Ch12. Filters and Tuned Amplifiers

Introduction

- ❑ Passive LC filters

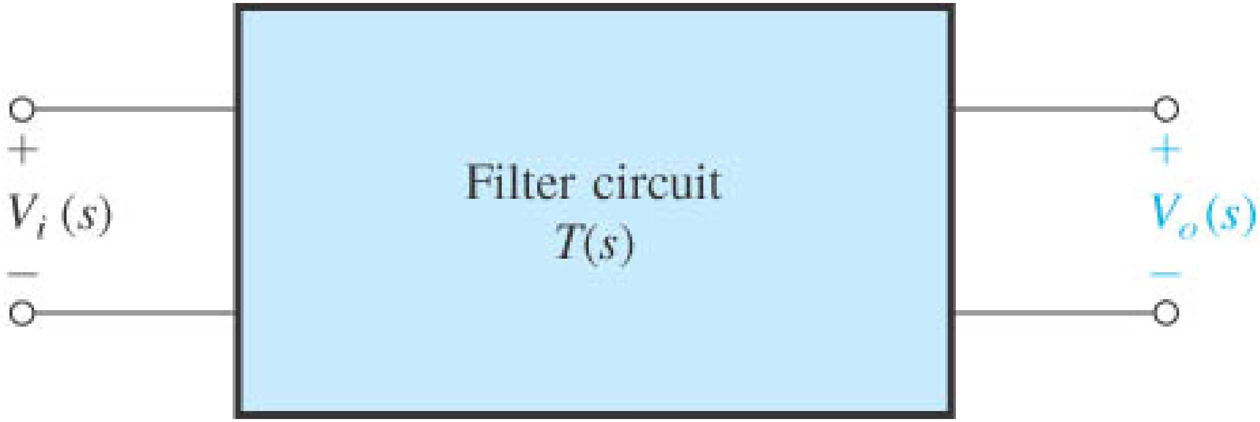
- ❑ Electronic Filter → Active filter
  - – Active RC filters
  - – Switched capacitor circuits

  → **Advantages: No inductors!**

  Inductors are large and physically bulky for low frequency applications (such as those used in passive LC filters)

Filter Transmission

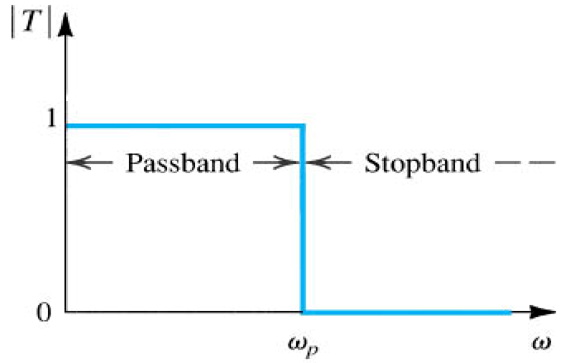
❑ Filter - a two port device



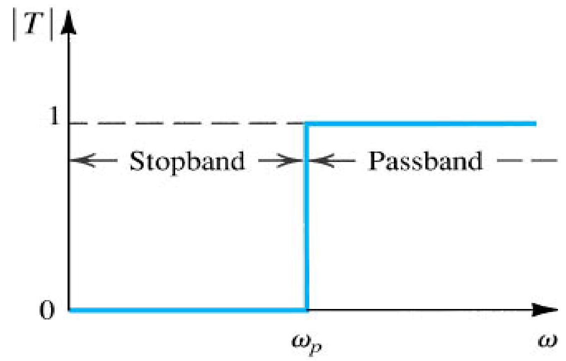
- Transfer function: $T(s) \equiv \dfrac{V_o(s)}{V_i(s)}$
- Transfer transmission: $T(j\omega) = \underbrace{|T(j\omega)|}_{\text{magnitude}} e^{\overbrace{j\phi(\omega)}^{\text{phase}}}$ $\quad (s = j\omega)$
- Gain function: $G(\omega) \equiv 20\log|T(j\omega)|$ dB
- Attenuation function: $A(\omega) \equiv -20\log|T(j\omega)|$ dB
- Input Output relation: $|V_o(j\omega)| = |T(j\omega)||V_i(j\omega)|$

Filter Types

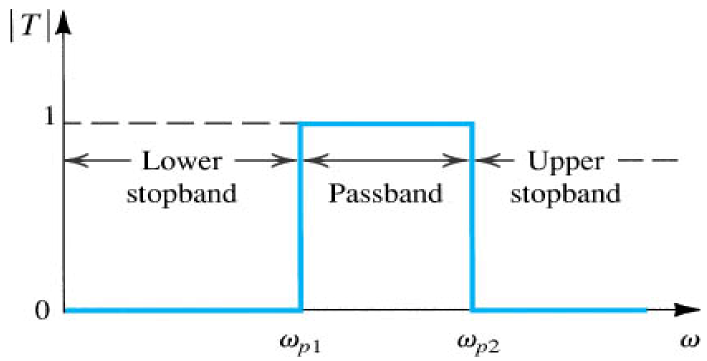
- Frequency-selection function
  - passing → passband: $|T| = 1$
  - stopping → stopband: $|T| = 0$
- Brick-wall response



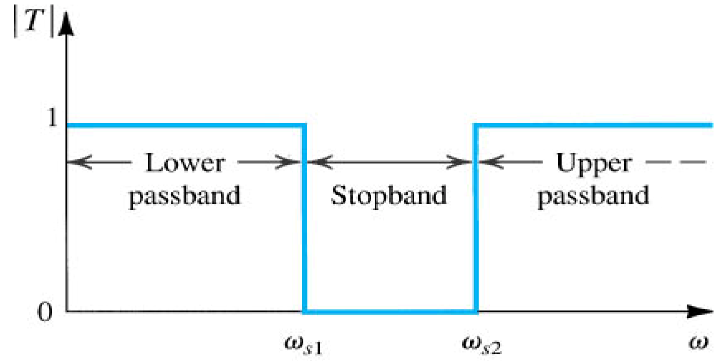
(a) Low-pass (LP)



(b) High-pass (HP)

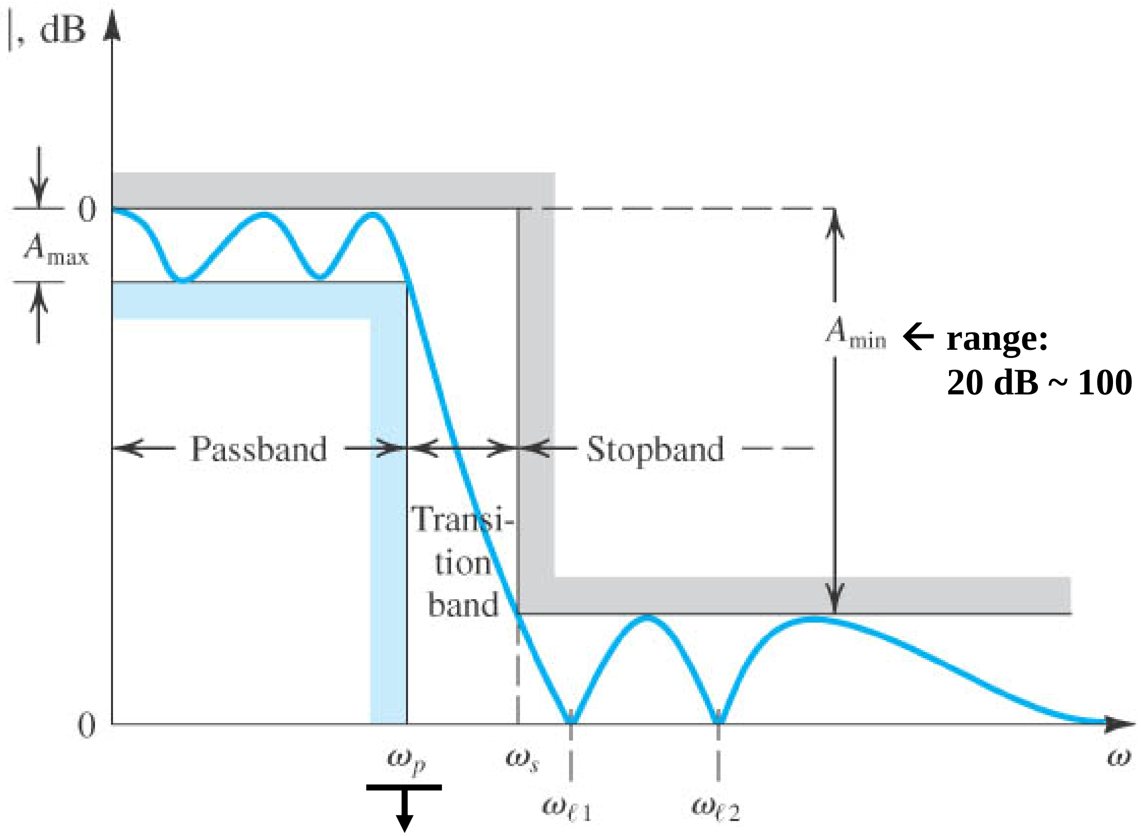


(c) Bandpass (BP)



(d) Bandstop (BS)

