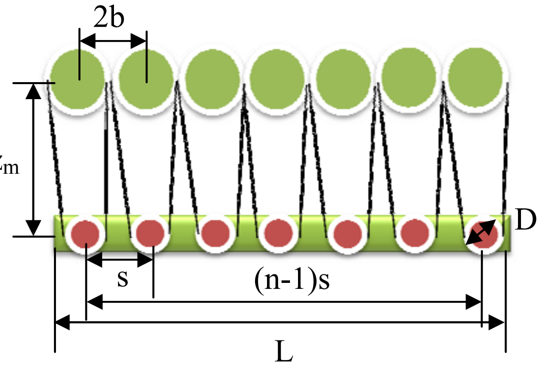
Assume multiple round plumes merge when half widths of the individual plume get in touch together.

Let the multiple round port as the equivalent slot of width, B, and length, L

$$BL = nA_0 = n\times\frac{\pi}{4}D^2$$

$$L = (n-1)s + D \approx ns$$

$$s \approx \frac{L}{n} \qquad (1)$$



•Point where merging begins

$$\rightarrow\ 2b = s \quad \rightarrow\ 2(0.1\,z_m) = s$$

$$\therefore\ z_m/s = 5 \qquad (2)$$

Substitute (1) into (2)

$$\therefore\ \frac{z_m}{(L/n)} = 5$$

$$\frac{z_m}{L} = \frac{5}{n} \qquad (3)$$

	Merging Height	Equi-dilution Height
n=10;	$z/L = \frac{5}{10} = 0.50$	$z/L = \frac{3.41}{10} = 0.341$
n=20;	$z/L = \frac{5}{20} = 0.25$	$z/L = \frac{3.41}{20} = 0.171$
n=50;	$z/L = \frac{5}{50} = 0.1$	$z/L = \frac{3.41}{50} = 0.068$

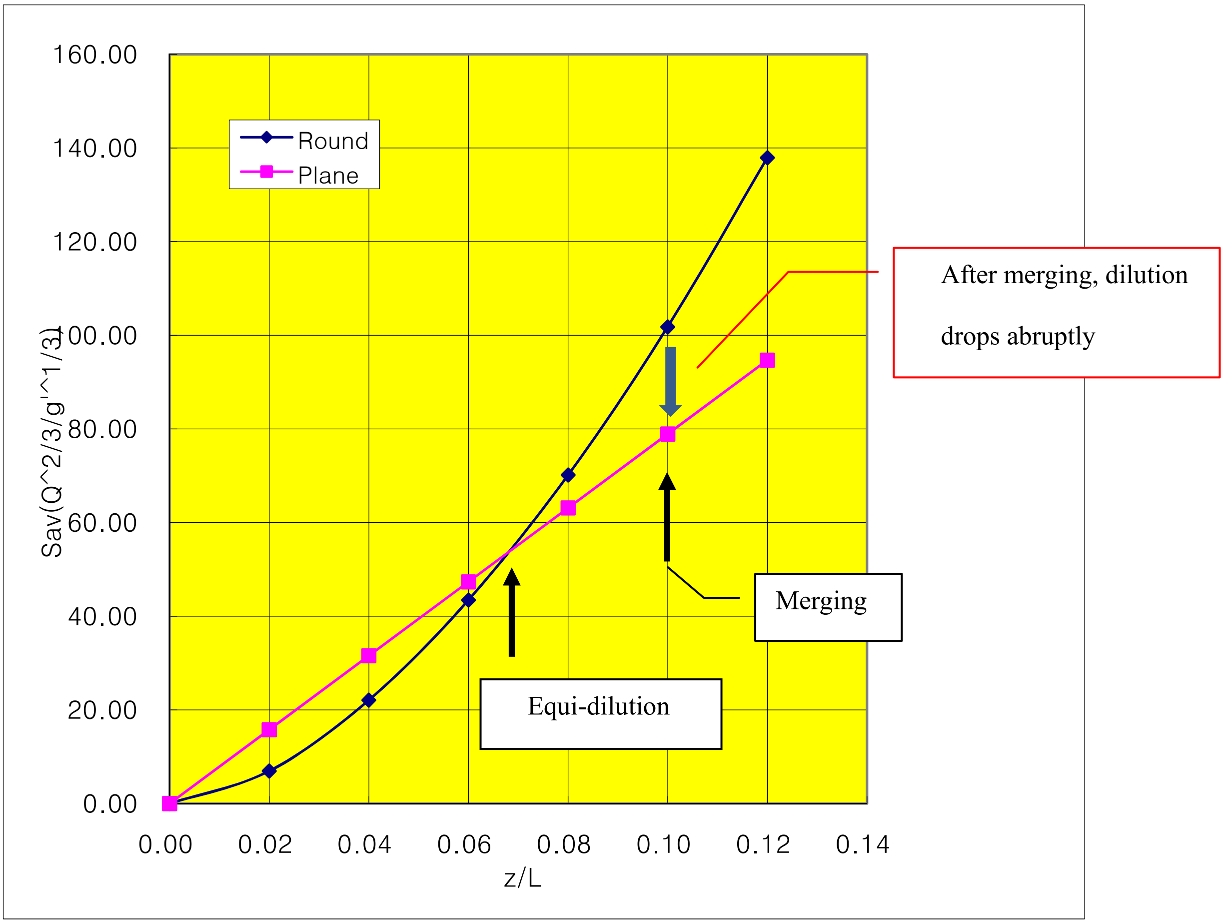
Dilution of Multiple Round Plumes and Plane Plume

