Aircraft Structures

CHAPTER 8.
Thin-walled beams

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8.1 Basic equation for thin-walled beams

❖ Typical aeronautical structures

● Typical aeronautical structures --- Light-weight, thin walled, beam-like structure
← complex loading environment
(combined axial, bending, shearing, torsional loads)

- Closed or open sections, or a combination of both : profound implications for the structural response (shearing and torsion)

- Thin-walled beams : specific geometric nature of the beam will be exploited to simplify the problem's formulation and solution process

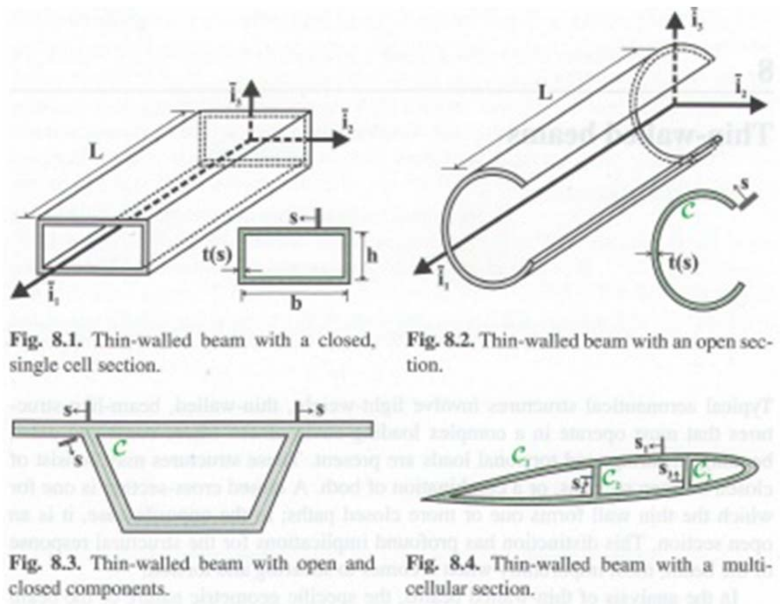


Fig. 8.1. Thin-walled beam with a closed, single cell section.

Fig. 8.2. Thin-walled beam with an open section.

Fig. 8.3. Thin-walled beam with open and closed components.

Fig. 8.4. Thin-walled beam with a multi-cellular section.

8.1 : closed section

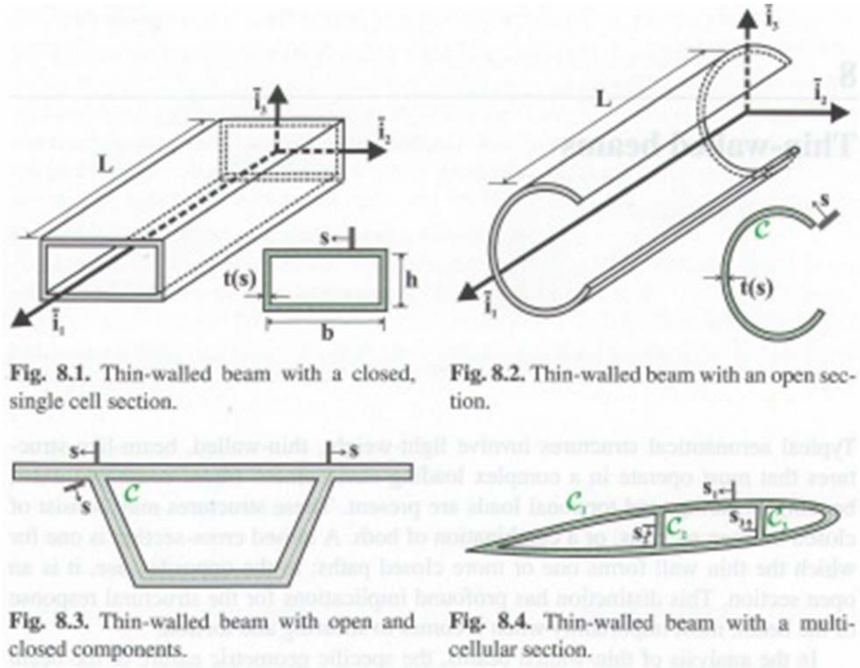
8.2 : open section

8.3 : combination of both

8.4 : multi-cellular section

8.1 Basic equation for thin-walled beams

8.1.1 The thin wall assumption



C : geometry of the section, along the mid-thickness of the wall

s : length along the contour, orientation along C

t(s) : wall thickness

- The thin wall assumption --- wall thickness is assumed to be much smaller than the other representative dimensions.

$$\frac{t(s)}{b} \ll 1, \quad \frac{t(s)}{h} \ll 1, \quad \frac{t(s)}{\sqrt{b^2+h^2}} \ll 1 \quad (8.1)$$

8.1 Basic equation for thin-walled beams

8.1.1 The thin wall assumption

- The thin-walled beam must also be long to enable the beam theory to be a reasonable approximation

$$\frac{\sqrt{b^2+h^2}}{L} \ll 1$$

8.1.2 Stress flows

- The stress components acting in the plane of the cross-section are assumed to be negligible as compared to the others.

$$\sigma_3 \ll \sigma_1 \;, \quad \tau_{23} \ll \tau_{12} \;, \quad \tau_{23} \ll \tau_{13}$$

  - Only non-vanishing components : axial stress σ_1
    transverse shear stress τ_{12}, τ_{13}

- It is preferable to use the stress components parallel and normal to *C*.

  - τ_n, τ_s, rather than Cartesian components.

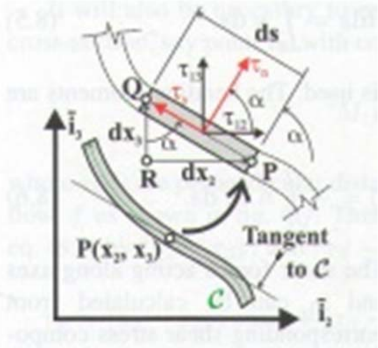


Fig. 8.5. Geometry of a differential element of the wall.

$$\tau_n = \tau_{12}\cos\alpha + \tau_{13}\sin\alpha = \tau_{12}\frac{dx_3}{ds} - \tau_{13}\frac{dx_2}{ds} \quad \text{(8.2a)}$$

$$\tau_s = -\tau_{12}\sin\alpha + \tau_{13}\cos\alpha = \tau_{12}\frac{dx_2}{ds} - \tau_{13}\frac{dx_3}{ds} \quad \text{(8.2b)}$$

$$\cos\alpha = \frac{dx_3}{ds}, \sin\alpha = -\frac{dx_2}{ds}$$

Sign convention for *s*

8.1.2 Stress flows

➢ Principle of reciprocity of shear stress → normal shear stress

- τ_n must vanish at the two edges of the wall because the outer surfaces are stress free.
- No appreciable magnitude of this stress component can build up since the wall is very thin.
- τ_n vanishes through the wall thickness.
- The only non-vanishing shear stress component : τ_s , tangential stress

Inverting Eq. (8.2a), (8.2b), and $\tau_n \simeq 0$

$$\tau_{12} \approx \tau_s \frac{dx_2}{ds} \quad , \quad \tau_{13} \approx \tau_s \frac{dx_3}{ds} \qquad (8.3)$$

8.1 Basic equation for thin-walled beams

8.1.2 Stress flows

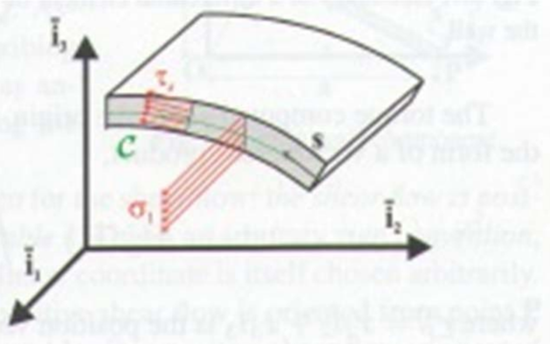


Fig. 8.6. Uniform distributions of axial and shear stresses across the wall thickness.

Thin-walled beams :

It seems reasonable to assume that τ_s is uniformly distributed across the wall thickness since the wall is very thin.

Concept of "stress flow"

$$n(x_1, s) = \sigma_1(x_1, s)t(s) \quad (8.4a)$$

$$f(x_1, s) = \tau_s(x_1, s)t(s) \quad (8.4b)$$

n : "axial stress flow," "axial flow"

f : "shearing stress flow," "shear flow"

- Only necessary to integrate a stress flow along *C*, instead of over an area, to compute a force.

8.1.3 Stress resultant

- Integration over the beam's cross-sectional area → integration along curve *C*
- Infinitesimal area of the cross-section $dA = tds$

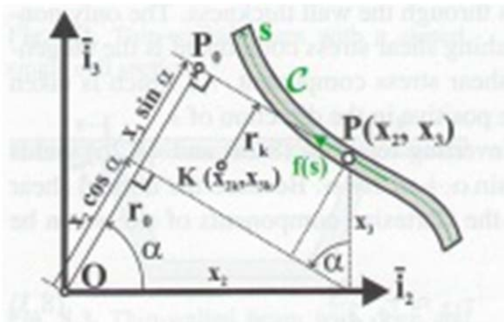


Fig. 8.7. Geometry of a differential element of the wall.

- axial force

$$N_1(x_1) = \int_A \sigma_1 dA = \int_C \sigma_1 tds = \int_C nds \qquad (8.5)$$

Axial flow

- bending moments

$$M_2(x_1) = \int_C n x_3 ds \quad M_3(x_1) = -\int_C n x_2 ds \qquad (8.6)$$

- shear forces

$$V_2(x_1) = \int_C f \frac{dx_2}{ds} ds \quad V_3(x_1) = \int_C f \frac{dx_3}{ds} ds \qquad (8.7)$$

8.1.3 Stress resultant

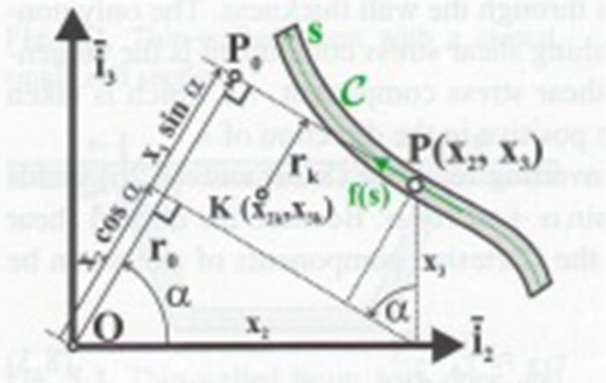


Fig. 8.7. Geometry of a differential element of the wall.

- Torque about origin O,

$$\vec{M}_O(x_1) = \int_C \vec{r}_P \times f d\vec{s}$$

$$\vec{r}_P = x_2 \bar{i}_2 + x_3 \bar{i}_3$$: position vector of point P

$$d\vec{s} = dx_2 \bar{i}_2 + dx_3 \bar{i}_3$$: increment in curvilinear coord.

$$\vec{M}_O(x_1) = \int_C (x_2 dx_3 - x_3 dx_2) f \bar{i}_1 = \int_C (x_2 \frac{dx_3}{ds} - x_3 \frac{dx_2}{ds}) f \bar{i}_1 ds$$

At point P_o,

$$r_O = x_2 \cos\alpha + x_3 \sin\alpha = x_2 \frac{dx_3}{ds} - x_3 \frac{dx_2}{ds} \qquad (8.8)$$

8.1.3 Stress resultant

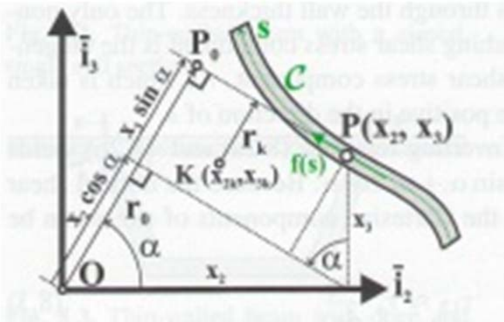


Fig. 8.7. Geometry of a differential element of the wall.

- Magnitude of the torque

$$M_{1O}(x_1) = \int_C f r_O \, ds \ , \qquad r_O \neq |\vec{r}_P| \qquad (8.9)$$

--- torque = magnitude of the force X perpendicular distance from the point to the line of action of the force

- Torque about an arbitrary point *K*, of the cross-section

$$M_{1k}(x_1) = \int_C f r_k \, ds \quad (8.10) \quad \text{and,} \quad r_k = (x_2 - x_{2k})\cos\alpha + (x_3 - x_{3k})\sin\alpha = r_O - x_{2k}\frac{dx_3}{ds} + x_{3k}\frac{dx_2}{ds}$$

  - r_k : perpendicular distance from *K* to the line of action of the shear flow (8.11)

8.1.4 Sign conventions

variable *s*,

$$x_2(s) = a\left(1-\frac{s}{l}\right), \quad x_3(s) = b\frac{s}{l}, \quad l = \sqrt{a^2+b^2}$$

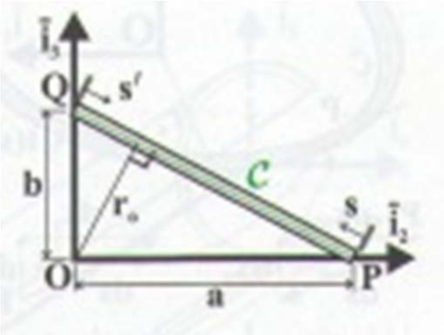


Fig. 8.8. Thin wall component.

The perpendicular distance from *O*, to the tangent curve *C*, denote r_o, becomes

$$r_O = x_2\frac{dx_3}{ds} - x_3\frac{dx_2}{ds} = a\left(1-\frac{s}{l}\right)\frac{b}{l} - b\frac{s}{l}\left(-\frac{a}{l}\right) = \frac{ab}{l} \quad (8.12)$$

variable *s*',

$$x_2(s') = a\frac{s'}{l}, \quad x_3(s) = b\left(1-\frac{s'}{l}\right)$$

r'_o becomes,

$$r'_O = x_2\frac{dx_3}{ds'} - x_3\frac{dx_2}{ds'} = a\frac{s'}{l}\left(-\frac{b}{l}\right) - b\left(1-\frac{s'}{l}\right)\frac{a}{l} = \frac{ab}{l} \quad (8.13)$$

8.1.4 Sign conventions

- The sign convention for the torque is independent of the choice of the curvilinear variable, *s*

s: counterclockwise, *s'*: clockwise

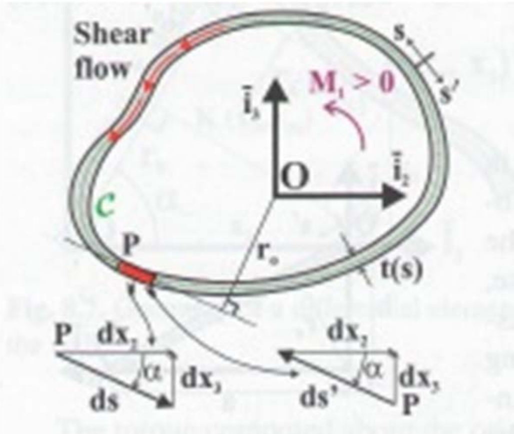


Fig. 8.9. Sign conventions for the shear flow, $f(s)$.

$$f'(s') = -f(s) \qquad r'_O(s') = -r_O(s)$$

However, the resulting torque is unaffected by this choice.

$$M_{1O}(x_1) = \int_C f r_O \, ds = \int_C f' r'_O \, ds'$$

8.1.5 Local equilibrium equation

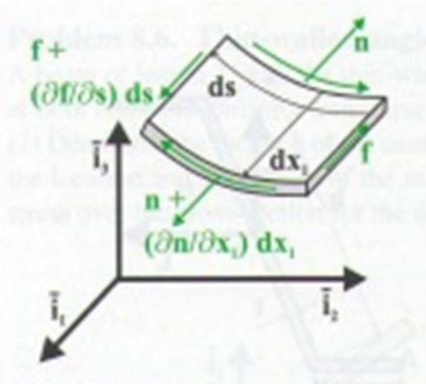


Fig. 8.10. Equilibrium of a differential element of the wall.

