22   Analytic Functions

22.1   Mapping

$$w = f(z) = u(x,y) + iv(x,y) \qquad (z = x + iy)$$

f : mapping of D into w-plane.
$w_0 = f(z_0)$: the image of z_0 with respect to f.

Example 1. Mapping $w = z^2$

i) polar coordinates: $z = r(\cos\theta + i\sin\theta)$

$w = R(\cos\phi + i\sin\phi) = z^2 = r^2(\cos 2\theta + i\sin 2\theta)$
moduli $R = r^2$, arguments $\phi = 2\theta$.

Fig.306 $\quad \begin{pmatrix} 1 \le |z| \le 3/2 \\ \pi/6 \le \theta \le \pi/3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \le |w| \le 4/9 \\ \pi/3 \le \theta \le 2\pi/3. \end{pmatrix}$

ii) Cartesian coordinates: $z = x + iy$
$u = Re(z^2) = x^2 - y^2, v = Im(z^2) = 2xy.$
$x = c = const. u = c^2 - y^2. v = 2cy. v^2 = 4c^2(c^2 - a).$
$y = k = const. \Rightarrow v^2 = 4k^2(k^2 + u)$

22.2   Conformal Mapping.

Definition : A mapping that preserves angles between any oriented curves both in magnitude and in sense.

Oriented curve :   C : z(t)=x(t)+iy(t) : parametric representation

Example 2. circle : $x^2 + y^2 = 1. \Rightarrow x = \cos t, y = \sin t.$

$$\therefore x(t) = \cos t + i\sin t.$$

Smooth C :
   $z(t)$ is differentiable and the derivative $\dot{z} = dz/dt$ is continuous and nowhere zero.
positive sense :
   increasing values of the parameter t. $\rightarrow$ defines an orientation of C.

angle of intersection α of two curves C_1 and C_2 :

the angle between the oriented tangents at the intersection point.

Theorem 1 (Conformality of mapping by analytic functions).

The mapping defined by an analytic function $f(z)$ is conformal except at critical points, i.e. points at which the derivative $f'(z)$ is zero.

Proof. . $\dot{z}(t) = dz/dt = \dot{x}(t) + i\dot{y}(t)$ is tangent to C.

$$\left(\because \lim_{\Delta t} \frac{z_1 - z_0}{\Delta t} \right)$$

$$C^* : \text{the image of } C.$$

$$w = f(z(t)) \Rightarrow \dot{w} = f[z(t)]\dot{z}(t)$$

$$\text{tangent direction of } C^*$$

$$\arg \dot{w} = \arg \dot{f} + \arg \dot{z}$$

The mapping rotates all directions at a point z_0 in the domain of analyticity of f through the same angle $\arg f'(z_0)$, which exists as long as $f'(z_0) \neq 0$.

Example 3. Conformality of $w = z^n$.

$$w = z^n, n = 2, 3, \cdots \text{conformal except at } z = 0.$$

$$w' = nz^{n-1} \text{ at } z = 0.w' = 0$$

$$0 \leq \theta \leq \pi/n \to^{f(z)} \to v \geq 0.$$

Example 4. Mapping $x = z + 1/z$. Joukowski airfoil.

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$u = (r + \frac{1}{r})\cos\theta, v = (r - \frac{1}{r})\sin\theta.$$

circle :
$|z| = r = \text{const} \neq 1 \to$ ellipses $x^2/a^2 + y^2/b^2 = 1$
$|z| = 1 \to -2 \leq u \leq 2$ of u-axis.

$$w' = 1 - \frac{1}{z^2} = \frac{(z-1)(z+1)}{z^2} \qquad z = \pm 1, w' = 0 : \text{ not conformal.}$$

22.3 Exponential Function.

(1)
$$e^z = e^x(\cos y + i \sin y)$$

(2)
$$(e^z)' = e^z$$

requirements :
(a) e^z should reduce to the real e^x, when $z = x$ is real.
(b) e^z should be an entire function, i.e. analytic for all z.
(c) similar to calculus, its derivative should be

Proof. .
(a) $y = 0, \cos 0 = 1 \,\&\, \sin 0 = 0, e^z = e^x$.
(b) $u = e^x \cos y, v = e^x \sin y$.

