Transfer Function and Block Diagram Approach to Modeling Dynamic Systems

The Concept of Transfer Function

Consider the linear time-invariant system defined by the following differential

equation :

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y$$
$$= b_0 u^{(m)} + b_1 u^{(m-1)} + \cdots + b_{m-1}\dot{u} + b_m u \quad (n \geq m)$$

Where y is the output of the system, and x is the input. And the Laplace transform of the equation is,

$$(a_0 S^n + a_1 S^{n-1} + \cdots + a_{n-1} S + a_n) Y(s)$$
$$= (b_0 S^m + b_1 S^{m-1} + \cdots + b_{m-1} S + b_m) U(s)$$

The Concept of Transfer Function

The ratio of the Laplace transform of the output (response function) to the Laplace Transform of the input (driving function) under the **assumption that all initial conditions are Zero.**

$$Transfer\ Function = \frac{Y(s)}{U(s)} = G(s) = \frac{b_0 S^m + b_1 S^{m-1} + \cdots + b_{m-1} S + b_m}{a_0 S^n + a_1 S^{n-1} + \cdots a_{n-1} S + a_n}$$

$$= \frac{P(s)}{Q(s)} \qquad (n \geq m)$$

Comments on Transfer Function

1. A mathematical model.

2. Property of system itself.
   - Independent of the input function and initial condition
   - Denominator of the transfer function is the characteristic polynomial,
   - TF tells us something about the intrinsic behavior of the model.

3. ODE equivalence
   - TF is equivalent to the ODE. We can reconstruct ODE from TF.

4. One TF for one input-output pair. : Single Input Single Output system.
   - If multiple inputs affect $\rightarrow$ Obtain TF for each input

$$\ddot{x} + 6\dot{x} + 20x = 7f(t) + 3g(t)$$

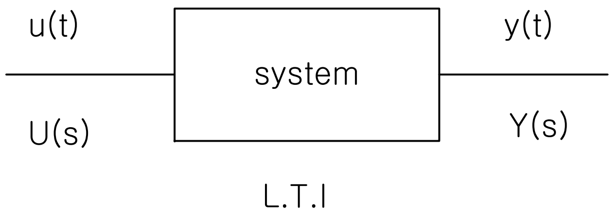
-If multiple outputs

$$\dot{x} = -3x + 2y$$

$$\dot{y} = -y - x + 3u(t)$$

Comments on Transfer Function

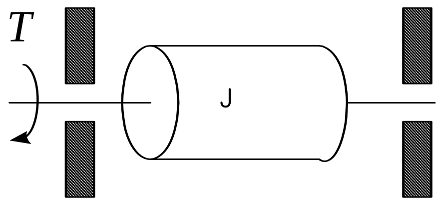
5. Analytic method and Experimental method



$$G(s) = \frac{Y(s)}{U(s)}$$

6. Different systems may have identical T.F

ex)



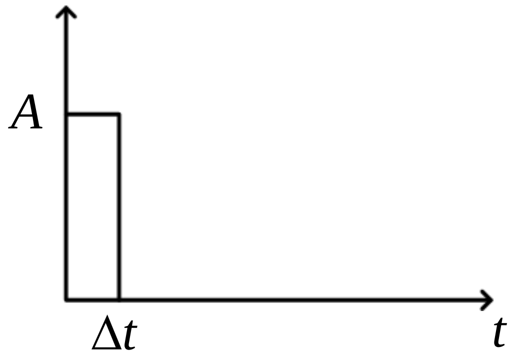
$$J\dot{\omega} + b\omega = T \qquad \frac{Y}{U} = \frac{\omega}{T} = \frac{1}{Js + b}$$

Electrical System $$\frac{Y}{U} = \frac{R}{Ls + R} = \frac{1}{\frac{L}{R}s + 1}$$ (if b=1, J=L/R)

Comments on Transfer Function

$$Y(s)=G(s)u(s)$$

Impulse Input



The Strength of an impulsive input : $A \cdot \Delta t$

The Dirac Delta function $\delta(t)$: **An Impulsive function with a strength aquals to unity**

$$\int_{0-}^{0+} \delta(t)\,dt = 1$$

$$\mathcal{L}[\,\delta(t)\,] = 1$$

$u(t)=\delta(t)$ $\quad Y(s) = G(s)$ **Transfer Function = unit impulse response**