Filter Specification



Passband Ripple →
range: 0.05 dB ~ 3 dB

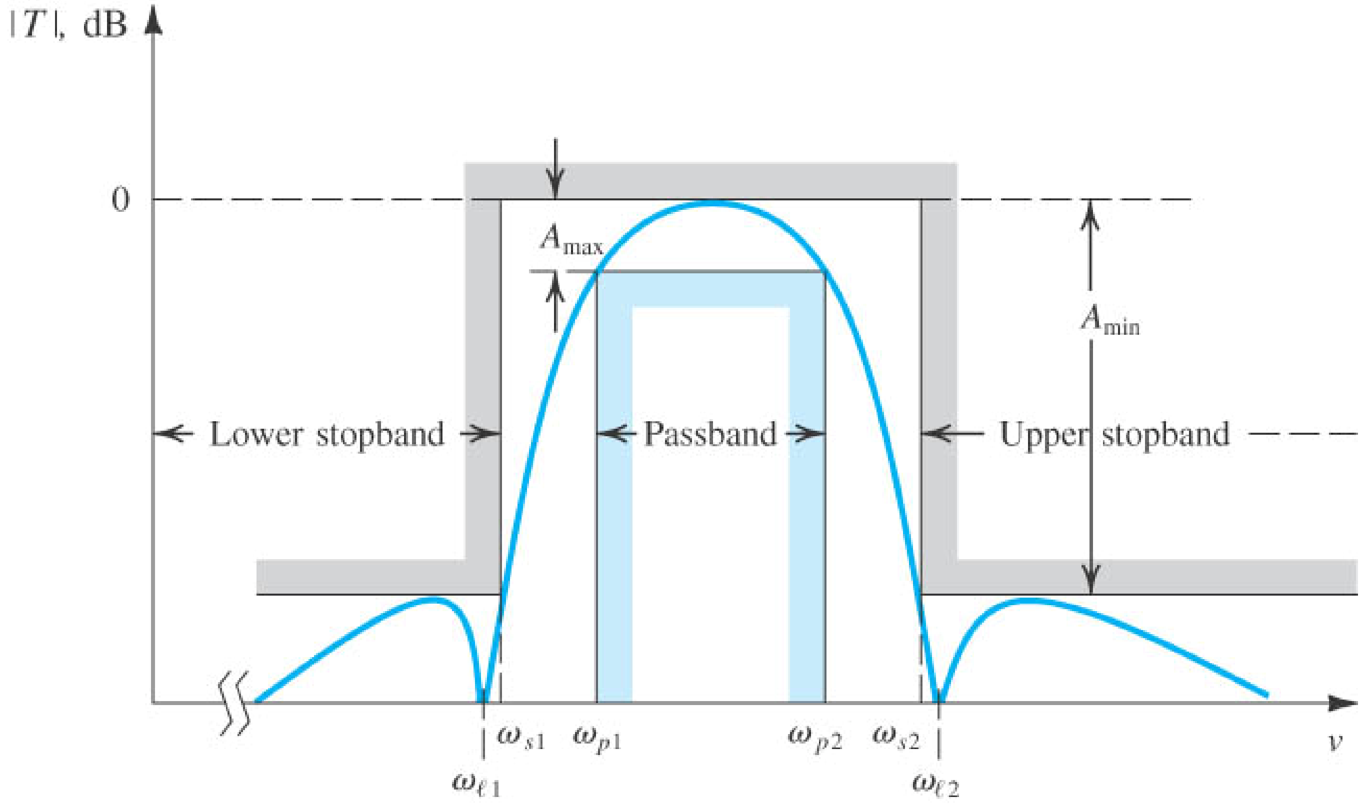
← range:
20 dB ~ 100 dB

Ripple bandwidth

- ω_p : Passband edge
- A_{max}: Maximum allowed variation in passband tansmission
- ω_s : Stopband edge
- A_{min} : Minimum required stopband attenuation
- $\dfrac{\omega_S}{\omega_P}$: Selectivity factor

Filter Specification

- Filter approximation
  — The process of obtaining a transfer function that meets given specifications
  — Performed using computer programs(Snelgrove, 1982;Ouslis and Sedra, 1995), filter design table(Zverev, 1967) or closed-form expressions(Section 12.3)



The Filter Transfer Function

- Filter Transfer Function $T(s)$

  - $$T(s) = \frac{a_M s^M + a_{M-1} s^{M-1} + \cdots + a_0}{s^N + b_{N-1} s^{N-1} + \cdots + b_0}$$

    – N : Filter order

    – If $N \geq M$, stable

    – $a_{0,}, \ldots, a_M$ & $b_0, \ldots, b_{N-1}$: real numbers

    → **Poles need be in conjugate pairs or negative real numbers**

  - $$T(s) = \frac{a_M (s - z_1)(s - z_2) \cdots (s - z_M)}{(s - p_1)(s - p_2) \cdots (s - p_N)}$$

    – $z_1, \ldots., z_M$ = transfer-function zeros = transmission zeros

    – $p_1, \ldots, p_N$ = transfer-function poles = natural modes

    → All poles must lie in left half plane.

The Filter Transfer Function (zeros)

- Since in the stopband the transmission is zero or small

  → the zeros are usually, placed on the $j\omega$ axis at stopband frequencies

1. zeros at $s = + j\omega_{l1}$ & $+ j\omega_{l2}$

   also at $s = -j\omega_{l1}$ & $-j\omega_{l2}$

   — Numerator polynomial

   $(s + j\omega_{l1})(s - j\omega_{l1})(s + j\omega_{l2})(s - j\omega_{l2})$

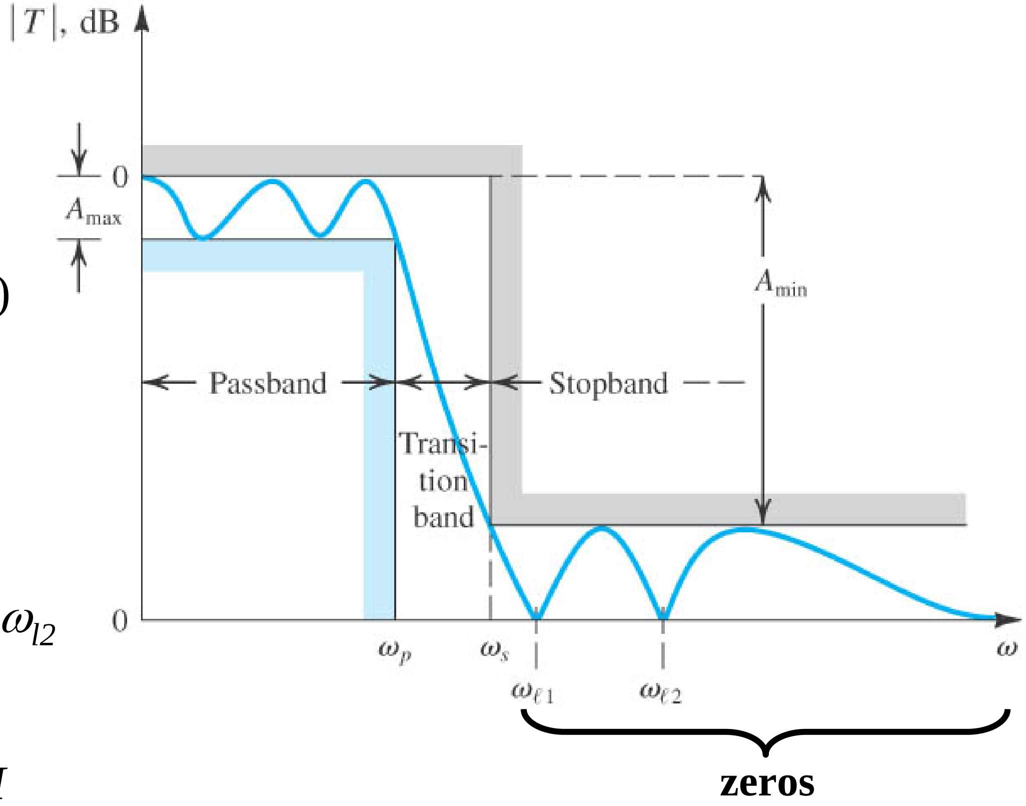
   $= (s^2 + \omega_{l1}{}^2)(s^2 + \omega_{l2}{}^2)$

   → for $s = j\omega$,

   $(s^2 + \omega_{l1}{}^2)(s^2 + \omega_{l2}{}^2)$

   $= (-\omega^2 + \omega_{l1}{}^2)(-\omega^2 + \omega_{l2}{}^2)$

   which is zero at $\omega = \omega_{l1}$ and $\omega = \omega_{l2}$

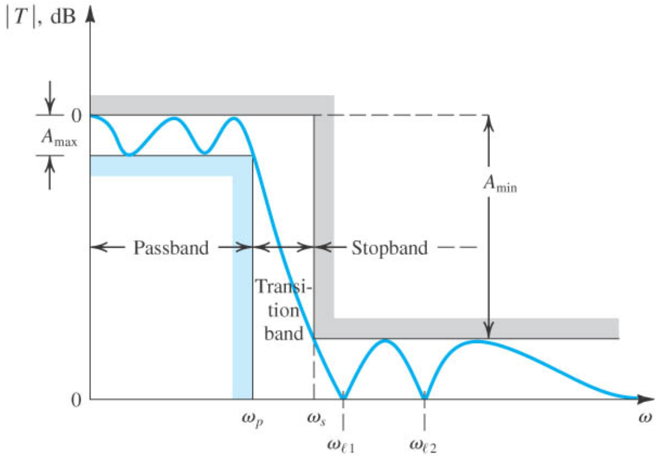


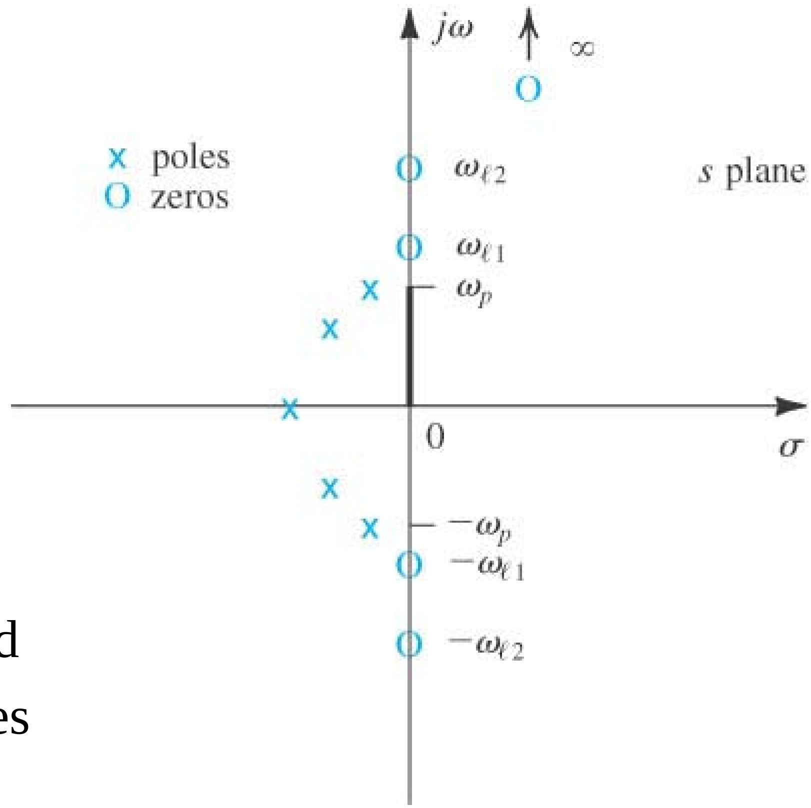
2. zeros at $s = \infty$

   the numbers of zeros at $s = \infty = N—M$

   $\because$ as $s \to \infty$, $\quad T(s) \to \dfrac{a_M}{s^{N-M}}$ → Denominator make zeros at infinity

The Filter Transfer Function (ex. 1 : LPF)