- A differential element of the thin-walled beam

--- all the forces acting along axis $\bar{i}_1$

$$-nds + \left(n + \frac{\partial n}{\partial x_1} dx_1 \right) ds - f dx_1 + \left(f + \frac{\partial f}{\partial s} ds \right) dx_1 = 0$$

After simplification,

$$\frac{\partial n}{\partial x_1} + \frac{\partial f}{\partial s} = 0 \qquad (8.14)$$

- Any change in axial stress flow, *n*, along the beam axis must be equilibrated by a corresponding change in shear flow, *f*, along curve *C* that defines the cross-section

8.2 Bending of thin-walled beams

• A thin-walled beam subjected to axial forces and bending moments
--- Euler-Bernoulli assumptions are applicable for either open or closed cross-sections

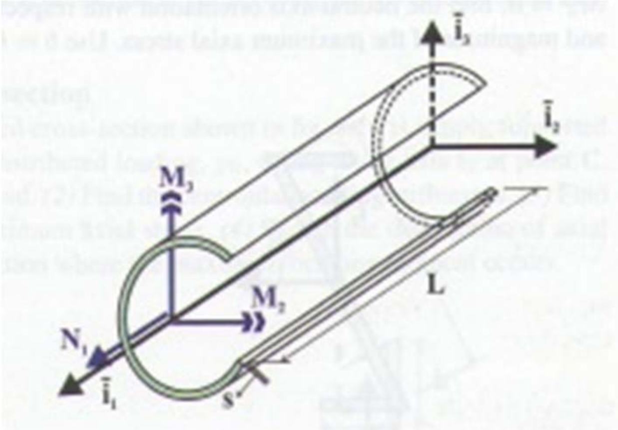


Fig. 8.11. Thin-walled beam subjected to axial forces and bending moments.

- Assuming a displacement field in the form of Eq. (6.1)
  Stain field given by Eq. (6.2a) – (6.2c)

--- axial stress distribution, from Eq. (6.15)

$$\sigma_1 = E\left[\frac{N_1}{S} - \frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta H} M_2 - \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta H} M_3\right] \qquad (8.15)$$

$$S = \int_A E dA \ , \qquad \Delta H = H_{22}^c H_{33}^c - (H_{23}^c)^2$$

$$H_{22}^c = \int_A E x_3^2 dA \ , \quad H_{33}^c = \int_A E x_2^2 dA \ , \quad H_{23}^c = \int_A E x_2 x_3 dA$$

- axial flow distribution using Eq. (8.4a)

$$n(x_1, s) = E(s)t(s)\left[\frac{N_1(x_1)}{S} - \frac{x_2(s)H_{23}^c - x_3(s)H_{33}^c}{\Delta H} M_2(x_1) - \frac{x_2(s)H_{22}^c - x_3(s)H_{23}^c}{\Delta H} M_3(x_1)\right] \qquad (8.16)$$

8.3 Shearing of thin-walled beams

- Bending moments in the thin-walled beams are accompanied by transverse shear force → give rise to shear flow distribution

  - evaluated by introducing the axial flow, given by Eq. (8.16) into the local equilibrium eqn., Eq. (8.14)

$$\frac{\partial f}{\partial s} = -Et\left[\frac{1}{S}\frac{dN_1}{dx_1} - \frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta H}\frac{dM_2}{dx_1} - \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta H}\frac{dM_3}{dx_1}\right] \qquad (8.17)$$

  - sectional equilibrium eqns, Eq. (6.16), (6.18), (6.20) substituting into (8.17), and assuming that $p_1, q_2, q_3 = 0$

$$\frac{\partial f}{\partial s} = -E(s)t(s)\left[-\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta H}V_3 + \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta H}V_2\right] \qquad (8.18)$$

  - Integration -> shear flow distribution arising from V_2, V_3

$$f(s) = c - \int_0^s Et\left[-\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta H}V_3 + \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta H}V_2\right]ds \qquad (8.19)$$

c: integration constant corresponding to the value at $s = 0$

The procedure to determine this depends on whether cross-section is closed or open.

- Since H^c_{∞}, V_2, V_3 are function of x_1 alone

$$f(s) = c + \frac{Q_3(s)H^c_{23} - Q_2(s)H^c_{33}}{\Delta H}V_3 - \frac{Q_3(s)H^c_{22} - Q_2(s)H^c_{23}}{\Delta H}V_2 \qquad (8.20)$$

where "stiffness static moment" or "stiffness first constant"

$$Q_2(s) = \int_0^s Ex_3(s)tds \qquad Q_3(s) = \int_0^s Ex_2(s)tds \qquad (8.21)$$

--- static moments for the portion of the cross-section from s = 0 to s

8.3.1 Shearing of open sections

Principle of reciprocity of shear stress $\tau_{12} = \tau_{21}, \tau_{23} = \tau_{32}, \tau_{13} = \tau_{31},$

→ shear flow vanishes at the end points of curve *C*

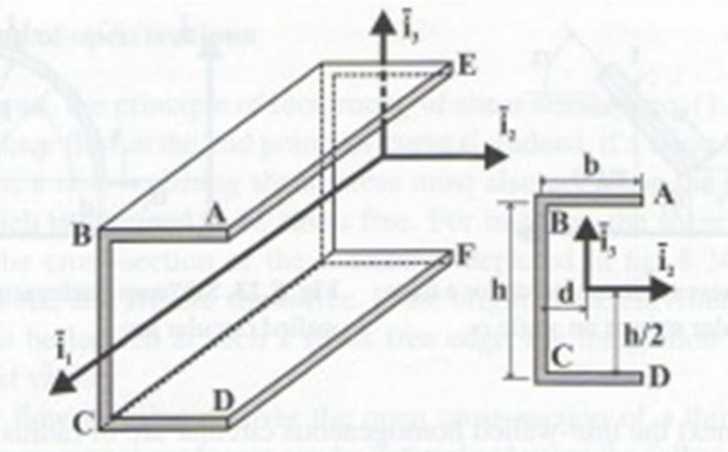


Fig. 8.24. Cantilevered beam with a C-channel cross-section.

Shear flow must vanish at point A and D since edges AE and DF are stress free.

If the origin of *s* is chosen to be located at such a stress free edge, the integration constant *c* in

$$f(s) = c + \frac{Q_3(s)H_{23}^c - Q_2(s)H_{33}^c}{\Delta H} V_3 - \frac{Q_3(s)H_{22}^c - Q_2(s)H_{23}^c}{\Delta H} V_2$$

must vanish.

8.3 Shearing of thin-walled beams

8.3.1 Shearing of open sections

Procedure to determine the shear flow distribution over cross-section

1. Compute the location of the centroid of the cross-section, and select a set of centroid axes, $\bar{i}_1$ and $\bar{i}_2$, and compute the sectional centroidal bending stiffness H_{22}^c, H_{33}^c and H_{23}^c. (principal centroidal axes $\rightarrow H_{23}^c = 0$)
2. Select suitable curvilinear coord. *s* to describe the geometry of cross-section.
3. Evaluate the 1st stiffness moments using

$$Q_2(s) = \int_0^s Ex_3(s)tds \qquad Q_3(s) = \int_0^s Ex_2(s)tds \qquad (8.21)$$

4. *f(s)* is determined by $f(s) = c + \dfrac{Q_3(s)H_{23}^c - Q_2(s)H_{33}^c}{\Delta H} V_3 - \dfrac{Q_3(s)H_{22}^c - Q_2(s)H_{23}^c}{\Delta H} V_2$ (8.20)

8.3.2 Evaluation of stiffness static moments

homogeneous, thin-walled rectangular strip oriented at an angle α

$$Q_2(s) = \int_0^s Ex_3 t ds = E\int_0^s (d_3 + s\sin\alpha) t ds = Est(d_3 + \frac{s}{2}\sin\alpha) \qquad (8.22)$$

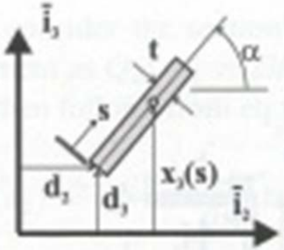


Fig. 8.22. Stiffness static moments for a thin-walled rectangular strip at an angle α.

Young's modulus $\times$ the area of strip $\times$ coord. of the centroid of the local area

similar result of the other stiffness static moment,

$$Q_3(s) = Est(d_2 + \frac{s}{2}\cos\alpha) \qquad (8.23)$$

Since the strip is made of a homogeneous material, *E* factors Out of integral.

$$Q_2(s) = E \underbrace{\int_0^s x_3 t ds}_{\text{Area static moment}}$$

8.3.2 Evaluation of stiffness static moments

Thin-walled homogeneous circular arc of radius R

$$ds = Rd\theta$$

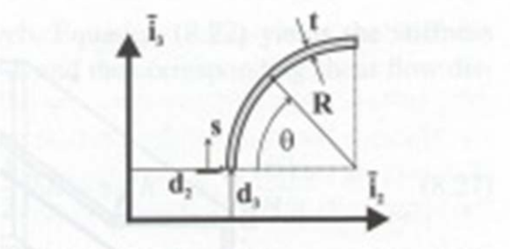


Fig. 8.23. Stiffness static moments for a thin-walled circular arc.

$$Q_2(s) = \int_0^s Ex_3 t ds = Et\int_0^\theta (d_3 + R\sin\theta)Rd\theta = EtR^2\left(\frac{d_3}{R}\theta + 1 - \cos\theta\right)$$

$$Q_3(s) = EtR^2\left[\left(1 + \frac{d_2}{R}\right)\theta - \cos\theta\right] \qquad (8.24)$$

stiffness static moment = E $\times$ area $\times$ distance to the area centroid

$$Q_2(s) = EAx_3 \qquad Q_3(s) = EAx_2$$

- ”Parallel axis theorem”, but in this case, only the transport term remains since the static moment about the area centroid itself is zero, by definition.

8.3.3 Shear flow distributions in open sections

- Example 8.1 Shear flow distribution in a C-channel

  - uniform thickness *t*, vertical web height *h*, flange width *b*, subject to a vertical shear force V_3
  - centroid: $$d = \frac{b}{\left(2+\frac{h}{b}\right)}$$
  - symmetric about axis $\bar{i}_2$, principal axes of bending, $H_{23}^c = 0$

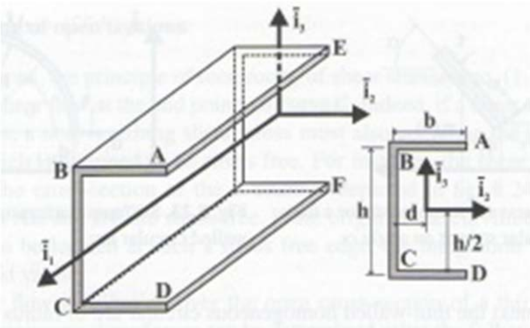


Fig. 8.24. Cantilevered beam with a C-channel cross-section.

$$f(s) = c - \frac{Q_2(s)}{H_{22}^c} V_3 \quad (8.25)$$

$$H_{22}^c = E\left[\frac{th^3}{12} + 2bt\left(\frac{h}{2}\right)^2\right] = E\left(\frac{h^3}{12} + \frac{bh^2}{2}\right)t$$

8.3.3 Shear flow distributions in open sections

➢ Example 8.1 Shear flow distribution in a C-channel

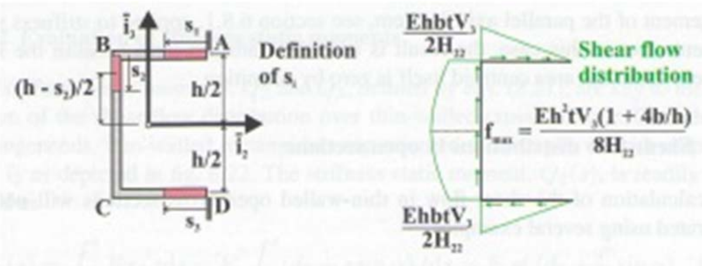


Fig. 8.25. Distribution of shear flow over the C-channel cross-section.

$$f(s_1) = c_1 - \frac{Q_2(s_1)}{H_{22}^c} V_3 = 0 - \frac{Ets_1 \frac{h}{2}}{H_{22}^c} V_3 = -\frac{Ehts_1}{2} \frac{V_3}{H_{22}^c} \qquad (8.26)$$

Because, $f(s_1 = 0) = 0$

$$Q_2(s_2) = Ets_2 \frac{h - s_2}{2} \qquad f(s_2) = c_2 - \frac{h - s_2}{2} ts_2 \frac{EV_3}{H_{22}^c} = -\frac{1}{2}[bh + s_2(h - s_2)] \frac{tEV_3}{H_{22}^c} \qquad (8.27)$$

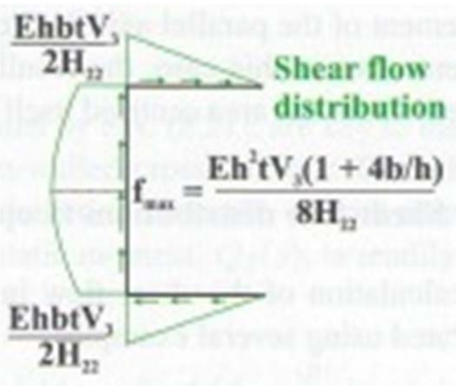
Because, $f(s_2 = 0) = f(s_1 = b)$

$$f(s_3) = c_3 + \frac{Ets_3 \frac{h}{2}}{H_{22}^c} V_3 = \frac{hs_3}{2} \frac{tEV_3}{H_{22}^c} \qquad (8.28)$$

8.3.3 Shear flow distributions in open sections

- Example 8.1 Shear flow distribution in a C-channel

  - upper and lower flange: linearly distributed, 0 at the edges
  - vertical web: varies in a quadratic manner, shear flow and the stress pointing upward
  - max. shear flow: mid-point of the vertical web



8.3.3 Shear flow distributions in open sections

- Example 8.2 Shear flow continuity conditions

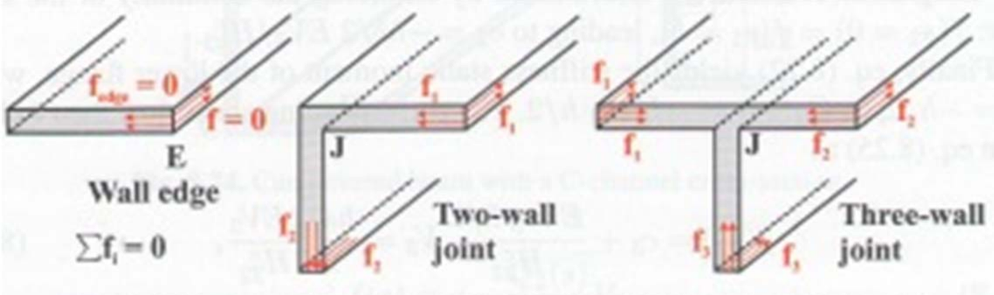


Fig. 8.26. Equilibrium condition at the junction of two or more thin walls.

- 2-wall joint : equilibrium of forces along the beam's axis → $-f_1+f_2=0$, or $f_1=f_2$
  : The shear flow must be continuous at the junction **J**

- 3-wall joint : $-f_1-f_2-f_3=0$, or more generally

$$\sum f_i = 0 \qquad (8.29)$$

- "sum of the shear flows converging to a joint must vanish.

8.3.5 Shear center for open sections

- Problem is not precisely defined --- Whereas the magnitudes of the transverse shear forces are given, their lines of action are not specified.
  -> It is not possible to verify the torque equilibrium of the cross section.

➢ Definition of the shear center

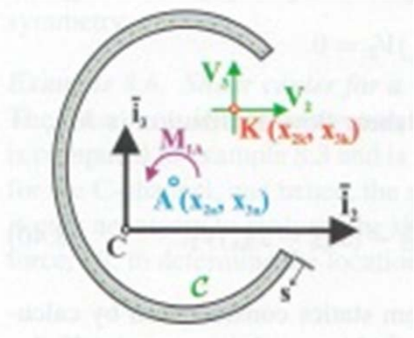


Fig. 8.30. Thin-walled open cross-section subjected to shear forces.