$u(s)=1$

Collision, Impulse Response and Transfer Function

Impulse: large force in a very short time

Newton's law of motion

$$F = ma = m\frac{dv}{dt} = \frac{d}{dt}(mv) \quad \rightarrow \quad F \cdot dt = d(mv)$$

The impulse-momentum principle

$$F = ma = m\frac{dv}{dt} = \frac{d}{dt}(mv) \quad \rightarrow \quad F \cdot dt = d(mv)$$

$$\int_{t_1}^{t_2} F dt = \int_{v_1}^{v_2} d(mv) = mv_2 - mv_1$$

The law of conservation of momentum

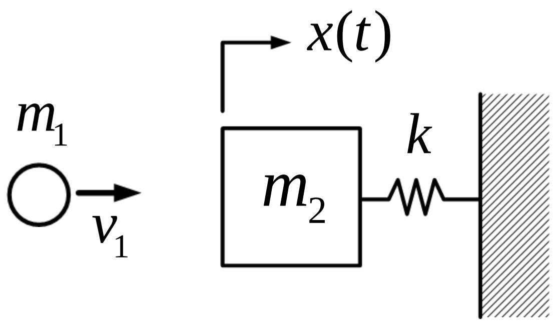
And if There is no input force, then we get

$$d(mv) = 0, \quad mv = const.$$ The law of conservation of momentum

And also,

$$I\omega = const.$$ The law of conservation of angular momentum

Inelastic collision



m_1 **Becomes embedded in mass** m_2 **after collision**

No external force, conservation of momentum

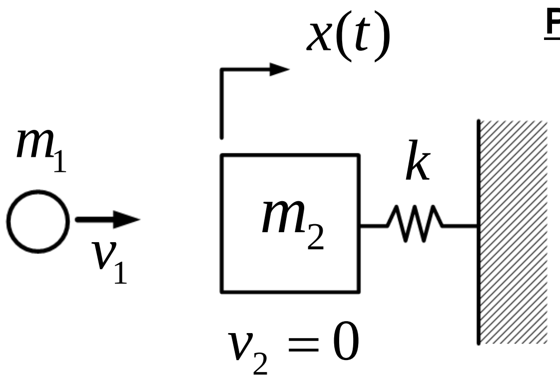
$$\int_{t_1}^{t_2} F\,dt = (m_1 + m_2)v(0+) - m_1 v_1(0) = 0\ ,\quad v(0+) = \frac{m_1}{m_1 + m_2} v_1$$

The equation of motion for the combined mass

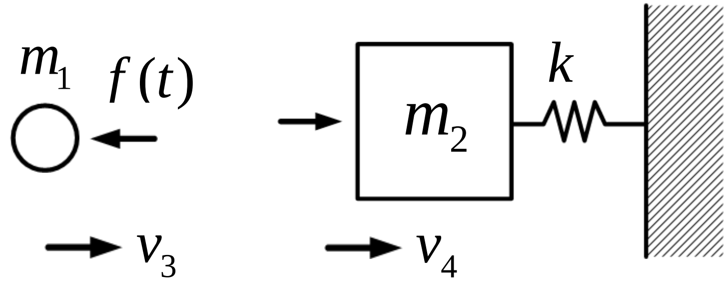
$$(m_1 + m_2)\ddot{x} + kx = 0,\quad v(0+) = \dot{x}(0+) = \frac{m_1}{m_1 + m_2} v_1$$

$$\Rightarrow x(t) = v(0+)\frac{1}{w}\sin(w_n t),\ w_n = \sqrt{\frac{k}{m_1 + m_2}}$$

Elastic collision



Perfectly elastic collision



No external force : conservation of momentum

$$m_1 v_1 + m_2 \cancel{v_2} = m_1 v_3 + m_2 v_4$$

Perfectly elastic collision : No energy loss, kinetic energy is conserved

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_3^2 + \frac{1}{2} m_2 v_4^2$$

$$\rightarrow \; v_3 = \frac{m_1 - m_2}{m_1 + m_2} v_1 \; , \quad v_4 = \frac{2 m_1}{m_1 + m_2} v_1$$

