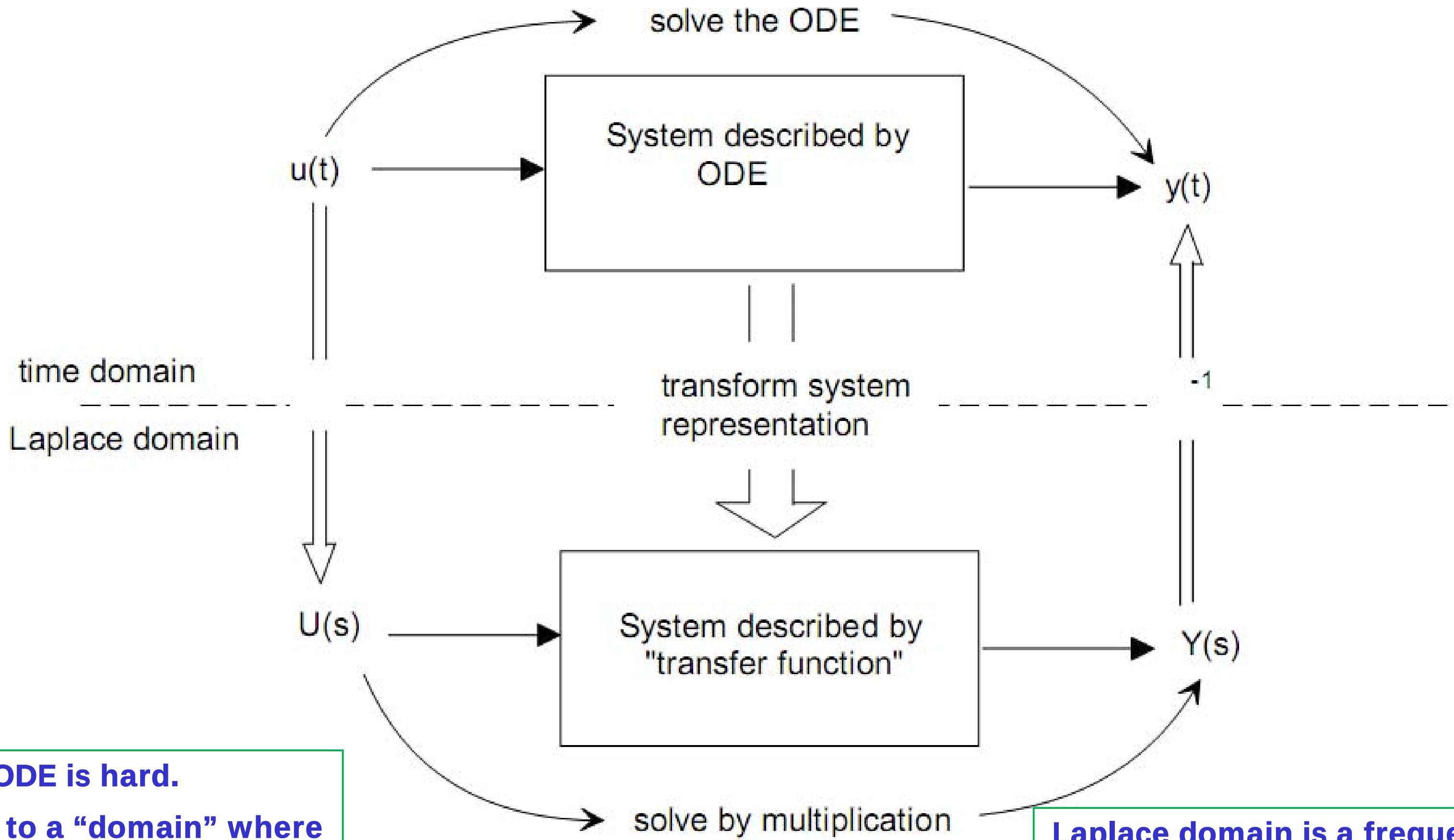
Laplace Transformation I

Why use Laplace Transform?

Algebraic Manipulation of ODE



Solution of ODE is hard.

Tranform in to a “domain” where it’s easier to solve

Solve in the new domain

Perform “inverse” transform.

Laplace domain is a frequency domain.

Integration, Differentiation becomes multiplication, division.

