

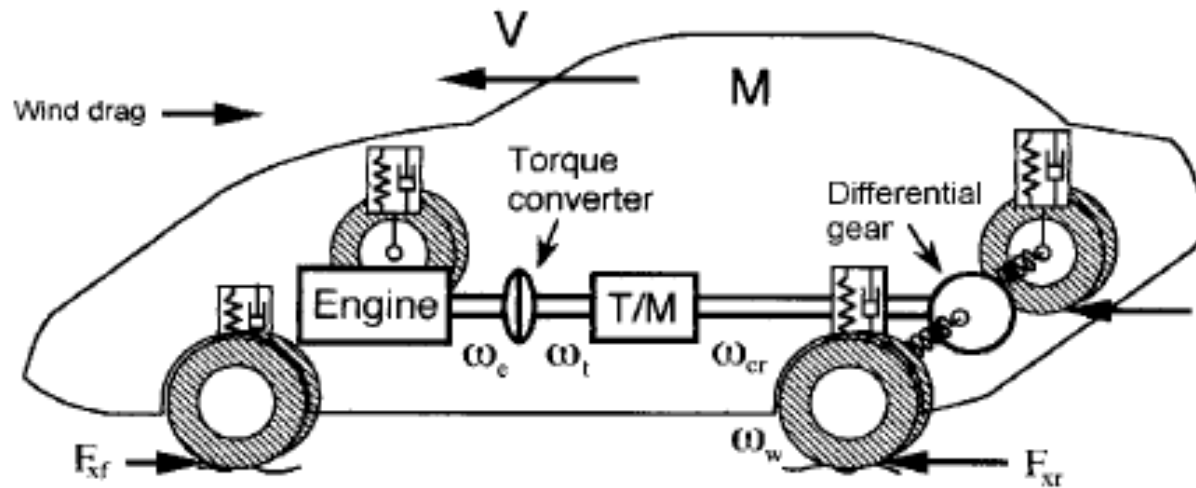
# Mechanical Systems I

# Vehicle Suspension

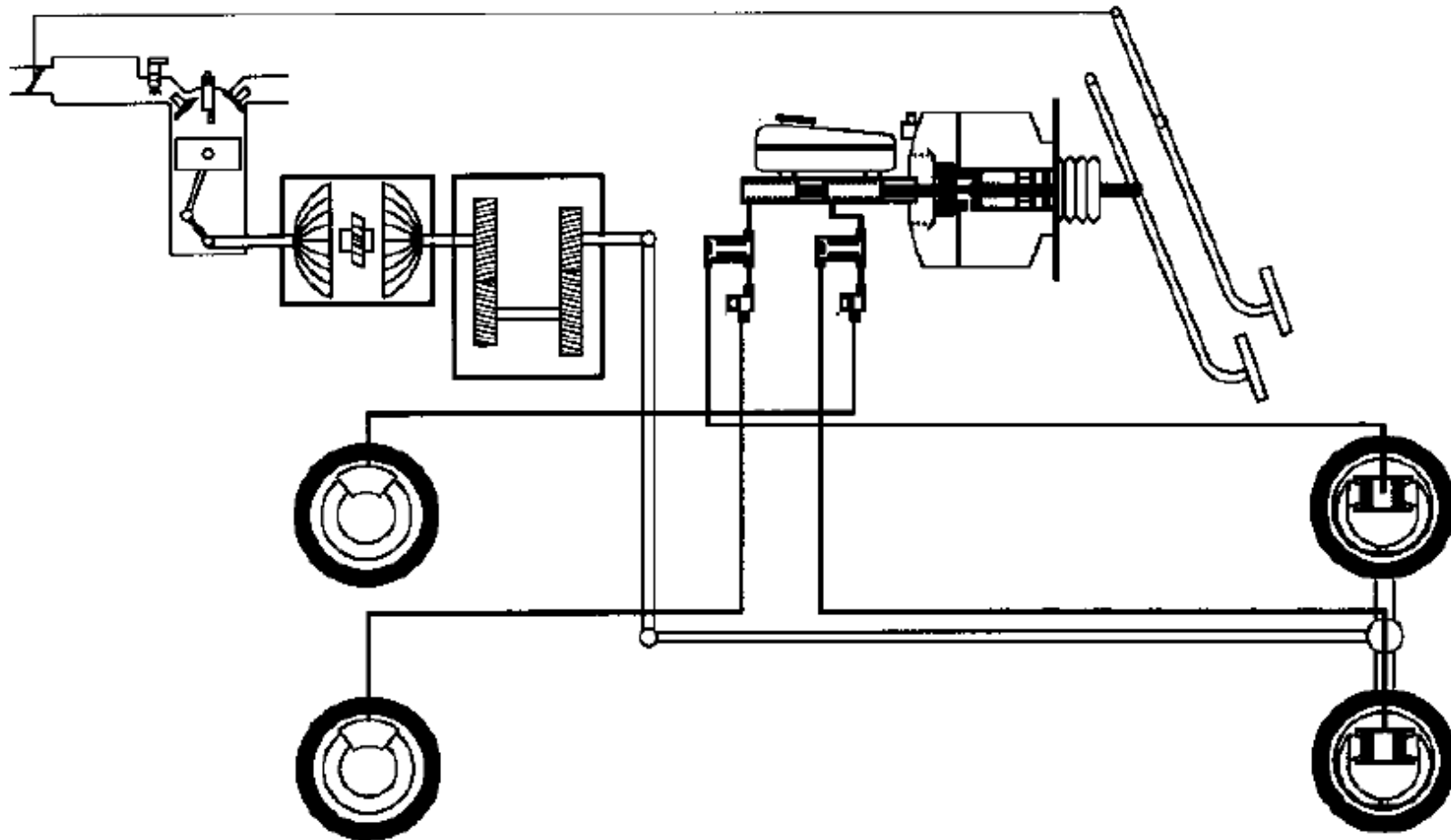