n	50	**D**	0.3 m
L	500 m	B	0.071 m
s	10 m	A	3.534 m^2
Q	$3m^{3/s}$	W	0.849 m/s

		Round	**Plane**
z(m)	**z/L**	**$0.15z^{5/3}$**	**$0.34z(L/n)^{2/3}$**
0	0.00	0.00	0.00
10	0.02	6.96	15.78
20	0.04	22.10	31.56
30	0.06	43.45	47.34
40	0.08	70.18	63.13
50	0.10	101.79	78.91
60	0.12	137.94	94.69
70	0.14	178.34	110.47
80	0.16	222.80	126.25
90	0.18	271.12	142.03
100	0.20	323.17	157.81

	Eqi-Dilution	Merging
	3.41/n	5/n
z/L	0.068	0.100

Dilution for *n*=50



2.2.3 Buoyant Jets

Driven by both momentum and buoyancy

- Definition of buoyant jet

= a jet whose density initially differs by an amount $\Delta \rho_0$ from the density of the receiving water

positive or negative buoyancy

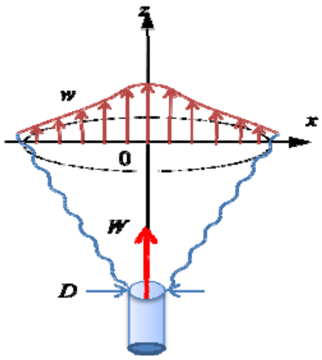
→ A buoyant jet has jet like characteristics depending on its initial volume and momentum fluxes, and plume like characteristics depending on its initial buoyancy flux.

→ Far enough from the source, a buoyant jet will always turn into a plume.

1) Centerline velocity

Consider round buoyant jet that is discharged vertically upward which slightly less dense than its surroundings

~ If the receiving water is stagnant, and homogeneous, then the only parameters that can determine the flow in the jet or plume is $Q,\ M,\ B,\ z$



Apply dimensional analysis

→ two independent dimensionless parameters

$$\frac{M^{1/2} z}{Q}, \quad \frac{B^{1/2} z}{M^{3/4}}$$

Introduce a characteristic length l_M which is a measure of dominancy of B or M

$$l_M = \frac{M^{3/4}}{B^{1/2}} = \frac{[L^4 t^{-2}]^{3/4}}{[L^4 t^{-3}]^{1/2}} = \frac{[L^3 t^{-3/2}]}{[L^2 t^{-3/2}]} = [L]$$

Then $$\frac{B^{1/2} z}{M^{3/4}} = \frac{z}{l_M} \qquad \text{(a)}$$

$$\frac{M^{1/2}z}{Q} = \frac{z}{l_Q} \qquad \text{(b)}$$

Centerline velocity:

$$w_m = f(Q,\ M,\ B,\ z)$$

Flow variables including w_m must be a function of two dimensionless variables (a, b).

$$w_m \frac{M^{1/4}}{B^{1/2}} = \phi(\frac{z}{l_Q},\ \frac{z}{l_M}) \qquad \text{(A)}$$

