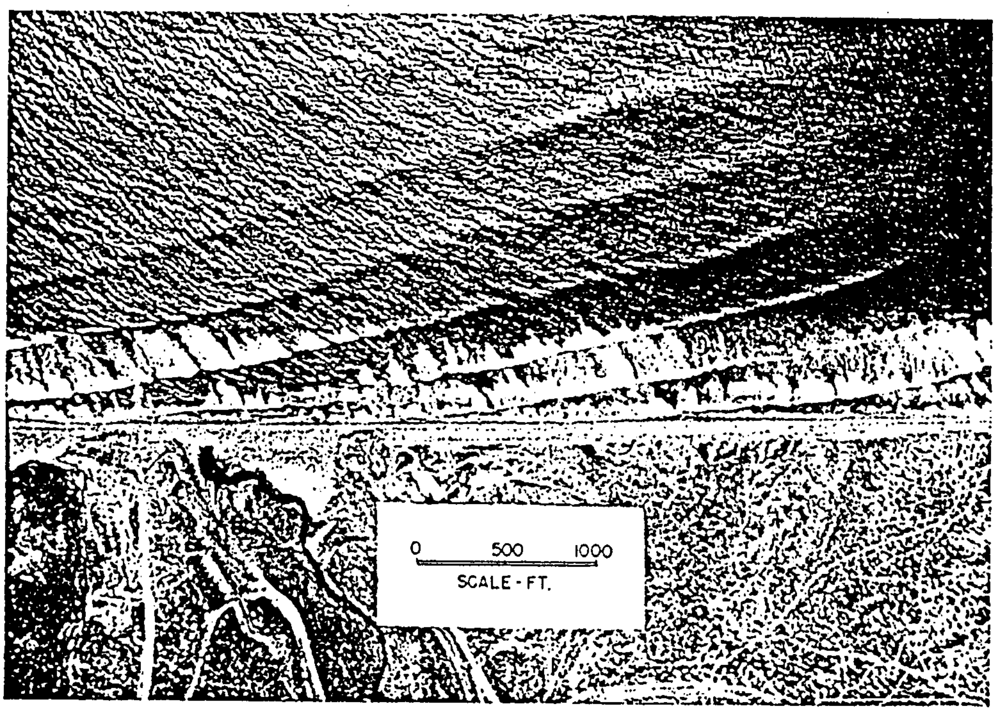
Chap 2. Statistical Properties and Spectra of Sea Waves

2.1 Random Wave Profiles and Definitions of Representative waves

2.1.1 Spatial Surface Forms of Sea Waves

- Long-crested waves: Wave crests have a long extent
  (swell, especially in shallow water)
- Short-crested waves: Wave crests do not have a long extent, but instead consists of
  short segments (wind waves in deep water)