# Vehicle Suspension

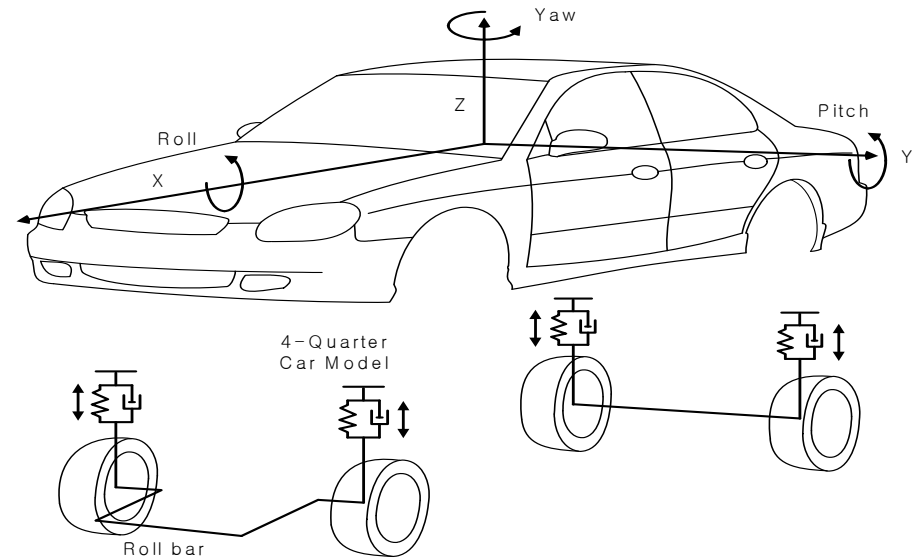
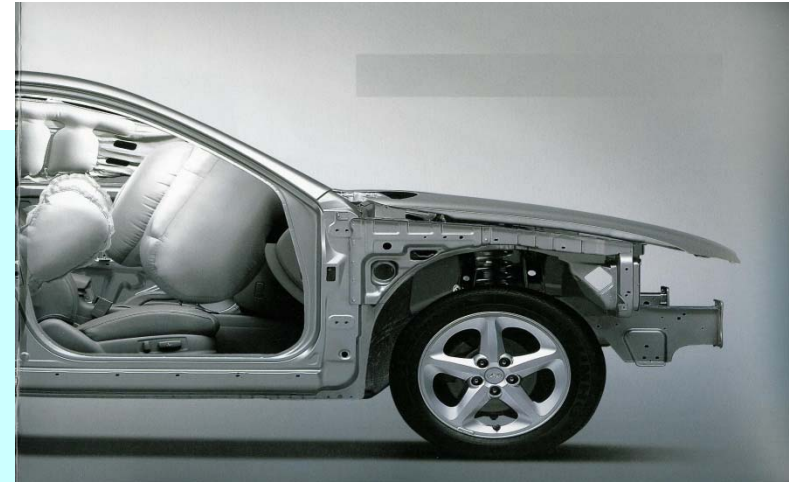