- subjected to horizontal and vertical shear force V_2, V_3 with lines of action passing through K, (x_{2K}, x_{3K}), no external torque applied, $M_{1K}=0$

8.3.5 Shear center for open sections

- 3 equipollence conditions

1 Integration of the horizontal component of the shear flow over cross -section must equal the applied horizontal shear force

$$\int_C f\left(\frac{dx_2}{ds}\right)ds = V_2$$

will be satisfied since it simply corresponds to the definition of shear force $V_2(x_1) = \int_C f\frac{dx_2}{ds}ds$

2 Integration of the vertical component of the shear flow over cross-section must equal the applied vertical shear force

$$\int_C f\left(\frac{dx_3}{ds}\right)ds = V_3$$

8.3.5 Shear center for open sections

- 3 equipollence conditions

  3 Torque generated by the distributed shear flow is equivalent to the externally applied torque, about the same point.

  --- does require the line of action of the the applied shear forces about point *K*, the torque,

$$M_{1k} = \int_C f r_k ds \qquad (8.10)$$

torque generated by the external forces w.r.t. point *K* = 0

$$M_{1k} = 0 + 0 \bullet V_2 + 0 \bullet V_3$$

$$M_{1k} = \int_C f r_k ds = 0 \qquad (8.39)$$

8.3.5 Shear center for open sections

--- point K cannot be an arbitrary point, its coords must satisfy the torque equipollence condition

$$M_{1k} = \int_C f r_k \, ds = 0 \qquad (8.39)$$

"Definition of the shear center location"

8.3.5 Shear center for open sections

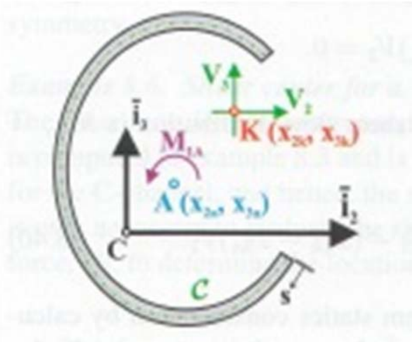


Fig. 8.30. Thin-walled open cross-section subjected to shear forces.

- Alternative definition

  Perpendicular distance from an arbitrary point *A* to the line of action $r_a = r_O - x_{2a}\dfrac{dx_3}{ds} + x_{3a}\dfrac{dx_2}{ds}$

  $\left(x_{2a}, x_{3a}\right)$: coord. of point *A*

  Subtracting this equation from Eq. (8.11)

$$r_k = r_a - (x_{2k} - x_{2a})\frac{dx_3}{ds} + (x_{3k} - x_{3a})\frac{dx_2}{ds}$$

Substituting into the torque equipollence condition, Eq. (8.39)

$$\int_C f r_a ds - (x_{2k} - x_{2a})\left[\int_C f\frac{dx_3}{ds}ds\right] + (x_{3k} - x_{3a})\left[\int_C f\frac{dx_2}{ds}ds\right]$$

$$= \int_C f r_a ds - (x_{2k} - x_{2a})V_3 + (x_{3k} - x_{3a})V_2 = 0$$

8.3.5 Shear center for open sections

Torque generated about point *A* by the shear flow distribution

$$M_{1a} = \int_C f r_a ds = (x_{2k} - x_{2a})V_3 - (x_{3k} - x_{3a})V_2 \qquad (8.40)$$

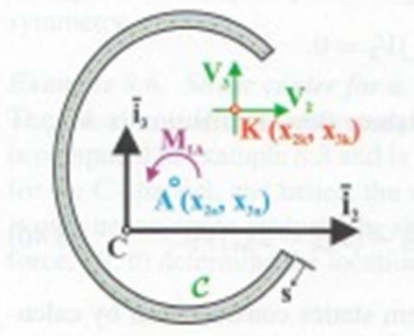


Fig. 8.30. Thin-walled open cross-section subjected to shear forces.

--- moment at *A* due to force and moment resultant at point *K*

$$M_{1a} = M_{1k} + (x_{2k} - x_{2a})V_3 - (x_{3k} - x_{3a})V_2$$

$$M_{1k} = 0 \quad \text{By Eq. (8.39)}$$

Eqs. (8.39), (8.40) ---
Torque generated by the shear flow distribution associated with transverse shear force must vanish w.r.t. the shear center.

8.3 Shearing of thin-walled beams

8.3.5 Shear center for open sections

- Summary
  - A beam bends without twisting if and only the transverse shear loads are applied at the shear center.
  - If the transverse loads are not applied at the shear center, the beam will both bend and twist.
  - If the cross-section features a plane of symmetry, the shear center must lie in that plane of symmetry.

8.3.5 Shear center for open sections

- Example 8.6 Shear center for a C-channel

  - axis i_2: axis of symmetry -> shear center lies at a point along this axis
  - It is necessary to evaluate the shear flow distribution by V_3, to determine the shear center location
  - Resultant force in each segment: by Eqs. (8.30) – (8.32)

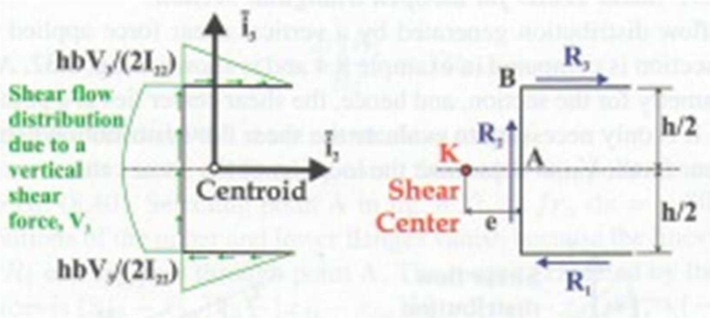


Fig. 8.31. Shear center in C-channel section.

$$R_1 = \int_0^b f(s_1)ds_1 = \frac{hb^2t}{4}\frac{EV_3}{H_{22}^c}$$

$$R_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} f(s_2)ds_2 = V_3$$

$$R_3 = \int_0^b f(s_3)ds_3 = \frac{hb^2t}{4}\frac{EV_3}{H_{22}^c} = R_1$$

- 3 equipollence conditions

$$R_1 - R_1 = 0$$

$$R_2 = V_3$$

8.3.5 Shear center for open sections

$$\int_C f r_k \, ds = -R_1 \frac{h}{2} + R_2 e - R_1 \frac{h}{2} = 0$$

$$e = \frac{hR_1}{R_2} = \frac{h^2 b^2 t}{4} \frac{E}{H_{22}^c} = \frac{3b}{6 + \dfrac{h}{b}} \qquad (8.41)$$

- Example 8.8 Shear center for a thin-walled right-angle section

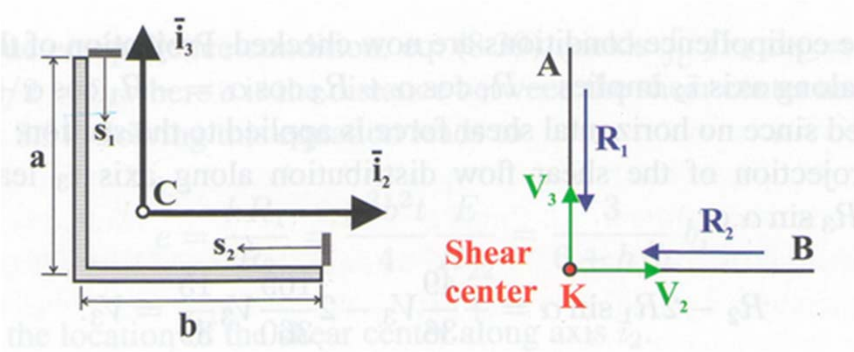


Figure 8.33 - Shear center in thin-walled right-angle section

Lines of actions of two resultant of the shear flow distributions, R_1 and R_2, will intersect at point K → procedures no torque about this point → must then be the shear center

8.3.7 Shearing of closed sections

Same governing equation $f(s) = c - \int_0^s Et\left[-\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta H} V_3 + \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta H} V_2\right] ds$ (8.19)

still applies, but no boundary condition is readily available to integrate this equation.

Exception: axis of symmetry

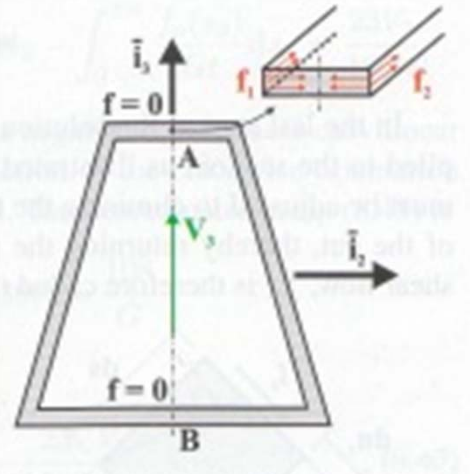


Fig. 8.34. Trapezoidal section subjected to a shear force.

If V_3 acts in the plane of symmetry $(\bar{i}_1, \bar{i}_3)$

→ mirror image of shear flow distribution

point A : joint equilibrium condition

$$f_1 + f_2 = 0$$

symmetry condition : $f_1 = f_2$

$$f_1 = f_2 = 0$$

shear flow vanishes at *A* and similarly *B*

8.3.7 Shearing of closed sections

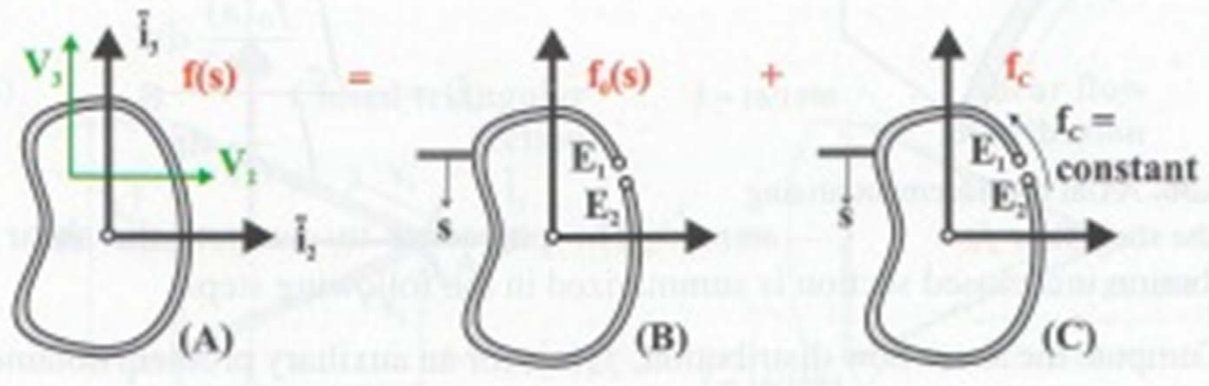


Fig. 8.35. (A): a general closed section. (B): the auxiliary problem created by cutting the section open. (C): the constant closing shear flow.

1st step : Beam is cut along its axis at an arbitrary point.

→ "auxiliary problem," shear flow distribution $f_o(s)$

2nd step : $f_o(s)$ creates a shear strain γ_s → infinitesimal axial strain du_1

$$du_1 = \gamma_s ds = \frac{\tau_s}{G} ds = \frac{f_0(s)}{Gt} ds \qquad (8.43)$$

8.3.7 Shearing of closed sections

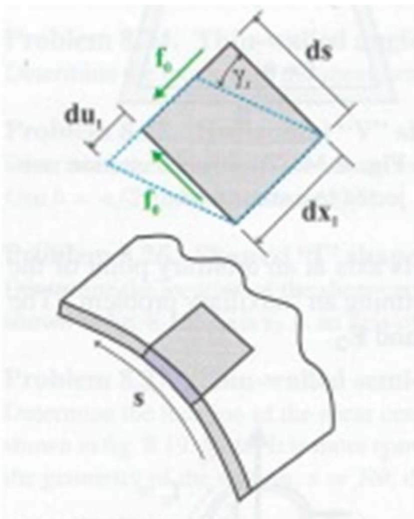


Fig. 8.36. Axial displacement arising from the shear flow f_0.

3rd step : total relative axial displacement at the cut

$$u_0 = \int_C \frac{f_0(s)}{Gt} ds$$

4th step : f_c is applied to eliminate the relative axial displacement, thereby returning the section to its original, closed state (f_c : "closing shear flow")

total shear flow $f(s) = f_0(s) + f_c(s)$

$$u_t = \int_C \frac{f_0(s) + f_c}{Gt} ds = 0 \qquad (8.44)$$

displacement compatibility eqn. for the closed section

$$f_c = -\frac{\int_C \frac{f_0(s)}{Gt} ds}{\int_C \frac{1}{Gt} ds} \qquad (8.45)$$

8.3.7 Shearing of closed sections

- Summary
  - $f_0(s)$ for an auxiliary problem
  - $f_c(s)$ by →→→→→→→→ $f_c = -\dfrac{\int_C \dfrac{f_0(s)}{Gt} ds}{\int_C \dfrac{1}{Gt} ds}$
  - $f(s) = f_0(s) + f_c(s)$

8.3.7 Shearing of closed sections

- Example 8.9 Shear flow distribution in a closed triangular section

  - shear flow distribution for open section: already computed in Example. 8.4

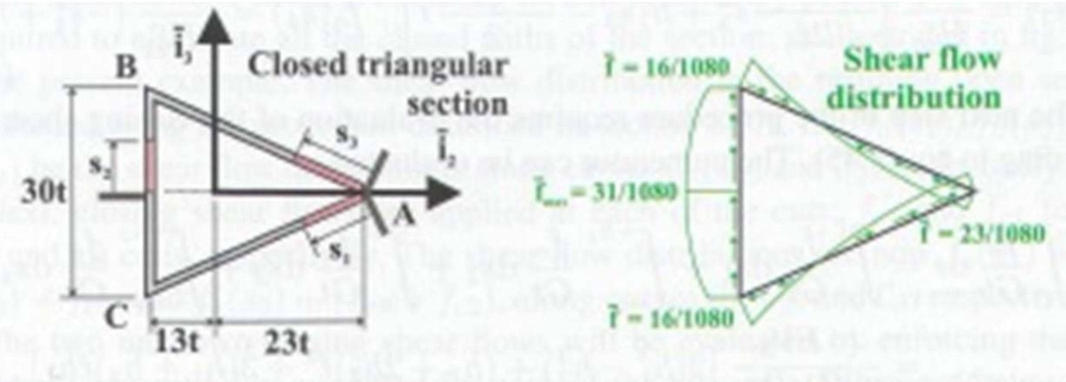


Fig. 8.37. Non-dimensional shear flow distribution in a closed triangular section.

$$f_0(s_1) = \frac{13}{360}\left(\frac{s_1}{39t}\right)^2 \frac{V_3}{t}$$

$$f_0(s_2) = \frac{13}{360}\frac{V_3}{t} + \frac{1}{72}\left[1 - \left(\frac{s_2}{15t}\right)^2\right]\frac{V_3}{t} \quad (8.46)$$

$$f_0(s_3) = -\frac{13}{360}\left(\frac{s_3}{39t}\right)^2 \frac{V_3}{t}$$

  - constant closing shear flow : by Eq. (8.45)

$$\int_C \frac{f_0}{Gt} ds = \int_0^{39t} \frac{f_0(s_1)}{Gt} ds_1 + \int_{-15t}^{15t} \frac{f_0(s_2)}{Gt} ds_2 - \int_0^{39t} \frac{f_0(s_3)}{Gt} ds_3 = \frac{23V_3}{10Gt}$$

$$\int_C \frac{ds}{Gt} = \frac{1}{Gt}(39t + 30t + 39t) = \frac{108}{G}$$

$$f_c = -\frac{\dfrac{23V_3}{10Gt}}{\dfrac{108}{G}} = -\frac{23V_3}{1080t} \quad (8.47)$$

8.3.7 Shearing of closed sections

- Example 8.9 Shear flow distribution in a closed triangular section

  - final shear flow distribution $f(s) = f_0(s) + f_c$

  - Both shear flow in the auxiliary section and the closing shear flow are (+) when pointing along the local curvilinear variable

8.3.8 Shearing of multi-cellular sections

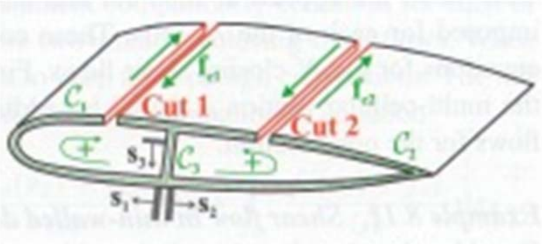


Fig. 8.39. A thin-walled, multi-cellular section.