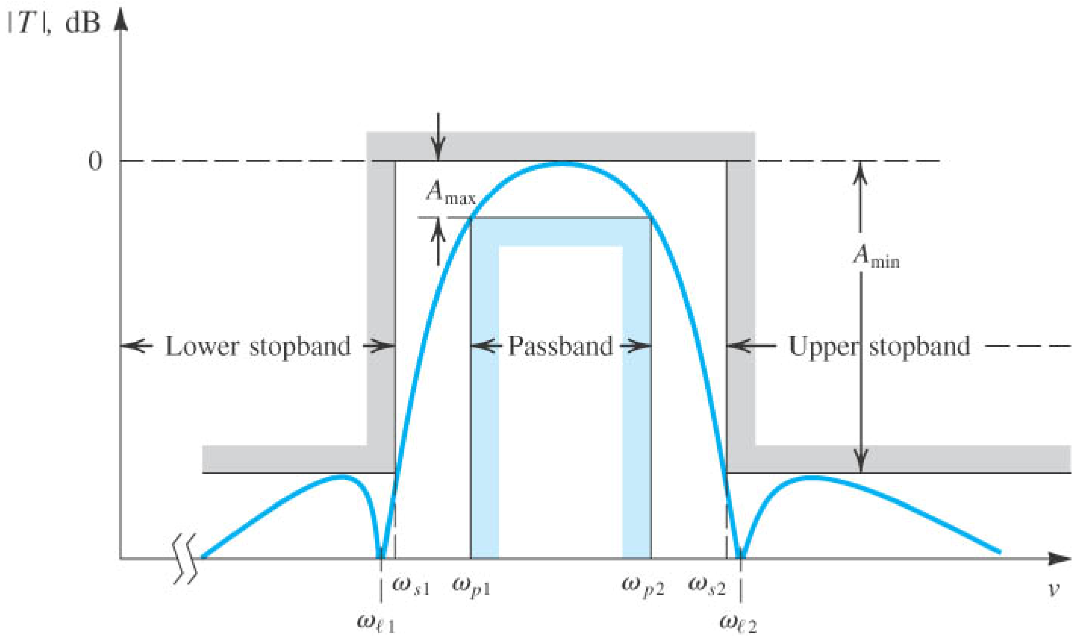
- Number of **poles** =5

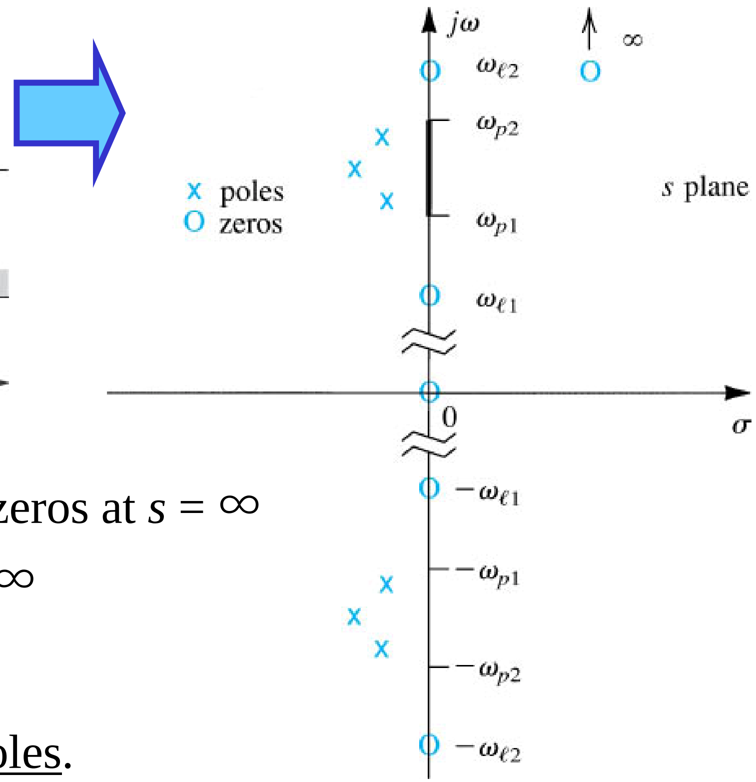
- Two pairs of complex-conjugate poles
  and real-axis pole
  → all the poles lie in the vicinity of passband
  → high transmission at pass band frequencies

- **Zeros** : $s = \pm j\omega_{l1}$ & $\pm j\omega_{l2}$ & ∞

- $T(s) = \dfrac{a_4(s^2+\omega_{l1}^2)(s^2+\omega_{l2}^2)}{s^5+b_4s^4+b_3s^3+b_2s^2+b_1s+b_0}$

The Filter Transfer Function (ex. 2 : BPF)



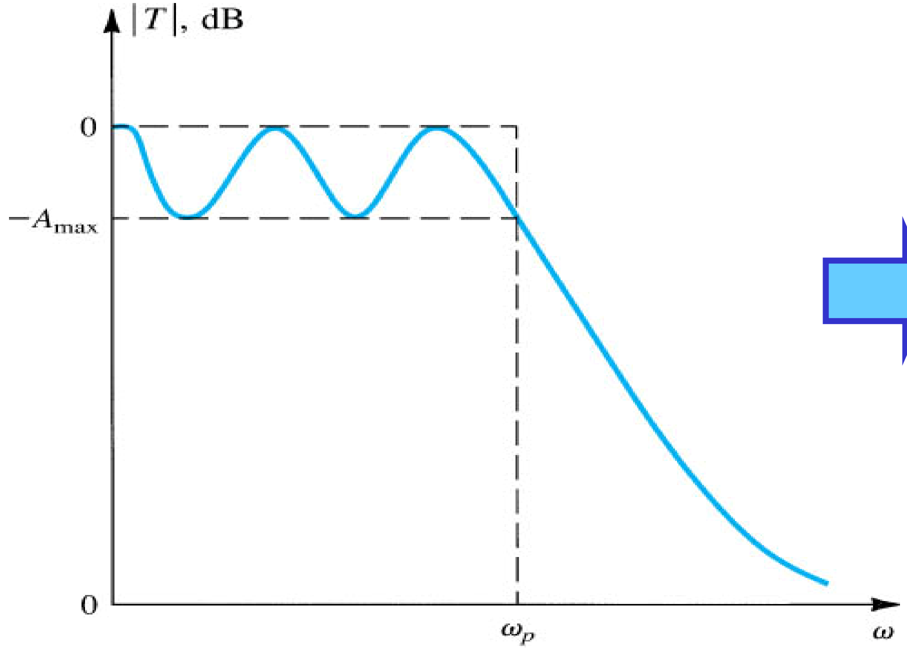


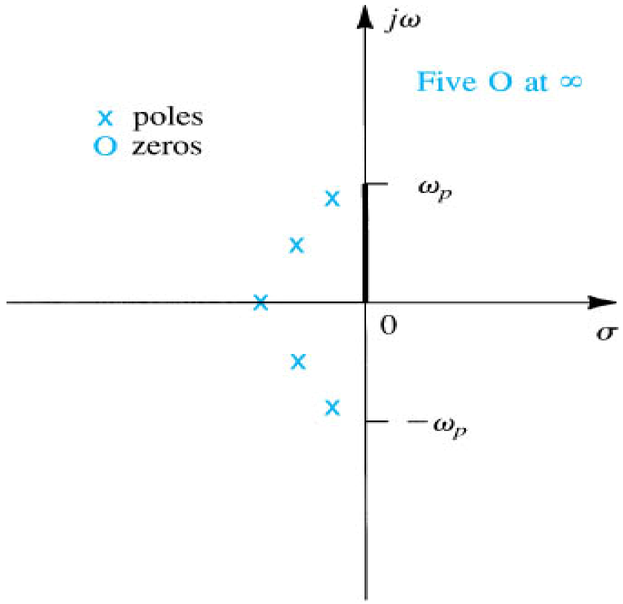
- It has one or more zeros at s = 0 and one or more zeros at $s = \infty$
- Assuming that only one zero exists at s = 0 & $s = \infty$

  → $s = \pm j\omega_{l1}, s = \pm j\omega_{l2}, s = 0, s = \infty$

- Number of poles =6 → <u>Zeros set the number of poles</u>.
- $T(s) = \dfrac{a_5 s(s^2 + \omega_{l1}^2)(s^2 + \omega_{l2}^2)}{s^6 + b_5 s^5 + \ldots + b_0}$

The Filter Transfer Function (ex. 3 : all-pole LPF)





- It is possible that all zeros are at $s = \infty$

$$T(s) = \frac{a_0}{s^N + b_{N-1}s^{N-1} + \cdots + b_0}$$ ➔ all-pole filter

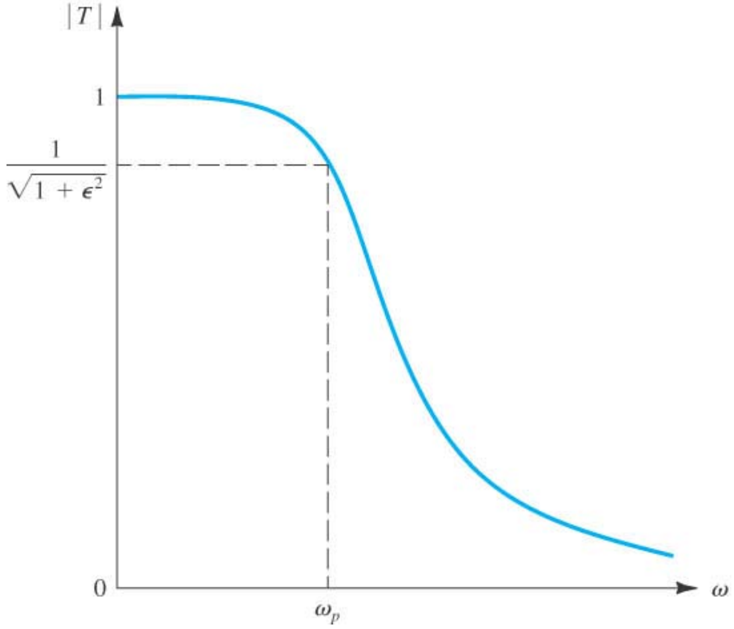
- The more selective the required filter response is, the higher its order must be, and the closer its natural modes are to the $j\omega$ axis.