Suppose flow that has both *M* and *B*, but no initial volume flux *Q*, then solution for w_m must be of the form

$$w_m \frac{M^{1/4}}{B^{1/2}} = f\left(\frac{z}{l_M}\right) = f\left(\frac{z\,B^{1/2}}{M^{3/4}}\right) \qquad \text{(B)}$$

There are two limiting solutions:

i) Close to source: $z/l_M \ll 1$ → jetlike solution

M dominates and B is not important. → B should be vanished.

$$w_m \frac{M^{1/4}}{B^{1/2}} = c_1 \left(\frac{z\,B^{1/2}}{M^{3/4}}\right)^{-1} = c_1 \left(\frac{M^{3/4}}{z\,B^{1/2}}\right)^{1}$$

$$\boxed{w_m = c_1 \frac{M^{1/2}}{z}}$$

[Cf] Recall solution for pure jet

$$w_m \frac{Q}{M} = 7.0 \frac{l_Q}{z}$$

$$\rightarrow \quad w_m = 7.0 \frac{M}{Q} \frac{l_Q}{z} = 7.0 \frac{M}{Q} \frac{Q/M^{1/2}}{z} = 7.0 \frac{M^{1/2}}{z}$$

ii) Far from source: $z/l_M \gg 1 \rightarrow$ plumelike solution

B dominates and M is not important. $\rightarrow$ M should be vanished.

$$w_m \frac{M^{1/4}}{B^{1/2}} = c_2 \left(\frac{M^{3/4}}{zB^{1/2}} \right)^{1/3}$$

$$w_m = c_2 \frac{B^{1/3}}{z^{1/3}}$$

[Cf] Recall plume solution

$$w_m = 4.7 \frac{B^{1/3}}{z^{1/3}} \qquad \leftarrow \text{Table 2.3}$$

• Ratios of characteristic length scales

i) z/l_M :
- $z/l_M \ll 1 \rightarrow$ jetlike solution
- $z/l_M \gg 1 \rightarrow$ plumelike solution

ii) z/l_Q :
- $z/l_Q \sim 1 \rightarrow$ flow is controlled by the jet exit geometry
- $z/l_Q \gg 1 \rightarrow$ flow is fully developed

iii) l_Q/l_M : $l_Q/l_M \sim 1 \rightarrow$ flow is similar to a plume from the outset

→ Jet Richardson number, R_0

$$R_0 = \frac{l_Q}{l_M} = \frac{\dfrac{Q}{M^{1/2}}}{\dfrac{M^{3/4}}{B^{1/2}}} = \frac{Q\,B^{1/2}}{M^{5/4}}$$

$$= \frac{Q(g_0'\,Q)}{M^{5/4}} = \frac{\frac{\pi}{4}D^2W(g_0'\,\frac{\pi}{4}D^2\,W)^{1/2}}{\left(\frac{\pi}{4}D^2\,W^2\right)^{5/4}} = \left(\frac{\pi}{4}\right)^{1/4}\frac{\sqrt{g_0'\,D}}{W} = \left(\frac{\pi}{4}\right)^{1/4}\frac{1}{F_d}$$

where F_d = jet densimetric Froude number $= \dfrac{W}{\sqrt{g_0'\,D}}$

Momentum/buoyancy

→ $0 \le R_0 \le 1$

[Cf] plume Richardson number, $R_p = \dfrac{\mu B^{1/2}}{m^{5/4}}$

2) Asymptotic solution for buoyant jet

Introduce new variables, $\bar{\mu}$, dimensionless jet volume flux

$$\bar{\mu} \equiv \frac{\mu B^{1/2}}{R_p M^{5/4}} = \frac{\mu}{Q}\frac{\left(\dfrac{QB^{1/2}}{M^{5/4}}\right)}{R_p} = \frac{\mu}{Q}\left(\frac{R_0}{R_p}\right) \tag{2.41}$$

Define new dimensionless distance from jet orifice, ζ as

$$\zeta \equiv \frac{c_p}{R_p}\frac{z}{l_M} = c_p\frac{z}{l_Q}\frac{(l_Q/l_M)}{R_p} = c_p\frac{z}{l_Q}\left(\frac{R_0}{R_p}\right) \tag{2.42}$$

Then Eq. (2.41) for the jet volume flux becomes

i) For pure jet

Eq. (2.18): $\frac{\mu}{Q} = 0.25\frac{z}{l_Q}$, for $\frac{z}{l_M} \ll 1$ (a)