The law of conservation of momentum

Mass m_1 **:** $\int_{0-}^{0+} f(t)\,dt = m_1 v_3 - m_1 v_1 = m_1 (v_3 - v_1) = m_1 \left(\frac{m_1 - m_2}{m_1 + m_2} - 1 \right) v_1$

$$= m_1 \frac{-2m_2}{m_1 + m_2} v_1$$

Mass m_2 **:** $\int_{0-}^{0+} f(t)\,dt = m_2 v_4 - m_2 \cancel{v_2} = \frac{2m_1 m_2}{m_1 + m_2} v_1$

$$= \int_{0-}^{0+} A\delta(t) = A$$ **Linear impulse**

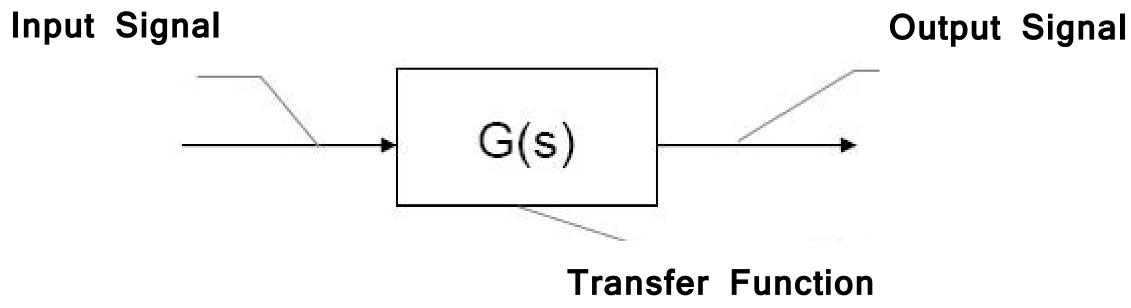
We can solve $x(t)$ $\quad m_2 \ddot{x}(t) + kx = 0 \quad x(0+) = 0,\ v(0+) = \dot{x}(0+) = \frac{2m_1}{m_1 + m_2} v_1$

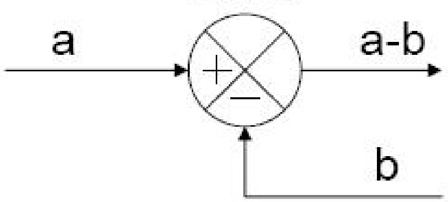
Or LT: $\quad m_2 (s^2 X(s) - sx(0) - \dot{x}(0)) + k\,X(s) = 0$

$$m_2 \ddot{x}(t) + kx = A\delta(t) = \frac{2m_1 m_2}{m_1 + m_2} v_1 \delta(t) \qquad x(0-) = \dot{x}(0-) = 0$$

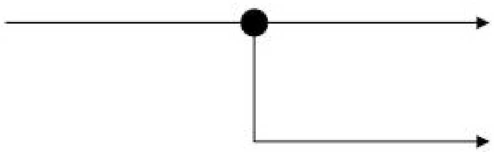
$$m_2 (s^2 X(s)) + k\,X(s) = \frac{2m_1 m_2}{m_1 + m_2} v_1$$

