26 Series, Convergence tests

26.1 Derivatives of Analytic Functions.

Theorem 1 (Derivatives of an analytic function)
If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of derivatives at a point z_0 in D are given by the formulas
(1')

$$f'(z_0) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^2}dz,$$

(1")

$$f''(z_0) = \frac{2}{2\pi i}\oint_c \frac{f(z)}{(z-z_0}^{3} dz,$$

and in general
(1)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz \qquad (n = 1, 2, \cdots);$$

here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D.
Proof.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

By Cauchy's integral formula ;

$$\begin{aligned}\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} &= \frac{1}{2\pi i \Delta z}\left[\oint \frac{f(z)}{z-(z_0+\Delta z)}dz - \oint \frac{f(z)}{z-z_0}dz\right] \\ &= \frac{1}{2\pi i \Delta z}\oint \frac{f(z)\{z-z_0-[z-(z_0+\Delta z)]\}}{[z-(z_0+\Delta z)][z-z_0]}dz\end{aligned}$$

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}dz$$

We consider the difference between these two integrals.

$$\begin{aligned}\oint_c \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}dz - \oint_c \frac{f(z)}{(z-z_0)^2}dz &= \oint_c \frac{f(z)[z-z_0-(z-z_0-\Delta z)]}{(z-z_0-\Delta z)(z-z_0)^2}dz \\ &= \oint \frac{f(z)\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}dz\end{aligned}$$

Being analytic, the function $f(z)$ is continuous on C, hence bounded in absolute value, $|f(z) \leq K$, Let d be the smallest distance from z_0 to the points of C.

$$|z - z_0|^2 \geq d^2, \qquad \text{hence} \qquad \frac{1}{|z - z_0|^2} \leq \frac{1}{d^2}$$

By the triangle inequality,

$$d \leq |z - z_0| = |z - z_0 - \Delta z + \Delta z| \leq |z - z_0 - \Delta z| + |\Delta z|$$

let $|\Delta z| \leq d/2$, so that $-|\Delta z| \geq -d/2$

$$\left| \oint_c \frac{f(z)\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2} dz \right| \leq KL|\Delta z| \cdot \frac{1}{d} \cdot \frac{1}{d^2}$$

This approaches zero as $\Delta z \to 0$

Example 1. Evaluation of line integrals.

for any contour enclosing the point πi (ccw)

$$\oint_c \frac{\cos z}{(z - \pi i)^2} dz = 2\pi i (\cos z)'|_{z=\pi i} = -2\pi i \sin \pi i = 2\pi \sinh \pi$$

Example 2. for any contour enclosing the point $-i$ (ccw)

$$\begin{aligned} \oint_c \frac{z^4 - 3z^2 + 6}{(z + i)^3} dz &= \pi i (z^4 - 3z^2 + 6)''|_{z=-i} = \pi i (4z^3 - 6z)'|_{z=-i} \\ &= \pi i (12z^2 - 6)|_{z=-i} = \pi i(-12 - 6) = -18\pi i \end{aligned}$$

Example 3. for any contour for which 1 lies inside and $\pm 2i$ lie outside (ccw)

$$\begin{aligned} \oint \frac{e^z}{(z - 1)^2(z^2 + 4)} dz &= 2\pi i \left(\frac{e^z}{z^2 + 4} \right)' \Bigg|_{z=1} = 2\pi i \frac{e^z(z^2 + 4) - e^z(2z)}{(z^2 + 4)^2} \Bigg|_{z=1} \\ &= 2\pi i \frac{e(5) - e(2)}{25} = \frac{6e\pi}{25} i \approx 2.050i \end{aligned}$$

26.2 Cauchy's Inequality. Liouville's and Morera's Theorems.

Choose for C a circle of radius r and center z_0 with $|f(z)| \leq M$ on C

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \oint \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} M \cdot \frac{1}{r^{n+1}} 2\pi r$$

(2)

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n} \qquad : \text{Cauchy's inequality}$$

Theorem 2 Liouville's theorem

If an entire function $f(z)$ is bounded in absolute value for all z, then $f(z)$ must be a constant.
Proof. By assumption, $|f(z)|$ is bounded, say, $|f(z)| < k$ for all z. Using Cauchy's inequality, $|f'(z_0)| < k/r$. Since $f(z)$ is entire, this is true for every r, so that we can take r as large as we please and conclude that $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) = 0$ for all z, and $f(z)$ is constant.

Theorem 3 Morera's theorem (Converse of Cauchy's integral theorem)

If $f(z)$ is continuous in a simply connected domain D and if
(3)

$$\oint_c f(z)dz = 0$$

for every closed path in D, then $f(z)$ is analytic in D.
Proof. If $f(z)$ is analytic in D, then

$$F(z) = \int_{z_0}^{z} f(z^*)dz^*$$

is analytic in D and $F'(z) = f(z)$. In the proof we used only the continuity of $f(z)$ and the property that its integral around every closed path in D is zero ; from these assumptions we conclude that $F(z)$ is analytic. By theorem 1, the derivative of $F(z)$ is analytic, that is $f(z)$ is analytic in D.