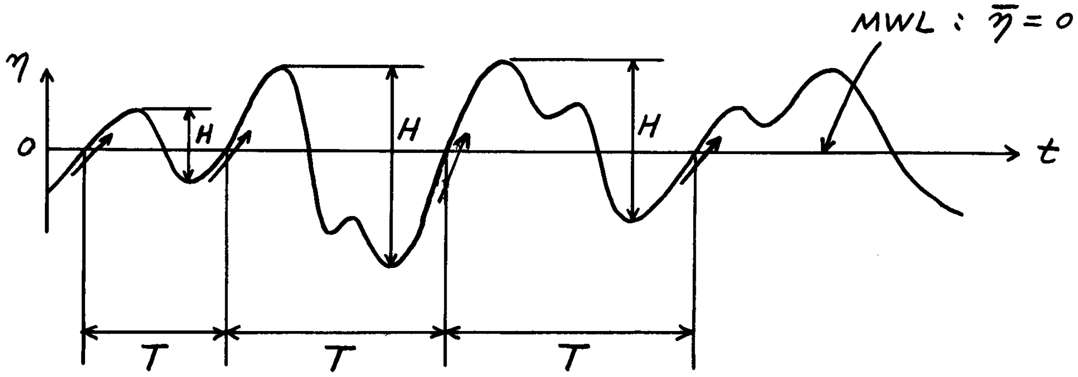


2.1.2 Definition of Representative Wave Parameters

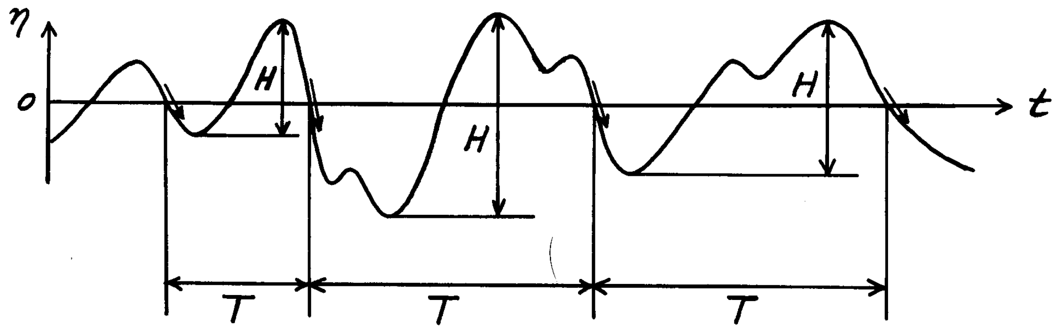
In nature, no sinusoidal wave exists
→ Wave forms are irregular (or random)
→ Difficult to define individual waves
→ Zero-crossing method is used.

Assume that we measured $\eta(t)$ at a point.

Zero-upcrossing:

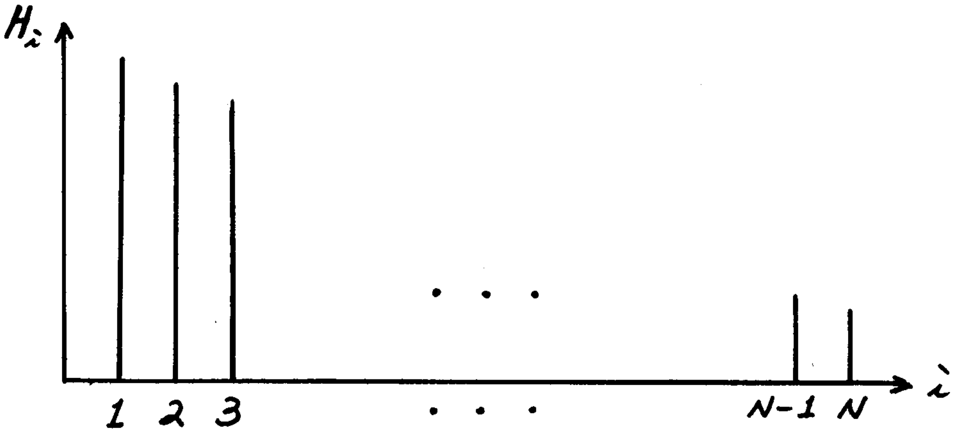


Zero-downcrossing:



Statistically, zero-upcrossing and zero-downcrossing are equivalent if the wave record is long enough. But zero-upcrossing is more commonly used.

Arrange the wave heights in descending order.



(a) Highest wave: $H_{\max}$, $T_{\max}$

Note: $T_{\max}$ is not the maximum wave period in the record, but the wave period corresponding to $H_{\max}$.

(b) Highest one-tenth wave: $H_{1/10}$, $T_{1/10}$

Average of the highest $N/10$ waves

$$H_{1/10} = \frac{1}{N/10} \sum_{i=1}^{N/10} H_i$$

$T_{1/10}$ = average of the wave periods corresponding to the highest $N/10$ waves

(c) Significant wave, or highest one-third wave: $H_{1/3}$, $T_{1/3}$ (or H_s, T_s)

$$H_{1/3} = \frac{1}{N/3} \sum_{i=1}^{N/3} H_i$$

(d) Mean wave: $\overline{H}$ (or H_1), $\overline{T}$

$$\overline{H} = \frac{1}{N} \sum_{i=1}^{N} H_i$$

2.2 Distribution of Individual Wave Heights and Periods

2.2.1 Wave Height Distribution

Gaussian stochastic (linear) theory for narrow-band spectra (range of periods is small) suggests Rayleigh distribution of H_i (discrete) = H (continuous) for large $N \to \infty$.