Eq. (2.41): $\frac{\mu}{Q} = \bar{\mu}\frac{R_p}{R_o}$ (b)

Eq. (2.42): $\frac{z}{l_Q} = \frac{1}{c_p}\frac{R_p}{R_o}\zeta$ (c)

Substituting (b) & (c) into (a) yields

$$\bar{\mu}\frac{R_p}{R_o} = 0.25\left(\frac{1}{c_p}\frac{R_p}{R_o}\zeta\right) \quad \text{(d)}$$

$$c_p = b_3 \big/ \sqrt{b_2} = 0.15 \big/ \sqrt{0.35} = 0.25$$

Substitute $c_p = 0.25$ into (d), then we get

$$\bar{\mu} = \zeta, \quad \zeta \ll 1 \quad (2.43)$$

ii) For pure plume, $\zeta \gg 1$

Eq. (2.30): $\mu = b_3 B^{1/3} z^{5/3}$ (A)

Eq. (2.41): $\mu = \bar{\mu} Q \frac{R_p}{R_0}$ (B)

Eq. (2.42): $z = \frac{1}{c_p} l_Q \frac{R_p}{R_0}\zeta$ (C)

Substituting (B) & (C) into (A) gives

$$\bar{\mu} Q \frac{R_p}{R_0} = b_3 B^{1/3} \left(\frac{1}{c_p} l_Q \frac{R_p}{R_0} \zeta \right)^{5/3}$$

$$\bar{\mu} = \frac{b_3 B^{1/3}}{Q} \frac{l_Q^{\,5/3}}{c_p^{\,5/3}} \frac{R_p^{\,2/3}}{R_o^{\,2/3}} \zeta^{5/3} = \frac{b_3 R_p^{\,2/3}}{c_p^{\,5/3}} \frac{B^{1/3} l_Q^{\,5/3}}{Q\, R_0^{\,2/3}} \zeta^{5/3}$$

$$= \frac{b_3 R_p^{\,2/3}}{c_p^{\,5/3}} \frac{B^{1/3} \dfrac{Q^{5/3}}{M^{5/6}}}{Q \dfrac{Q^{5/3} B^{1/3}}{M^{5/6}}} \zeta^{5/3}$$

$$\boxed{\bar{\mu} = \frac{b_3 R_p^{\,2/3}}{c_p^{\,5/3}} \zeta^{5/3}} \qquad \text{(D)}$$

Recall

$$\mu = b_3 B^{1/3} z^{5/3} \qquad (2.30)$$

$$m = b_2 B^{2/3} z^{4/3} \qquad (2.29)$$

$$c_p = b_3 / \sqrt{b_2} \qquad (2.31)$$

Eq. (2.29): $m^{4/5} = b_2^{\,5/4} B^{5/6} z^{5/3}$ (E)

Divide Eq. (2.30) by Eq. (E)

$$\frac{\mu}{m^{5/4}} = \frac{b_3 B^{1/3} z^{5/3}}{b_2^{\,5/4} B^{5/6} z^{5/3}} = \frac{b_3}{b_2^{\,5/4}} \frac{1}{B^{1/2}}$$

$$\frac{b_3}{b_2^{\,5/4}} = \mu \frac{B^{1/2}}{m^{5/4}} = R_p \qquad \text{(F)}$$

Substitute (F) and Eq. (2.31) into (D)

$$\frac{b_3 R_p^{\ 2/3}}{c_p^{\ 5/3}} = \frac{b_3\left(b_3^{\ 2/3} / b_2^{\ 5/6}\right)}{(b_3^{\ 5/3} / b_2^{\ 5/6})} = 1$$

