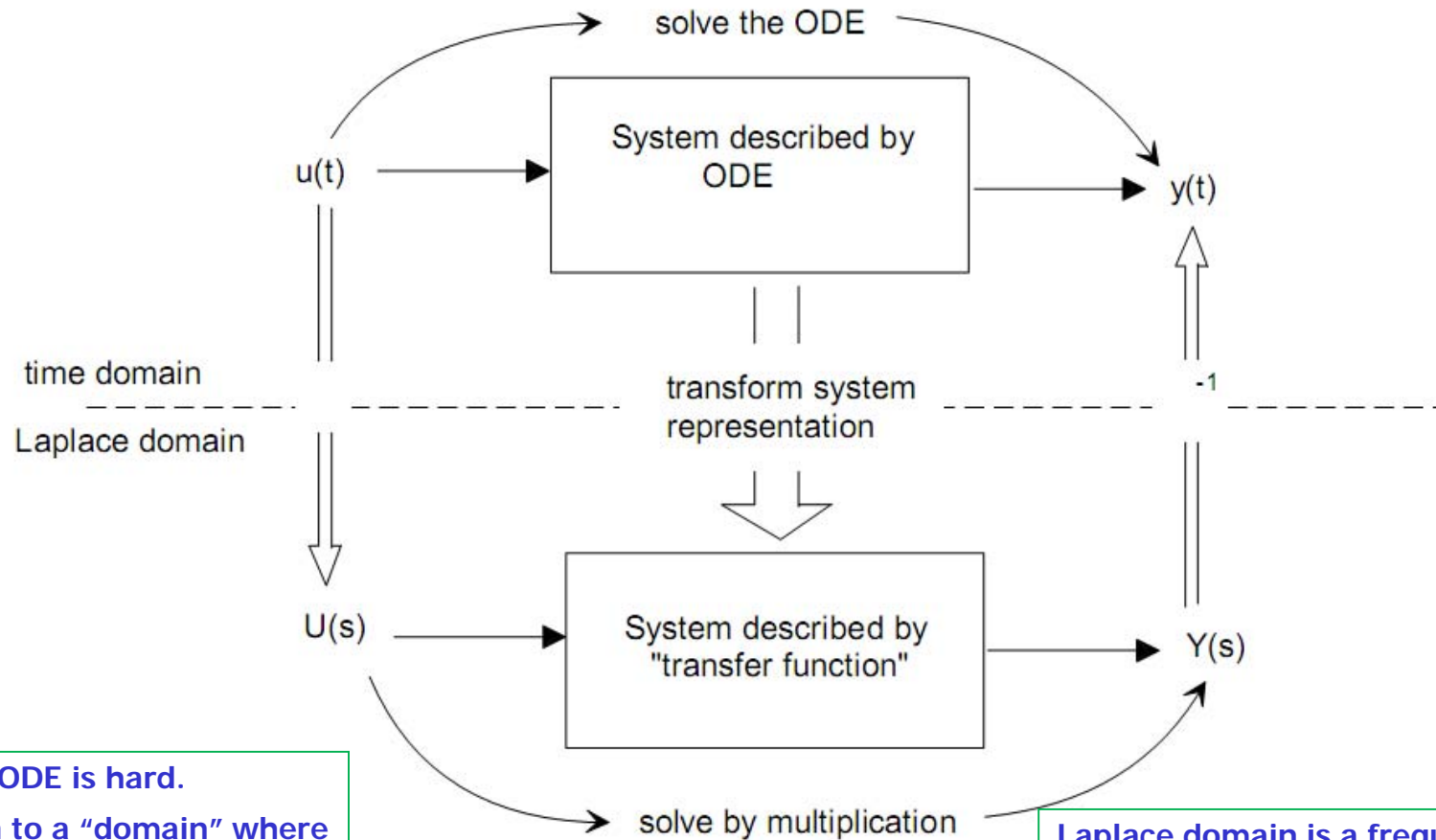


Laplace Transformation II

Why use Laplace Transform?

Algebraic Manipulation of ODE



Solution of ODE is hard.
Transform in to a "domain" where it's easier to solve
Solve in the new domain
Perform "inverse" transform.

Laplace domain is a frequency domain.
Integration, Differentiation becomes multiplication, division.