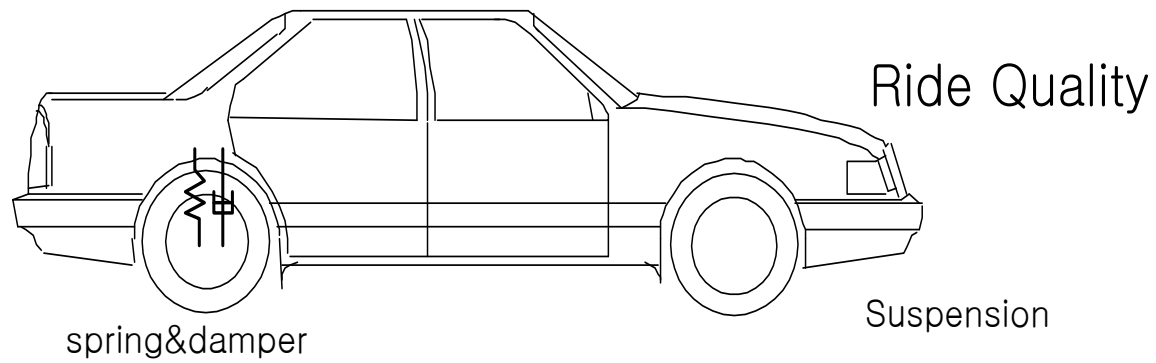
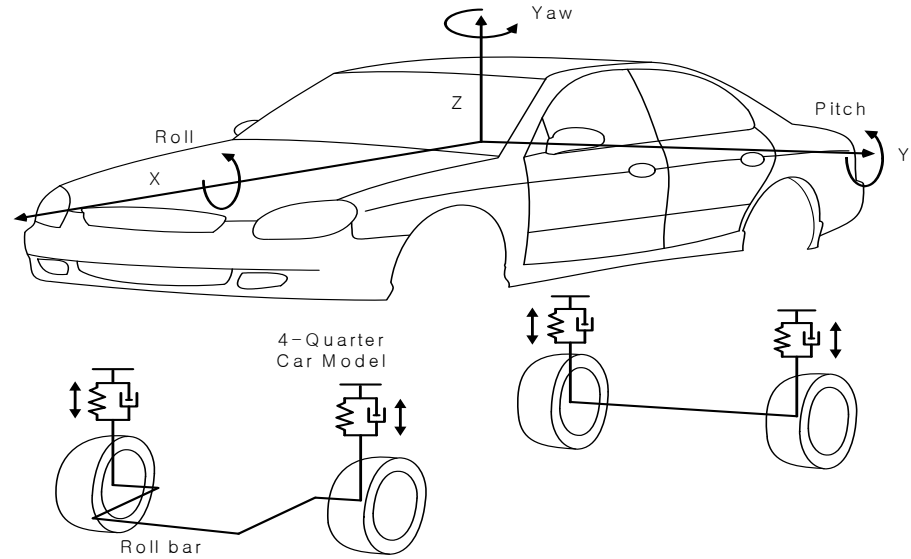


- Engine Model
- Torque Converter
- Transmission
- Axle Shaft
- Differential Gear



# Vehicle Suspension





# Newton's Laws

1) First law : conservation of momentum

no external force

→ no momentum change

linear momentum :  $mv$

angular momentum :  $J\omega$

2) Second law :  $\sum F = ma = m \frac{dv}{dt}$

$$\sum T = J\alpha = J \frac{d\omega}{dt}$$

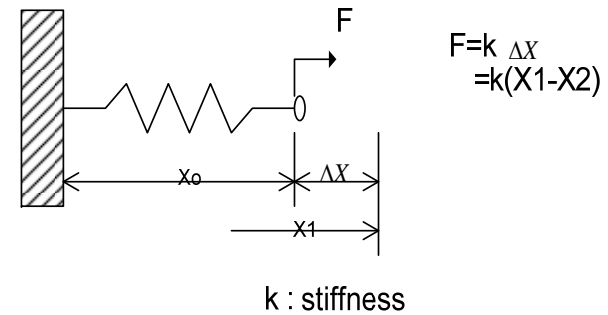
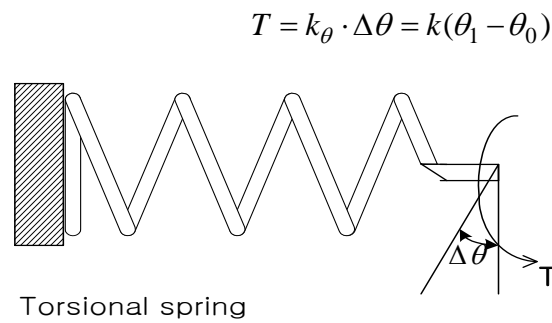