$$\frac{\partial u}{\partial x} = e^x \cos y, \frac{\partial v}{\partial y} = e^x \cos y \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \frac{\partial v}{\partial x} = e^x \sin y \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(c) $(e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x = e^x \cos y + ie^x \sin y = e^z$

Further Properties.
(3) functional relation : $e^{z_1+z_2} = e^{z_1} e^{z_2}$ for $z_1 = x_1 + iy_1 \,\&\, z_2 = x_2 + iy_2$

$$
\begin{aligned}
e^{z_1} e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1)e^{x^2}(\cos y_2 + i \sin y_2) \\
&= e^{x_1+x_2}[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)] \\
&= e^{x_1+x_2}[\cos(y_1 + y_2) + i \sin(y_1 + y_2)] = e^{z_1+z_2}
\end{aligned}
$$

$$z_1 = x \,\&\, z_2 = iy;$$

(4)
$$e^z = e^x \cdot e^{iy}$$

(5) Euler formula :
$$e^{iy} = \cos y + i \sin y$$

(6) polar form :
$$x = re^{i\theta}$$

(7)
$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$$

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(8)

$$e^{\pi i/2} = \cos \pi/2 + i \sin \pi/2 = i$$
$$e^{\pi i} = \cos \pi + i \sin \pi = -1$$
$$e^{-\pi i/2} = \cos(-\pi/2) + i \sin(-\pi/2) = -i$$
$$e^{-\pi i} = \cos(-\pi) + i \sin(-\pi) = -1$$

(9) From (5)
$$|e^{iy}| = |\cos y + i \sin y| = (\cos^2 y + \sin^2 y)^{1/2} = 1$$

(10) From (9) and(1)

$$|e^z| = e^x, \quad \arg e^z = y \pm 2n\pi \quad (n = 0, 1, 2, \cdots.)$$

(11)

$$e^z \neq 0 \ , \ \forall z$$

Periodicity of e^z with period $2\pi i$
(12)
$$e^{z+2\pi i} = e^z \ \forall z$$

(13) fundamental region of e^z : $-\pi < y \leq \pi$.
: e^z maps a fundamental region bijectively onto the entire plane.

$0 < y \leq \pi$ of the fundamental strip $\to$ mapped onto the upper half-plane.
$0 < \arg w \leq \pi$

$\left(\begin{array}{l} \text{left half of the strip goes insede the unit disk } |w| \leq 1, \ (\because e^x \leq 1 \text{ for } x \leq 0) \\ \text{right half goes outside, } |x| > 1. \end{array}\right.$

Example 5. Function values. Solution of equations.

(1) $e^z = e^x(\cos y + i \sin y)$

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i \sin 0.6) = 4.055(0.825 - 0.565i) = 3.347 - 2.290i.$$
$$|e^{1.4-0.6i}| = e^{1.4} = 4.055 \arg e^{1.4-0.6i} = -0.6$$

(3) $e^{z_1+z_2} = e^{z_1} \cdot z^{z_2}$

$$e^{2+i} = e^2(\cos 1 + i \sin 1) \& e^{4-i} = e^4(\cos 1 - i \sin 1)$$
$$e^2 \cdot e^4(\cos^2 1 + \sin^2 1) = e^6 = e^{(2+i)+(4-i)}$$

Equation $e^z = 3 + 4i = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

$$e^x = |e^z| = 5, x = \ln 5 = 1.609$$

$$\begin{pmatrix} e^x \cos y = 3 \\ e^x \sin y = 4 \end{pmatrix} \Rightarrow \begin{pmatrix} \cos y = 0.6 \\ \sin y = 0.8 \end{pmatrix} y = 0.927$$

Ans. $z = 1.609 + 0.927i \pm 2n\pi i \ (n = 0, 1, 2, \cdots)$

22.4   Trigonometric Functions. Euler formulas.

$$e^{ix} = \cos x + i \sin x, \quad e^{-ix} = \cos x - i \sin x$$
$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \quad \& \quad \sin z = \frac{1}{2i}(e^{ix} - e^{-ix})$$

(1)
$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin \frac{1}{2i}(e^{iz} - e^{-iz})$$

(2)
$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

(3)
$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

(4)
$$(\cos z)' = -\sin z, \quad (\sin z)' = \cos z, \quad (\tan z)' = \sec^2 z$$