- Procedure similar to that used for a single closed section must be developed. One cut per cell is required.

- Shear flow distribution in the resulting open sections is evaluated using the procedure in sec. 8.3.1 $f_0(s_1)$, $f_0(s_2)$, $f_0(s_3)$ along C_1, C_2, C_3
- Closing shear flows are applied at each cut. : f_{c1}, f_{c2}
- Then, shear flow distribution : $f_0(s_1)+f_{c1}$, $f_0(s_2)+f_{c2}$, $f_0(s_3)+(f_{c1}+f_{c2})$, along C_1 , C_2 , C_3 .

8.3.8 Shearing of multi-cellular sections

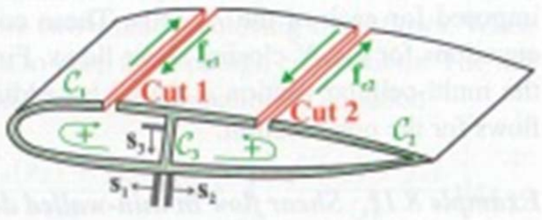


Fig. 8.39. A thin-walled, multi-cellular section.

front cell : clockwise / aft cell : counterclockwise

$$u_{t1} = \int_{C_1} \frac{f_0(s_1) + f_{c1}}{Gt} ds_1 + \int_{C3} \frac{f_0(s_3) + (f_{c1} + f_{c2})}{Gt} ds_3 = 0$$

$$u_{t2} = \int_{C_2} \frac{f_0(s_2) + f_{c2}}{Gt} ds_2 + \int_{C3} \frac{f_0(s_3) + (f_{c1} + f_{c2})}{Gt} ds_3 = 0$$

$$\left[\int_{C_1+C_3} \frac{1}{Gt} ds\right] f_{c1} + \left[\int_{C_3} \frac{1}{Gt} ds\right] f_{c2} = -\int_{C_1+C_3} \frac{f_0(s)}{Gt} ds$$

$$\left[\int_{C_3} \frac{1}{Gt} ds\right] f_{c1} + \left[\int_{C_2+C_3} \frac{1}{Gt} ds\right] f_{c2} = -\int_{C_2+C_3} \frac{f_0(s)}{Gt} ds$$

- Extension to multi-cellular section with *N* closed cells
  - Open section by *N* cut, one per cell: shear flow distribution in open section by the procedure in sec 8.3.1
  - Closing shear flows are applied at each cut and displacement compatibility conditions are imposed: *N* simultaneous equations.
  - Total shear flow distribution is found by adding the closing shear flow to that for the open section.

8.3.8 Shearing of multi-cellular sections

- Example 8.11 Shear flow in thin-walled double-box section

  - multi-cellular, thin-walled, double-box section subjected to a vertical shear force, V_3
  - right cell wall thickness $2t$, while the remaining three walls of the left cell wall thickness t

  - Due to symmetry, i_2: principal axis of bending -> $H_{23}^c = 0$
  - bending stiffness based on thin-wall assumption

$$H_{22}^c = E\left[2\left(\frac{2tb^3}{12}\right)+\frac{tb^3}{12}+2(bt+b\times 2t)\left(\frac{b}{2}\right)^2\right]=\frac{23}{12}tb^3E$$

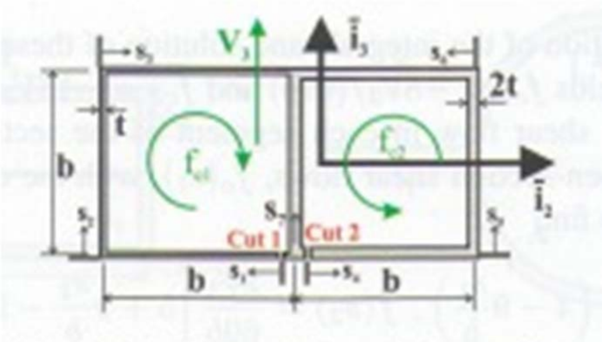


Fig. 8.40. A thin-walled double-box section.

  - 1st step: transformed into an open section by cutting the two lower flanges

8.3.8 Shearing of multi-cellular sections

- shear flow distribution for open section

$$f_0(s_1) = \frac{6V_3}{23b}\frac{s_1}{b},\ f_0(s_3) = \frac{6V_3}{23b}\left(1-\frac{s_3}{b}\right),\ f_0(s_4) = \frac{12V_3}{23b}\frac{s_4}{b}$$

$$f_0(s_2) = \frac{6V_3}{23b}\left[1+\left(1-\frac{s_2}{b}\right)\frac{s_2}{b}\right],\ f_0(s_5) = \frac{12V_3}{23b}\left[1+\left(1-\frac{s_5}{b}\right)\frac{s_5}{b}\right]$$

$$f_0(s_6) = \frac{12V_3}{23b}\left(1-\frac{s_6}{b}\right),\quad -f_0(s_7) = \frac{12V_3}{23b}\left(1-\frac{s_7}{b}\right)\frac{s_7}{b}$$

- 2nd step: closing shear flows, f_{c1}, f_{c2} , are added to the left and right cells

- axial displacement compatibility at left cell

$$u_{t1} = \int_0^b \frac{f_0(s_1)+f_{c1}}{Gt}ds_1 + \int_0^b \frac{f_0(s_2)+f_{c1}}{Gt}ds_2 + \int_0^b \frac{f_0(s_3)+f_{c1}}{Gt}ds_3$$

$$-\int_0^b \frac{f_0(s_7)-f_{c1}-f_{c2}}{G\times 2t}ds_7 = \frac{b}{Gt}\left(\frac{7f_{c1}}{2}+\frac{f_{c2}}{2}+\frac{12V_3}{23b}\right)=0$$

8.3.8 Shearing of multi-cellular sections

- axial displacement compatibility at right cell

$$u_{t2} = \int_0^b \frac{f_0(s_4)+f_{c2}}{G\times 2t} ds_4 + \int_0^b \frac{f_0(s_5)+f_{c2}}{G\times 2t} ds_5 + \int_0^b \frac{f_0(s_6)+f_{c2}}{G\times 2t} ds_6$$
$$-\int_0^b \frac{f_0(s_7)-f_{c1}-f_{c2}}{G\times 2t} ds_7 = \frac{b}{Gt}\left(\frac{f_{c1}}{2}+2f_{c2}+\frac{12V_3}{23b}\right)=0$$

- sol. of two simultaneous eqn.: $f_{c1} = -8\frac{V_3}{69b}, f_{c2} = -16\frac{V_3}{69b}$

- total shear flow in each segment of the section

$$f(s_1) = -\frac{2V_3}{69b}\left(4-9\frac{s_1}{b}\right),\ f(s_2) = \frac{2V_3}{69b}\left[5+9\frac{s_2}{b}-9\left(\frac{s_2}{b}\right)^2\right]$$
$$f(s_3) = \frac{2V_3}{69b}\left(5-9\frac{s_3}{b}\right),\ f(s_1) = -\frac{4V_3}{69b}\left(4-9\frac{s_4}{b}\right)$$
$$f(s_5) = \frac{4V_3}{69b}\left[5+9\frac{s_5}{b}-9\left(\frac{s_5}{b}\right)^2\right],\ f(s_6) = \frac{4V_3}{69b}\left(5-9\frac{s_6}{b}\right) \quad (8.50)$$
$$f(s_7) = \frac{12V_3}{69b}\left[2+3\frac{s_7}{b}-3\left(\frac{s_7}{b}\right)^2\right]$$

8.3.8 Shearing of multi-cellular sections

- shear flows in the webs vary quadratically, while those in flanges linearly

- Net resultant of the shear flows in the flanges must vanish because no shear forces is externally applied in the horizontal direction.

- Resultant of the shear flows in the webs must equal the externally applied vertical shear force, V_3

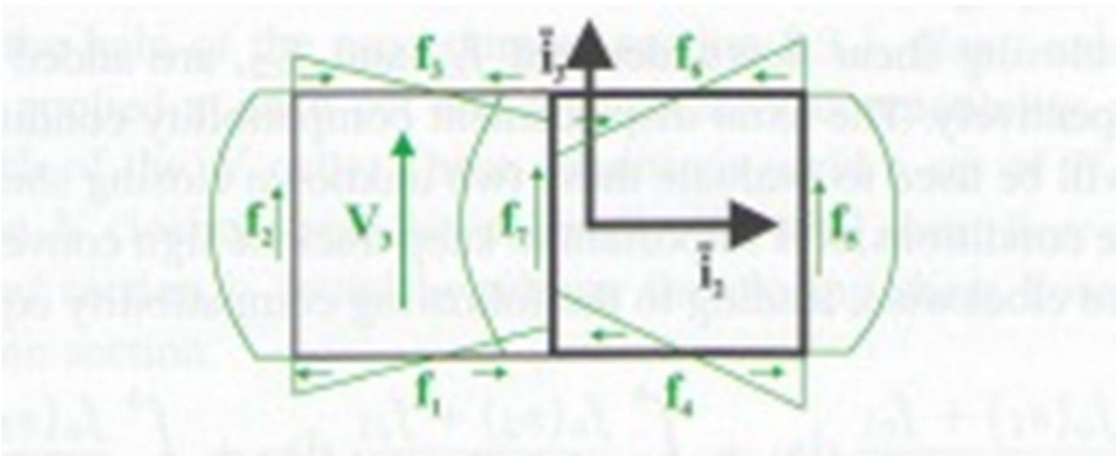


Fig. 8.41. Shear flow in the thin-walled double-box section.

8.4 The shear center

8.4 Shear Center

- **Chap. 6… Assumption that transverse loads are applied in "such a way that the beam will bend without twisting"**
  - More precise statement : the lines of action of all transverse loads pass through the shear center
  - If the sear forces are not applied at the shear center, the beam will undergo both bending and twisting

8.4.1 Calculation of the shear center location

- Involves two linearly independent loading cases

① $(\cdot)^{[2]}$, unit shear force $V_2^{[2]}=1$, no shear force along $\bar{i}_3$, $V_3^{[2]}=0$
$\rightarrow$ shear flow $f^{[2]}(s)$

② $(\cdot)^{[3]}$, $V_3^{[3]}=1, V_2^{[3]}=0 \rightarrow f^{[3]}(s)$

- from Eq.(8.7), shear forces equipollent to $f^{[2]}(s)$

$$V_2^{[2]}=\int_c f^{[2]}\frac{dx_2}{ds}ds=1, \quad V_3^{[2]}=\int_c f^{[3]}\frac{dx_3}{ds}ds=0 \qquad (8.51)$$

8.4 The shear center

- shear center location $K(x_{2K}, x_{3K})$: Eq (8.10) $\rightarrow$

$$M_{1K} = \int_c f^{[2]} r_K ds = \int_c f^{[2]} (r_0 - x_{2K} \frac{dx_3}{ds} + x_{3K} \frac{dx_2}{ds}) ds$$

r_K : distance from K to the tangent to contour C , Eq. (8.11)

- Rearranging

$$-x_{2K} \left[\underbrace{\int_c f^{[2]} \frac{dx_3}{ds} ds}_{0} \right] + x_{3K} \left[\underbrace{\int_c f^{[3]} \frac{dx_2}{ds} ds}_{1} \right] = \int_c f^{[2]} r_0 ds$$

by Eq.(8.51)

$$\rightarrow x_{3K} = -\int_c f^{[2]} r_0 ds \qquad (8.52)$$

similarly, $x_{2K} = \int_c f^{[3]} r_0 ds$ (8.53)

8.4 The shear center

- alternate torque equipollence condition, Eq.(8.40)

$$x_{3K} = x_{3a} - \int_c f^{[2]} r_a ds \qquad (8.54)$$

$$x_{2K} = x_{2a} - \int_c f^{[3]} r_a ds \qquad (8.55)$$

(x_{2a}, x_{3a}) : coordinate of an arbitrary point *A*

- General procedure for determination of the shear center
  ① compute the x-s centroid and select a set of centroidal axes (sometimes convenient with principal centroidal axes)

  ② compute $f^{[2]}(s)$ corresponding to $V_2^{[2]} = 1,\ V_3^{[2]} = 0$

  ③ compute $f^{[3]}(s)$ corresponding to $V_2^{[3]} = 0,\ V_3^{[3]} = 1$

  ➔ according to Sections 8.3.1 or 8.3.7

  ④ compute the coordinate of shear center using Eqs (8.52) and (8.53) or (8.54) and (8.55)

- If the x-s exhibits a plane of symmetry, simplified

plane $(\bar{i}, \bar{i}_2)$ is a plane of symmetry, the s.c. must be located in that plane.

$\rightarrow x_{3K} = 0$, Eq. (8.52) can be bypassed.

8.4 The shear center

- Example 8.12 Shear center of a trapezoidal section

  - closed trapezoidal section
  - shear flow distribution generated by a vertical shear force, V_3
    : sum of the shear flow distribution in the auxiliary open section and the closing shear flow $f(s) = f_0(s) + f_c$

$$f_0(s_1) = \frac{EV_3}{H_{22}^c}\left[\frac{h_2 - h_1}{2l} s_1^2 - h_2 s_1\right], \quad f_0(s_2) = \frac{EV_3}{H_{22}^c}\left[s_2^2 - h_1^2 - (h_1 + h_2) l\right],$$

$$f_0(s_3) = \frac{EV_3}{H_{22}^c}\left[\frac{h_2 - h_1}{2l} s_3^2 + h_1 s_3 - \frac{h_1 + h_2}{2} l\right], \quad f_0(s_4) = \frac{EV_3}{H_{22}^c}\left[-s_4^2 + h_2^2\right] \tag{8.48}$$

$$f_c = \frac{EV_3}{H_{22}^c} \frac{2(h_1^3 - h_2^3) + (h_1 + 2h_2) l^2 + 3(h_1 + h_2) l h_1}{6(l + h_1 + h_2)} \tag{8.49}$$

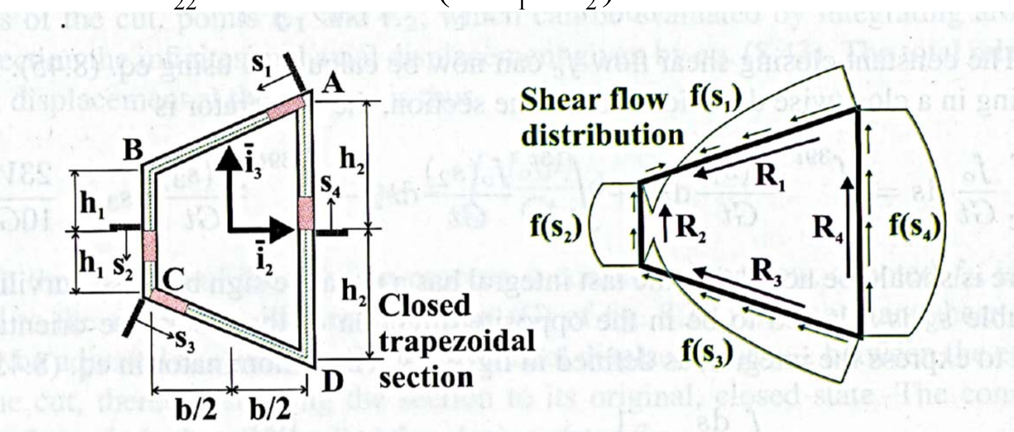


Fig. 8.38. Thin-walled trapezoidal section subjected to a vertical shear force, V_3

- location of the shear center: by Eq. (8.49)

$$x_{2k} = \int_C \left(\bar{f}_o^{[2]}(s) + \bar{f}_c^{[2]} \right) r_o ds$$

$$\bar{f}_o^{[2]}(s) = f_o(s)/V_3, \bar{f}_c^{[2]} = f_c/V_3, V_3 = 1$$

- Evaluation of integral

$$x_{2k} = \frac{b}{4} \frac{h_2 - h_1}{l} \frac{1-(h_1+h_2)/l}{1+(h_1+h_2)/l} \frac{1+l(h_2^2-h_1^2)/(h_2^3-h_1^3)}{1+l(h_2-h_1)(h_2^3+h_1^3)/(l(h_2^3-h_1^3))}$$