# Three Basic Elements in Modeling Mechanical Systems

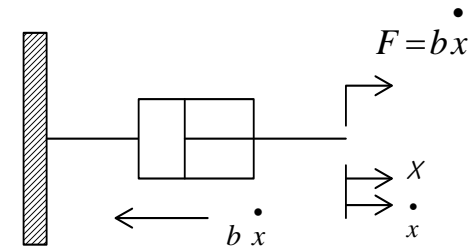
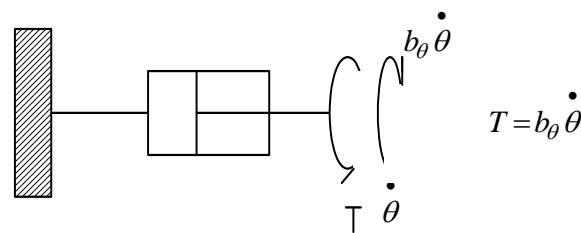
i) Inertial elements ( kinetic energy ):

masses:  $M$                       moments of inertial :  $J$

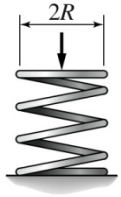
ii) Spring elements ( Potential energy )



iii) Damper elements ( Energy dissipation )



# Spring Elements



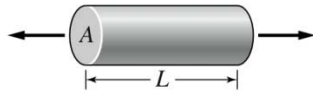
$$k = \frac{Gd^4}{64nR^3}$$

$d = \text{wire diameter}$

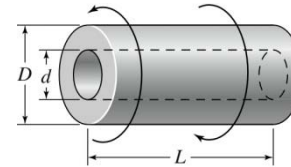
$n = \text{number of coils}$



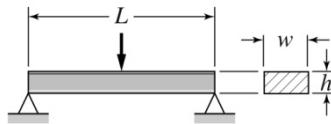
$$k_T = \frac{Ed^4}{64nD}$$



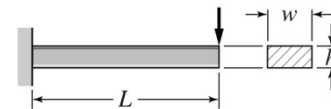
$$k = \frac{EA}{L}$$



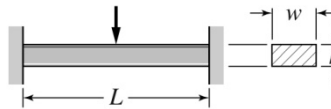
$$k_T = \frac{\pi G(D^4 - d^4)}{32L}$$



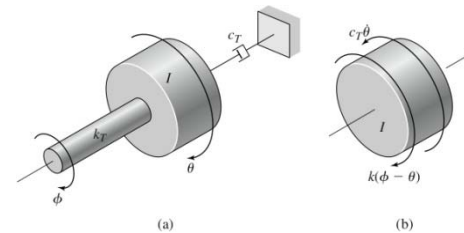
$$k = \frac{4Ewh^3}{L^3}$$



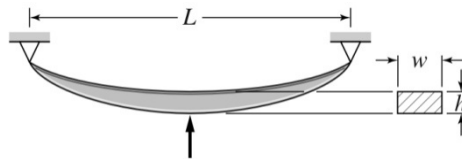
$$k = \frac{Ewh^3}{4L^3}$$



$$k = \frac{16Ewh^3}{L^3}$$

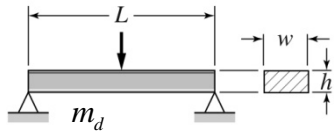
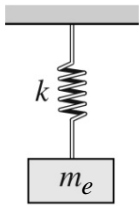


$$k_T = \frac{\pi GD^4}{32L}$$

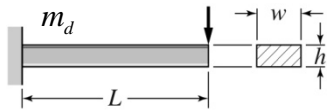


$$k = \frac{2Ewh^3}{L^3}$$

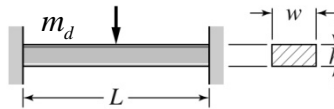
## Equivalent masses of common elements



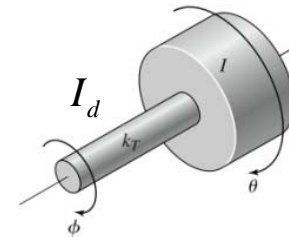
$$k = \frac{4Ewh^3}{L^3} \quad m_e = 0.38m_d$$



$$k = \frac{Ewh^3}{4L^3} \quad m_e = 0.23m_d$$



$$k = \frac{16Ewh^3}{L^3} \quad m_e = 0.5m_d$$



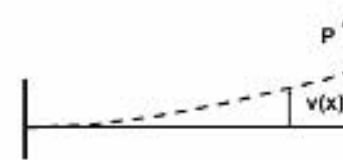
$$I_e = I + I_d / 3$$

$$k_T = \frac{\pi GD^4}{32L}$$

# Tip-Loaded Cantilever: Lateral Deflections

## curvature / moment relations:

$$\begin{aligned}v''(x) &= \frac{1}{EI} M(x) \\ &= \frac{1}{EI} (P(L-x)) \Rightarrow \\ v'(x) &= \frac{P}{EI} (Lx - x^2/2 + C_1) \Rightarrow \\ v(x) &= \frac{P}{EI} (Lx^2/2 - x^3/6 + C_1x + C_2)\end{aligned}$$



## geometric boundary conditions

$$\begin{aligned}\phi(0) = v'(0) &= 0 \Rightarrow C_1 = 0 \\ v(0) &= 0 \Rightarrow C_2 = 0 \\ v(x) &= \frac{Px^2}{6EI} (3L - x)\end{aligned}$$