Laplace Transformation

· Definition: $\mathcal{L}\,[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}dt = F(s)$

$\mathcal{L}$: $f(t) \Rightarrow F(s)$, $s=\sigma + j\omega$ (complex variable)

$f(t)$: a time function such that $f(t)=0$ for $t<0$

· Inverse Laplace Transformation

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

Existence of Laplace Transformation

- f(t) Laplace – transformable

  if i) f(t) piecewise-continuous

  ii) f(t) of exponential order as t approaches infinity

  – $e^{\alpha t}\left|f(t)\right|$ bounded, α exist.

  – or $e^{-\sigma t}\left|f(t)\right|$ approaches zero as t approaches infinity.

- If $\lim_{t\to\infty} e^{-\sigma t}\left|f(t)\right| = \begin{cases} 0 & for\ \sigma > \sigma_c \\ \infty & for\ \sigma < \sigma_c \end{cases}$

  the σ_c : the abscissa of convergence

Existence of Laplace Transformation

example : **1)** $t,\ \sin\omega t,\ t\sin\omega t\ldots$

$$\lim_{t\to\infty} e^{-\sigma t}\left|t\sin\omega t\right| = \begin{cases} 0 & if\ \sigma > 0 \\ \infty & if\ \sigma < 0 \end{cases}$$

the abscissa of convergence $\sigma_c = 0$

2) $e^{-ct},\ te^{-ct},\ e^{-ct}\sin\omega t,\quad c = const.$

$$\lim_{t\to\infty} e^{-\sigma t}\left|te^{-ct}\right| = \begin{cases} 0 & if\ \sigma > -c \\ \infty & if\ \sigma < -c \end{cases}$$

the abscissa of convergence $\sigma_c = -c$

- $e^{t^2},\quad te^{t^2}$ **does not possess L. T.**
- $f(t) = \begin{cases} e^{t^2} & for\quad 0 \le t \le T < \infty \\ 0 & for\quad t < 0,\ \mathrm{T} < \mathrm{t} \end{cases}$ **L[f(t)] exists.**

  - **The signals that can be physically generated always have corresponding Laplace transforms**

Laplace Transformation of simple function

- **Exponential function**

$$f(t) = \begin{cases} 0 & for \quad t < 0 \\ Ae^{-\alpha t} & for \quad t \geq 0 \end{cases}$$

A, α : constants

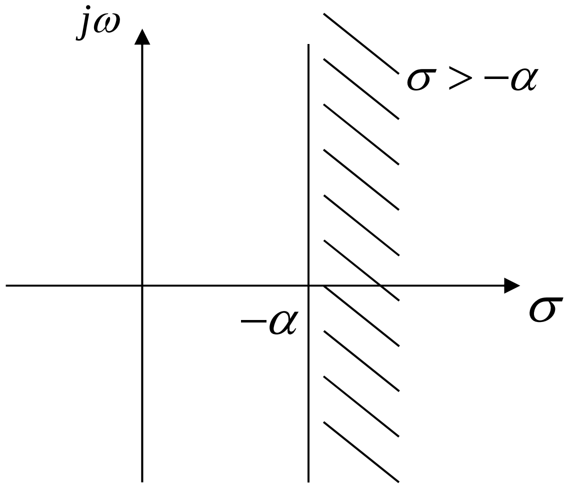
$$\mathcal{L}[Ae^{-\alpha t}] = \int_{0-}^{\infty} Ae^{-\alpha t} \cdot e^{-st} dt$$

$$= \int_{0-}^{\infty} Ae^{-(\alpha+s)t} dt$$

$$= A[-\frac{1}{s+\alpha} e^{-(s+\alpha)\cdot\infty} + \frac{1}{s+\alpha} e^{0}]$$

$$= A[-\frac{1}{s+\alpha} e^{-(\sigma+\alpha)\cdot\infty - j\omega\cdot\infty} + \frac{1}{s+\alpha}] \qquad (s = \sigma + j\omega)$$

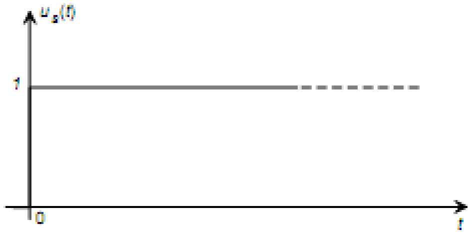
$$= A[0 + \frac{1}{s+\alpha}] = \frac{A}{s+\alpha}$$