Butterworth and Chebyshev Filters

- In this section, we present two functions that are frequently used in **approximating the transmission characteristics** of low-pass filters.
  : Closed-form expressions

Butterworth Filters : Filter Shape and $|T(j\omega)|$

❑ The Butterworth Filter



- Monotonically decreasing transmission
- All the transmission zero at $\omega = \infty$ → **all-pole**
- The magnitude function for an Nth-order Butterworth filter with a passband edge ω_P is

$$|T(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2\left(\frac{\omega}{\omega_P}\right)^{2N}}}$$

→ at $\omega = \omega_P$,

$$|T(j\omega_p)| = \frac{1}{\sqrt{1+\varepsilon^2}}$$

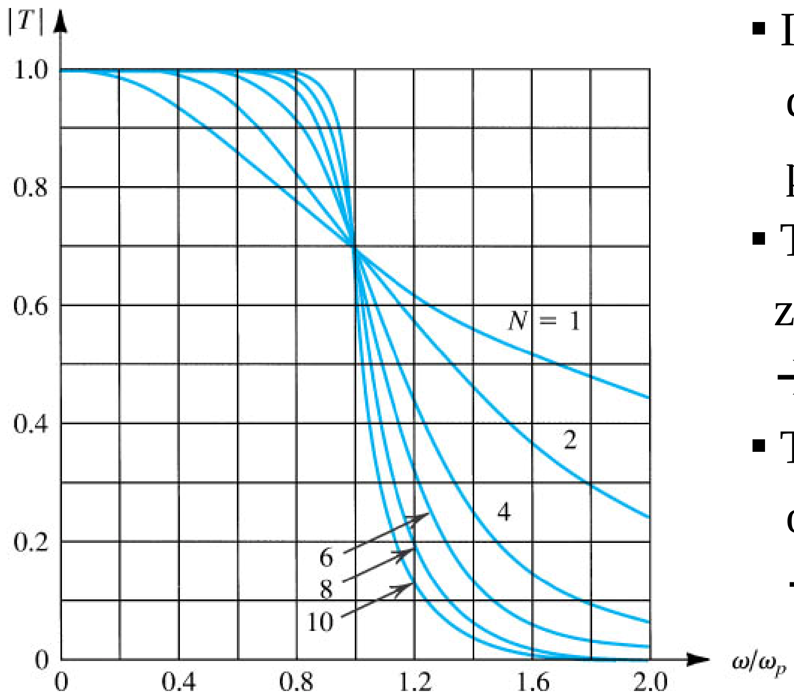
- Thus, the parameter ε determines the maximum variation in passband transmission,

$$A_{\max} = 20\log\sqrt{1+\varepsilon^2}$$

- Conversely, given A_{max},

$$\varepsilon = \sqrt{10^{A_{\max}/10} - 1}$$

The Butterworth Filter : N effects



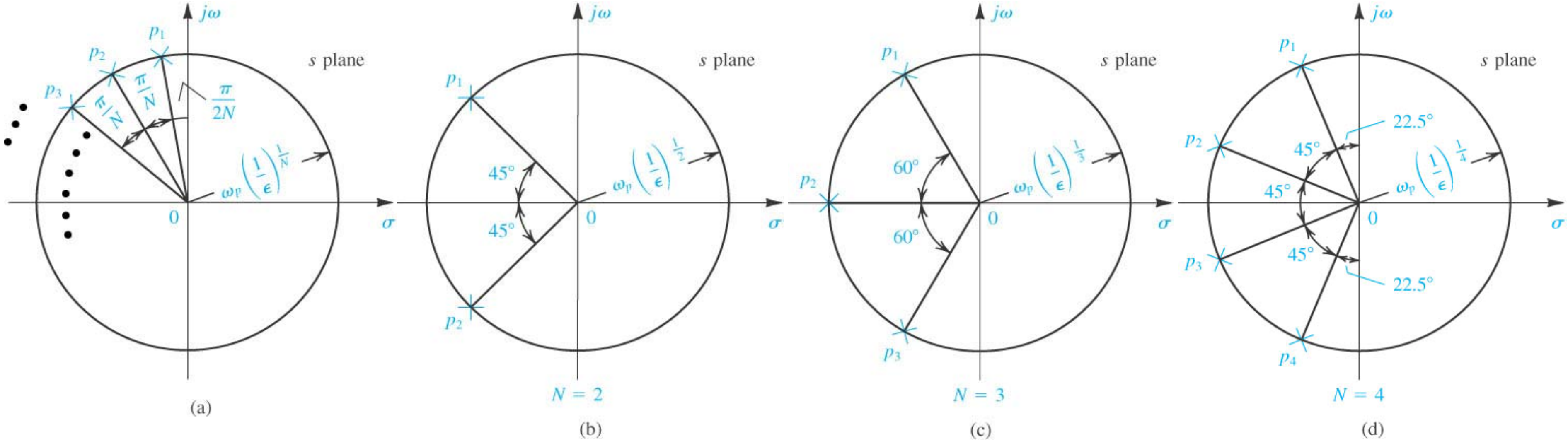
- In the Butterworth response the maximum deviation in passband transmission occurs at the passband edge, ω_P, only
- The first $2N$—1 derivatives of $|T|$ relative to ω are zero at $\omega = 0$
  → very flat near $\omega = 0$ (**maximally flat** response)
- The degree of passband flatness increases as the order N is increased
  → as the order N is increased the filter response approaches the ideal **brick-wall**

- At the edge of the stopband, $\omega = \omega_S$, attenuation is

$$A(\omega_S) = -20\log\left[1\Big/\sqrt{1+\varepsilon^2(\omega_S/\omega_P)^{2N}}\right] = 10\log\left[1+\varepsilon^2(\omega_S/\omega_P)^{2N}\right]$$

- The required filter order = the lowest integer value of N that yields $A(\omega_S) \geq A_{\min}$

The Butterworth Filter : Poles



- The natural modes of an N^{th}-order Butterworth filter can be determined from the graphical construction above.
- Natural modes lies on a circle of radius $\boldsymbol{\omega_P(1/\varepsilon)^{1/N}}$

  → **same frequency of** $\boldsymbol{\omega_0 = \omega_P(1/\varepsilon)^{1/N}}$
- Space by equal angles of $\pi/\boldsymbol{N}$, with the first mode at an angle $\pi/\boldsymbol{2N}$ from the $+j\omega$ axis.
- Transfer function is

$$T(s) = \frac{K\omega_0^N}{(s-p_1)(s-p_2)\cdots(s-p_N)}$$

→ K is a constant dc gain of the filter

The Butterworth Filter

- How to find a Butterworth transfer function

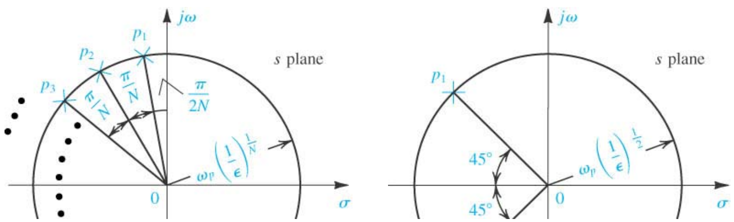
  1. Determine ε.

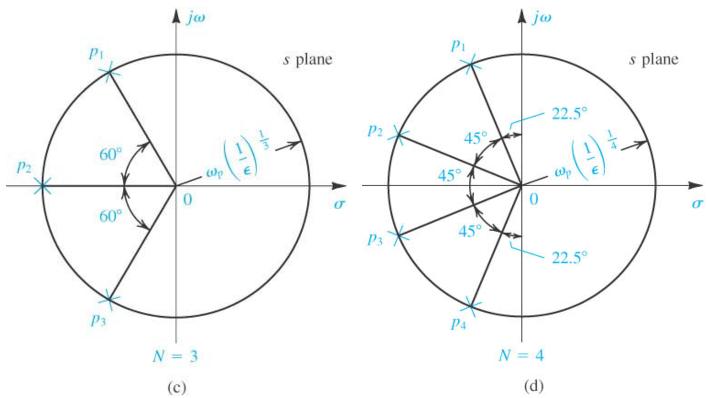
  $$\varepsilon = \sqrt{10^{A_{\max}/10} - 1}$$

  2. Determine the required filter order as the lowest integer value of N that results in $A(\omega_S) \geq A_{\min}$.

  $$A(\omega_S) = -20\log\left[1 \Big/ \sqrt{1 + \varepsilon^2 (\omega_S/\omega_P)^{2N}}\right] = 10\log\left[1 + \varepsilon^2 (\omega_S/\omega_P)^{2N}\right]$$

  3. Determine the N natural modes.

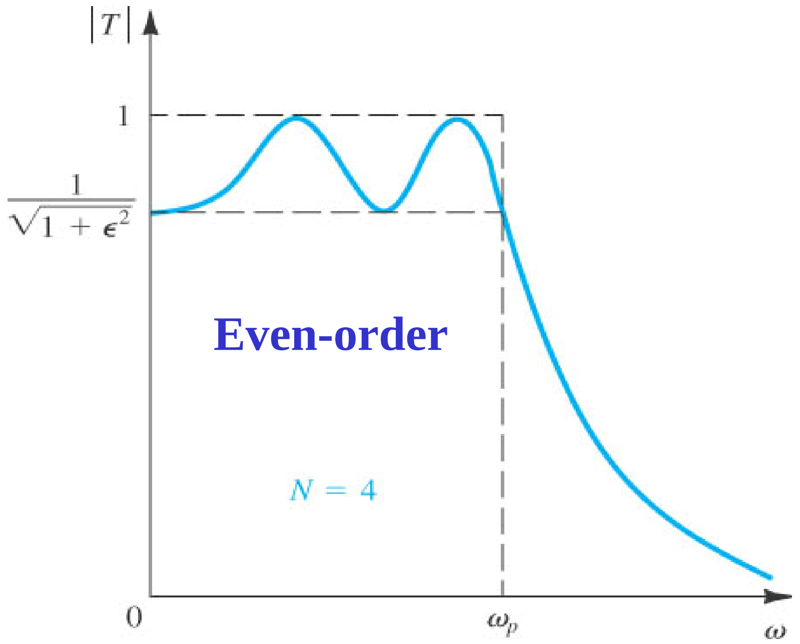


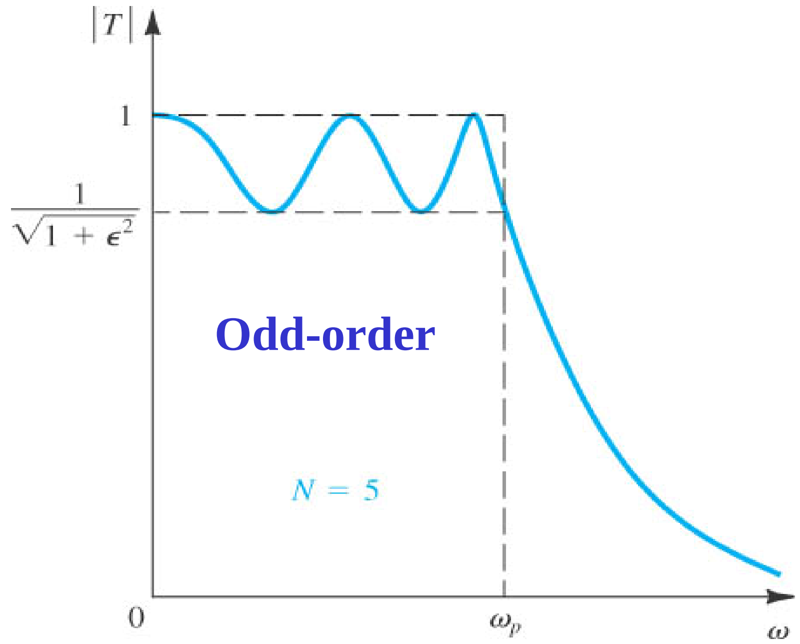


  4. Determine $T(s)$

  $$T(s) = \frac{K\omega_0^N}{(s - p_1)(s - p_2)\cdots(s - p_N)}$$

The Chebyshev Filter





- **Equi-ripple response** (A_{max} = the peak ripple) in the passband and a monotonically decreasing transmission in the stopband.
- The odd-order filter, |T(0)|=1

  The even-order filter exhibits its maximum magnitude deviation at $\omega = 0$.
- Total number of passband maxima and minima equals the order of the filter, *N*.
- All the zeros are at $\omega = \infty$. → **all-pole filter**

The Chebyshev Filter : Equation

- The magnitude of the transfer function with a passband edge ω_P is

$$|T(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 \cos^2[N\cos^{-1}(\omega/\omega_P)]}} \qquad \text{for} \quad \omega \le \omega_P$$

$$|T(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 \cosh^2[N\cosh^{-1}(\omega/\omega_P)]}} \qquad \text{for} \quad \omega \ge \omega_P$$

$$|T(j\omega_P)| = \frac{1}{\sqrt{1+\varepsilon^2}} \qquad \text{for} \quad \omega = \omega_P$$

- Maximum passband ripple :

$$A_{\max} = 10\log(1+\varepsilon^2)$$

conversely,

$$\varepsilon = \sqrt{10^{A_{\max}/10} - 1}$$

- The attenuation at the stopband edge($\omega = \omega_S$) is

$$A(\omega_S) = 10\log[1+\varepsilon^2 \cosh^2(N\cosh^{-1}(\omega_S/\omega_P))]$$

→Required order N calculation by

finding the lowest integer value of N that yields $A(\omega_S) \ge A_{\min}$.