Probability density function:

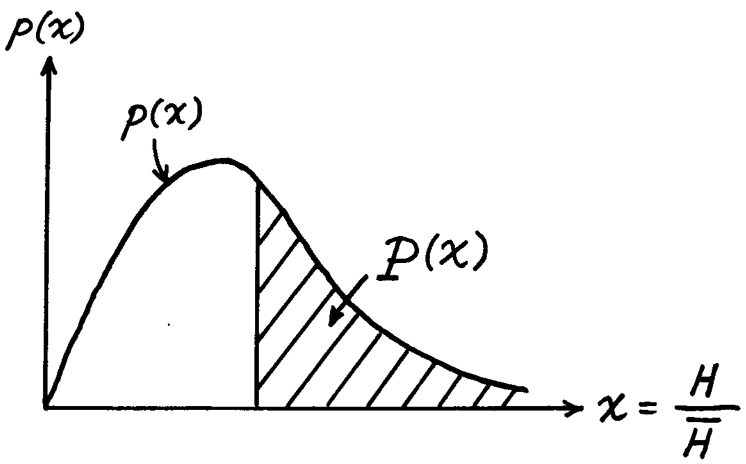
$$p(x) = \frac{\pi}{2} x \exp\left(-\frac{\pi}{4} x^2\right) \quad \text{with} \quad x = \frac{H}{\overline{\overline{H}}}$$

satisfying $\int_0^\infty p(x)dx = 1$

Probability of exceedance:

$$P(x) = \int_x^\infty \frac{\pi}{2} \xi \exp\left(-\frac{\pi}{4}\xi^2\right) d\xi = \exp\left(-\frac{\pi}{4}x^2\right)$$

$$= \text{probability (arbitrary } \tilde{x} = \frac{\tilde{H}}{\overline{\overline{H}}} > \text{given } x)$$



Field data indicate that Rayleigh distribution based on restricted assumptions (linear + narrow-band) yields good agreement with data.

2.2.2 Relations between Representative Wave Heights

Exceedance probability based on Rayleigh distribution

$$P(x_N) = \exp\left(-\frac{\pi}{4}x_N^2\right) = \frac{1}{N}$$

$$x_N = \frac{2}{\sqrt{\pi}}(\ln N)^{1/2} \quad \text{for given} \quad N \quad (\text{e.g.,} \quad N = 3, 10)$$

By definition

$$x_{1/N} = \frac{H_{1/N}}{\overline{H}} = \frac{\int_{x_N}^{\infty} xp(x)dx}{\int_{x_N}^{\infty} p(x)dx} = \frac{1}{1/N}\int_{x_N}^{\infty} xp(x)dx$$

Using $\dfrac{dP}{dx} = -p(x)$ and $P(x) = \exp\left(-\dfrac{\pi}{4}x^2\right) = \int_x^{\infty} p(\xi)d\xi$

$$\begin{aligned}
x_{1/N} &= N\int_{x_N}^{\infty}\left(-x\frac{dP}{dx}\right)dx \\
&= N\left\{[-xP]_{x_N}^{\infty} - \int_{x_N}^{\infty}(-P)dx\right\} \\
&= N\left\{x_N P(x_N) + \int_{x_N}^{\infty} Pdx\right\} \\
&= N\left\{x_N \exp\left(-\frac{\pi}{4}x_N^2\right) + \int_{x_N}^{\infty}\exp\left(-\frac{\pi}{4}x^2\right)dx\right\}
\end{aligned}$$

using complementary error function, $\operatorname{erfc}(x) = \int_x^{\infty} e^{-t^2}dt$,

$$x_{1/N} = N\left\{x_N \exp\left(-\frac{\pi}{4}x_N^2\right) + \int_{\frac{\sqrt{\pi}}{2}x_N}^{\infty}\exp(-t^2)\frac{2}{\sqrt{\pi}}dt\right\}$$

$$x_{1/N} = x_N + \frac{2}{\sqrt{\pi}}N\operatorname{erfc}\left(\frac{\sqrt{\pi}}{2}x_N\right) \qquad \text{with} \quad x_N = \frac{2}{\sqrt{\pi}}(\ln N)^{1/2}$$

See Table 10.1 (p 373) for $H_{1/N} / \overline{H}$ vs $N = 100, 50, 20, 10, \cdots$

$$H_s = H_{1/3} = 1.6\overline{H}\,, \quad H_{1/10} = 1.27 H_s\,, \quad H_{1/100} = 1.67 H_s$$