- Due to the symmetry of the problem, $x_{3k} = 0$
- If $h_2 = h_1$, $x_{3k} = 0$ by symmetry

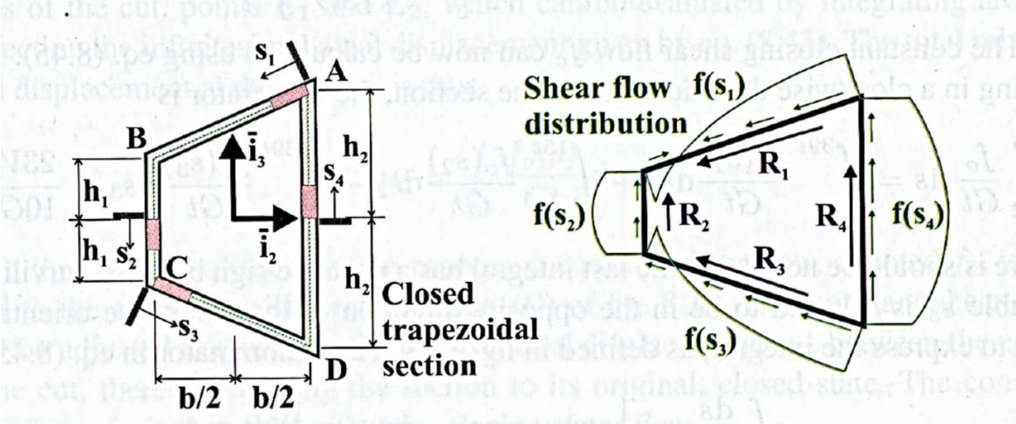


Fig. 8.38. Thin-walled trapezoidal section subjected to a vertical shaer force, V_3

8.5 Torsion of thin-walled beams

8.5 Torsion of thin-walled beams

- Chap. 7… Saint-Venant's theory of torsion for x-s of arbitrary shape. solution of PDE is required to evaluate the warping or stress function. However, approximate solution can be obtained for thin-walled beams

8.5.1 Torsion of open section

- Sec. 7.4 … Torsional behavior of beams with thin rectangular x-s
- Sec. 7.5 … Thin-walled, open x-s of arbitrary shape, shear stresses are linearly distributed through the thickness,
  torsional stiffness ~ (wall thickness)^3 (Eq. (7.61)), very limited torque carrying capability

8.5.2 Torsion of closed section

❖ **Fig. 8.50… thin-walled, closed x-s of arbitrary shape subjected to an applied torque, assumed to be in a state of uniform torsion, axial strain and stress components vanish → *n(s)* =*0***

- local equilibrium eqn. for a differential element, Eq.(8.14) →

$$\frac{\partial f}{\partial s} = 0 \qquad (8.59)$$

→ shear flow must remain constant along curve *C*

$$f(s) = f = const. \qquad (8.60)$$

- constant shear flow distribution generates a torque M_1

$$M_1 = \int_c f(s) r_0(s) ds = f \underbrace{\int_c r_0(s) ds}_{2A \text{ (Eq. (8.56))}}$$

$$M_1 = 2Af \quad (A : \text{enclosed area by } C) \qquad (8.61)$$

"Bredt-Batho formula"

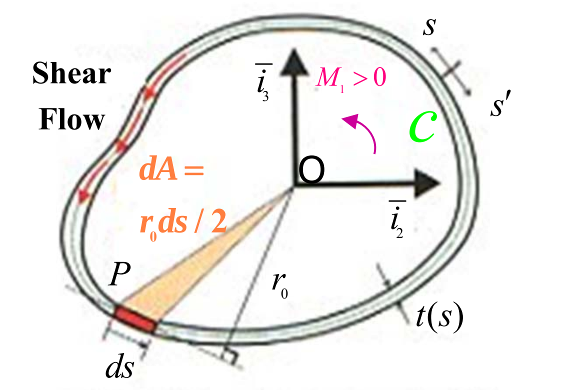


Fig. 8.50. Thin-walled tube of arbitrary cross-sectional shape

8.5 Torsion of thin-walled beams

- shear stress τ_s resulting from torque M_1

$$\tau_s(s) = \frac{M_1}{2At(s)} \tag{8.62}$$

twist rate vs. applied torque… simple energy argument

- strain energy stored in a differential slice of the beam of length dx_1

$$dA = \left[\frac{1}{2}\int_c r_s \tau_s t ds\right] = \left[\frac{1}{2}\int_c \frac{\tau_s^2}{G} t ds\right] dx_1 \tag{8.63}$$

- introducing shear stress distribution, Eq.(8.62)

$$dA = \left[\frac{1}{2}\frac{M_1^2}{4A^2}\int_c \frac{ds}{Gt(s)}\right] dx_1 \tag{8.64}$$

- work done by the applied torque

$$dW = \frac{1}{2}M_1 d\Phi_1 = \left[\frac{1}{2}M_1\frac{d\Phi_1}{dx_1}\right]dx_1 = \left[\frac{1}{2}M_1\kappa_1\right]dx_1 \tag{8.65}$$

where twist rate $\kappa_1 = \dfrac{d\Phi_1}{dx_1}$

8.5 Torsion of thin-walled beams

- 1st law of thermodynamics… $dW = dA$

$$\kappa_1 = \frac{M_1}{4A^2} \int_c \frac{ds}{Gt} \tag{8.66}$$

→ proportionality between M_1 and x_1, torsional stiffness

$$H_{11} = \frac{4A^2}{\displaystyle\int_c \frac{ds}{Gt}} \tag{8.67}$$

- arbitrary shaped closed x-s of const. wall thickness, homogeneous material

$$H_{11} = \frac{4GtA^2}{l}, \quad l: \text{ Perimeter of } C \tag{8.68}$$

… maximum H_{11} → thin-walled circular tube (maximize the numerator)

8.5 Torsion of thin-walled beams

- Sign convention

  A : area enclosed by curve *C* that defines the section's configuration

$$2A = \int_C r_0(s)ds$$

$r_0(s)$: perpendicular distance from the origin, *O*, to the tangent to *C*, its sign depends on the direction of the curvilinear variable, *s*

A is (+) when *s* describes *C* while leaving *A* to the left

(-) in the opposite.

$$f > 0, A > 0 \rightarrow M_1 = 2Af > 0$$

- s' : clockwise direction, $f' = -f,\ A' = -A$

$$M_1 = 2A'f' = 2Af > 0$$

8.5.3 Comparison of open and closed sections

- Closed section : shear stress is uniformly distributed through the thickness
- Open section : shear stress is linearly distributed
- Torsional stiffness $\propto$ (enclosed area)^2 for closed section, Eq(8.67)
- Torsional stiffness $\propto$ (thickness)^2 for open section, Eq(7.64)

- Fig.8.51. Circular shape, thin-walled tube of mean radius R_m

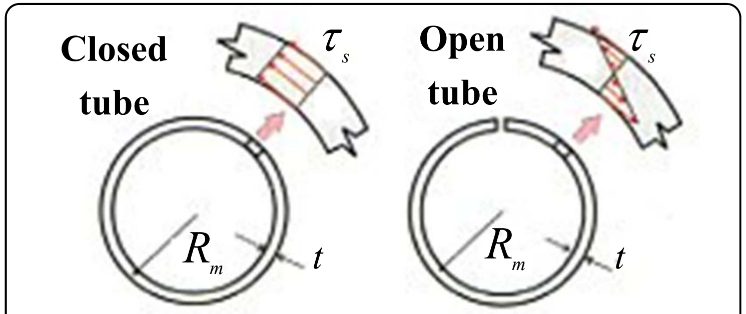


Fig. 8.51 A thin-walled open tube and closed tube

$$H_{11}^{open} = 2\pi G R_m t^3 / 3 \quad \text{, Eq(7.64)}$$

$$H_{11}^{closed} = 2\pi G R_m^3 t \quad \text{, Eq(7.19)}$$

$$\frac{H_{11}^{closed}}{H_{11}^{open}} = 3\left(\frac{R_m}{t}\right)^2 \qquad (8.69)$$

- Maximum shear stress $\tau_{\max}$ subjected to t he same torque, M_1

$$\tau_{\max}^{open} = G\kappa_1^{open} t = G\frac{M_1 t}{H_{11}^{open}} = \frac{3M_1}{2\pi R_m t^2} \qquad \tau_{\max}^{closed} = R_m G\kappa_1^{closed} = G\frac{M_1 R_m}{H_{11}^{closed}} = \frac{M_1}{2\pi R_m^2 t}$$

$$\frac{\tau_{\max}^{open}}{\tau_{\max}^{closed}} = 3\left(\frac{R_m}{t}\right) \qquad (8.70)$$

8.5 Torsion of thin-walled beams

- Example : R_m =20t

① H_{11} $\cdots$ that of closed section will be 1,200 times larger than that of the open section

② $\tau_{\max}$ $\cdots$ that of open section will be 60 times larger than that of the closed section

→ closed section can carry a 60 times larger torque

8.5.4 Torsion of combined open and closed sections

- x-s presenting a combination of open and closed curves (Fig. 8.52)

  - twist rate is identical for $\begin{cases} \text{the trapezoidal box} \\ \text{rectangular strips} \end{cases}$

- Torques they carry $\cdots \begin{cases} M_1^{box} = H_{11}^{box}\kappa_1 \\ M_1^{strip} = H_{11}^{strip}\kappa_1 \end{cases}$

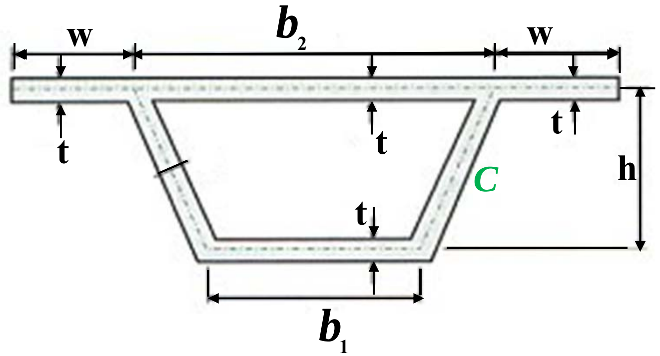


Fig. 8.52 Thin-walled trapezoidal beam with overhangs

8.5 Torsion of thin-walled beams

- Torsional stiffness $\cdots\begin{cases} H_{11}^{box} = 4GtA^2/l & \text{Eq.(8.68)} \\ H_{11}^{strip} = Gwt^2/3 & \text{Eq.(7.64)} \end{cases}$

- Total torque $M_1 = M_1^{box} + 2M_1^{strip}$

$$M_1 = H_{11}^{box}\left(1+2\frac{H_{11}^{strip}}{H_{11}^{box}}\right)\kappa_1 = H_{11}^{box}\left(1+\frac{2}{3}\frac{wl}{(b_1+b_2)^2}\left(\frac{t}{h}\right)^2\right)\kappa_1$$

… for thin-walled section, $\dfrac{t}{h} \ll 1$, $H_{11} \simeq H_{11}^{box}$

→ torsional stiffness of the section is nearly equal to that of the closed trapezoidal box alone.

$$M_1^{box} = H_{11}^{box}\kappa_1 \simeq H_{11}^{box}\frac{M_1}{H_{11}^{box}} = M_1$$

$$M_1^{strip} = H_{11}^{strip}\kappa_1 \simeq \frac{H_{11}^{strip}}{H_{11}^{box}}M_1$$

- Max. shear stress … from Eqs. (8.62), (7.65)

$$\tau_{\max}^{box} = \frac{M_1^{box}}{2At} \simeq \frac{1}{2At} M_1$$

$$\tau_{\max}^{strip} = \frac{3M_1^{strip}}{20t^2} \simeq \frac{3}{wt^2} \frac{H_{11}^{strip}}{H_{11}^{box}} M_1$$

- ratio

$$\frac{\tau_{\max}^{strip}}{\tau_{\max}^{box}} = \frac{l}{b_1 + b_2} \left(\frac{t}{h} \right)$$

… the max. shear stress in the strip is far smaller than that in the trapezoidal box

8.5 Torsion of thin-walled beams

8.5.5 Torsion of multi-cellular sections

- 4-cell, thin-walled x-s subjected to a torque M_1 (Fig. 8.53)
  - only uniform torsion exists, and hence the axial stress flow vanishes
  : Eq.(8.14) reduces to $\frac{\partial f}{\partial s}=0$
  → shear flow is constant

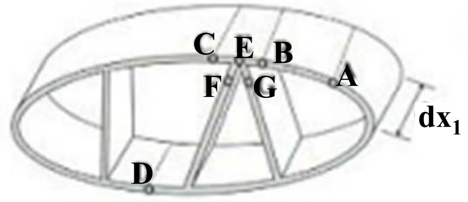


Fig. 8.53 A thin-walled, multi-cellular section under torsion

- Free-body diagrams of the portion of the section
  - Fig. 8.54-(1) … axial stress flow=0, $f_A = f_B$
  - Fig. 8.54-(2) … $f_C = f_D$
  - Fig. 8.54-(3) … $f_C + f_F + f_G - f_B = 0$, $\sum f_i = 0$ (8.71)
  … "the sum of the shear flows going into a joint must vanish"

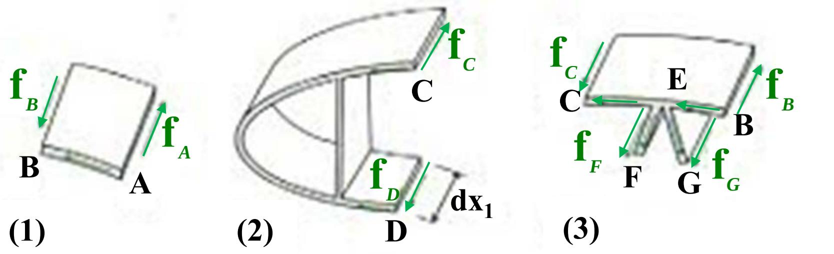


Fig. 8.54 Free-body diagrams of the thin-walled, multi-cellular section.

8.5.5 Torsion of multi-cellular sections

- Const. shear flows are assumed to act in each cell of the section (Fig. 8.55)

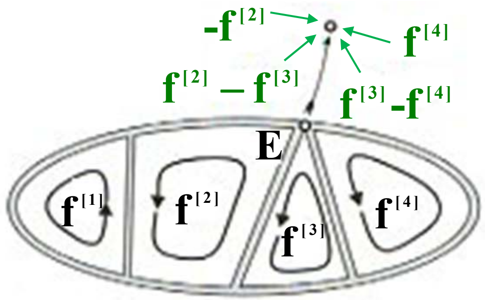


Fig. 8.55 Shear flows in each cell of a thin-walled, multi-cellular section

- Determination of the const. shear flow in each cell

$$M_1 = \sum_{i=1}^{N} M_1^{[i]} = 2\sum_{i=1}^{N} A^{[i]} f^{[i]} \qquad (8.72)$$

- Const. shear flows are assumed to act in each cell of the section
- Determination of the const. shear flow in each cell

① total torque = sum of the torques carried by each individual cell
"Bredt-Batho formula"

② compatibility condition … twist rates of the various cells are identified.

$$\kappa_1^{[1]} = \kappa_1^{[2]} = \cdots = \kappa_1^{[i]} = \cdots = \kappa_1^{[N]} \tag{8.73}$$

- Eq.(8.66) →

$$\kappa_1^{[i]} = \int_{C^{[i]}} \frac{M_1^{[i]}}{4(A^{[i]})^2} \frac{ds}{Gt} = \int_{C^{[i]}} \frac{2A^{[i]} f^{[i]}}{4(A^{[i]})^2} \frac{ds}{Gt}$$
$$= \frac{1}{2A^{[i]}} \int_{C^{[i]}} \frac{f^{[i]}}{Gt} ds \tag{8.74}$$

- Eqs.(8.72), (8.73) … N_{cells} eqn.s for N_{cells} shear flows

- Example 8.17 Two-cell cross-section

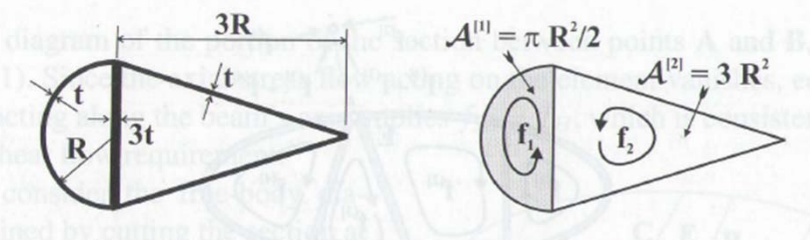


Fig. 8.56. A two-cell thin-walled section under torsion.