Laplace Transformation

• Definition: $\mathcal{L} [f(t)] = \int_{0^-}^{\infty} f(t)e^{-st} dt = F(s)$ Integration from 0-

$\mathcal{L}: f(t) \Rightarrow F(s)$, $s = \sigma + j\omega$ (complex variable)

$f(t)$: a time function such that $f(t)=0$ for $t < 0$

$$\mathcal{L}[1(t)] = \frac{1}{s}$$

$$\mathcal{L}[1(t-t_0)] = \frac{e^{-st}}{s}$$

$$\mathcal{L}[Ae^{-\alpha t}] = \frac{A}{s + \alpha}$$

$$\mathcal{L}[t] = \frac{1}{s^2}$$

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[\delta(t-t_0)] = e^{-t_0 s} \cdot 1$$

$$\mathcal{L}[A \sin \omega t] = \frac{A\omega}{s^2 + \omega^2}$$

$$\mathcal{L}[A \cos \omega t] = \frac{As}{s^2 + \omega^2}$$

Useful Theorems

Theorem 1. Linearity

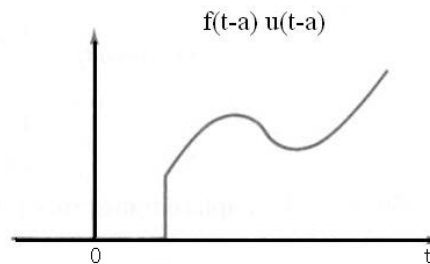
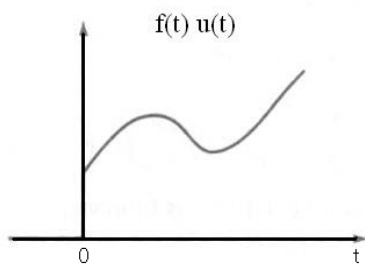
$$\mathcal{L}[af(t)] = aF(s)$$

Theorem 2. Superposition

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

Theorem 3. Translation in time.

$$\begin{aligned}\mathcal{L}[f(t-a)u(t-a)] &= \int_0^{\infty} f(t-a)u(t-a)e^{-st} dt \quad (a > 0) \\ &= \int_{-a}^{\infty} f(\tau)u(\tau)e^{-s(\tau+a)} d\tau \quad (\text{let } t-a = \tau) \\ &= \int_{-a}^{\infty} f(\tau)u(\tau)e^{-s\tau} e^{-sa} d\tau = e^{-sa} F(s) \quad (\because f(\tau)u(\tau) = 0 \text{ for } \tau < 0)\end{aligned}$$



Useful Theorems

Theorem 4. Complex differentiations

$$\mathcal{L}[tf(t)] = -\frac{d}{ds} F(s)$$

$$\mathcal{L}[1] = \frac{1}{s} \quad \mathcal{L}[t \cdot 1] = -\frac{d}{ds} F(s) = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = \mathcal{L}[t \cdot t] = -\frac{d}{ds} \left(\frac{1}{s^2} \right) = \frac{2}{s^3} \quad \mathcal{L}[tf(t)] = -\frac{d}{ds} F(s)$$

proof) let. $F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt$

$$\frac{d}{ds} F(s) = \int_0^{\infty} \frac{\partial}{\partial s} [f(t)e^{-st}] dt = -\int_0^{\infty} tf(t)e^{-st} dt = -\mathcal{L}[tf(t)]$$

similarly, $\mathcal{L}[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$ $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

Theorem 5. Translation in the s-domain

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

$$\mathcal{L}[e^{at} \cos \omega t] = \frac{(s-a)}{(s-a)^2 + \omega^2}$$

Useful Theorems

Theorem 6. Real Differentiation

$$Df(t) = \frac{d}{dt} f(t) \qquad \mathcal{L}\left[\frac{d}{dt} f(t)\right] = s \cdot F(s) - f(0-)$$

proof) let.
$$\begin{aligned} \mathcal{L}\left[\frac{d}{dt} f(t)\right] &= \int_{0-}^{\infty} \frac{d}{dt} f(t) e^{-st} dt \\ &= f(t) e^{-st} \Big|_0^{\infty} - \int_{0-}^{\infty} f(t) e^{-st} dt (-s) \\ &= -f(0-) + s \cdot F(s) \\ &= \boxed{s \cdot F(s) - f(0-)} \end{aligned}$$

similarly,
$$\begin{aligned} \mathcal{L}[D^2 f(t)] &= \mathcal{L}[D \cdot Df(t)] = \mathcal{L}[Df'(t)] \\ &= s \{s \cdot F(s) - f(0-)\} - f'(0-) = s^2 \cdot F(s) - s \cdot f(0-) - f'(0-) \end{aligned}$$

$$\begin{aligned} \mathcal{L}\left[\frac{d^n}{dt^n} f(t)\right] &= s^n \cdot F(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0-) \\ &\quad \dots - f^{(n-1)}(0-) \end{aligned}$$

Useful Theorems

Theorem 7. Real Integration

$$\int_0^t f(t)dt = D^{-1}f(t) - D^{-1}f(0)$$

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \int_0^\infty \int_0^t f(\tau)d\tau \cdot e^{-st} dt$$