- Increasing the order N of the Chebyshev filter causes its magnitude function to approach the ideal brick-wall low-pass response.

The Chebyshev Filter

- The poles are

$$p_k = -\omega_P \sin\left(\frac{2k-1}{N}\frac{\pi}{2}\right)\sinh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right) + j\omega_P \cos\left(\frac{2k-1}{N}\frac{\pi}{2}\right)\cosh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right) \quad k = 1,2,\cdots,N$$

- The transfer function is

$$T(s) = \frac{K\omega_P^N}{\varepsilon 2^{N-1}(s-p_1)(s-p_2)\cdots(s-p_N)}$$

- How to find the transfer function
  1. Determine ε
  2. Determine the order required, A(ω_S)
  3. Determine the poles, p_k
  4. Determine the transfer function, $T(s)$

First-Order and Second-Order Filter Functions

- Nth-order response is very hard to visualize → Simple filter transfer functions
  — first and second order

- Cascade design.
  — possible for the design of active filters (utilizing op amps and RC circuits).
  — OP-amp output : low impedance

- High-order transfer function T(s) can be factored into the product of first-order and second-order functions.

First-Order Filter Function

❑ First-Order Filter Function

$$T(s) = \frac{a_1 s + a_0}{s + \omega_0}$$

→ bilinear transfer function

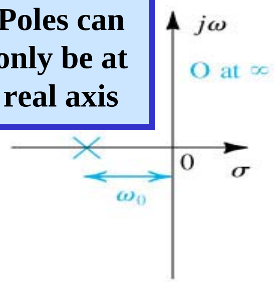
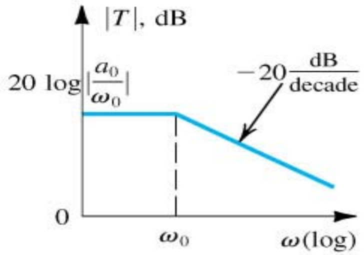
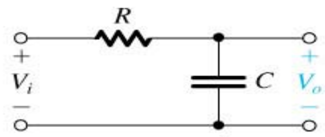
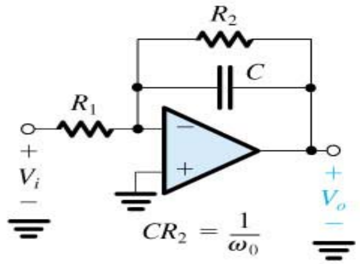
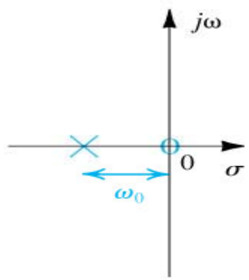
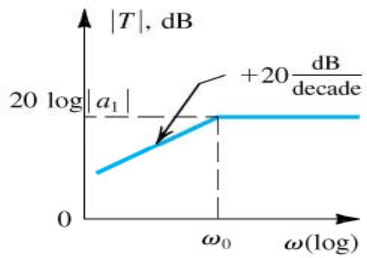
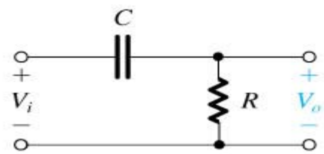
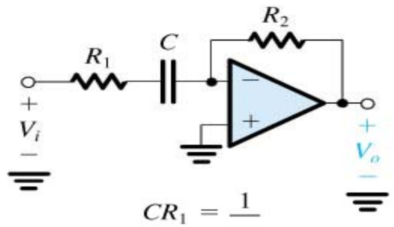
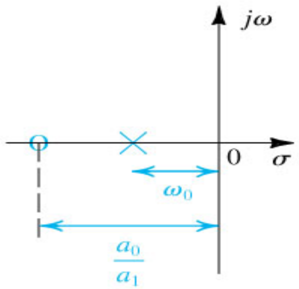
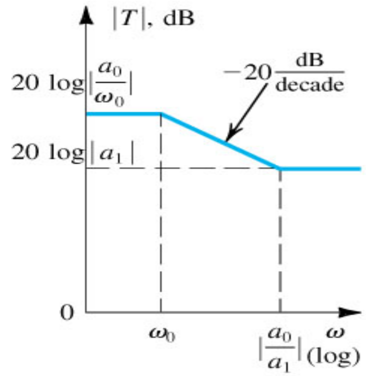
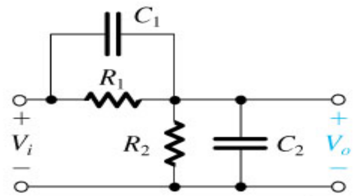
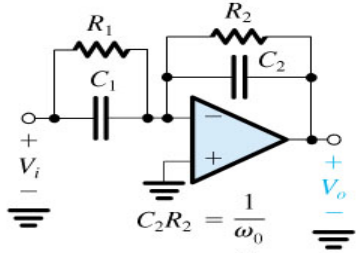
— natural mode at $s = -\omega_0$

— transmission zero at $s = -\frac{a_0}{a_1}$

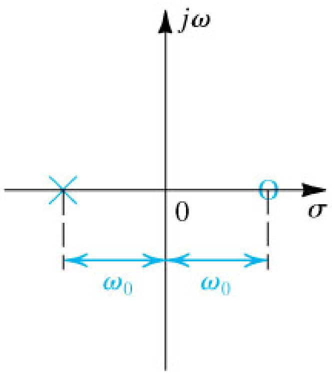
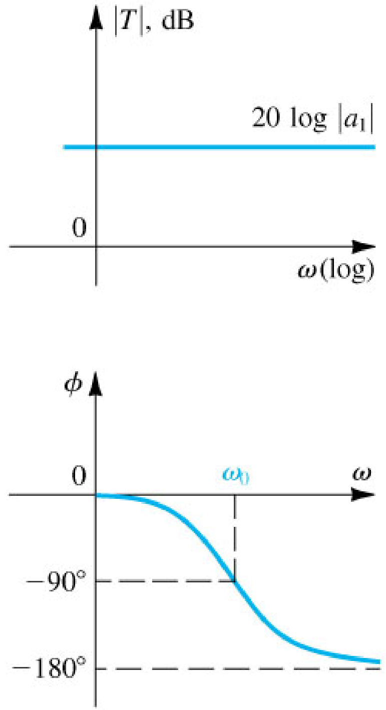
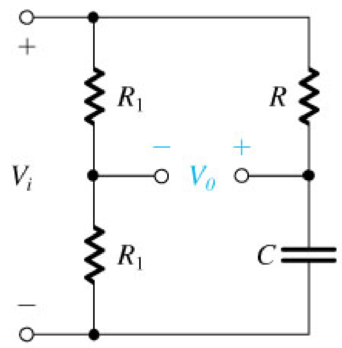
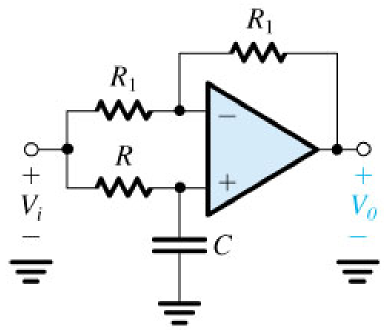
— high-frequency gain = a_1

— The numerator coefficients, a_0 and a_1, determine the type of filter

First-Order Filter Function

Filter Type and $T(s)$	s-Plane Singularities	Bode Plot for $\lvert T\rvert$	Passive Realization	Op Amp–RC Realization
(a) Low pass (LP) $T(s) = \frac{a_0}{s+\omega_0}$	**Poles can only be at real axis** 		 $CR = \frac{1}{\omega_0}$ DC gain $= 1$	 $CR_2 = \frac{1}{\omega_0}$ DC gain $= -\frac{R_2}{R_1}$
(b) High pass (HP) $T(s) = \frac{a_1 s}{s+\omega_0}$			 $CR = \frac{1}{\omega_0}$ High-frequency gain $= 1$	 $CR_1 = \frac{1}{\omega_0}$ High-frequency gain $= -\frac{R_2}{R_1}$
(c) General $T(s) = \frac{a_1 s + a_0}{s+\omega_0}$			 $(C_1 + C_2)(R_1 /\!/ R_2) = \frac{1}{\omega_0}$ $C_1R_1 = \frac{a_1}{a_0}$ DC gain $= \frac{R_2}{R_1 + R_2}$ HF gain $= \frac{C_1}{C_1 + C_2}$	 $C_2R_2 = \frac{1}{\omega_0}$ $C_1R_1 = \frac{a_1}{a_0}$ DC gain $= -\frac{R_2}{R_1}$ HF gain $= -\frac{C_1}{C_2}$

First-Order Filter Function

$T(s)$	Singularities	$\lvert T \rvert$ and ϕ	Passive Realization	Op Amp–RC Realization
All pass (AP) $T(s) = -a_1 \frac{s - \omega_0}{s + \omega_0}$ $a_1 > 0$			 $CR = 1/\omega_0$ Flat gain $(a_1) = 0.5$	 $CR = 1/\omega_0$ Flat gain $(a_1) = 1$

- Although the transmission is constant, its phase shows frequency selectivity
- All-pass filters are used as phase shifters and in systems that require phase shaping

Second-Order Filter Function

❑ Second-Order-Filters

$$T(s) = \frac{a_2 s^2 + a_1 s + a_0}{s^2 + (\omega_0 / Q)s + \omega^2{}_0}$$

ω_0 / Q

(i) pole position

(ii) physical

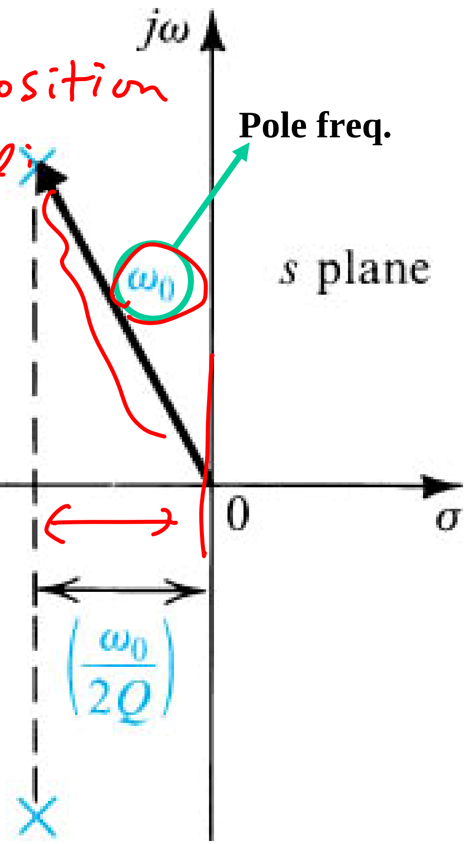
(iii) filter ch.