2.2.3 Distribution of Wave Periods

Not well established.
Local wind waves (~10 s) + Swell (~15 s) $\rightarrow$ two peaks (bi-modal)
or two main direction

Typically, $T_{\max} \cong T_{1/10} \cong T_{1/3} \cong (1.1 \sim 1.3)\overline{T}$

2.3 Spectra of Sea Waves

2.3.1 Frequency Spectra

Free surface oscillation at a point:

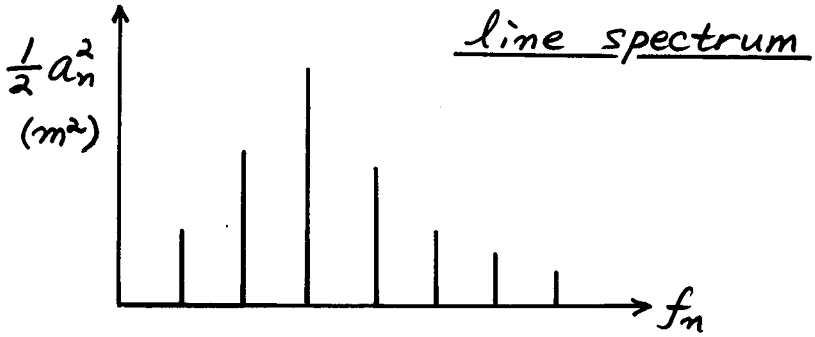
$$\eta(t) = \sum_{n=1}^{\infty} a_n \cos(2\pi f_n t + \varepsilon_n)$$

where

a_n = amplitude of wave with frequency $f_n = \dfrac{1}{T_n}$

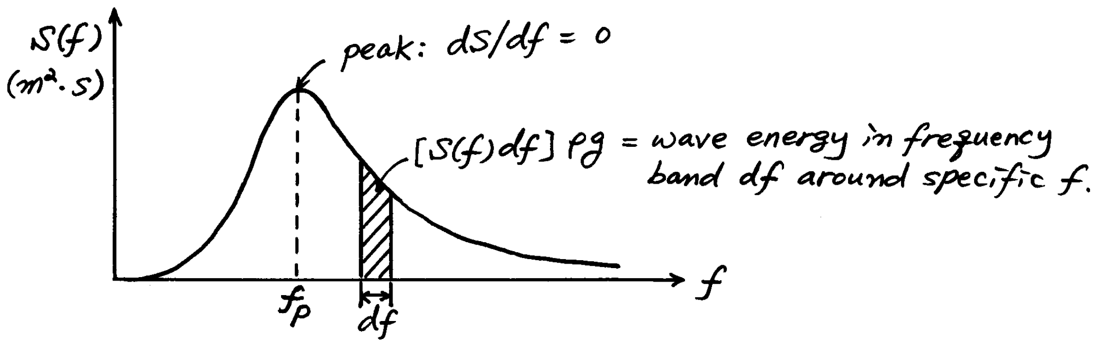
ε_n = phase of wave with frequency f_n

Energy of wave with frequency $f_n = \frac{1}{2}\rho g a_n^2$



Real sea waves $\rightarrow$ infinite number of frequency components

$\rightarrow$ (continuous) frequency spectrum



Note: $\frac{1}{2}a_n^2 = S(f_n)\Delta f$

$a_n = \sqrt{2S(f_n)\Delta f}$

Standard spectra

$\uparrow$

ensemble average of large number of wave records

(1) Bretschneider-Mitsuyasu spectrum

Fully developed wind waves in deep water
(energy input from wind = energy dissipation due to breaking)

$$S(f) = 0.257 H_s^2 T_s^{-4} f^{-5} \exp[-1.03(T_s f)^{-4}] \qquad (2.10)$$

for given H_s and T_s.

Modified by Goda (1988): 0.257→0.205, 1.03→0.75 as in Eq. (2.11).