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$$

$$= -\frac{1}{s}e^{-st} \int_0^t f(\tau)d\tau \Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st} f(t)dt$$

$$= \frac{1}{s} \int_0^t f(\tau)d\tau \Big|_{t=0} + \frac{1}{s} \int_0^\infty e^{-st} f(t)dt = \boxed{\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}}$$

Theorem 8. Complex Integration

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$$

Useful Theorems

Theorem 9. Final value Theorem

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Theorem 10. Initial value Theorem

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

Inverse Laplace Transformation

$$\mathcal{L}: f(t) \rightarrow F(s)$$

$$\mathcal{L}^{-1}: F(s) \rightarrow f(t)$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} F(s)e^{st} ds \quad (\text{c : real constant})$$

* Inverse Laplace Transformation by Partial Fraction Method

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots + a_0}{s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0} \quad (n \geq m)$$

$$s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0 \quad \Rightarrow \text{real complex conjugate, } a, b : \text{real num.}$$

$$= (s - c_1)(s - c_2) \dots (s^2 + d_1 s + d_2)$$

$$\Rightarrow F(s) = \frac{\alpha_1}{s - c_1} + \frac{\alpha_2}{s - c_2} + \dots + \frac{\beta_1 s + \beta_2}{s^2 + d_1 s + d_2} + \dots$$

Examples of Inverse Laplace Transformation

$$\text{ex)} \quad F(s) = \frac{1}{(s+2)^2(s+3)} = \frac{a}{(s+2)} + \frac{b}{(s+2)^2} + \frac{c}{(s+3)}$$

$$a(s+2)(s+3) + b(s+3) + c(s+2)^2 = 1$$

let $s = -2$, then $b = 1$, let $s = -3$, then $c = 1$

$$a(s+2) + b + c \cdot \frac{s+2}{s+3} = \frac{1}{s+3} \quad \xrightarrow{\frac{d}{ds}} \quad a + c \cdot \frac{(s+3) - (s+2)}{s+3} = \frac{-1}{(s+3)^2}$$

$$\therefore a = -1, b = 1, c = 1, \quad F(s) = \frac{-1}{(s+2)} + \frac{1}{(s+2)^2} + \frac{1}{(s+3)}$$

=> Inverse Laplace Transformation

$$f(t) = -e^{-2t} + te^{-2t} + e^{-3t}$$

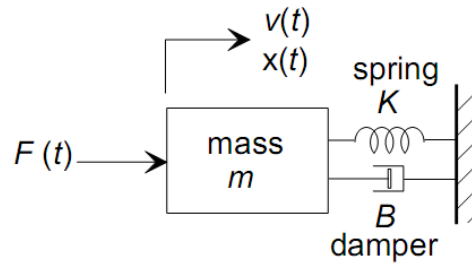
fraction method)

(partial

$$\text{ex)} \quad F(s) = \frac{10}{s^2 + 6s + 25} = \frac{10 \times \frac{1}{4} \times 4}{(s+3)^2 + 4^2} = \frac{10}{4} \frac{4}{(s+3)^2 + 4^2}$$

$$\therefore f(t) = \frac{10}{4} \sin 4te^{-3t}$$

Solution of Differential Equation by Laplace Transformation



$$m\ddot{x} + B\dot{x} + Kx = F.$$

$$y'' + 2y' + 4y = 1 \quad y(0) = 0, \quad y'(0) = 2$$

$$\text{L.T: } s^2Y(s) - sy(0) - y'(0) + 2\{sY(s) - y(0)\} + 4Y(s) = \frac{1}{s}$$

$$(s^2 + 2s + 4)Y(s) = \frac{1}{s} + 2 = \frac{2s + 1}{s}$$

$$Y(s) = \frac{2s + 1}{s(s^2 + 2s + 4)} = \frac{1}{4s} - \frac{1}{4} \frac{s + 1 - 1}{(s + 1)^2 + (\sqrt{3})^2}$$

$$\therefore y(t) = \frac{1}{4} - \frac{1}{4} \cos \sqrt{3}te^{-t} + \frac{1}{4\sqrt{3}} \sin \sqrt{3}te^{-t}$$

Laplace Transform Table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at} u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

Nise Ch.2

Laplace Transform Theorems

Nise Ch.2

Item no.	Theorem	Name
1.	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$	Definition
2.	$\mathcal{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathcal{L}[e^{-at} f(t)] = F(s + a)$	Frequency shift theorem
5.	$\mathcal{L}[f(t - T)] = e^{-sT} F(s)$	Time shift theorem
6.	$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0-)$	Differentiation theorem
10.	$\mathcal{L}\left[\int_{0-}^1 f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \rightarrow 0} sF(s)$	Final value theorem ¹
12.	$f(0+) = \lim_{s \rightarrow \infty} sF(s)$	Initial value theorem ²

End of Laplace Transform