Block Diagram



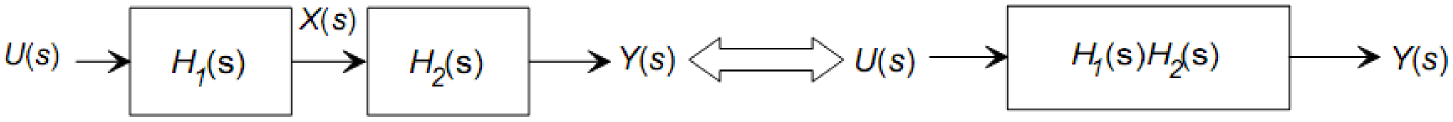


Summing Point

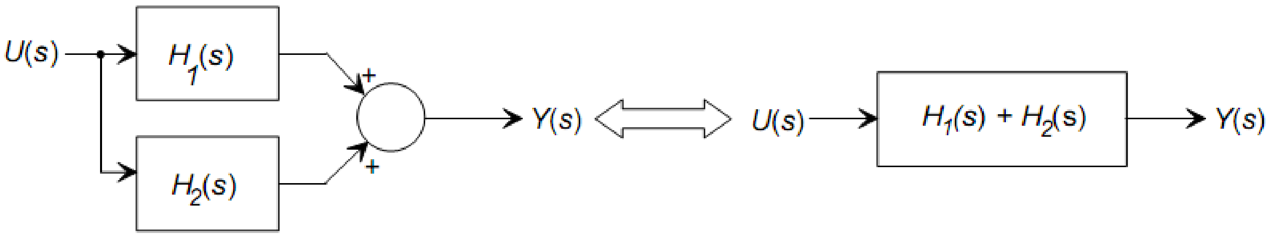


Pick Off Point

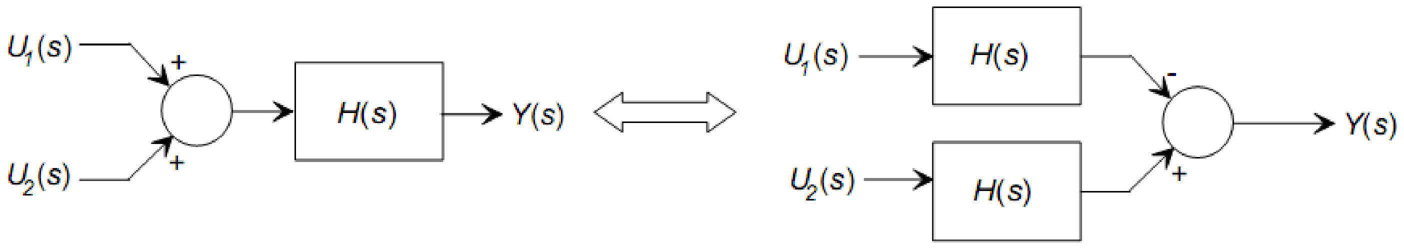
Series(Cascade) Connection



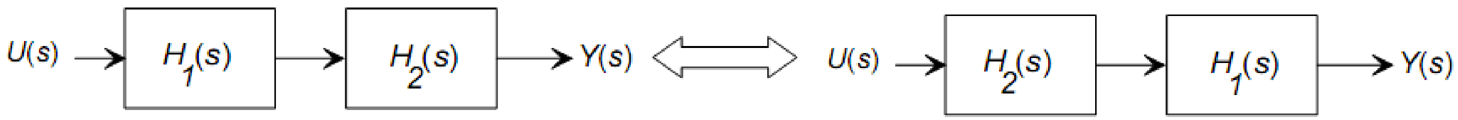
Parallel Connection



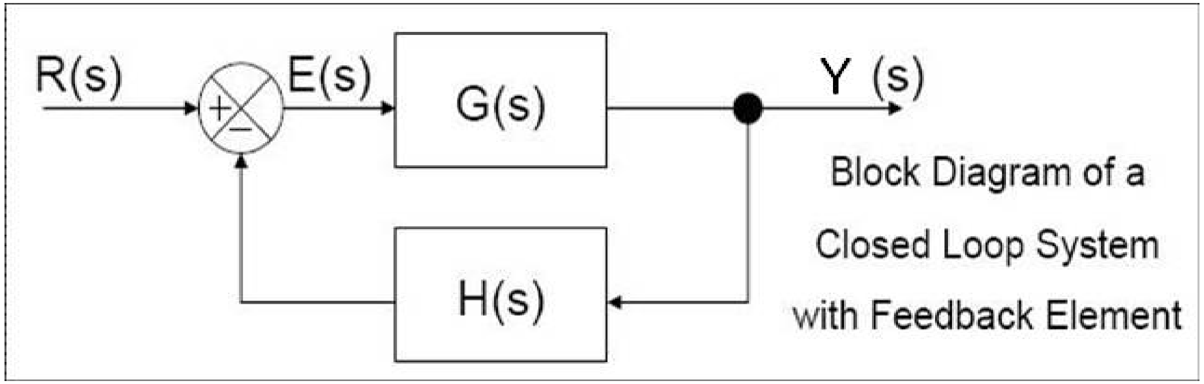
Associative Rule



Commutative Rule



Closed Loop Transfer Function



$$Y(s) = G(s)E(s) = G(s)[R(s) - H(s)Y(s)]$$