(2) JONSWAP (JOint North Sea WAve Project) spectrum

Growing wind seas in deep water

$$S(f) = \beta_J H_s^2 T_p^{-4} f^{-5} \exp[-1.25(T_p f)^{-4}] \gamma^{\exp[-(T_p f - 1)^2 / 2\sigma^2]}$$

where

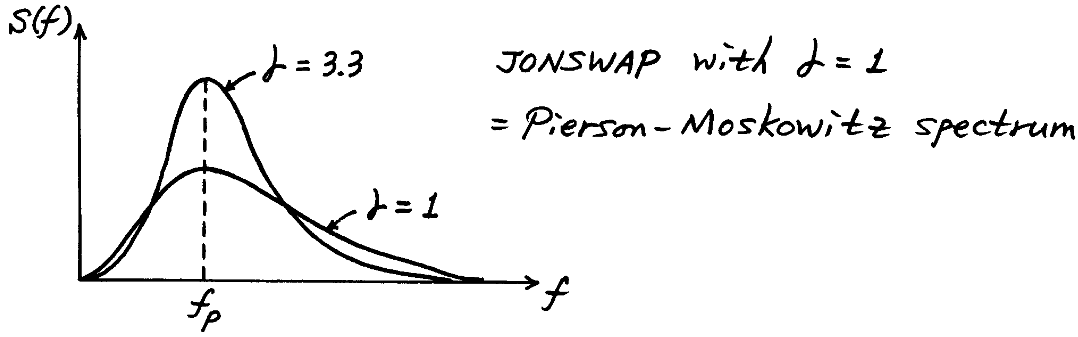
$$\beta_J = \beta_J(\gamma) \qquad (2.13)$$

$$T_p = T_p(T_s, \gamma) \qquad (2.14)$$

$$\sigma = \begin{cases} \sigma_a \approx 0.07 & \text{for } f \le f_p \\ \sigma_b \approx 0.09 & \text{for } f > f_p \end{cases}$$

peak enhancement factor $\gamma = 1 \sim 7$ (typically 3.3)

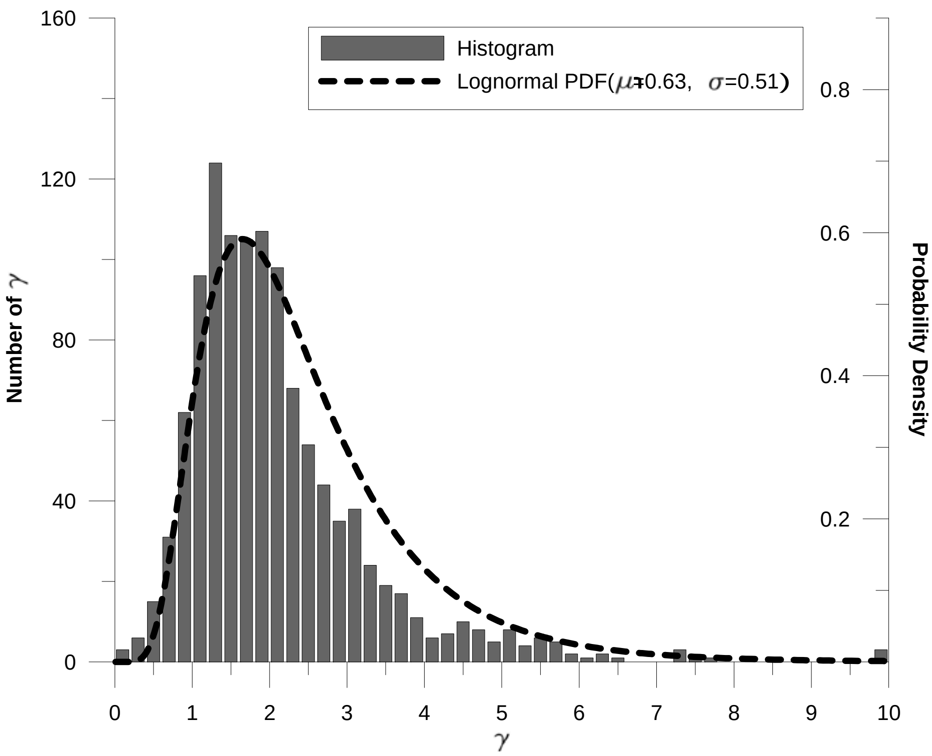
Need to specify H_s, T_s, σ_a, σ_b, and γ (sharper spectral peak as $\gamma \uparrow$).