(5) Euler's formula
$$e^{iz} = \cos z + i \sin z, \qquad \text{for all } z.$$

Example 6. Real and Imaginary parts. Absolute value. Periodicity

(6)
$$\text{(a) } \cos z = \cos x \cosh y - i \sin x \sinh y.$$
$$\text{(b) } \sin z = \sin x \cosh y + i \cos x \sinh y.$$

(7)
$$\text{(a) } |\cos z|^2 = \cos^2 x + \sinh^2 y.$$
$$\text{(b) } |\sin z|^2 = \sin^2 x + \sinh^2 y.$$

$$\begin{aligned} \cos z &= \frac{1}{2}[e^{i(x+iy)} + e^{-i(x+iy)}] \\ &= \frac{1}{2}e^{-y}(\cos x + i \sin x) + \frac{1}{2}e^{y}(\cos x - i \sin x) \\ &= \frac{1}{2}(e^{y} + e^{-y})\cos x - \frac{1}{2}i(e^{y} - e^{-y})\sin x. \end{aligned}$$

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(8)

$$\cosh y = \frac{1}{2}(e^y + e^{-y})$$

$$\sinh y = \frac{1}{2}(e^y - e^{-y})$$

$$\therefore \cos z = \cos x \cosh y - i \sin x \sinh y$$

$$
\begin{aligned}
\therefore \sin z &= \frac{1}{2i}[e^{i(x+iy)} - e^{-i(x+iy)}] \\
&= \frac{1}{2i}e^{-y}(\cos x + i \sin x) - \frac{1}{2i}e^{y}(\cos z - i \sin x) \\
&= \frac{1}{2}(e^y + e^{-y})\sin x + \frac{1}{2}i(e^y - e^{-y})\cos x
\end{aligned}
$$

$$\therefore \sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\text{since} \quad \cosh^2 y - \sinh^2 y = \frac{1}{4}(e^{2y} + e^{-2y} + 2 - e^{-2y} + 2) = 1$$

From(6a)

$$
\begin{aligned}
|\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
&= \cos^2 x(1 + \sinh^2 y) + \sin^2 x \sinh^2 y \\
&= \cos^2 x + \sinh^2 y
\end{aligned}
$$

From(6b)

$$
\begin{aligned}
|\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x(1 + \sinh^2 y) + \cos^2 x \sin^2 y \\
&= \sin^2 x + \sinh^2 y
\end{aligned}
$$

For instance,

$$\cos(2 + 3i) = \cos 2 \cdot \cosh 3 - i \sin 2 \cdot \sinh 3 = -4.190 - 9.109i$$

$$\cos z \quad \& \quad \sin z \text{ are periodic with period } 2\pi.$$

$$\tan z \quad \& \quad \cot z \text{ are periodic with period } 2\pi.$$

Example 7. Solution of equations. zeros of $\cos z$ and $\sin z$.

(a)

$$\cos z = 5$$

$$5 = \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\therefore e^{2iz} - 10e^{iz} + 1 = 0$$

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$$e^{iz} = e^{-y+ix} = 5 \pm \sqrt{25 - 1} = 9.899 \text{ and } 0.101$$

$$\therefore e^{-y} = 9.899 \text{ or } 0.101, \quad e^{ix} = 1$$

$$x = 2n\pi \quad y = \pm 2.292i.$$

$$\text{Ans. } z = \pm 2n\pi \pm 2.292i$$

(b)

$$\cos z = 0 \Rightarrow \cos x = 0 \quad \& \quad \sinh y = 0$$

$$\text{Ans. } z = \pm \frac{1}{2}(2n+1)\pi \qquad (n = 0, 1, 2, \cdots)$$

(c)

$$\sin z = 0 \Rightarrow \sin x = 0 \quad \& \quad \sinh y = 0$$

$$\text{Ans. } z = \pm 2n\pi \qquad (n = 0, 1, 2, \cdots)$$

(9)   General formulas

$$\cos(z_1 \pm z_2) = \cos z_1 \cdot \cos z_2 \mp \sin z_1 \cdot \sin z_2$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cdot \cos z_2 \pm \sin z_2 \cdot \cos z_1$$

(10)

$$\cos^2 z + \sin^2 z = 1$$

Proof. $\cos^2 z + \sin^2 z = \dfrac{1}{4}(e^{2iz} + 2 + e^{-2iz}) - \dfrac{1}{4}(e^{2iz} - 2 + e^{-2iz}) = 1$