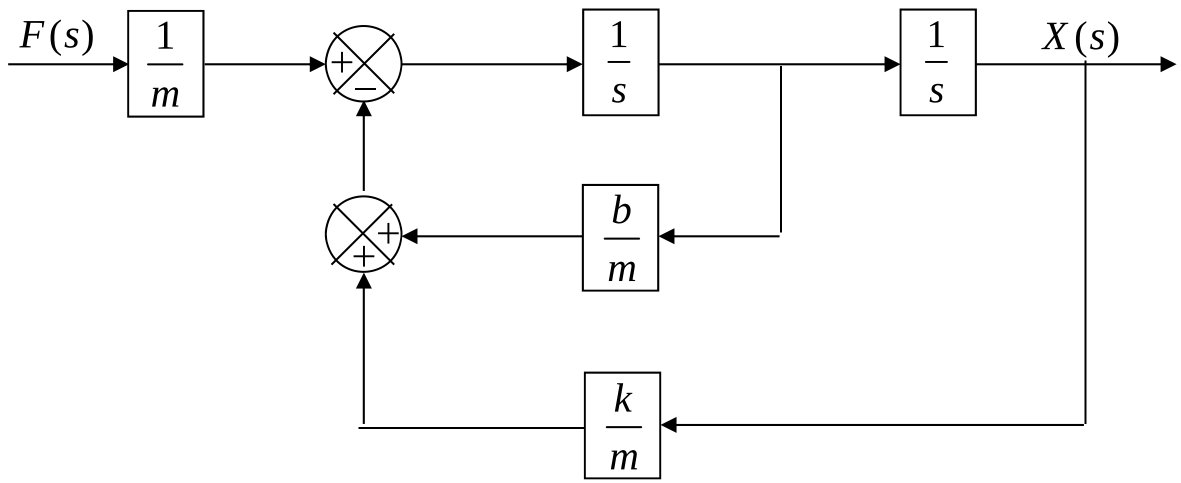
$$m\ddot{x} + b\dot{x} + kx = \overline{h}$$

$$[1 + G(s)H(s)]Y(s) = G(s)R(s)$$

$$\therefore Transfer\ Function = \frac{Y(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)H(s)}$$

Closed Loop Transfer Function

ex)



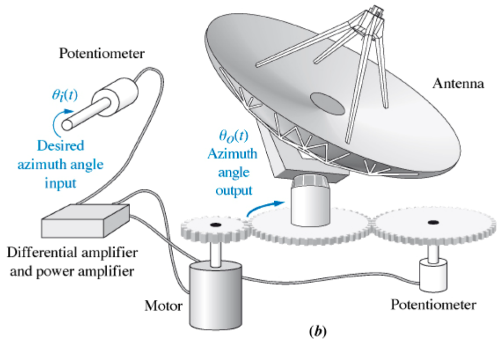
$$\frac{1}{m}F(s)-\frac{k}{m}X(s)-\frac{b}{m}sX(s)=s^2X(s)$$

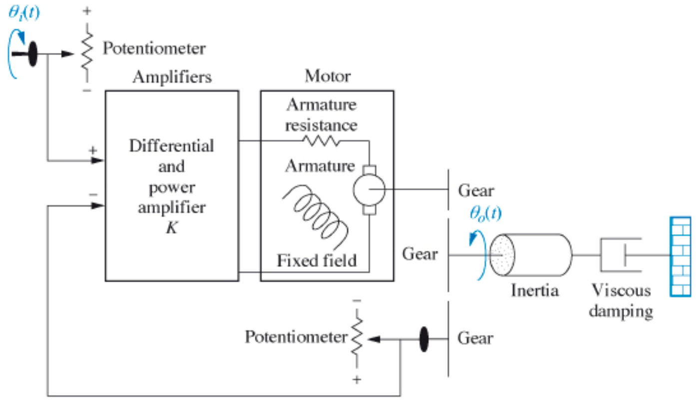
$$F(s)-\left[kX(s)+bsX(s)\right]=ms^2X(s)$$