Suh et al. (2010, Coastal Engineering 57, 375-384) showed that for deepwater waves around the Korean Peninsula, the probability density function of γ is given by a lognormal distribution:

$$p(\gamma) = \frac{1}{0.53\sqrt{2\pi}\gamma} \exp\left\{ -\frac{1}{2}\left[\frac{\ln\gamma - 0.58}{0.53}\right]^2 \right\}$$

with its mean equal to 2.14, which is somewhat smaller than 3.3 in the North Sea.



(3) TMA spectrum (Bouws et al., 1985, J. Geophys. Res., 90, C1)

↓

Includes effects of finite water depth

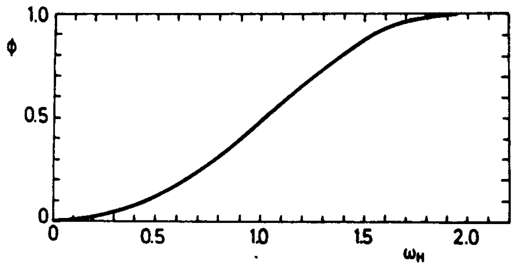
$$S_{TMA} = S_J \phi_k(f, h)$$

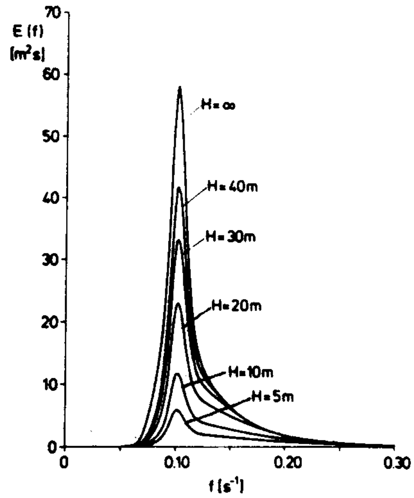
↑

Kitaigordskii shape function (depth effect)

$$\phi_k(f,h) = \begin{cases} 0.5\omega_h^2 & \text{for } \omega_h < 1 \\ 1 - 0.5(2-\omega_h)^2 & \text{for } 1 \le \omega_h \le 2 \\ 1 & \text{for } \omega_h > 2 \end{cases}$$

$$\omega_h = 2\pi f (h/g)^{1/2}$$





2.3.2 Directional Wave Spectra

(1) General

frequency spectrum $\rightarrow$ assumes waves with many different frequencies
but single direction.

However, real sea waves consist of many component waves with different frequency and different direction. Therefore, we need directional wave spectrum:

$$S(f,\theta)=S(f)G(\theta\,|\,f)$$

$\uparrow$

directional spreading function

$\downarrow$

directional distribution of wave energy

$\downarrow$

varies with frequency f .

$$\int_{-\pi}^{\pi}\underbrace{G(\theta\,|\,f)}_{\uparrow}d\theta=1$$

represents relative magnitude of directional spreading of wave energy

$$\therefore \text{ Total energy } = \int_0^\infty\int_{-\pi}^{\pi}S(f,\theta)d\theta df$$

$$= \int_0^\infty\int_{-\pi}^{\pi}S(f)G(\theta\,|\,f)d\theta df$$

$$= \int_0^\infty S(f)df$$

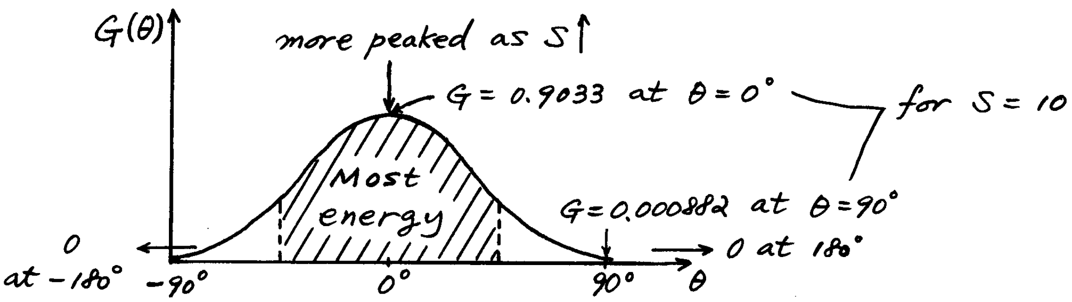
(2) Mitsutasu-type directional spreading function

Based on field measurements,

$$G(\theta \,|\, f) = G_0 \cos^{2s}\left(\frac{\theta}{2}\right) \leftarrow \text{ symmetric about } \theta = 0$$

θ = wave angle from principal direction ($\theta = 0$)

See Fig. 2.11 for the variation of G versus θ.