- Two-cell cross- section (Fig. 8.56): highly idealized airfoil structure
- Eq. (8.72): total torque carried by the section is the sum of the torques carried in each cell

$$M_1 = 2\sum_{i=1}^{N_{cell}} A^{[i]} f^{[i]} = \pi R^2 f^{[1]} + 6R^2 f^{[2]} \qquad (8.75)$$

- Eq. (8.73): twist rates for the two cells are identical.
  twist rate for the front cell

$$\kappa_1^{[1]} = \frac{1}{2A^{[1]}}\int_{C_1} \frac{f}{Gt(s)} ds = \frac{1}{2G\pi R^2/2}\left[\frac{f^{[1]}}{t}\pi R + \frac{f^{[1]} - f^{[2]}}{3t} 2R\right] = \frac{1}{G\pi Rt}\left[f^{[1]}\pi + \frac{2}{3}(f^{[1]} - f^{[2]})\right]$$

twist rate for the aft cell

$$\kappa_1^{[2]} = \frac{1}{2A^{[2]}}\int_{C_2} \frac{f}{Gt(s)} ds = \frac{1}{2G3R^2}\left[\frac{f^{[2]} - f^{[1]}}{3t} 2R + f^{[2]} 2\sqrt{10}\frac{R}{t}\right] = \frac{1}{6GRt}\left[\frac{2}{3}(f^{[2]} - f^{[1]}) + f^{[2]} 2\sqrt{10}\right]$$

- Example 8.17 Two-cell cross-section

  - Equating the two twist rate -> second eqn. for the shear flow

$$\frac{1}{\pi}\left[\pi f^{[1]}+\frac{2}{3}(f^{[2]}-f^{[1]})\right]=\frac{1}{6}\left[\frac{2}{3}(f^{[2]}-f^{[1]})+f^{[2]}2\sqrt{10}\right]$$

  - which simplifies to

$$f^{[1]}=1.04f^{[2]}$$

  - This can be used to solve for $f^{[1]}$ and $f^{[2]}$

  - largest contribution to the torsional stiffness comes from the outermost closed sections, which is the union of the frount and aft cells.

    The largest shear flow circulates in this outmost section, leaving the spar nearly unloaded.

  - torsional stiffness

$$H_{11}=\frac{M_1}{\kappa_1^{[1]}}=\frac{(\pi R^2 1.04+6R^2)f^{[2]}}{1/(\pi GRt)[1.04\pi+2/3(1.04-1)]f^{[2]}}=2.81\pi GR^3 t$$

8.6 Coupled bending-torsion problems

❖ **Chap. 6… arbitrary x-s subjected to complex loading conditions**

2 important restrictions

① no torques

② transverse shear forces are assumed to be applied in such a way that the beam will bend without twisting

→ Now can be removed

- Fig. 8.65 … concentrated transverse load P_2 acting at the tip and it point of application, *A*, with coord. (x_{2a}, x_{3a}) ,

  $p_1(x_1)$, $p_2(x_1)$, $p_3(x_1)$ $\cdots$ distributed loads

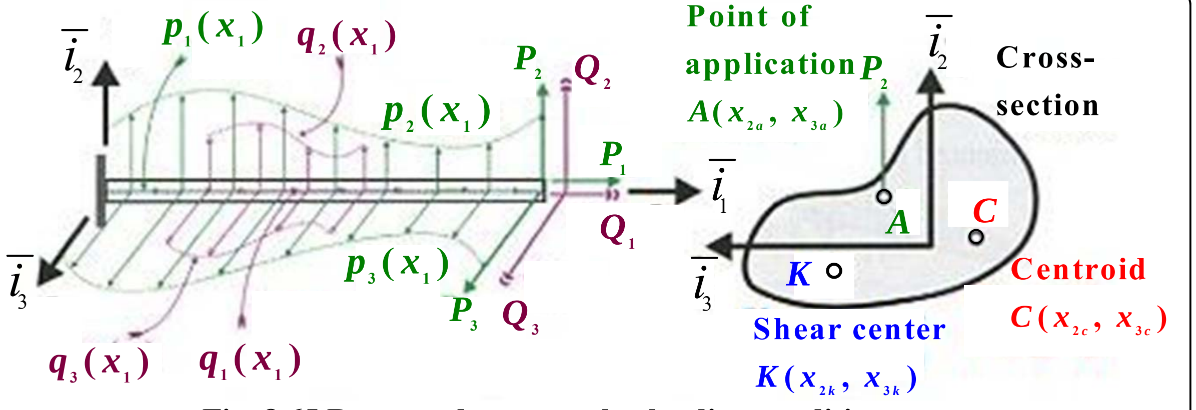


Fig. 8.65 Beam under a complex loading condition

8.6 Coupled bending-torsion problems

- Solution procedure

① Compute location of the centroid, $C(x_{2c}, x_{3c})$

② Compute orientation of the principal axes of bending $\bar{i}_1^*, \bar{i}_2^*, \bar{i}_3^*$
and the principal bending stiffness (sec. 6.6)

③ Compute location of the shear center, $\kappa(x_{2\kappa}, x_{3\kappa})$ (sec. 8.4)

④ Compute torsional stiffness (chap. 7, or sec. 8.5.2)

⑤ Solve the extensional problems Eqs. (6.31), (6.32) in principal centroidal axes of bending planes

⑦ Compute torsional problem

$$\frac{d}{dx_1^4}\left(H_{11}^* \frac{d\Phi_1^*}{dx_1^*}\right) = -[g_1^*(x_1^*) + (x_{2a}^* - x_{2\kappa}^*)p_3^*(x_1^*) - (x_{3a}^* - x_{3\kappa}^*)p_2^*(x_1^*)] \quad (8.76)$$

B.C. $\Phi_1^* = 0$ at root

$$H_{11}^* \frac{d\Phi_1^*}{dx_1^*} = Q_1^* + (x_{2a}^* - x_{2\kappa}^*)P_3^* - (x_{3a}^* - x_{3\kappa}^*)P_2^* \quad @\ at\ tip \qquad (8.77)$$

… : axis system defined by the principal centroidal axes of bending

→ More convenient to recast the governing eqn. in a coord. system for which axis $\bar{i}_1^*$ is aligned with the axis of a beam

8.6 Coupled bending-torsion problems

- Knowledge of centroid and shear center → complete decoupling of a problem
  → 4 independent problems $\begin{cases} \text{axial problem} \\ \text{bending problem} \\ \text{torsional problem} \end{cases}$
- If no torque and all transverse loads are applied at the s.c.
  → R.H.S of Eq(8.77) =0 → $\Phi_1(x_1) = 0$, the beam does not twist
  If not, the beam twists, rigid body rotation $\Phi_1(x_1)$ about the s.c.

8.6 Coupled bending-torsion problems

- Example 8.18 Wing subjected to aerodynamic lift and moment

  - Wing coupled bending-torsion problem (Fig. 8.66)
  - principal axes of bending i_2 and i_3 : their origin at shear center
  - axis i_1 : along the locus of the shear centers of all the cross-sections
    -> straight line called the "elastic axis"
  - aerodynamic loading : lift per unit span L_{AC}, applied at the aerodynamic center
    aerodynamic moment per unit span M_{AC}

  - differential eqn for bending in plane (i_2 , i_3)

$$\frac{d^2}{dx_1^2}\left(H_{22}^c \frac{d^2\bar{u}_3}{dx_1^2}\right) = L_{AC} \qquad (8.79)$$

BC: $\bar{u}_3 = \frac{d\bar{u}_3}{dx_1} = 0$ at the root, $\frac{d^2\bar{u}_3}{dx_1^2} = \frac{d^3\bar{u}_3}{dx_1^3} = 0$ at the unloaded tip

  - governing eqn for torsion

$$\frac{d}{dx_1}\left(H_{11}\frac{d\Phi_1}{dx_1}\right) = -\left(M_{AC} + eL_{AC}\right)$$

BC: $\Phi_1 = 0$ at the root, $\frac{d\Phi_1}{dx_1} = 0$ at the tip

e: distance from the aerodynamic center to the shear center

- Example 8.18 Wing subjected to aerodynamic lift and moment

  - typical transport aircraft: e = 25 – 35% chord
  - it is convenient to select the origin of the axes at the s.c., rather than at the centroid.: bending problem is decoupled from the axial problem. beam will rotate about the origin of the axes system.

  - The rotation Φ_1 of the section is, in fact, the geometric angle of attack of the airfoil.
  - lift, L_{AC} , is a function of the angle of attack

  - aerodynamic problem: computation of the lift as a function of the angle of attack
  - elastic problem: computation of wing deflection and twist as a function of the applied loads
  - aeroelasticity: study of this interaction

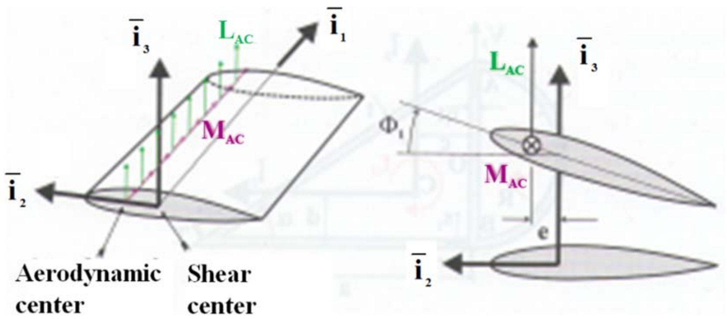


Fig.8.66. The wing bending torsion coupled problem

8.7 Warping of thin-walled beams under torsion

❖ **Thin-walled beam subjected to an applied torque**

→ Shear stress generated

→ Out-of-plane deformations, “warping”, in x-s : magnitude is typically small, but dramatic effect on the torsional behavior

➢ Particularly pronounced for non-uniform torsion of open sections

↰ Twist rate varies along the span

↔ Contrasts with Saint-Venant theory

↰ Uniform torsion, constant twist rate

8.7 Warping of thin-walled beams under torsion

8.7.1 Kinematic description

❖ **Fig 8.70 : thin-walled beam subjected to a tip concentrated torque Q_1**

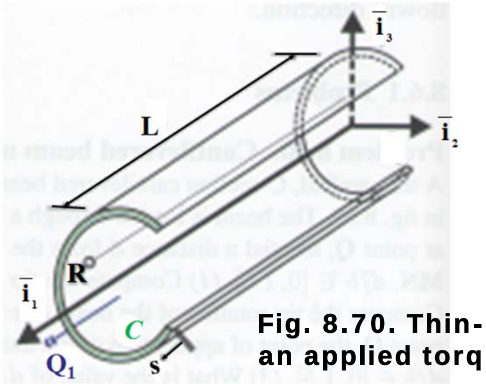


Fig. 8.70. Thin-walled beam subjected to an applied torque.

➢ Displacement field

- Similar to that for Saint-Venant solution
- Each x-s is assumed to rotate like a rigid body about R ("center of twist", (x_{2r}, x_{3r})) ← unknown yet

$$u_1(x_1,s)=\underbrace{\Psi(s)}_{\text{Unknown warping function}}\underbrace{\kappa_1(x_1)}_{\text{Twist rate}} \tag{8.80a}$$

$$u_2(x_1,s)=-(x_3-x_{3r})\Phi_1(x_1) \tag{8.80b}$$

$$u_3(x_1,s)=(x_2-x_{2r})\Phi_1(x_1) \tag{8.80c}$$

8.7 Warping of thin-walled beams under torsion

- Strain field

$$
\begin{aligned}
&\varepsilon_1 = \frac{\partial u_1}{\partial x_1} = \Psi(s)\frac{d\kappa_1}{dx_1} && \gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 \\
&\varepsilon_2 = \frac{\partial u_2}{\partial x_2} = 0 && \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \left[\frac{d\Psi}{dx_2} - (x_3 - x_{3r})\right]\kappa_1 \\
&\varepsilon_3 = \frac{\partial u_3}{\partial x_3} = 0 && \gamma_{13} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = \left[\frac{d\Psi}{dx_3} + (x_2 - x_{2r})\right]\kappa_1
\end{aligned}
\qquad (8.81)
$$

- Non-uniform torsion is assumed $\rightarrow \dfrac{d\kappa_1}{dx_1} \neq 0$

→ axial strain ≠ 0

In-plane strain components =0 since rigid body rotation assumed

Shear strain components → partial derivatives of warping function and twist rate

8.7 Warping of thin-walled beams under torsion

8.7.2 Stress-strain relations

❖ **Non-vanishing components of the stress**

$$\sigma_1 = E\varepsilon_1 = E\Psi(s)\frac{d\kappa_1}{dx_1}$$

$$\tau_{12} = G\gamma_{12} = \left[\frac{d\Psi}{dx_2} - (x_3 - x_{3r})\right]G\kappa_1 \qquad (8.82)$$

$$\tau_{13} = G\gamma_{13} = \left[\frac{d\Psi}{dx_3} + (x_2 - x_{2r})\right]G\kappa_1$$

➢ Only non-vanishing shear stress component for thin-walled beams → τ_s

$$\tau_s = \tau_{12}\frac{dx_2}{ds} + \tau_{13}\frac{dx_3}{ds}$$

$$= \Bigg[\underbrace{\frac{d\Psi}{dx_2}\frac{dx_2}{ds} + \frac{d\Psi}{dx_3}\frac{dx_3}{ds}}_{\text{Total derivative of } \Psi \text{ w.r.t. } s} + \underbrace{(x_2 - x_{2r})\frac{dx_3}{ds} - (x_3 - x_{3r})\frac{dx_2}{ds}}_{\text{Distance from the twist center to the tangent to C, Eq.(8.11)}}\Bigg]G\kappa_1$$

$$\tau_s = \left(\frac{d\Psi}{ds} + r_r\right)G\kappa_1 \qquad (8.83)$$

→ for open and closed sections

8.7.3 Warping of open sections

❖ **Shear stress distribution in open-section**

→ linearly distributed across the wall thickness and 0 along the wall mid-line

➢ $\tau_s = 0$ along curve C, Eq.(8.83)

$$\tau_s = \left(\frac{d\Psi}{ds} + r_r \right) G\kappa_1 = 0 \tag{8.84}$$

➢ Warping function relation

$$\frac{d\Psi}{ds} = -r_r = -\left(r_o - x_{2r}\frac{dx_3}{ds} + x_{3r}\frac{dx_2}{ds} \right) \tag{8.85}$$

➢ Purely geometric function, $\Gamma(s)$

$$\frac{d\Gamma}{ds} = -r_o \tag{8.86}$$

➢ Warping function

$$\Psi(s) = \Gamma(s) + x_{2r}x_3 - x_{3r}x_2 + c_1 \tag{8.87}$$

8.7.3 Warping of open sections

- **Uniform torsion,** $\dfrac{d\kappa_1}{dx_1} = 0$ → axial strain/stress = 0

→ c_1 and (x_{2r}, x_{3r}) cannot be determined, simply represents a rigid body displacement field, does not affect the state of stress/strain

- Non-uniform torsion

$$\begin{cases} \text{varying applied torque} \\ \text{constrained warping displacement at a boundary or at some point} \end{cases}$$

→ non-vanishing axial strain/stress although acted upon by a torque alone

but, still $N_1, M_2, M_3 = 0$

- Axial force $N_1 = 0$ → Eq.(8.82a), (8.87)