Then (D) becomes

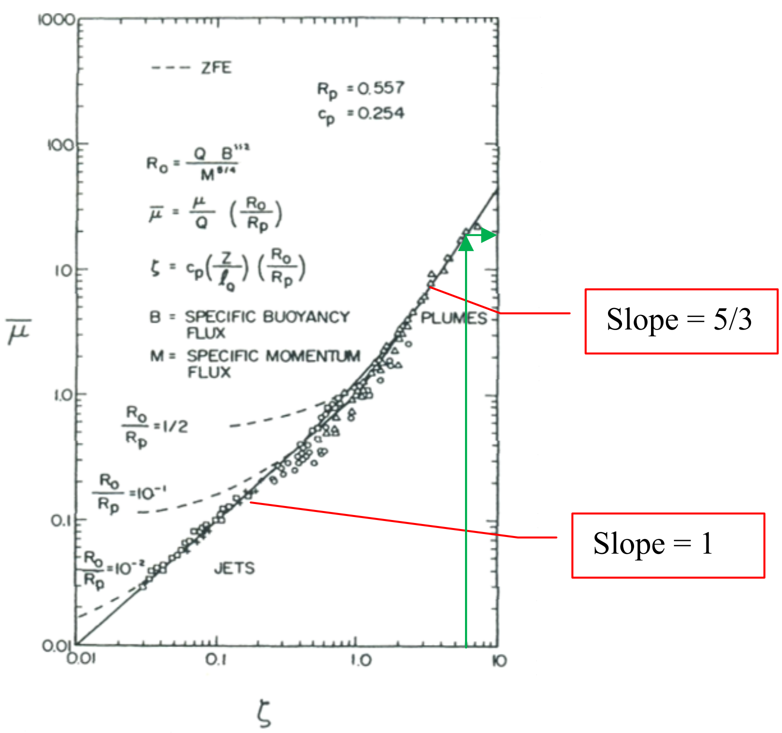
$$\bar{\mu} = \zeta^{5/3}, \qquad \zeta \gg 1 \tag{2.44}$$

[Re] Derivation of (2.30) from (2.44)

$$\frac{\mu}{Q}\left(\frac{R_0}{R_p}\right) = \left\{c_p \frac{z}{l_Q}\frac{R_0}{R_p}\right\}^{5/3}$$

$$\frac{\mu}{Q} = \frac{c_p^{\ 5/3}}{R_p^{\ 2/3}}\left(\frac{z}{l_Q}\right)^{5/3} R_o^{\ 2/3} = \frac{(0.25)^{5/3}}{(0.557)^{2/3}}\frac{z^{5/3}}{(Q / M^{1/2})^{5/3}}\left(\frac{Q\,B^{1/2}}{M^{5/4}}\right)^{2/3}$$

$$= 0.15\frac{B^{1/3} z^{5/3}}{Q}$$



[Example 2.3]

A freshwater is discharged with velocity of $3\ \text{m/s}$ vertically upward into the sea. In this case the flow is a buoyant jet with $Q = 1\ \text{m}^3/\text{s}$, $M = 3\ \text{m}^4/\text{s}^2, B = 0.257\ \text{m}^4/\text{s}^3$.

[Sol]

$$l_Q = \frac{Q}{M^{1/2}} = \frac{1}{\sqrt{3}} = 0.577\ \text{m}$$

$$l_M = \frac{M^{3/4}}{B^{1/2}} = \frac{(3)^{0.75}}{\sqrt{0.257}} = 4.50\ \text{m} \rightarrow \text{Flow becomes plumelike very quickly.}$$

$$R_o = \frac{l_Q}{l_M} = \frac{0.577}{4.50} = 0.128$$

$$R_p = 0.557 \rightarrow \frac{R_o}{R_p} = \frac{0.128}{0.557} = 0.23$$

Eq. (2.42):

$$\zeta\,|_{z=60} = 0.25\frac{z}{l_Q}\left(\frac{R_o}{R_p}\right) = 0.25\left(\frac{60}{0.577}\right)\left(\frac{0.128}{0.557}\right) = 6.0$$

From Fig. 2.7 or Eq. (2.44)

$$\bar{\mu} = \zeta^{5/3} = (6.0)^{5/3} \cong 20$$

Eq. (2.41):

$$\frac{\mu}{Q} = \bar{\mu}\left(\frac{R_p}{R_o}\right) = 20\left(\frac{0.557}{0.128}\right) = 87$$

~ identical with plume results; plume behavior assumption in Exam. 2.2 seems to be O.K.

[Cf] Comparing this result with that in Exam. 2.1., we see that a jet with buoyancy gets significantly more dilution than pure jet ($87 / 26 \approx 3.5$).