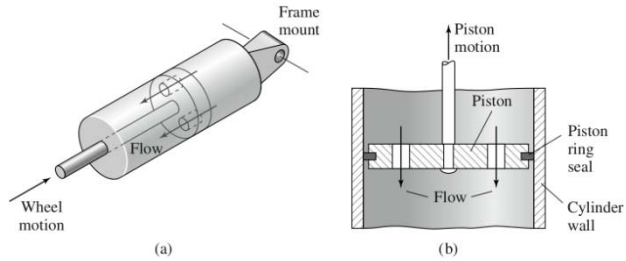
## tip deflection and rotation:

$$\begin{aligned}\Delta \equiv v(L) &= \frac{PL^3}{3EI} \\ \Phi \equiv v'(L) &= \frac{PL^2}{2EI}\end{aligned}$$

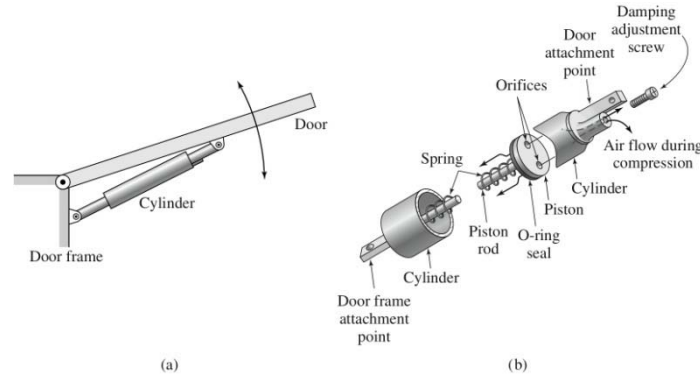
## stiffness and modulus:

$$\begin{aligned}k \equiv \frac{P}{\Delta} &= \frac{3EI}{L^3} \\ E &= \frac{kL^3}{3I} = \frac{PL^3}{3I\Delta}\end{aligned}$$

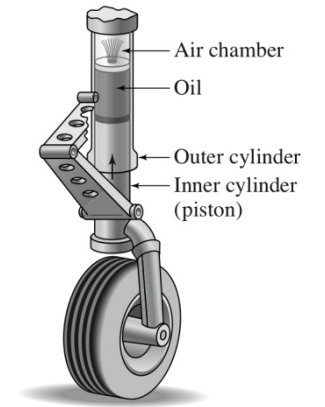
# Damping Elements



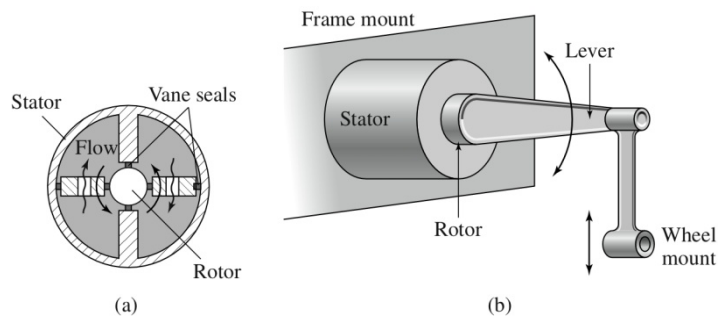
**Piston damper**



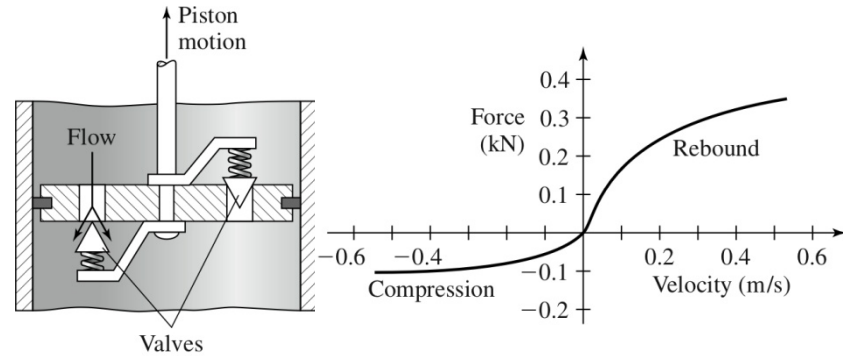
**Pneumatic door closer**



**Oleo strut**



**Rotary damper**



**Damper with spring loaded valves**

# Examples of Modeling Mechanical Systems

**ex1)**  $t = 0, \quad \omega(0) = \omega_0$

**equation of motion :**

$$J \frac{d\omega}{dt} = -b\omega$$

$$J \frac{d\omega}{dt} + b\omega = 0 \rightarrow \frac{d\omega}{dt} + \frac{b}{J} \omega = 0$$