↳ $\int_c \sigma_1 t ds = 0$

8.7.3 Warping of open sections

$$\int_C E\Gamma t ds + x_{2r}\underbrace{\int_C E x_3 t ds}_{=0} - x_{3r}\underbrace{\int_C E x_2 t ds}_{=0} + c_1 \underbrace{\int_C E t ds}_{= S \text{ (axial stiffness)}} = 0$$

origin of the axes is selected to be at the centroid

$$c_1 = -\frac{1}{S}\int_C E\Gamma t ds \qquad (8.88)$$

- Bending moment $M_2 = \int_C \sigma_1 x_3 t ds = 0$

$$\int_C E\Gamma x_3 t ds + x_{2r}\underbrace{\int_C E x_3^2 t ds}_{= H_{22}^C} - x_{3r}\underbrace{\int_C E x_2 x_3 t ds}_{= H_{23}^C = 0} + c_1 \underbrace{\int_C E x_3 t ds}_{=0} = 0$$

(principal centroidal axes of bending)

8.7.3 Warping of open sections

$$x_{2r} = -\frac{1}{H_{22}{}^{C}} \int_C E\Gamma x_3 t ds \tag{8.89}$$

- $M_3 = 0$

$$x_{3r} = -\frac{1}{H_{33}{}^{C}} \int_C E\Gamma x_2 t ds \tag{8.90}$$

- Example 8.20 Warping of a C-channel

  - C-channel cross section subjected to a tip torque (Fig. 8.24)
  - axes in the figure: principal centroidal axes of bending i_2 and i_3
  - axis i_2 : axis of symmetry
  - 1st step : compute the purely geometric function, $\Gamma(s)$

$$\frac{d\Gamma}{ds} = -r_o \qquad (8.86)$$

r_o : normal distance from the origin of the axes to the tangent of the curve C

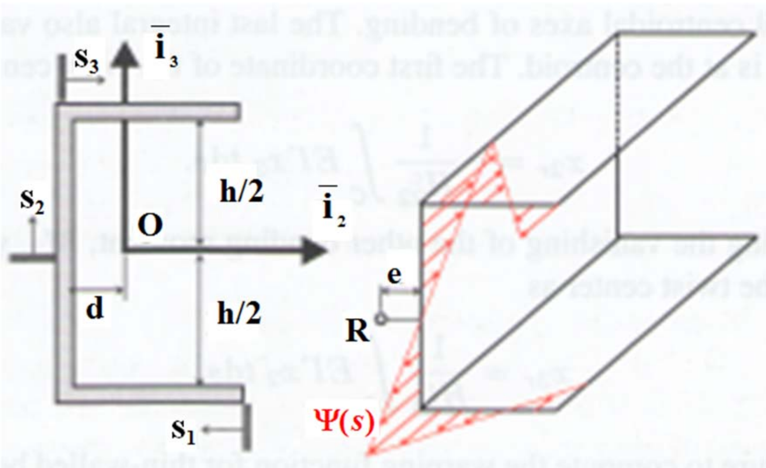


Fig. 8.71. The warping function for a C-channel

- For the lower flange (s_1)

where, $r_0 = -h/2$

$$\Gamma(s_1) = -hs_1/2 + c$$

applying boundary condition, $\Gamma(s_1) = 0 \;\; at \;\; s_1 = 0$ then,

$$\Gamma(s_1) = -hs_1/2$$

$$r_0 = -d \;\; and \;\; r_0 = -h/2$$

$$\Gamma(s_2) = ds_2 + h(b+d)$$

$$\Gamma(s_3) = hs_3/2 + h(b+2d)/2$$

-2nd step: evaluate the integration constants

$$c_1 = -\frac{Et}{S}\left[\int_0^b \Gamma(s_1)ds_1 + \int_{-h/2}^{h/2}\Gamma(s_2)ds_2 + \int_0^b \Gamma(s_3)ds_3\right] = -\frac{h}{2}(b+d)$$

- Final step: coord. of the twist center

$$x_{2r} = -\frac{Et}{H_{22}^c}\left[\int_0^b \Gamma(s_1)(-\frac{h}{2})ds_1 + \int_{-h/2}^{h/2}\Gamma(s_2)s_2 ds_2 + \int_0^b \Gamma(s_3)\frac{h}{2}ds_3\right]$$

$$= -d - \frac{h^2b^2t}{4}\frac{E}{H_{22}^c}$$

$$x_{3r} = -\frac{Et}{H_{22}^{c}}\left[\int_0^b \Gamma(s_1)(b-d-s)ds_1 + \int_{-h/2}^{h/2}\Gamma(s_2)(-d)ds_2 + \int_0^b \Gamma(s_3)(s-d)ds_3\right] = 0$$

$$= -d - \frac{h^2 b^2 t}{4}\frac{E}{H_{22}^{c}}$$

The warping function then follows from eq.(8.87) as

$$\Psi(s_1) = \frac{h}{2}\left(s_1 + e - b\right);\quad \Psi(s_2) = -es_2;\quad \Psi(s_3) = \frac{h}{2}\left(s_3 - e\right)$$

where,

$$e = h^2 b^2 tE\,/\,(4H_{22}^{c})$$

- The location of shear center coincides that of the twist center.

8.7.5 Warping of closed sections

- **Shear stress distribution** → constant through the wall thickness in closed section

$$\tau_s = \frac{M_1}{2At} = H_{11}\frac{\kappa_1}{2At} \qquad A = \text{area enclosed by curve } C \text{ (8.62)}$$

- Eq. (8.83) → $$\frac{d\Psi}{ds} = \frac{\tau_s}{G\kappa_1} - r_r = \frac{H_{11}}{2AGt} - r_r \qquad (8.94)$$

  ✓ governing equation for $\Psi(s)$ in closed sections

- Process of integration of Eq. (8.94) → similar to that for open section

  ① Purely geometric function $\Gamma(s)$

$$\frac{d\Gamma}{ds} = \frac{H_{11}}{2AGt} - r_o \qquad (8.95)$$

  arbitrary B.C. is used to integrate Eq.(8.95)

  ② c_1 and (x_{2r}, x_{3r}) can be determined by the vanishing of F_1, M_2, M_3

8.7.6 Warping of multi-cellular sections

- **Section 8.5.5** → shear flow distribution *f(s)* due to applied torque

$$f(s) = \vartheta(s)\kappa_1 \qquad \tau_s = \vartheta(s)\frac{\kappa_1}{t}$$

- Governing equation for the warping function

$$\frac{d\Psi}{ds} = \frac{\vartheta(s)}{Gt} - r_r \tag{8.97}$$

- Determination of the warping function --- exactly mirrors that for open and closed sections, except the following

$$\frac{d\Gamma}{ds} = \frac{\vartheta(s)}{Gt} - r_o \tag{8.98}$$

8.8 Equivalence of the shear and twist centers

❖ **Shear center** → defined by torque equipollence condition, Eq.(8.39)

❖ **Center of twist** → introduced for the analysis of thin-walled beams under torsion

Eq.(8.53) → Eq.(8.86)

$$x_{2k} = \int_C f^{[3]} r_o ds = -\int_C f^{[3]} \frac{d\Gamma}{ds}$$

➢ Integrating by parts

$$x_{2k} = \int_C \Gamma \frac{df^{[3]}}{ds} ds - \left[f^{[3]} \Gamma \right]_{boundary}$$

by Eq.(8.58)

$$x_{2k} = -\int_C \frac{Et}{H_{22}{}^C} x_3 \Gamma ds = -\frac{1}{H_{22}{}^C} \int_C E\Gamma x_3 t ds = x_{2r}$$

by Eq.(8.89)

similarly, $x_{3k} = x_{3r}$

→ Equivalence of the shear and twist center for open sections.
Equivalence also holds for closed sections direct consequence of Betti's reciprocity theorem. Eq.(10.117)

8.9 Non-uniform torsion

❖ Non-uniform torsion

→ both shear and axial stresses generated by differential warping

↳ Markedly different behavior from that under uniform torsion

- Axial stress distribution → uniform across the wall thickness

  axial flow $n_w = t\sigma_1$

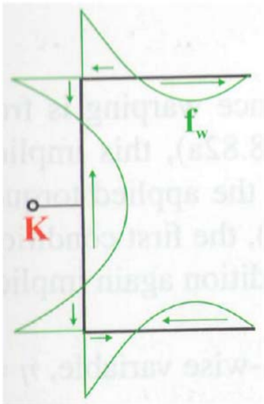
- Although the axial stress does not vanish, the resulting axial force and bending moment do vanish → local equilibrium equation, Eq.(8.14), is not necessarily satisfied

- For this local equilibrium to hold, a shear flow, f_w, "warping shear flow" is generated to satisfy the local equilibrium

$$\frac{\partial n_w}{\partial x_1} + \frac{\partial f_w}{\partial s} = 0$$

8.9 Non-uniform torsion

➢ Introducing Eq.(8.82a) for the case of open sections



$$\frac{\partial f_w}{\partial s} = -Et\Psi \frac{d^2\kappa_1}{dx_1^{\ 2}} \qquad (8.99)$$

→ can be integrated
by the procedure in Section 8.3

- Simple C-channel Fig. (8.75)

- Question of overall equilibrium

→ the warping shear flow generate resultant transverse shear force?

Eq.(8.7) → $V_{2w} = \int_C f_w \frac{dx_2}{ds} ds = -\int_C x_2 \frac{\partial f_w}{\partial s} ds + \left[x_2 f_w\right]_{boundary}$

Integrating by parts

$\|$

0 since $f_w = 0$ at the edge of the contour

Eq.(8.99) → $V_{2w} = \frac{d^2\kappa_1}{dx_1^{\ 2}} \int_C E\Psi x_2 t ds = 0$

Similarly, $V_{3w} = 0$

8.9 Non-uniform torsion

- Torque resultant about the shear center generated by the warping shear flow

Eq.(8.10) →

$$M_{1wk} = \int_C f_w r_k ds = -\int_C f_w \frac{d\Psi}{ds} ds \tag{8.100}$$

Integrating by parts

$$M_{1wk} = \int_C \Psi \frac{df_w}{ds} ds - \left[f_w \Psi \right]_{boundary} \tag{8.101}$$

Introducing Eq.(8.99)

$$M_{1wk} = -H_w \frac{d^2\kappa_1}{dx_1^{\ 2}} \qquad H_w = \int_C E\Psi^2 t ds \tag{8.102}$$

→ "warping stiffness"

- Total torque = that by the twist rate + that due to warping

$$M_{1k} = H_{11}\kappa_1 - H_w \frac{d^2\kappa_1}{dx_1^{\ 2}} \tag{8.104}$$

↓ generated by shear stress distribution

→ Additional contribution from the warping shear flow, =0 for uniform torsion

- Equilibrium equation for a differential element of the beam under torsional load

→ Eq.(7.15)

$$\frac{d}{dx_1}\left(H_{11}\frac{d\Phi_1}{dx_1} - H_w \frac{d^3\Phi_1}{dx_1^{\ 3}} \right) = -q_1 \tag{8.105}$$

8.9 Non-uniform torsion

- Example 8.23 Torsion of a cantilevered beam with **free** root warping

  - Uniform cantilevered beam of length *L* subjected to a tip torque, *Q*
  - Root condition: No twisting is allowed, but warping is free to occur
    -> attaching the beam's root to a diaphragm that prevents any root rotation, but does not constrain axial displacement
  - uniform properties along its length, Eq. (8.105) becomes

$$H_{11}\frac{d^2\Phi_1}{dx_1^2} - H_w\frac{d^4\Phi_1}{dx_1^{\;4}} = 0$$

  - at the root : no twist occurs, $\Phi_1 = 0$
  - free warping at the root : axial stress must vanish, $\frac{d^2\Phi_1}{dx_1^2} = 0$
  - at the tip: torque must equal the applied torque, $Q = H_{11}\frac{d\Phi_1}{dx_1} - H_w\frac{d^3\Phi_1}{dx_1^3}$
  - at the tip: axial stress must vanish once again, $\frac{d^2\Phi_1}{dx_1^2} = 0$
  - Introduction of non-dimensional span-wise variable, $\eta = x_1/L$
  - Governing eqn.:

$$\Phi_1'''' - \bar{k}^2\Phi_1'' = 0 \qquad (8.106)$$

8.9 Non-uniform torsion

- New BC's: at the root, $\Phi_1 = 0, \Phi_1'' = 0$
  at the tip , $\Phi_1'' = 0, \bar{k}^2 \Phi_1' - \Phi_1''' = QL^3 / H_w$

$$\bar{k} = \frac{H_{11} L^2}{H_w} \tag{8.107}$$

: ratio of the torsional stiffness to the warping stiffness

- General sol. of the governing differential eqn.:

$$\Phi_1 = C_1 + C_2 \eta + C_3 \cosh \bar{k} \eta + C_4 \sinh \bar{k} \eta \tag{8.108}$$

- Application of BC's:

$$\Phi_1 = \frac{QL}{H_{11}} \eta \tag{8.108}$$

-> identical to the uniform torsion solution

$$\kappa_1 = \frac{d\Phi_1}{dx_1} = \frac{Q}{L} = const$$

- torsional warping stiffness, H_w, disappears from the solution.

- Example 8.24 Torsion of a cantilevered beam with **constrained** root warping

  - Same uniform cantilevered beam, but the root section is now solidly fixed to prevent any wapring at the root
  -> at this built-in end, no twisting occurs $\Phi_1 = 0$
  no axial displacement $\kappa_1 = \dfrac{d\Phi_1}{dx_1} = 0$

  - Governing eqn. is the same, Eq. (8.106). But BC's are
  New BC's: at the root, $\Phi_1 = 0, \Phi_1' = 0$
  at the tip , $\Phi_1'' = 0, \bar{k}^2 \Phi_1' - \Phi_1''' = QL^3 / H_w$

  - General sol. is the same as Eq. (8.108)
  - Application of BC's:

$$\Phi_1 = \frac{QL}{H_{11}} \left[\eta - \frac{\sinh \bar{k} - \sinh \bar{k}(1-\eta)}{\bar{k} \cosh \bar{k}} \right]$$

uniform torsion

influence of the non-uniform torsion induced by the root warping constraint

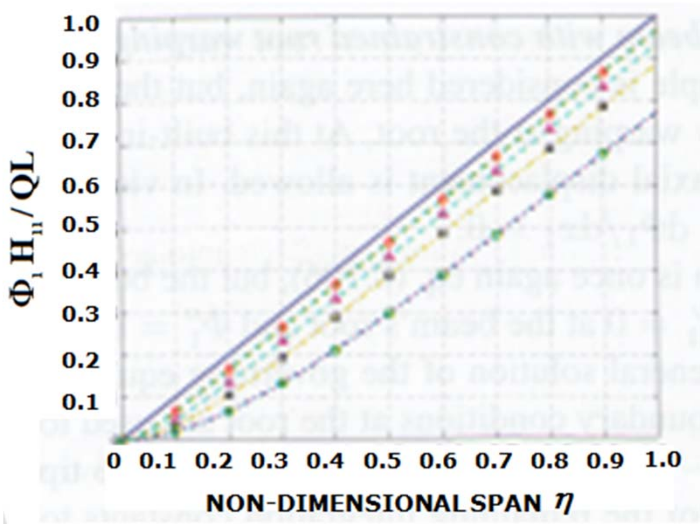


Fig. 8.76. Twist distribution for the closed rectangular section under non-uniform torsion. $\bar{k}$ =16.54 (◊), $\bar{k}$ =8.27 (Δ), $\bar{k}$ =5.04 (□), $\bar{k}$ =2.52(O).

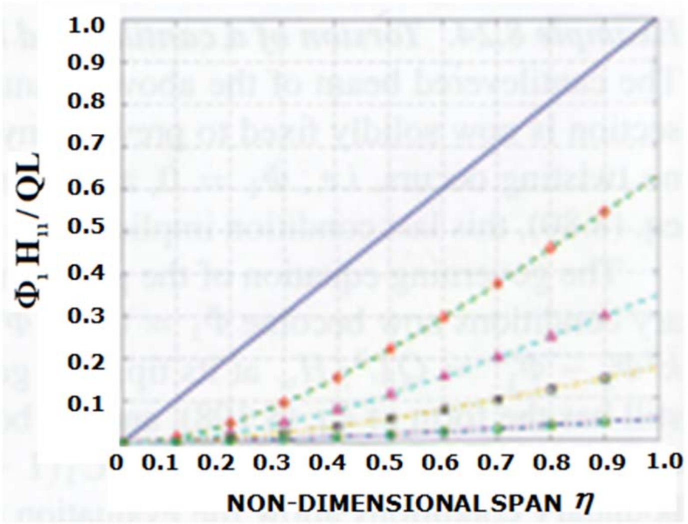


Fig. 8.77. Twist distribution for the C channel section under non-uniform torsion. $\bar{k}$ =2.65 (◊), $\bar{k}$ =1.33 (Δ), $\bar{k}$ =0.808 (□), $\bar{k}$ =0.404(O).