- Where ω_0 and Q determine the natural modes (poles) according to

$$p_1, p_2 = -\frac{\omega_0}{2Q} \pm j\omega_0 \sqrt{1 - (1/4Q^2)}$$

— ω_0 = pole frequency

- Q determines the distance of the poles from the $j\omega$ axis: the higher the value of Q the closer the poles are to the $j\omega$ axis
  → more selective
- Q < 0 → poles are in the RHP → oscillations
- Q = pole quality factor = pole Q

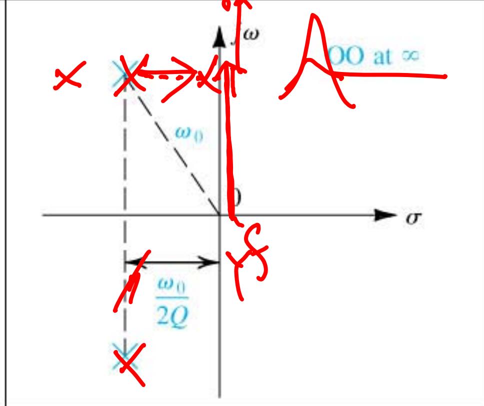


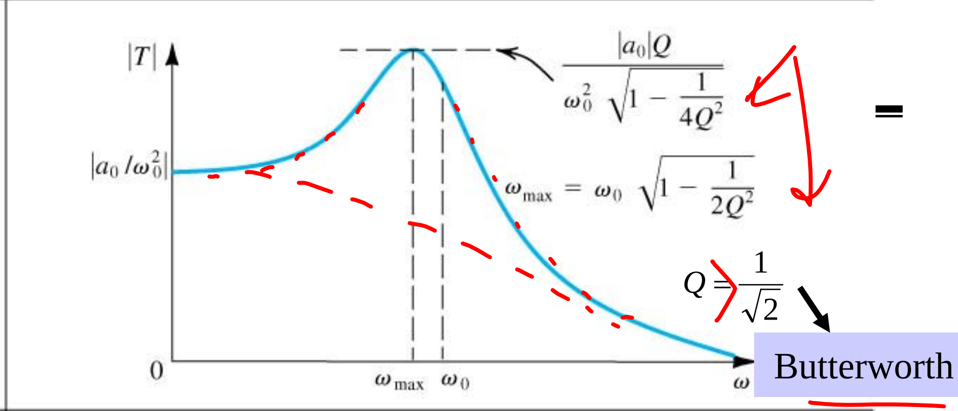
Filter Type and $T(s)$	s-Plane Singularities	$\lvert T \rvert$

(a) Low pass (LP)

$$T(s) = \frac{a_0}{\boxed{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}}$$

DC gain $= \frac{a_0}{\omega_0^2}$



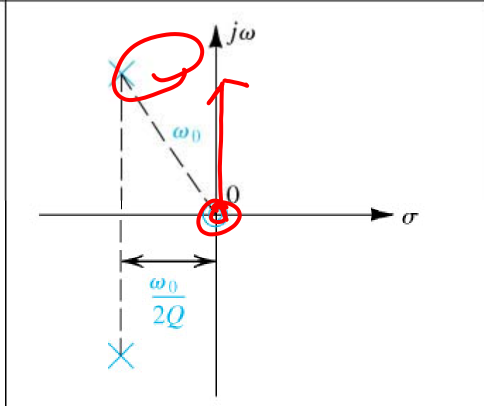


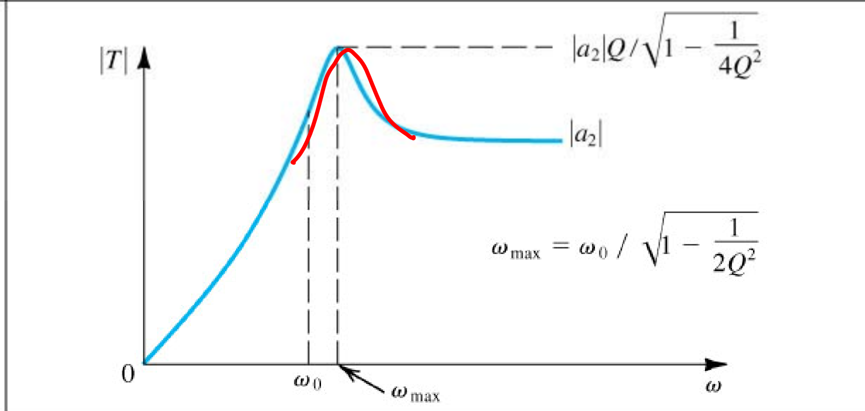
(b) High pass (HP)

as

$$T(s) = \frac{a_2 s^2}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

High-frequency gain $= a_2$

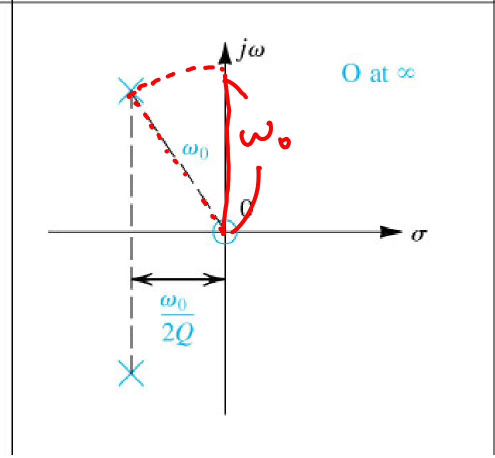


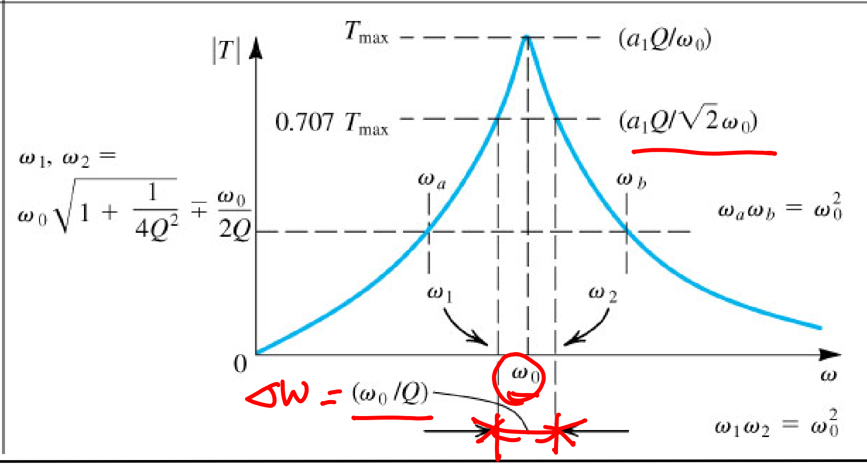


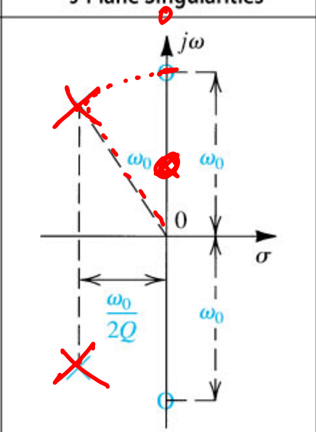
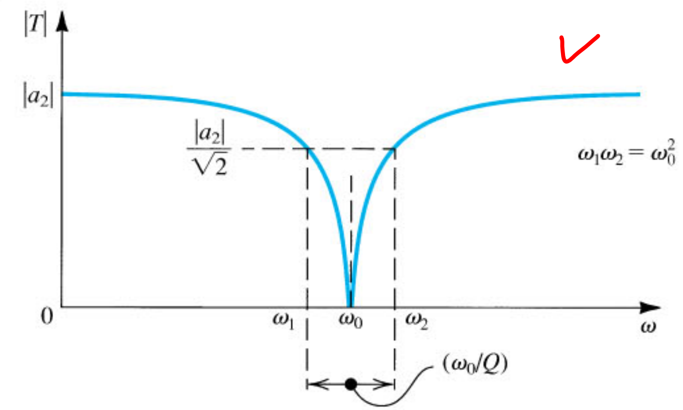
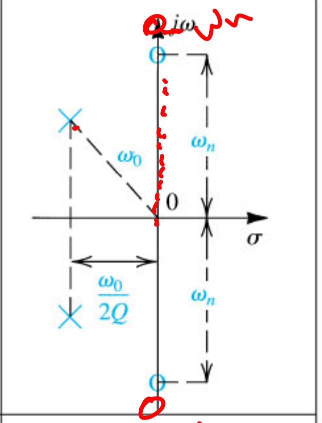
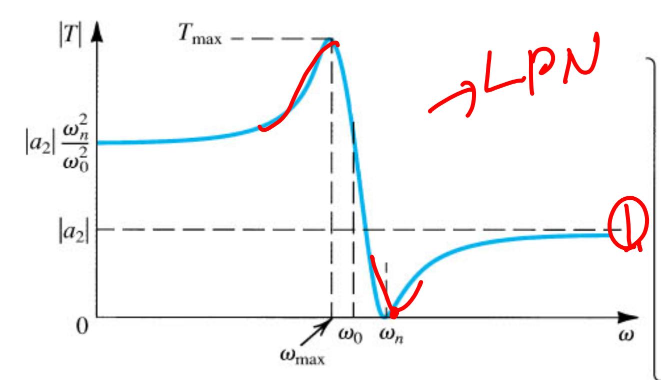
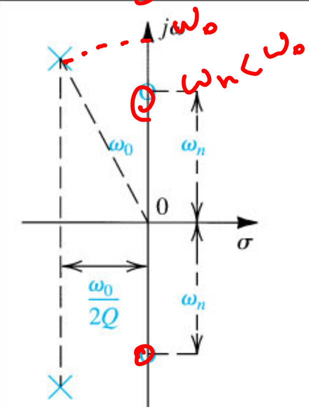
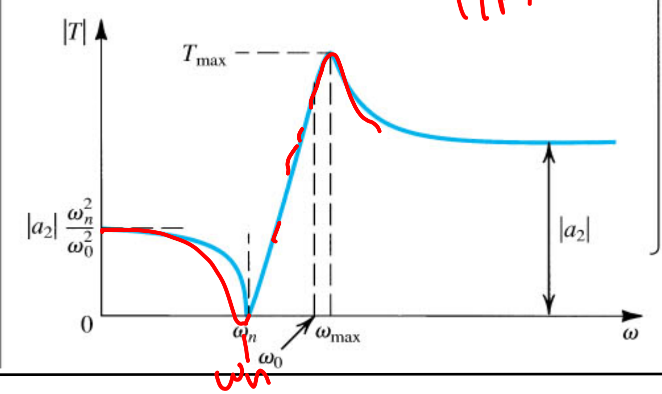
(c) Bandpass (BP)