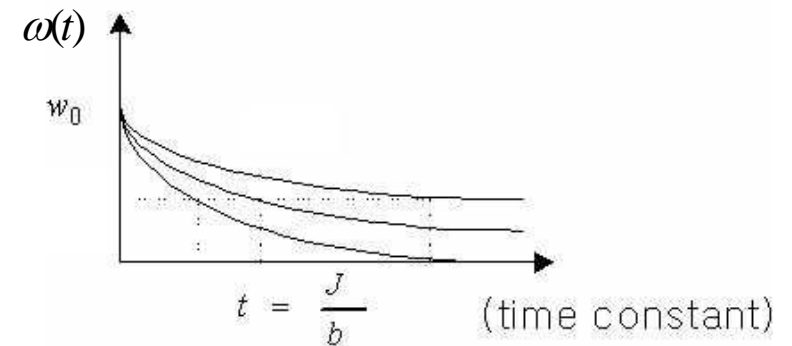
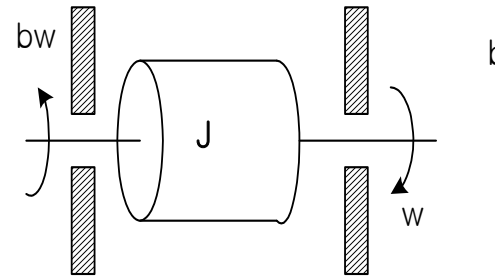
$$\text{let, } \omega(t) = ce^{\lambda t} \rightarrow \lambda e^{\lambda t} + \frac{b}{J} e^{\lambda t} = 0$$

$$\lambda = -\frac{b}{J}$$

$$\omega(t) = ce^{-\frac{b}{J}t}, \quad t = 0, \quad \omega(0) = \omega_0 = C$$

$$\therefore \omega(t) = \omega_0 e^{-\frac{b}{J}t}$$

$$\omega_0 e^{-1} = 0.368\omega_0$$

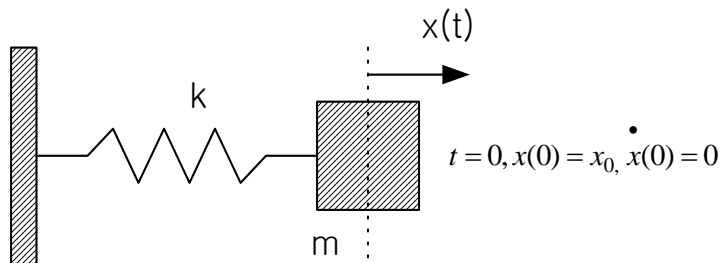


**T=Time constant: time to reach 63.2%**

# Examples of Modeling Mechanical Systems

## Spring-Mass

ex2)



$$m \frac{d^2 x}{dt^2} = \sum F = -kx$$

$$m\ddot{x} + kx = 0 \rightarrow \ddot{x} + \frac{k}{m}x = 0$$

$$x(t) = A \cos \sqrt{\frac{k}{m}}t + B \sin \sqrt{\frac{k}{m}}t$$

$$x(0) = A = x_0$$

$$\dot{x}(0) = -A \sqrt{\frac{k}{m}} \sin \sqrt{\frac{k}{m}}t + B \sqrt{\frac{k}{m}} \cos \sqrt{\frac{k}{m}}t = B \sqrt{\frac{k}{m}} = 0$$

$$x(t) = x_0 \cos \sqrt{\frac{k}{m}}t = x_0 \cos \omega_n t$$

- **Period**  $T = \frac{2\pi}{\sqrt{k/m}} [\text{sec}]$

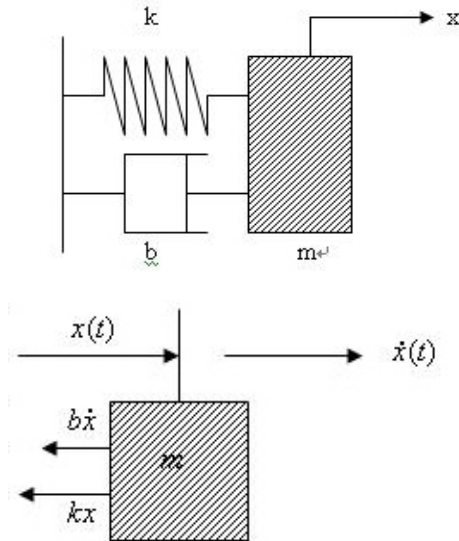
- **Frequency**  $f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} [Hz]$