8.10 Structural idealization

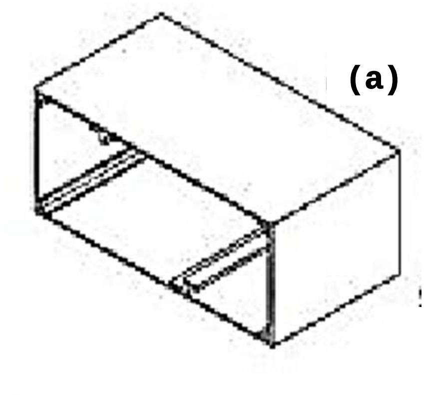
❖ **Actual thin-walled beam structures**

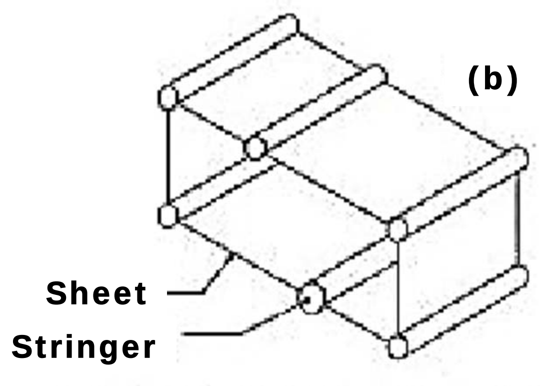
→ "stringers" added to increase the bending stiffness

➢ can be idealized by separating the axial and shear stress carrying components into distinct entities called
stringers
sheets

Axial stress → assumed to be carried only in the stringers

Shear stress → assumed to be carried only in the sheets





"box beam", "L" shaped longitudinal members located away from the centroid

→ much larger contribution to the bending stiffness

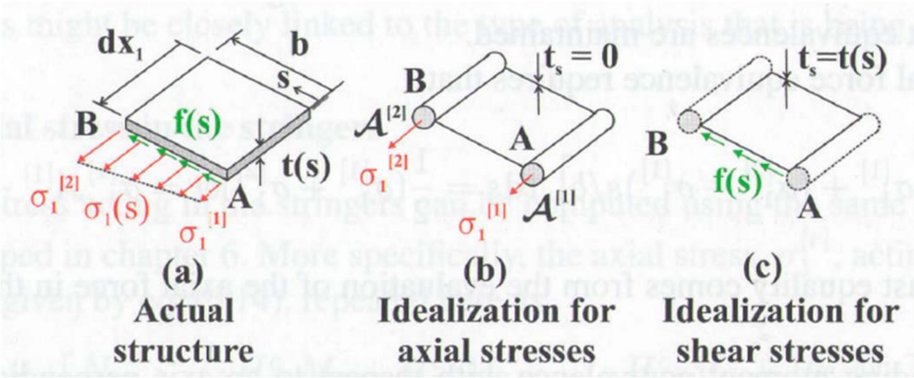
sheet-stringer idealization

→ considerably simplified analysis procedure for stress distribution

8.10 Structural idealization

8.10.1 Sheet-stringer approximation of a thin-walled beam

❖ **Figure 8.80**



→ no discrete “stringers” or with far smaller x-s area

- ✓ still possible to construct a sheet-stringer model

➢ Idealized structures

① Axial stresses are carried solely by the stringers

② Shear stresses are carried solely by the sheets

8.10 Structural idealization

- Approach to estimate the areas of the stringers

  ① Triangular equivalence method (sec. 6.8) → guarantee the same bending stiffness and centroid location

  ② Linear distribution of axial stress, $\sigma_1 = \sigma_1^{[1]} + (\sigma_1^{[1]} - \sigma_1^{[1]})\, s/b$

  - $\sigma_1^{[1]}$: stresses of point A
  - $\sigma_1^{[2]}$: stresses of point B
  - s : local position along the contour of width b

  → the areas $A^{[1]}$, and $A^{[2]}$, of the stringers need to be determined.

  2 constraints
  1) Axial stresses at A and B are the same as the actual
  2) Force and moment equivalences are maintained

- Force equivalence

$$F_1 = \int_0^b \left[\sigma_1^{[1]} + \left(\sigma_1^{[2]} - \sigma_1^{[1]}\right) s/b\right] t\,ds = \frac{1}{2}\left(\sigma_1^{[1]} + \sigma_1^{[2]}\right) bt = \sigma_1^{[1]} A^{[1]} + \sigma_1^{[2]} A^{[2]}$$

- Bending moment equivalence

$$M_A = \int_0^b \left[\sigma_1^{[1]} + \left(\sigma_1^{[2]} - \sigma_1^{[1]}\right) s/b\right] s t\,ds = \frac{b^2 t}{6}\left(\sigma_1^{[1]} + 2\sigma_1^{[2]}\right) = b\sigma_1^{[2]} A^{[2]}$$

solution

$$A^{[1]} = \frac{bt}{6}\left(2 + \frac{\sigma_1^{[2]}}{\sigma_1^{[1]}}\right), \qquad A^{[2]} = \frac{bt}{6}\left(2 + \frac{\sigma_1^{[1]}}{\sigma_1^{[2]}}\right) \tag{8.110}$$

8.10 Structural idealization

- 2 special cases

  ① Uniform axial stress $\sigma_1^{[1]} = \sigma^{[2]} \rightarrow A^{[1]}=A^{[2]}=bt/2$ (8.111)

  ② Pure bending $\sigma_1^{[1]} = -\sigma_1^{[2]} \rightarrow A^{[1]}=A^{[2]}=bt/6$ (8.112)

- Different stress distributions are considered, equivalent idealized area need to be recomputed

8.10.2 Axial stress in the stringers

- The same approach as developed in Chapter 6,

  axial stress $\sigma_1^{[r]}$ acting in the r-th stringer

$$\sigma_1^{[r]} = E^{[r]}\left[\frac{N_1}{S} + x_3^{[r]}\frac{H_{33}^C M_2 + H_{23}^C M_3}{\Delta H} - x_2^{[r]}\frac{H_{23}^C M_2 + H_{22}^C M_3}{\Delta H}\right] \quad (8.113)$$

Uniform stress is assumed in a small "lumped" case

→ net axial force = $A^{[r]}\sigma_1^{[r]}$

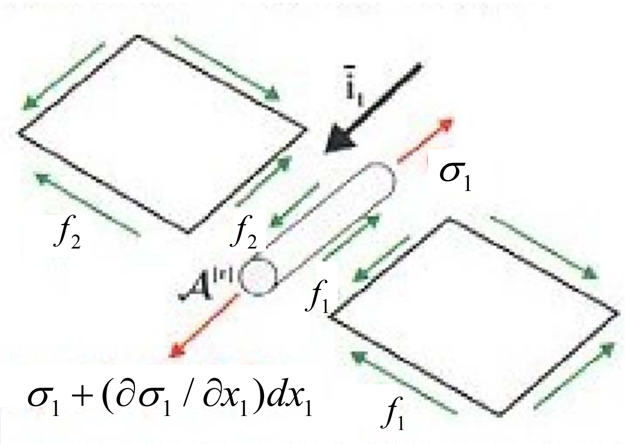
8.10 Structural idealization

8.10.3 shear flow in the sheet components

- Local equilibrium condition, Eq.(8.14) $\rightarrow$ $\partial f / \partial s = 0$, since no axial stress
  $\rightarrow$ $f = const.$ (8.114)

❖ Stringer equilibrium

- Figure 8.81



axial equilibrium for the r-th stringer

$$\left(\sigma_1 + \frac{\partial \sigma_1}{\partial x_1} dx_1 - \sigma_1 \right) A^{[r]} + f_2 dx_1 - f_1 dx_1 = 0$$

$$\Delta f^{[r]} = f_2 - f_1 = -A^{[r]} \frac{\partial \sigma_1}{\partial x_1} \qquad (8.115)$$

- Eq. (8.113) $\rightarrow$ (8.115)

$$\Delta f^{[r]} = -E^{[r]} A^{[r]} \left[\frac{H_{33}^C V_2 - H_{23}^C V_3}{\Delta H} - x_2^{[r]} \frac{H_{23}^C V_2 - H_{22}^C V_3}{\Delta H} \right] \qquad (8.116)$$

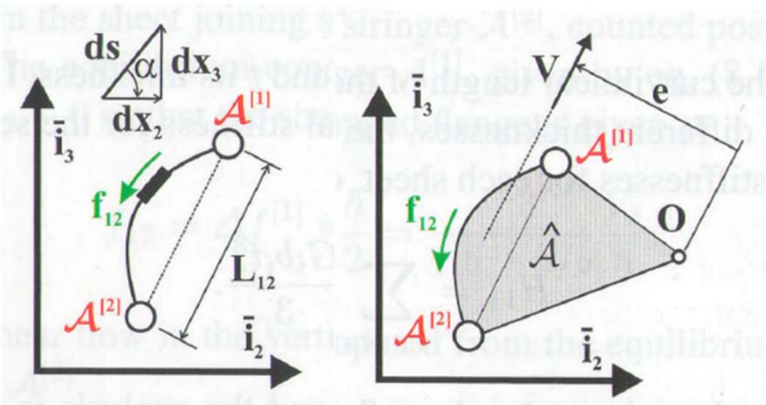
$$\Delta H = H_{22}^C H_{33}^C - (H_{23}^C)^2$$

- general thin-walled x-s → shear flow distribution is governed by a differential equation, Eq. (8.20)
  sheet-stringer idealization → shear flow distribution is governed by a difference equation, Eq. (8.116)

- Integration constant needs to be determined
  - open section → 0 at stress-free edge
  - closed section → Section 8.3.7

❖ Shear flow resultants

- Figure 8.82 → curved sheet carrying a constant shear flow f_{12}, and connecting 2 stringers, shear force resultant



$$V_3 = \int_1^2 i_3 \bullet f_{12} d\underline{s} = f_{12}\int_1^2 dx_3 = f_{12}(x_3^{[2]} - x_3^{[1]})$$

similarly, $V_2 = f_{12}(x_2^{[2]} - x_2^{[1]})$

$$V = \sqrt{V_2^2 + V_3^2} = f_{12}\sqrt{(x_2^{[2]} - x_2^{[1]})^2 + (x_3^{[2]} - x_3^{[1]})^2} = f_{12}L_{12}$$

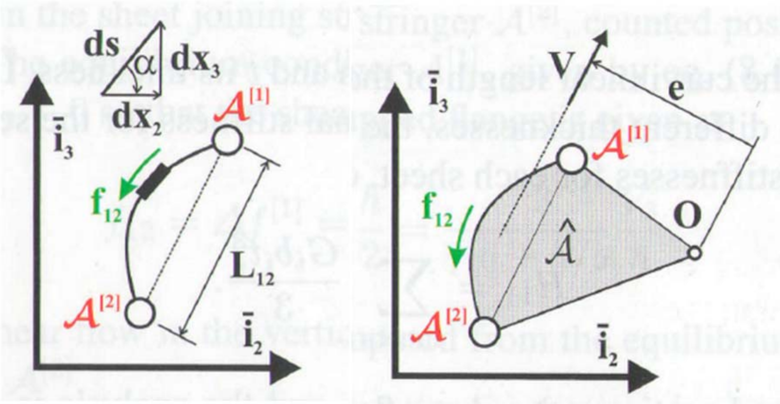
direction parallel to the line connecting the 2 stringers (8.118)

8.10 Structural idealization

- Moment resulting from the shear flow distribution w, r, t point O

$$M_0 = \int_1^2 f_{12} r_o ds = f_{12} \int_1^2 r_o ds = f_{12} \int_{\hat{A}} 2dA = f_{12} 2\hat{A}$$

$\hat{A}$: area of the sector defined by the 2 stringers (Fig. 8.82)



- Distance e of line of action from O

$$e = 2\hat{A}\frac{f_{12}}{V} = \frac{2\hat{A}}{L_{12}} \tag{8.119}$$

8.10 Structural idealization

8.10.4 Torsion of sheet-stringer sections

- Open section → linear shear stress distribution through thickness, inefficient at carrying torsional loads

$$H_{11} = G\frac{bt^3}{3}$$

If different thickness for individual sheets

$$H_{11} = \sum_{sheets} H_{11i} = \sum_{sheets} \frac{G_i b_i t_i^3}{3} \qquad (8.120)$$

8.10 Structural idealization

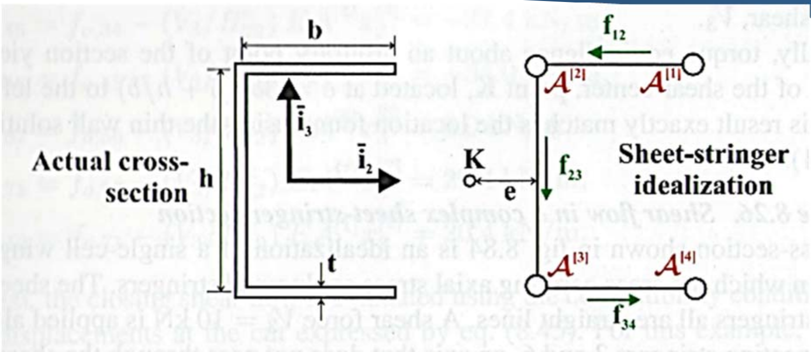


Fig. 8.83. Sheet-stringer model for C-channel section.

- Example 8.25 Shear flow in a sheet-stringer C-channel section

- C-channel section subjected to a shear load, V_3, and a bending moment, M_2
- i_2: axis of symmetry, principal centroidal axes.

- Under the bending moment, axial stress will be const. over the top flanges and bottom flanges, but will vary linearly in the web.
- Use Eqs. (8.111) and (8.112) to evaluate the stringers.

$$A^{[1]} = 1/2\,bt,\ A^{[2]} = 1/2\,bt + 1/6\,ht,$$

$$A^{[3]} = 1/2\,bt + 1/6\,ht,\ A^{[4]} = 1/2\,bt$$

- This idealization yields the same bending stiffness as that for the thin-walled section

$$H_{22}^{c} = \frac{1}{2}Ebht^2 + \frac{1}{12}Eth^3 = \frac{1}{12}Ebht^2\left(6 + \frac{h}{b}\right)$$

8.10 Structural idealization

- Equilibrium condition for stringer $A^{[1]}$, Eq. (8.116), yields

$$\Delta f^{[1]} = f_{12} - 0$$

- Shear flow in the upper flange

$$f_{12} = \Delta f^{[1]} = -\frac{V_3}{H_{22}^C} EA^{[1]} \frac{h}{2} = -\frac{3}{6 + h/b} \frac{V_3}{h}$$

- Shear flow in the vertical web

$$f_{23} = f_{12} - \frac{V_3}{H_{22}^C} EA^{[2]} \frac{h}{2} = -\frac{3}{6 + h/b} \frac{V_3}{h} - \frac{3 + h/b}{6 + h/b} \frac{V_3}{h} = -\frac{V_3}{h}$$

- Shear flow in the lower flange

$$f_{34} = -\frac{3}{\left(6 + h/b\right)} \frac{V_3}{h}, f_{34} = f_{12}$$

- Observation
  - shear flow is const. in each sheet in contrast with the thin-wall solution (Fig. 8.25)
  - Max. shear flow in the sheet-stringer idealization

$$f_{\max} = \frac{V_3}{h}$$

8.10 Structural idealization

Max. shear flow in the thin-wall solution

$$f_{\max} = \frac{3}{2}\frac{\left(1+4b/h\right)}{\left(1+6b/h\right)}\frac{V_3}{h}$$

Thus, sheet-stringer idealization underestimates the true shear flow and thus is not conservative.

- Sheet-stringer idealization exactly satisfy overall equilibrium requirements.
- Torque equipollence about an arbitrary point of the section yields the location of the shear center, *K*. This result exactly matches the location found using the thin-wall solution.