$$T(s) = \frac{a_1 s}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

Center-frequency gain $= \frac{a_1 Q}{\omega_0}$





Filter Type and $T(s)$	s-Plane Singularities	$\lvert T \rvert$
(d) Notch ω_n $T(s) = a_2 \dfrac{s^2 + \omega_0^2}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$ DC gain = High-frequency gain = a_2 ω_0		
(e) Low-pass notch (LPN) $T(s) = a_2 \dfrac{s^2 + \omega_n^2}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$ $\omega_n \geq \omega_0$ DC gain $= a_2 \dfrac{\omega_n^2}{\omega_0^2}$ High-frequency gain $= a_2$		
(f) High-pass notch (HPN) $T(s) = a_2 \dfrac{s^2 + \omega_n^2}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$ $\omega_n \leq \omega_0$ DC gain $= a_2 \dfrac{\omega_n^2}{\omega_0^2}$ High-frequency gain $= a_2$		

$$\omega_{\max} = \omega_0 \sqrt{\dfrac{\left(\dfrac{\omega_n^2}{\omega_0^2}\right)\left(1 - \dfrac{1}{2Q^2}\right) - 1}{\dfrac{\omega_n^2}{\omega_0^2} + \dfrac{1}{2Q^2} - 1}}$$

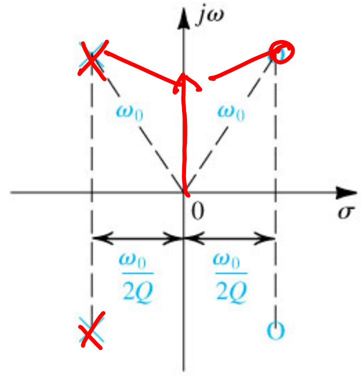
$$T_{\max} = \frac{|a_2| \, |\omega_n^2 - \omega_{\max}^2|}{\sqrt{(\omega_0^2 - \omega_{\max}^2)^2 + \left(\dfrac{\omega_0}{Q}\right)^2 \omega_{\max}^2}}$$

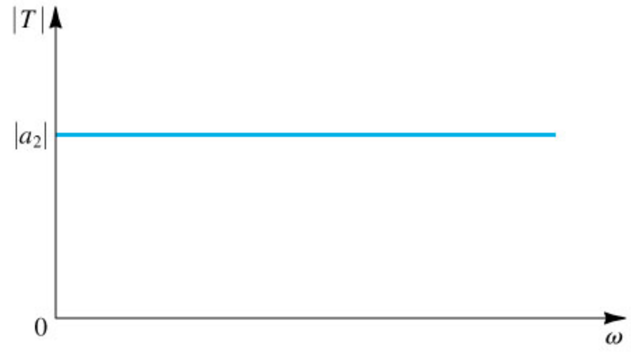
Second-Order Filter Function

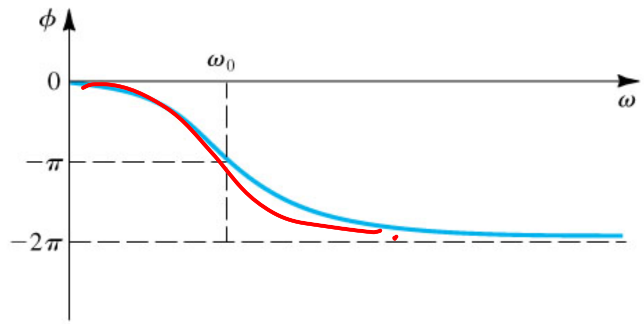
(g) All pass (AP)

$$T(s) = a_2 \frac{s^2 - s\frac{\omega_0}{Q} + \omega_0^2}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

Flat gain $= a_2$







Second-Order Filter Function

- LP case: The peak occurs only for $Q > \frac{1}{\sqrt{2}}$

  $Q = \frac{1}{\sqrt{2}}$ → Butterworth, or maximally flat
- HP case: Transmission zeros at s=0

  Dual to LP
- BP case: Transmission zeros at s=0 and $s=\infty$

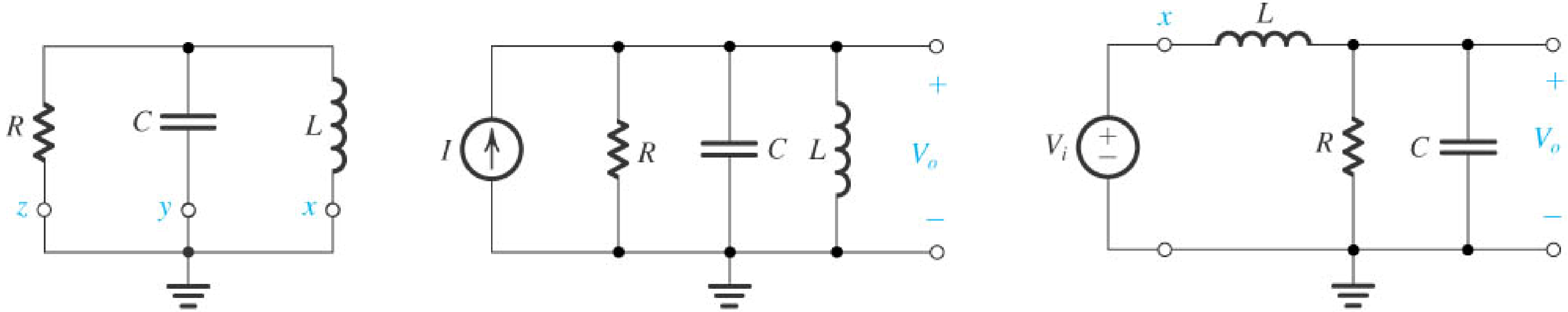
  Magnitude response peaks at $\omega = \omega_o$=center frequency

  3dB: ω_1, ω_2= $\omega_o\sqrt{1+\frac{1}{4Q^2}} \pm \frac{\omega_o}{2Q}$

  BW=$\omega_2 - \omega_1 = \frac{\omega_o}{Q}$:as Q↑ BW↓ more selective

The Second-order LCR Resonator : Natural Modes

❑ The Resonator Natural Modes



- The natural modes can be determined by applying an excitation that does not change the natural structure of the circuit

$$\frac{V_o}{I}=\frac{1}{Y}=\frac{1}{\frac{1}{sL}+sC+\frac{1}{R}}=\frac{\frac{s}{C}}{s^2+s(\frac{1}{CR})+\frac{1}{LC}} = \frac{as}{s^2+\frac{\omega_o}{Q}s+\omega_o^2}$$

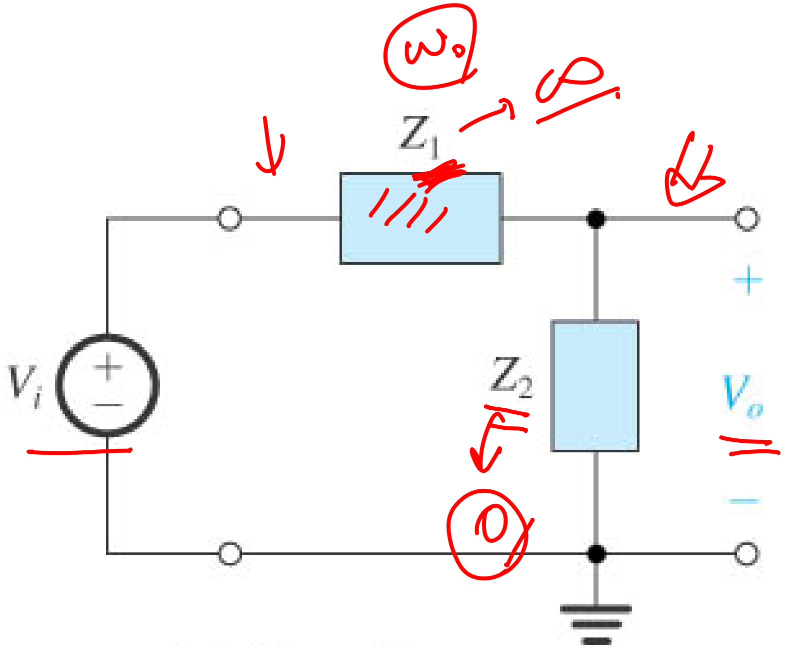
ω_o, Q

$$\omega_o^{\,2}=\frac{1}{LC}\quad;\quad\frac{\omega_o}{Q}=\frac{1}{CR}\quad;\quad\omega_0=\frac{1}{\sqrt{LC}}\quad;\quad Q=\omega_o CR = \omega_o \frac{L}{R}$$

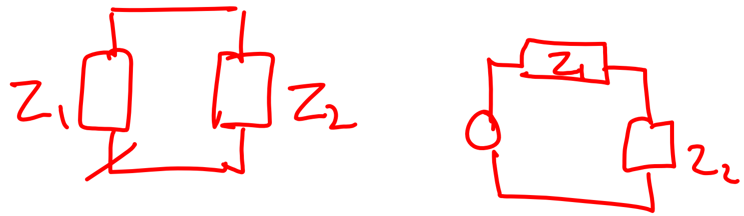
$\omega_o L = \frac{1}{\omega C}$

The Second-order LCR Resonator : Adding Zeros

❑ Realization of Transmission Zeros



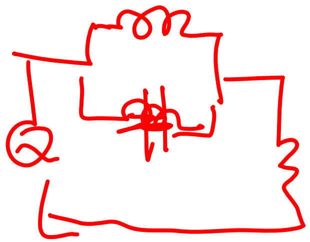
(a) General structure



- $T(s) = \dfrac{V_o(s)}{V_i(s)} = \dfrac{Z_2(s)}{Z_1(s)+Z_2(s)}$

$zeros:\ Z_2(s) = 0$ while $Z_1(s) \neq 0$

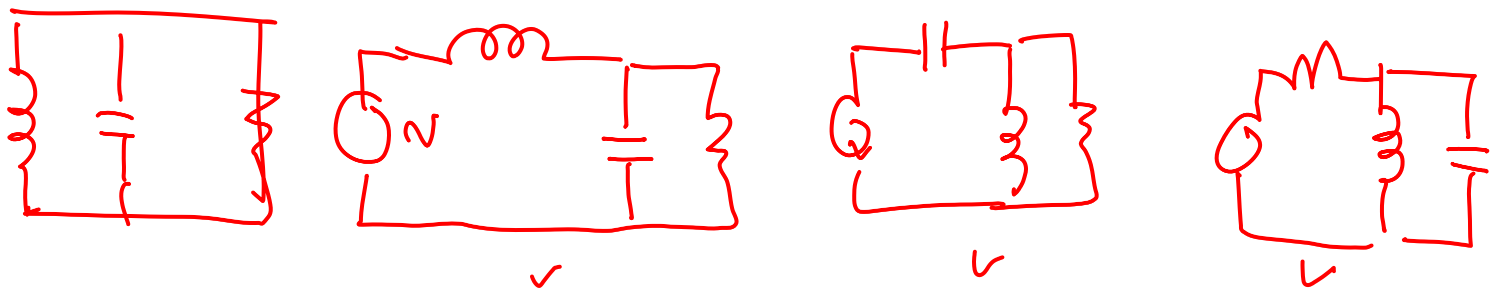
$Z_1(s) = \infty$ while $Z_2(s) \neq \infty$