Intuitively, $G = 0$ for $-180° \le \theta \le -90°$ and $90° \le \theta \le 180°$.

But, G is very small as long as s is large.

Must satisfy $\int_{-\pi}^{\pi} G_0 \cos^{2s}\left(\frac{\theta}{2}\right) d\theta = 1$

Symmetric about $\theta = 0$: $2G_0 \int_0^{\pi} \cos^{2s}\left(\frac{\theta}{2}\right) d\theta = 1$

$$G_0 = \frac{1}{2\int_0^{\pi} \cos^{2s}\left(\frac{\theta}{2}\right) d\theta} = \frac{1}{\pi} 2^{2s-1} \frac{[\Gamma(s+1)]^2}{\Gamma(2s+1)}$$

where Gamma function $\Gamma(n) = (n-1)!$ for integer n.

s depends on frequency f :

$$s = \begin{cases} s_{\max}(f/f_p)^5 & \text{for } f \le f_p \\ s_{\max}(f/f_p)^{-2.5} & \text{for } f > f_p \end{cases}$$

where $f_p = \dfrac{1}{T_p}$ = peak frequency, and roughly $T_p \approx 1.05 T_s$.

$s = s_{\max}$ at $f = f_p$, and s decreases as $|f - f_p|$ increases.

Hence, directional spreading is the narrowest near $f = f_p$. See Fig. 2.12 for $s_{\max} = 20$

where $f^* = f / f_p$ (=1 at peak frequency).

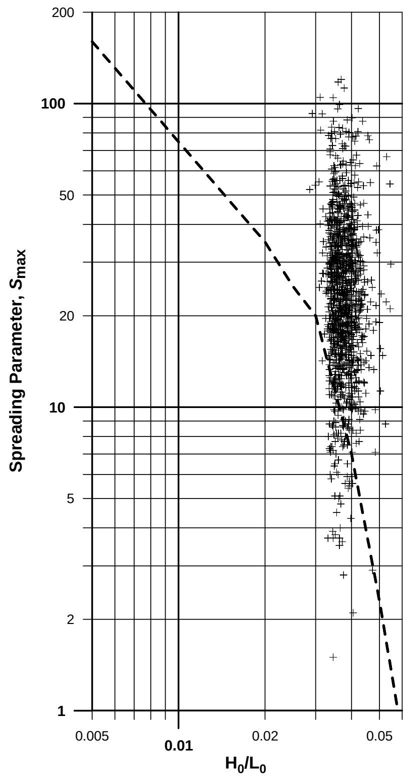
(3) Estimation of the spreading parameter $s_{\max}$

As $s_{\max}$ increases, more long-crested.

Tentatively, $\left\{ \begin{array}{ll} s_{\max} = 10 & \text{for wind waves} \\ s_{\max} = 25 \sim 75 & \text{for swell} \end{array} \right\}$ in deep water.

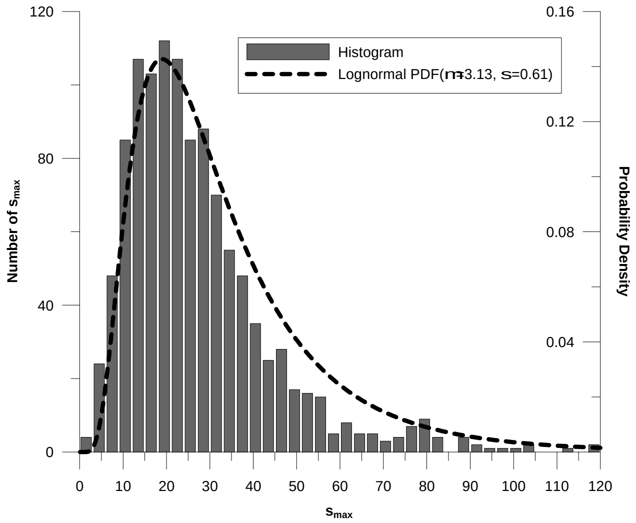
$s_{\max}$ increases as H_0 / L_0 decreases (see Fig. 2.13).