- **Natural frequency**

$$\omega_n = 2\pi f = \sqrt{\frac{k}{m}} [\text{rad/sec}]$$

# Examples of Modeling Mechanical Systems

## Spring-mass-damper



$$x(0) = x_0$$

$$\dot{x}(0) = \dot{x}_0$$

$$m \frac{d^2 x}{dt^2} = -kx - b\dot{x}$$

$$\ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = 0$$

$$\text{let, } \frac{k}{m} = \omega_n^2, \quad \frac{b}{m} = 2\zeta\omega_n, \quad \frac{b}{2\sqrt{mk}} = \zeta$$

Damping ratio =

Exponential decay

Natural frequency

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0, \quad x(t) = c \cdot e^{\lambda t} \quad \lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0$$

$$\therefore \lambda = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\omega_n$$



## Examples of Modeling Mechanical Systems

$$\ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = 0$$

$$x(0) = x_0$$

$$\dot{x}(0) = \dot{x}_0 = 0$$

$$\text{let, } \frac{k}{m} = \omega_n^2, \quad \frac{b}{m} = 2\zeta\omega_n, \quad \frac{b}{2\sqrt{mk}} = \zeta$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$$

### Laplace Transform

$$(s^2 X(s) - s \cdot x_0 - \dot{x}_0) + 2\zeta\omega_n (sX(s) - x_0) + \omega_n^2 X(s) = 0$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) X(s) - s \cdot x_0 - 2\zeta\omega_n x_0 = 0$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) X(s) = s \cdot x_0 + 2\zeta\omega_n x_0$$

$$X(s) = \frac{s \cdot x_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

## Examples of Modeling Mechanical Systems

$$X(s) = \frac{s \cdot x_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s \cdot x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$s^2 + 2\zeta\omega_n s + \omega_n^2$  : characteristic polynomial

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$  : characteristic equation

$s = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$  : characteristic roots

1)  $\zeta < 1$  underdamped

$$s = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2}$$

$$X(s) = \frac{(s + \zeta\omega_n) \cdot x_0}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} + \frac{\zeta\omega_n x_0}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$x(t) = e^{-\zeta\omega_n t} \left( x_0 \cos \sqrt{1 - \zeta^2} \omega_n t + \frac{\zeta}{\sqrt{1 - \zeta^2}} x_0 \sin \sqrt{1 - \zeta^2} \omega_n t \right)$$

## Examples of Modeling Mechanical Systems

2)  $\zeta > 1$  overdamped

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$X(s) = \frac{(s + 2\zeta\omega_n) \cdot x_0}{\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)}$$

$$x(t) = k_1 e^{(-\zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n)t} + k_2 e^{(-\zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n)t}$$

3)  $\zeta = 1$  critically damped

$$s = -\zeta\omega_n$$

$$X(s) = \frac{(s + 2\zeta\omega_n) \cdot x_0}{(s + \zeta\omega_n)^2}$$

$$x(t) = k_1 e^{(-\zeta\omega_n)t} + k_2 t e^{(-\zeta\omega_n)t}$$

# Examples of Modeling Mechanical Systems

1)  $\zeta < 1$       **Underdamped**

$$x(t) = e^{-\zeta\omega_n t} \left( x_0 \cos \sqrt{1-\zeta^2} \omega_n t + \frac{\zeta}{\sqrt{1-\zeta^2}} x_0 \sin \sqrt{1-\zeta^2} \omega_n t \right)$$

$$\begin{cases} \omega_n : & \text{natural frequency} \\ \zeta : & \text{damping ratio} \end{cases} \quad \omega_d = \sqrt{1-\zeta^2} \omega_n$$

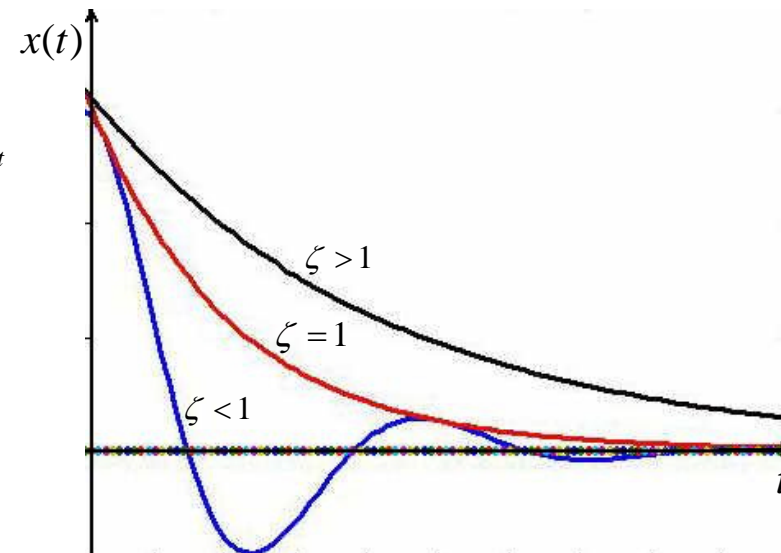
2)  $\zeta > 1$       **Overdamped**

$$x(t) = k_1 e^{(-\zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n)t} + k_2 e^{(-\zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n)t}$$