Laplace Transformation of simple function

- **step function**

$$f(t) = \begin{cases} A & t > 0 \\ 0 & t < 0 \end{cases}$$



$$\mathcal{L}[f(t)] = \int_{0-}^{\infty} Ae^{-st}dt = A[-\frac{1}{s}e^{-s\cdot\infty} + \frac{1}{s}e^{-s\cdot 0}]$$

$$= A[0 + \frac{1}{s}] \qquad if \ \mathrm{Re}[s] > 0$$

- **unit step input function**

$$1(t - t_0) = \begin{cases} 1 & t > t_0 \\ 0 & t < t_0 \end{cases}$$

$$1(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[1(t)] = \frac{1}{s}$$

Laplace Transformation of simple function

- Sinusoidal Functions

$$f(t)=\begin{cases} A\sin\omega t & t\geq 0 \\ 0 & t<0 \end{cases}$$

$$\cos\omega t=\frac{1}{2}(e^{j\omega t}+e^{-j\omega t})$$

$$\sin\omega t=\frac{1}{2j}(e^{j\omega t}-e^{-j\omega t})$$

$$\mathcal{L}[A\sin\omega t]=\int_{0-}^{\infty}\frac{A}{2j}(e^{j\omega t}-e^{-j\omega t})e^{-st}dt$$

$$=\frac{1}{2j}(\frac{1}{s-j\omega}-\frac{1}{s+j\omega})$$

$$=\frac{A\omega}{s^2+\omega^2}$$

$$\mathcal{L}[A\cos\omega t]=\frac{As}{s^2+\omega^2}$$

- **Ramp function**

$$f(t)=\begin{cases} At & t\geq 0 \\ 0 & t<0 \end{cases}$$

$$\mathcal{L}[At]=A\int_{0-}^{\infty}te^{-st}dt$$

$$=A\left(t\frac{e^{-st}}{-s}\bigg|_{0-}^{\infty}-\int_{0-}^{\infty}\frac{e^{-st}}{-s}dt\right)$$

$$=A\cdot\frac{1}{s}\int_{0-}^{\infty}e^{-st}dt=A\cdot\frac{1}{s^2}$$

Laplace Transformation of simple function

- **Pulse function**

$$f(t)=\begin{cases}\dfrac{A}{t_0} & for \quad 0\le t\le t_0\\ 0 \quad for \quad t<0,\ t_0<\mathrm{t}\end{cases}$$

$$\mathcal{L}[f(t)]=\mathcal{L}\left[\frac{A}{t_0}1(t)\right]-\mathcal{L}\left[\frac{A}{t_0}1(t-t_0)\right]$$

$$=\frac{A}{t_0}\frac{1}{s}(1-e^{-t_0 s})$$

$$f(t)=\frac{A}{t_0}1(t)-\frac{A}{t_0}1(t-t_0)$$

- **Impulse function**

$$\mathcal{L}[f(t)]=\lim_{t_0\to 0}\left[\frac{A}{t_0}\frac{1}{s}(1-e^{-t_0 s})\right]$$

$$f(t)=\begin{cases}\displaystyle\lim_{t_0\to 0}\frac{A}{t_0} & for \quad 0\le t<t_0\\ 0 \quad for \quad t<0, \quad t_0<t\end{cases}$$

$$=\lim_{t_0\to 0}\frac{\dfrac{d}{dt_0}\left[A(1-e^{-t_0 s})\right]}{\dfrac{d}{dt_0}(t_0 s)}=\frac{A\cdot s}{s}=A$$

Laplace Transformation of simple function

- **Unit impulse function ; impulse function of magnitude 1**

$$f(t)=\begin{cases} \lim_{t_0\to 0}\dfrac{1}{t_0} & for \quad 0\le t<t_0 \\ 0 \quad for & t<0, \quad t_0<t \end{cases}$$

$$\delta(t)=\begin{cases} \infty & t=0 \\ 0 & t\neq 0 \end{cases} \qquad \mathcal{L}\left[\delta(t)\right]=\int_0^\infty \delta(t)e^{-st}dt=\int_{0-}^{0+}\delta(t)dt=1$$

The unit-impulse function occuring at $t=t_0$

$$\delta(t)=\begin{cases} \infty & t=t_0 \\ 0 & t\neq t_0 \end{cases} \quad \mathcal{L}\left[\delta(t-t_0)\right]=\int_0^\infty \delta(t-t_0)e^{-st}dt=\int_{t_0-}^{t_0+}\delta(t-t_0)e^{-st_0}dt=e^{-t_0 s}\cdot 1$$

$$\delta(t-t_0)=\frac{d}{dt}1(t-t_0)$$