However, Suh et al. (2010, Coastal Engineering 57, 375-384) showed that for wind waves of $0.03 \le H_0 / L_0 \le 0.055$, the mean value of measured $s_{\max}$ is somewhat greater than that in Fig. 2.13.



They also showed that the probability density function of $s_{\max}$ is given by a lognormal function:

$$p(s_{\max}) = \frac{1}{0.61\sqrt{2\pi}\,s_{\max}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(s_{\max}) - 3.13}{0.61}\right]^2\right\}$$



As waves propagate to shallow water, they become long-crested due to refraction. In other words, $s_{\max}$ increases as h decreases. On the other hand, $s_{\max}$ increases more rapidly with decreasing h for a larger incident angle because of more refraction (see Fig. 2.14).

(4) Cumulative distribution curve of wave energy

Read text.

(5) Other directional spreading functions

Simplest:

$$G(\theta \mid f) \equiv G(\theta) = \begin{cases} \dfrac{2l!!}{\pi(2l-1)!!}\cos^{2l}(\theta-\theta_0) & \text{for } |\theta-\theta_0| \le \dfrac{\pi}{2} \\ 0 & \text{for } |\theta-\theta_0| > \dfrac{\pi}{2} \end{cases} \tag{2.30}$$

which is independent of f.

SWOP (Stereo Wave Observation Project):

$$G(\theta \mid f) = G(\theta \mid \omega)\,; \qquad \omega = 2\pi f \tag{2.31}$$

Eq. (2.30) with $l = 1$ and (2.31) are similar to Mitsuyasu-type with $s_{\max} = 10$, though the energy spread with f is not the same.

Wrapped normal function (Borgman, 1984):

$$G(\theta \mid f) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N} \exp\left\{-\frac{(n\sigma_\theta)^2}{2}\right\} \cos n(\theta-\theta_0)$$

θ_0 = mean wave direction

σ_θ = directional spreading parameter (broad directional spreading as $\sigma_\theta \uparrow$)

2.4 Relationship between Wave Spectra and Characteristic Wave Dimensions

$$\left.\begin{array}{l}\text{Wave - by - wave analysis} \rightarrow \text{time domain} \\ \text{Wave spectrum} \rightarrow \text{frequency domain}\end{array}\right] \leftarrow \text{ relates each other.}$$

$$\eta(t) = \sum_{n=1}^{\infty} a_n \cos(2\pi f_n t + \varepsilon_n)$$

$$\eta^2 = \left[\sum_{n=1}^{\infty} a_n \cos(2\pi f_n t + \varepsilon_n)\right]^2$$

$$\overline{\eta^2} = \frac{1}{T}\int_0^T \left[\sum_{n=1}^{\infty} a_n \cos(2\pi f_n t + \varepsilon_n)\right]^2 dt$$

Using orthogonality, $\frac{1}{T}\int_0^T \cos(m\sigma t)\cos(n\sigma t)dt = 0$ if $m \neq n$,

$$\begin{aligned}\overline{\eta^2} &= \frac{1}{T}\int_0^T \left[\sum_{n=1}^{\infty} a_n^2 \cos^2(2\pi f_n t + \varepsilon_n)\right] dt \\ &= \sum_{n=1}^{\infty} \frac{1}{2} a_n^2 \\ &= \int_0^{\infty} S(f)df \equiv m_0\end{aligned}$$

where m_0 = zeroth moment of $S(f)$

$\overline{\eta^2}$ = variance of $\eta(t)$

Defining $\eta_{rms} = \sqrt{\overline{\eta^2}}$ = root-mean-squared value of $\eta(t)$, we get

$$\eta_{rms} = \sqrt{\overline{\eta^2}} = \sqrt{m_0}$$

For Rayleigh distribution of H,

$$H_s \cong 4\eta_{rms} = 4\sqrt{m_0}$$

Since H_s and $4\sqrt{m_0}$ are not exactly the same, use

$H_s = H_{1/3}$ = average height of highest 1/3 waves from zero-crossing method

$H_{m0} = 4\sqrt{m_0}$ = spectral estimate of significant wave height

As for wave periods,

$$T_s \cong 0.95T_p$$

$\overline{T} \cong \sqrt{m_0 / m_2}$; $m_2 = \int_0^\infty f^2 S(f)df$ = 2nd moment of $S(f)$