3) Planar buoyant jets

→ use dimensionless normalized variables for 2-D flow

$$\bar{\mu} = \frac{\mu B^{1/3}}{R_p^{\ 1/2} M} = \frac{\mu}{Q}\left(\frac{R_0}{R_p}\right)^{1/2} \tag{2.45}$$

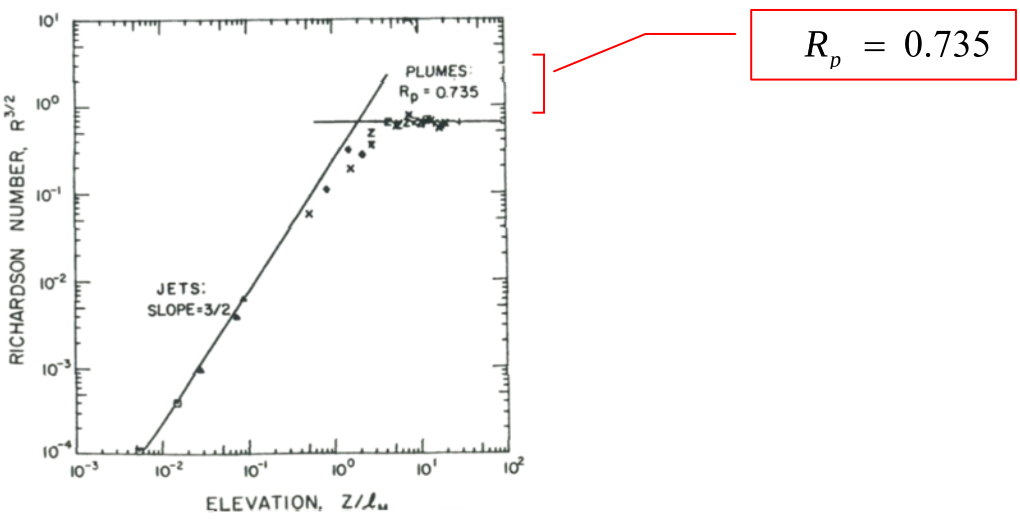
$$\zeta = \frac{c_p}{R_p}\frac{z}{l_M} = \frac{c_p z\, B^{2/3}}{R_p M} = c_p\left(\frac{z}{l_Q}\right)\left(\frac{R_0}{R_p}\right) \tag{2.46}$$

where R_p = plume Richardson number which is asymptotic value of local Richardson number for a plane buoyant jet, $R = \dfrac{\mu^2 \beta^{2/3}}{m^2}$

