24 Cauchy's integral theorem, Independence of path

24.1 Line integral

Under the assumption of continuous and smooth c, the line integral exist and value is independent of the choice of subdivisions and intermediate points ζ_m

First Method : Indefinite Integration and Substitution of Limits.

Theorem 1 (Indefinite integration of analytic functions)

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, and for all paths in D joining two points z_0 in D we have
(9)

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]$$

Simple connectedness is quit essential in Theorem 1.

Example 1.

$$\int_0^{1+i} z^2 dz = \frac{1}{3}z^3|_0^{1+i} = \frac{1}{3}(1-1+2i)(1+i) = \frac{1}{3}(-2+2i) = -\frac{2}{3} + \frac{2}{3}i$$

Example 2.

$$\int_{\pi i}^{\pi i} \cos z dz = \sin z|_{-\pi i}^{\pi i} = 2\sin \pi i = 2i \sinh \pi = 23.097i$$

$$(\because \sin iz = i \sinh z \text{ from (15) in sec 12.7 })$$

Example 3.

$$\int_{8+\pi i}^{8-3\pi i} e^{z/2} dz = e^{z/2}|_{8+\pi i}^{8-3\pi i} = 2(e^{4-3\pi i/2} - e^{4+\pi i/2}) = 0$$

since e^z is periodic with period $2\pi i$

Example 4.

$$\int_{-i}^{i} \frac{dz}{z} = \text{Ln } i - \text{Ln }(-i) = i\frac{\pi}{2} - (-i\frac{\pi}{2}) = i\pi.$$

D : simply connected   Ln z : 0 &, negative real axis are omitted in definition.

Second Method : Use of a Representation of the path.

Theorem 2 **(Integration by the use of the path)**

Let c be a piecewise smooth path, represented by $z = z(t)$, where $a \leq t \leq b$. Let $f(z)$ be a continuous function on c. Then,

(10)

$$\int_c f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt \quad (\dot{z} = dz/dt)$$

Proof. L.H.S of (10)
from (8)

$$\int_c f(z)dz = \int_c udx - \int_c vdy + i[\int_c udy + \int_c vdx] \quad ---1)$$

$$z = x + iy, \Rightarrow \dot{z} = \dot{x} + i\dot{y}$$

$$f = u + iv$$

$$\text{R.H.S. of (10)} \quad (dx = \dot{x}dt, dy = \dot{y}dt)$$

$$\begin{aligned}\int_a^b f[z(t)]\dot{z}(t) &= \int_a^b (u + iv)(\dot{x} + i\dot{y})dt \\ &= \int_c [udx - vdy + i(udy + vdx)] \\ &= \int_c (udx - vdy) + i\int_c (udy + vdx) \quad ---2)\end{aligned}$$

steps in applying Theorem 2

(a) Represent the path c in the form $z(t)$ $(a \leq t \leq b)$
(b) Calculate the derivative $\dot{z}(t) = dz/dt$.
(c) Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y).
(d) Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

Example 5. A basic result : Integral of $1/z$ around the unit circle.

(11)

$$\oint_c \frac{dz}{z} = 2\pi i \quad (c \text{ the unit circle, ccw})$$

Solution.

$$z(t) = \cos t + i \sin t = e^{it} \quad (0 \leq t \leq 2\pi)$$

$$\dot{z}(t) = ie^{it}, \qquad f[z(t)] = 1/z(t) = e^{-it}$$

$$\oint_c \frac{dz}{z} = \int_0^{2\pi} e^{-it} \cdot i \cdot e^{it}dt = i\int_0^{2\pi} dt = 2\pi i$$

Example 6. Integral of integer powers.

Let $f(z) = (z - z_0)^m$ where m is an integer and z_0 a constant.
Solution.

$$C : z(t) = z_0 + \rho(\cos t + i \sin t) = z_0 + \rho e^{it} \quad (0 \le t \le 2\pi)$$

$$(z - z_0)^m = \rho^m e^{imt}, dz = i\rho e^{it} dt$$

$$\oint_c (z - z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} i\rho e^{it} dt = i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t} dt$$

by the Euler formula

$$i\rho^{m+1}[\int_0^{2\pi} \cos(m+1)tdt + i \int_0^{2\pi} \sin(m+1)tdt]$$

If $m = -1, \rho^{m+1} = 1, \cos 0 = 1, \sin 0 = 0. \quad \therefore 2\pi i$
For $m \neq 1$,
(12)

$$\oint_c (z - z_0)^m dz = \begin{cases} 2\pi i & (m = -1) \\ 0 & (m \neq -1 \text{and integer}) \end{cases}$$

Dependence on path : a complex line integral depends not only on the endpoints of the path but in general also on the path itself.