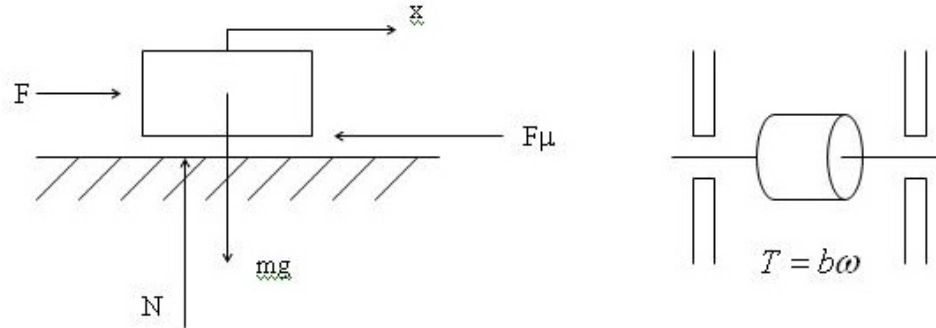
$$\lambda = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\omega_n$$

3)  $\zeta = 1$        $\lambda = -\zeta\omega_n$

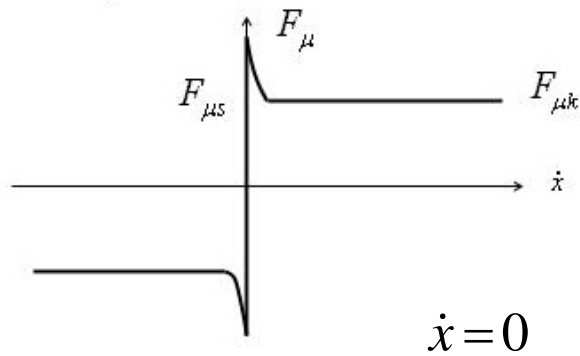
$$x(t) = k_1 e^{-\zeta\omega_n t} + k_2 t e^{-\zeta\omega_n t}$$



# Dry Friction (no lubricant)



- $F_{\mu s}$  = Static Friction Force
- $F_{\mu k}$  = Kinetic Friction Force



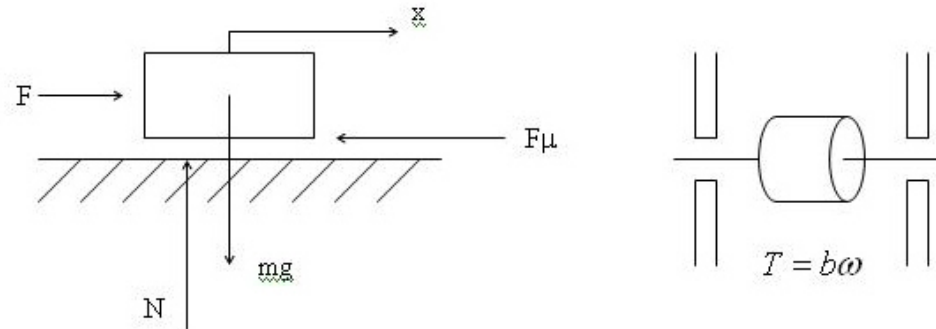
- $F_{\mu s} = \mu_s \cdot N$       $\mu_s$  : Static Friction Coefficient
- $F_{\mu k} = \mu_k \cdot N$       $\mu_k$  : Kinetic Friction Coefficient

$$\dot{x} = 0$$

$$\dot{x} \neq 0$$

$$F_{\mu} = \begin{cases} F & \text{if } F \leq F_{\mu s} \\ F_{\mu s} \operatorname{sgn}(F) & \text{if } F > F_{\mu s} \end{cases} \quad F_{\mu} = F_{\mu k} \operatorname{sgn}(\dot{x})$$

## Friction (with lubricant)



$$F_{\mu} = \begin{cases} b\dot{x} + G \cdot N \operatorname{sgn}(\dot{x}) & \forall \dot{x} > \varepsilon \\ F & \text{if } |\dot{x}| < \varepsilon \text{ and } |F| \leq (F_s + G \cdot N) \\ (F_s + G \cdot N) \operatorname{sgn}(F) & \text{otherwise} \end{cases} \quad \text{i.e., if } |\dot{x}| < \varepsilon \text{ and } |F| > (F_s + G \cdot N)$$

**b:** viscous friction coefficient

**G:** load-dependent factor

**N:** normal force

**F<sub>s</sub>:** the maximum static friction

**$\varepsilon$  :** a small bound for zero velocity detection