Chap 9. Description of Random Sea Waves

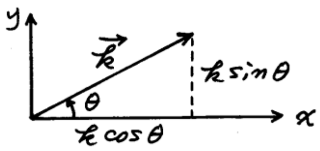
9.1 Profiles of Progressive Waves and Dispersion Relationship

9.2 Description of Random Sea Waves by Means of Variance Spectrum

Regular wave:

$$\eta = a\cos(kx\cos\theta + ky\sin\theta - 2\pi ft + \varepsilon)$$

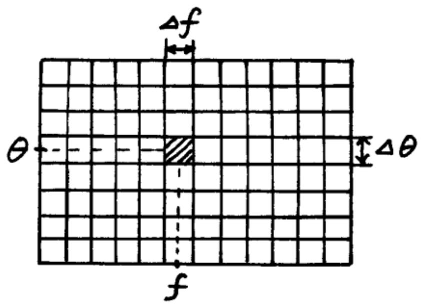
$$E = \frac{1}{2}\rho g a^2$$



$k = |\vec{k}|$

Random waves: superposition of many regular waves

$$\eta = \eta(x, y, t) = \sum_{n=1}^{\infty} a_n \cos(k_n x\cos\theta_n + k_n y\sin\theta_n - 2\pi f_n t + \varepsilon_n)$$



$$S(f, \theta)\Delta f\Delta\theta = \frac{1}{2}a_n^2$$

$\downarrow$ one-to-one correspondence between f and k by dispersion relation

$$S(k, \theta)\Delta k\Delta\theta = \frac{1}{2}a_n^2$$

Single point measurement ($x = y = 0$):

$$\eta = \eta(t) = \sum_{n=1}^{\infty} a_n \cos(2\pi f_n t + \varepsilon_n)$$

$$S(f)\Delta f = \frac{1}{2}a_n^2$$

9.3 Stochastic Process and Variance Spectrum

$$\downarrow$$

$$\eta(t) = \{\eta_1(t), \eta_2(t), \cdots, \eta_j(t), \cdots\} \leftarrow \{\ \} = \text{ensemble}$$

$$\uparrow$$

varies randomly with time

Assume

(1) stationarity: independent of time (valid for short duration $\leq$ 20~30 min)

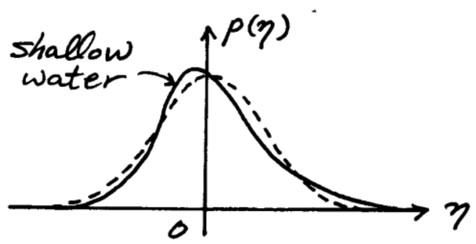
(2) ergodicity: Time-averaged statistics are equal to ensemble-averaged statistics

$$E[\eta(t)] = \overline{\eta_j(t)} = \lim_{t_0 \to \infty} \frac{1}{t_0} \int_0^{t_0} \eta_j(t)\,dt$$

$\uparrow$ ensemble-average (from many records) $\quad \uparrow$ time-average (from a single record)

$\therefore$ We need only one record

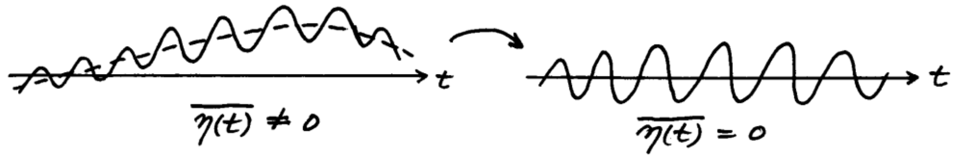
(3) Gaussian process: probability density of $\eta(t)$ is given by Gaussian (normal) distribution, but in shallow water, peaked crests and flatter troughs



Assume zero-mean process:

$$E[\eta(t)] = \overline{\eta(t)} = 0$$

If the wave record includes tidal variation, remove it.



Relation between $\Psi(\tau)$ and $S_0(f)$:

$$\left.\begin{aligned}\Psi(\tau) &= \int_{-\infty}^{\infty} S_0(f) e^{i2\pi f\tau} df \\ S_0(f) &= \int_{-\infty}^{\infty} \Psi(\tau) e^{-i2\pi f\tau} d\tau\end{aligned}\right\} \text{Fourier transform pair (Dean and Dalrymple's book)}$$

Redefining in the range of 0 to ∞ for both τ and f,

$$\left.\begin{aligned}\Psi(\tau) &= \int_{0}^{\infty} S(f) \cos(2\pi f\tau) df \\ S(f) &= 4\int_{0}^{\infty} \Psi(\tau) \cos(2\pi f\tau) d\tau\end{aligned}\right\} \text{Wiener - Khintchine relation}$$

For irregular wave profile,

$$\eta(t) = \sum_{n=1}^{\infty} a_n \cos(2\pi f_n t + \varepsilon_n)$$

$$\begin{aligned}\Psi(\tau) &= E[\eta(t+\tau)\eta(t)] \\ &= \overline{\eta_j(t+\tau)\eta_j(t)} \\ &= \lim_{t_0\to\infty} \frac{1}{t_0}\int_0^{t_0} \eta_j(t+\tau)\eta_j(t) dt \\ &= \lim_{t_0\to\infty} \frac{1}{t_0}\int_0^{t_0} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \cos[2\pi f_n (t+\tau)+\varepsilon_n]\cos(2\pi f_m t+\varepsilon_m) dt \\ &= \lim_{t_0\to\infty} \frac{1}{t_0}\int_0^{t_0} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m [\cos(2\pi f_n t+\varepsilon_n)\cos(2\pi f_m t+\varepsilon_m)\cos(2\pi f_n \tau) \\ &\qquad - \sin(2\pi f_n t+\varepsilon_n)\cos(2\pi f_m t+\varepsilon_m)\sin(2\pi f_n \tau)] dt \\ &= \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \cos(2\pi f_n \tau)\end{aligned}$$

Now

$$S(f) = 4\int_0^\infty \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \cos(2\pi f_n \tau)\cos(2\pi f \tau) d\tau$$
$$= \sum_{n=1}^{\infty} a_n^2 \int_0^\infty [\cos 2\pi(f_n + f)\tau + \cos 2\pi(f_n - f)\tau] d\tau$$

$$\int_0^\infty (\text{periodic function}) d\tau = 0$$

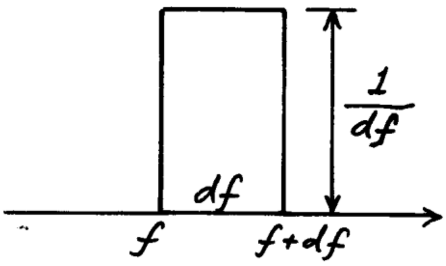
$\int_0^\infty (1) d\tau = \infty$ $\rightarrow$ Dirac delta function at $f_n = -f$ and $f_n = f$

$\uparrow$

take only this

Note: delta function is defined as the integral over $(-\infty, \infty)$. But we integrate over $(0, \infty)$. Therefore, take 1/2 of delta function.

$$S(f) = \sum_{n=1}^{\infty} \frac{1}{2} a_n^2 \delta(f_n - f)$$



$$S(f) = \frac{1}{df}\sum_{f}^{f+df} \frac{1}{2} a_n^2$$

$$m_0 = \overline{\eta^2} = \overline{\eta_j(t+0)\eta_j(t)} = \Psi(0) = \int_0^\infty S(f)\cos(2\pi f \cdot 0) df = \int_0^\infty S(f) df$$