Example 7. Integral of a nonanalytic function. Dependence on path.

$$f(z) = \text{Re } z = x \text{ from } 0 \text{ to } 1 + 2i.$$

(a) along c^* (b) along c consisting of c_1 and c_2

Solution.
(a)

$$c^* : z(t) = t + 2it (0 \le t \le 1)$$

$$\dot{z}(t) = 1 + 2i \& f[z(t)] = x(t) = t$$

$$\int_{c^*} \text{Re } zdz = \int_0^1 t(1 + 2i)dt = \frac{1}{2}(1 + 2i) = \frac{1}{2} + i$$

(b)

$$c_1 : z(t) = t, \dot{z}(t) = 1, f[z(t)] = x(t) = t \quad (0 \le t \le 1)$$
$$c_2 : z(t) = 1 + it, \dot{z}(t) = i, f[z(t)] = x(t) = 1 \quad (0 \le t \le 2)$$

$$\int_c \text{Re } zdz = \int_{c_1} \text{Re } zdz + \int_{c_2} \text{Re } zdz = \int_0^1 tdt + \int_0^2 1 \cdot idt = \frac{1}{2} + 2i$$

24.2 Bound for Absolute Value of Integrals.

$$\left|\int_c f(z)dz\right| \leq ML \qquad (ML-\text{inequality});$$

L : the length of C, $|f(z)| \leq M$ everywhere on C

Proof.

$$|S_n| = \left|\sum_{m=1}^{n} f(\zeta_m)\Delta z_m\right| \leq \sum_{m=1}^{n} |f(\zeta_m)||\Delta z_m| \leq M \sum_{m=1}^{n} |\Delta z_m|$$

$\sum_{m=1}^{n} |\Delta z_m|$ approaches the length L of the Curve C if n approaches infinity.

$$\therefore \left|\int_0 f(z)dz\right| \leq ML.$$

Example 8. Estimation of an integral. (upper bound)

$$\int_c z^2 dz. C: \text{ straight-line from } 0 \text{ to } 1+i.$$

Solution.

$$L = \sqrt{2} \text{ and } |f(z)| = |z^2| \leq 2 \text{ on } C.$$

$$\left|\int_c z^2 dz\right| \leq 2\sqrt{2} = 2.8284$$

24.3 Cauchy's Integral Theorem

1. A simple closed path is a closed path that does not intersect or touch itself.

2. A simple connected domain D in the complex plane is a domain such that every simple closed path in D encloses only points of D. A domain that is not simply connected is called multiply connected.

Theorem 3 Cauchy's integral theorem.

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

(1)

$$\oint_c f(z)dz = 0$$

Example 9. No singularities (Entire function)

$$\oint_c e^z dz = 0, \qquad \oint_c \cos z dz = 0, \qquad \oint_c z^n dz = 0 \qquad (n = 0, 1, \cdots)$$

for any closed path, since these function are entire (analytic for all z)

Example 10. Singularities outside contour.

$$\oint_c \sec z dz = 0, \qquad \oint_c \frac{dz}{z^2+4} = 0$$

where C is the unit circle, $\sec z = 1/\cos z$ is not analytic at $z = \pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside C ; none lies on C or inside C. Similarly for the second integral, whose integral, whose integrand is not analytic at $z = \pm 2i$ outside C.

Example 11. Nonanalytic function.

$$\oint_c \overline{z} dz = \int_0^{2\pi} e^{-it} \cdot i \cdot e^{it} dt = 2\pi i$$

where $C : z(t) = e^{it}$ is the unit circle. $f(z) = \overline{z}$: is not analytic

Solution.

$$\text{on } C \quad x = \cos t, \quad y = \sin t, \quad z = x + iy = \cos t + i \sin t = e^{it}$$

$$\dot{z}(t) = ie^{it}, \overline{z} = x - iy = \cos t - i \sin t = e^{-it}$$

Example 12. Analyticity sufficient, not necessary

$$\oint_c \frac{dz}{z^2} = 0 \text{ where } C \text{ is the unit circle}$$

$$\text{unit circle } z = e^{it} \quad dz = ie^{it} dt \quad z^{-2} = e^{-2it}$$

$$\begin{aligned} \oint_c \frac{dz}{z^2} &= \int_0^{2\pi} e^{-it} \cdot i \cdot e^{it} dt = i \int_0^{2\pi} e^{-it} dt = -e^{-it}|_0^{2\pi} = e^{-it}|_{2\pi}^0 \\ &= (\cos 0 - i \sin 0) - (\cos 2\pi - i \sin 2\pi) = 0 \end{aligned}$$

This result does not follow from Cauchy's theorem, because $f(z) = 1/z^2$ is not analytic at $z = 0$. Hence the condition that f be analytic in D is sufficient rather than necessary for $\oint_c f(z) dz = 0$ to be true

Example 13. Simple connectedness essential.

$$\oint_c \frac{dz}{z} = 2\pi i \text{ for ccw integration around the unit circle.}$$

C. lies the annulus $1/2 < |z| < 3/2$ where $1/z$ is analytic, but this domain is not simply connected, so that Cauchy's theorem cannot be applied. Hence the condition that the domain

D be simply connected is quite essential.

Cauchy's Proof
From (8) Sec. 13. 1
(8)

$$\begin{aligned}\lim_{n\to\infty} S_n &= \oint_c f(z)dz = \oint_c udx - \oint_c vdy + i[\oint_c udy + \oint_c vdx] \\ &= \oint_c (udx - vdy) + i\oint (udy + vdx)\end{aligned}$$

Green's theorem in the plane (sec 9.4 in chap. 9)

$$\iint_R (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dxdy = \oint_c (F_1dx + F_2dy)$$

Since $f'(z)$ is analytic in D, its derivative $f'(z)$ exists in D.
Since $f'(z)$ is assumed to be continuous, u and v have partial derivatives in D.

$$\oint_c (udx - vdy) = \iint_R (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dxdy$$

$$\text{Cauchy-Riemann equation } : \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\oint_c (udx - vdy) = \iint_R (-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x})dxdy = 0$$

$$\oint (udy + vdx) = \iint_R (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dxdy = \iint_R (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x})dxdy = 0$$

$$\therefore \oint_c f(z)dz = 0$$