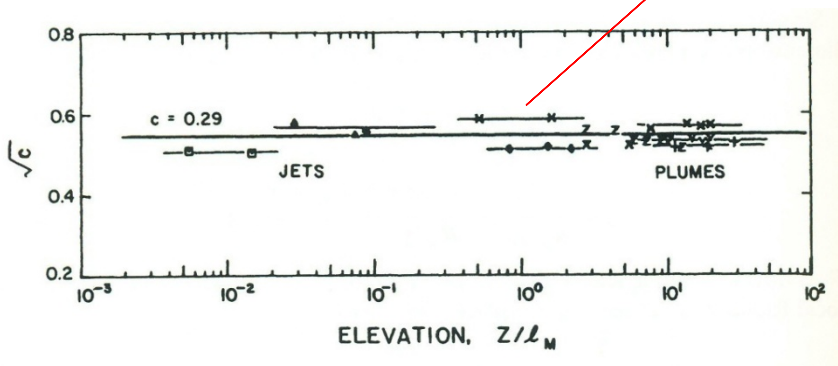
$$R_0 = \frac{Q^2 B^{2/3}}{M^2}$$

c_p = asymptotic value of the width parameter $c = \dfrac{\mu^2}{mz}$

• Kotsovinos (1975): experimental results for R and c → Fig. 2.8 & Fig. 2.9



$c_p = 0.29$



- Mean dilution in turbulent buoyant plane jets and plumes

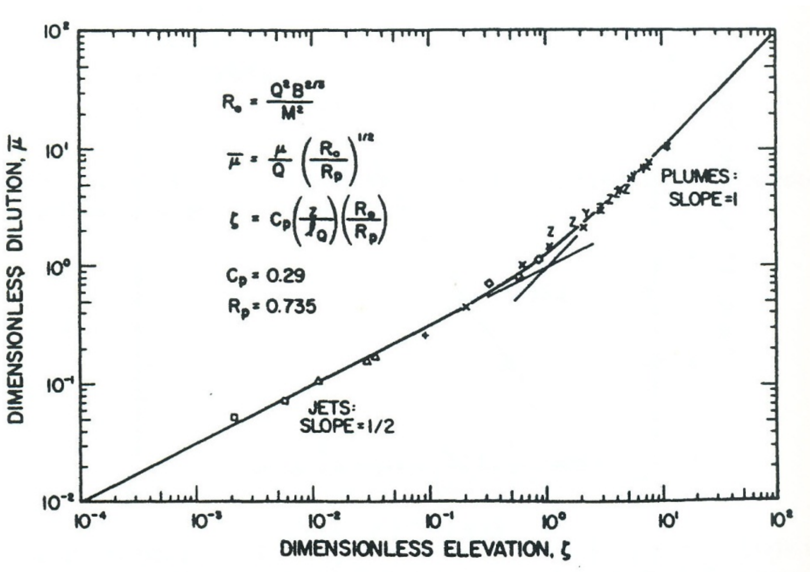
$$\bar{\mu} = \zeta^{1/2}, \quad \zeta \ll 1 \tag{2.49}$$

$$\bar{\mu} = \zeta, \quad \zeta \gg 1 \tag{2.50}$$

[Re] Dilution of plane jets and plumes (Table 2.2 & 2.3)

Plane jet: $\dfrac{\mu}{Q} = 0.5\left(\dfrac{z}{l_Q}\right)^{1/2}$

Plane plume: $\mu = 0.34B^{1/3}z$



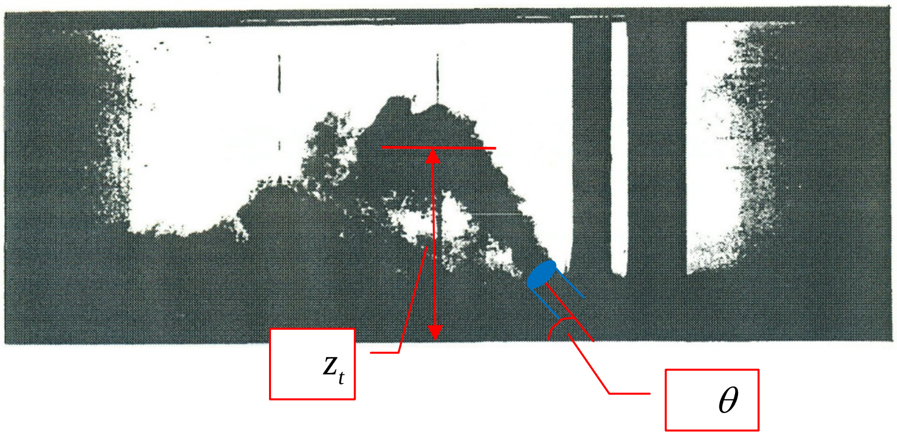
2.2.4 Angle of Jet Inclination

In many design configurations it is advantageous to use horizontal jets or jets at other inclinations than the vertical jet.

→ Advantage of horizontal jet vs. vertical jet: If a liquid jet is discharged into a shallow body of water, then to maximize the dilution path length of the jet can be extended by adopting horizontal jets, or jets at other inclinations.

• Negatively buoyant jets

→ avoid having the jet fall back on itself by angling up 60° from horizontal



1) For negatively buoyant jets directed vertically upward, the terminal height of rise is

$$\frac{z_t M^{1/2}}{Q} = f(R_o) \tag{2.52}$$

where z_t = terminal height of rise; R_o = jet Richardson number

$$R_o = \frac{QB^{1/2}}{M^{5/4}} \tag{2.51}$$

Assume the initial volume flux is not important → Q should be vanished.

Then, $f(R_o) \rightarrow (R_o)^{-1}$

$$\frac{z_t}{l_M} = \frac{z_t B^{1/2}}{M^{3/4}} = const. \quad (2.53)$$

Experiments by Abraham (1967), Turner (1966); const. = 1.5~2.1

2) For non-vertical negatively buoyant jets, the terminal height of rise is

$$z_t \sim (M \sin\theta)^{3/4} / B^{1/2} \quad (2.54)$$

where θ = angle from horizontal

• Dilution of inclined jets

$$S = f(\theta, R_0, z)$$

It is not possible to deduce general relationships for the form of the dilution function.

Horizontal buoyant jets rise exponentially with distance from the source.

→ For distance less than l_M from the jet orifice, the simple jet solutions may be used to predict an initial dilution.

→ Sec. 2.4.3 for dilution and trajectory