LP → ∞ → 0

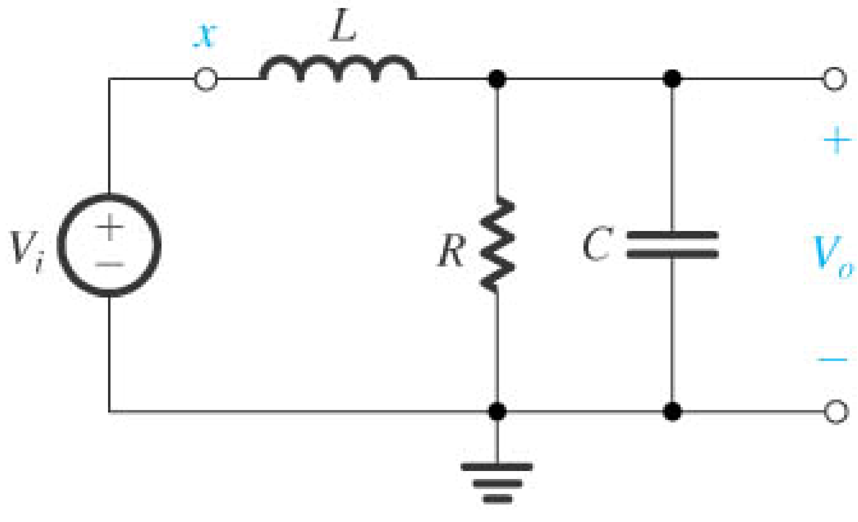
D → 0

∞ → 0



The Second-order LCR Resonator

❑ Realization of the Low-Pass Function



(b) LP

▪ zeros $sL = \infty$

$$\frac{1}{(sC + \frac{1}{R})} = 0$$

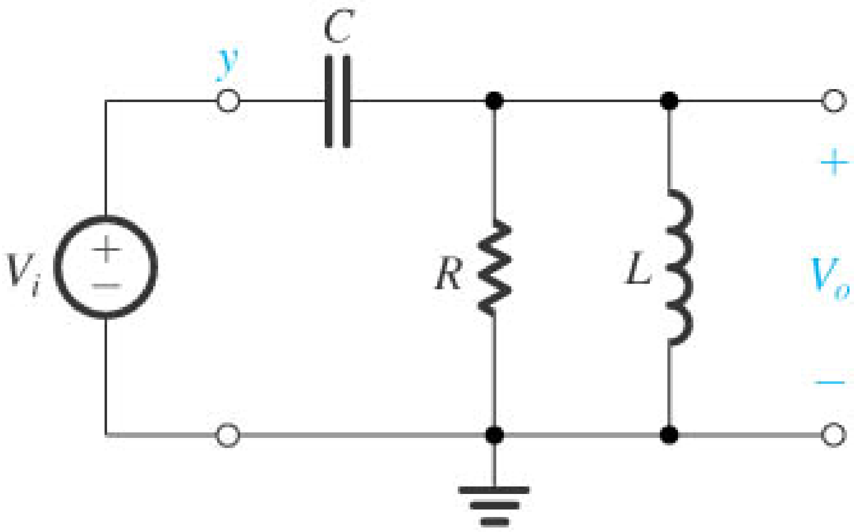
$\Rightarrow$ two zeros at $s = \infty$

$$T(s) \equiv \frac{V_o}{V_i} = \frac{Z_2}{Z_1 + Z_2} = \frac{Y_1}{Y_1 + Y_2} = \frac{\frac{1}{sL}}{\frac{1}{sL} + sC + \frac{1}{R}}$$

$$= \frac{1/LC}{s^2 + s(1/CR) + 1/LC}$$

The Second-order LCR Resonator

❑ Realization of the High-Pass Function



(c) HP

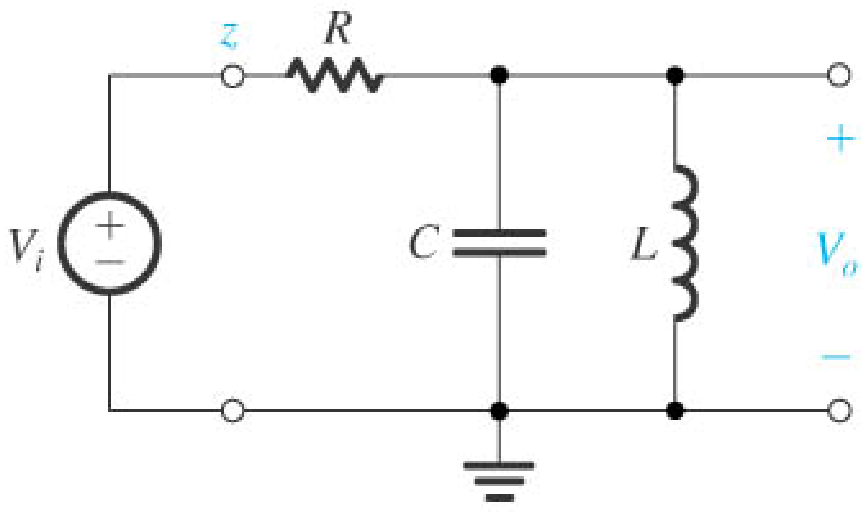
- zeros $\begin{cases} s = 0 : \text{Capacitor} \\ s = 0 : \text{Inductor} \end{cases}$

$$T(s) \equiv \frac{V_o}{V_i} = \frac{a_2 s^2}{s^2 + s(\frac{\omega_o}{Q}) + \omega_o^{\ 2}}$$

$$a_2 = 1$$

The Second-order LCR Resonator

❑ Realization of the Band-Pass Function



(d) BP

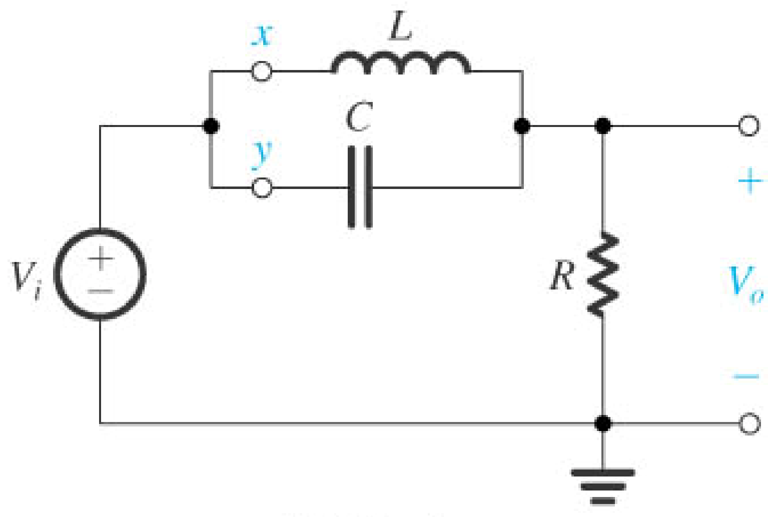
- zeros $\begin{cases} s = 0 & : \text{Inductor} \\ s = \infty & : \text{Capacitor} \end{cases}$

at ω0,
LC-tuned circuit exhibits an infinite impedance → no current flows
the center freq. gain is unity

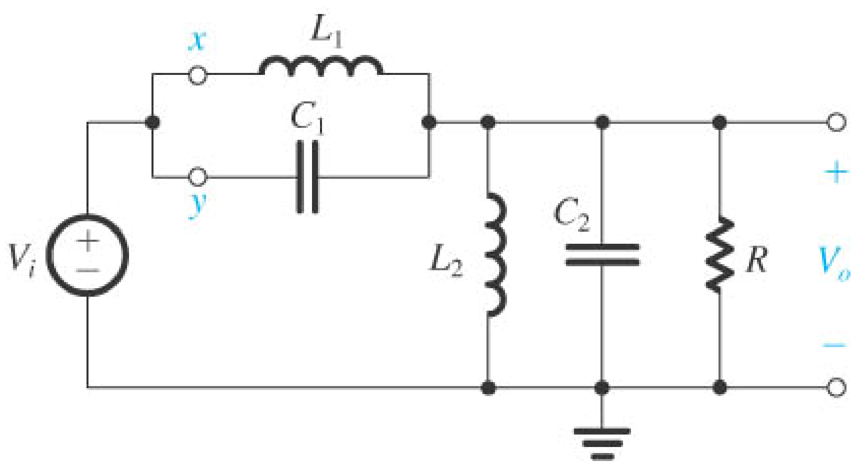
$$T(s) = \frac{Y_R}{Y_R + Y_L + Y_C} = \frac{\frac{1}{R}}{\frac{1}{R} + \frac{1}{sL} + sC}$$

$$= \frac{s(\frac{1}{CR})}{s^2 + s(\frac{1}{CR}) + (\frac{1}{LC})}$$

Realization of the Notch Functions



(e) Notch at ω_0



(f) General notch

- The impedance of the LC circuit becomes infinite at $\omega_0 = 1/\sqrt{LC}$
  → zero transmission
- The resistor does not introduce zeros.

$$T(s) = a_2 \frac{s^2 + \omega_0^{\ 2}}{s^2 + s(\omega_0 / Q) + \omega_0^{\ 2}}$$

- To obtain arbitrary ω_n

LPN

$$L_1 C_1 = \frac{1}{\omega_n^{\ 2}}$$

- L_1C_1 tank will introduce a pair of zeros at $\pm j\omega_n$, provided the L_2C_2 tank is not resonant at ω_n.
- The natural modes have not been altered,

$$C_1 + C_2 = C \quad ; \quad L_1 \| L_2 = L$$

- It is obtained from the original LCR resonator by lifting part of L and part of C off ground

Realization of the All-Pass Function

- The all-pass transfer function

$$T(s) = \frac{s^2 - s(\omega_o/Q) + \omega_o^2}{s^2 + s(\omega_o/Q) + \omega_o^2}$$

$$= 1 - \boxed{\frac{s2(\omega_o/Q)}{s^2 + s(\omega_o/Q) + \omega_o^2}}$$

bandpass function with a center-frequency gain of 2

- All pass realization with a flat gain of 0.5

$$T(s) = 0.5 - \frac{s(\omega_o/Q)}{s^2 + s(\omega_o/Q) + \omega_o^2}$$