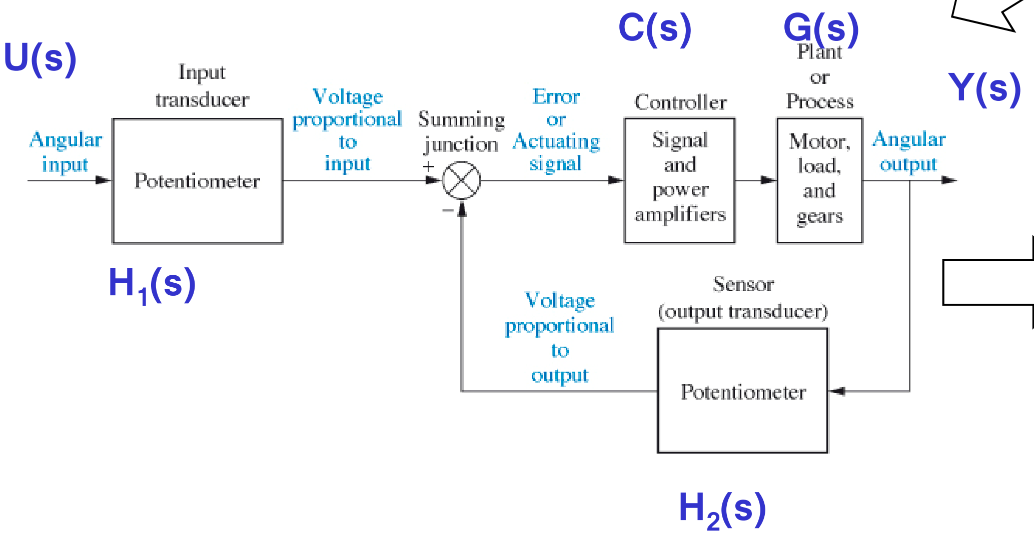
$$(ms^2+bs)X(s)=F(s)-kX(s),\qquad (ms^2+bs+k)X(s)=F(s)$$

$$\therefore\frac{X(s)}{F(s)}=\frac{1}{ms^2+bs+k}\quad Transfer\ Function$$

System → Schematic → Block Diagram →Transfer Functions



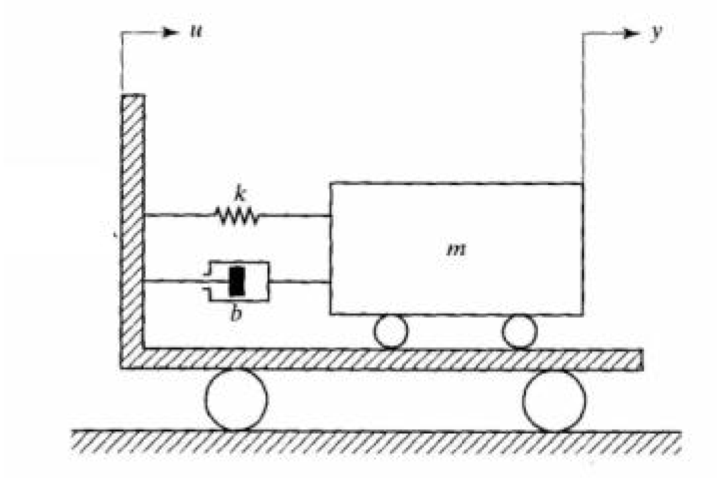




$$Y(s) = \frac{C(s)G(s)H_1(s)}{1+C(s)G(s)H_2(s)}\,U(s)$$

Example

ex)



$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

$$or \quad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

Assume the initial condition is 0,

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

$$Transfer\ Function = \frac{Y(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}$$

If, m=10kg, b=20N-s/m, k=100N/m

$$\frac{Y(s)}{U(s)} = \frac{20s+100}{10s^2+20s+100} = \frac{2s+10}{s^2+2s+10}, \qquad U(s) = \frac{1}{s}$$

$$\therefore Y(s) = \frac{2s+10}{s^2+2s+10}\frac{1}{s} = \frac{2s+10}{s^3+2s^2+10s}$$

Impulse response

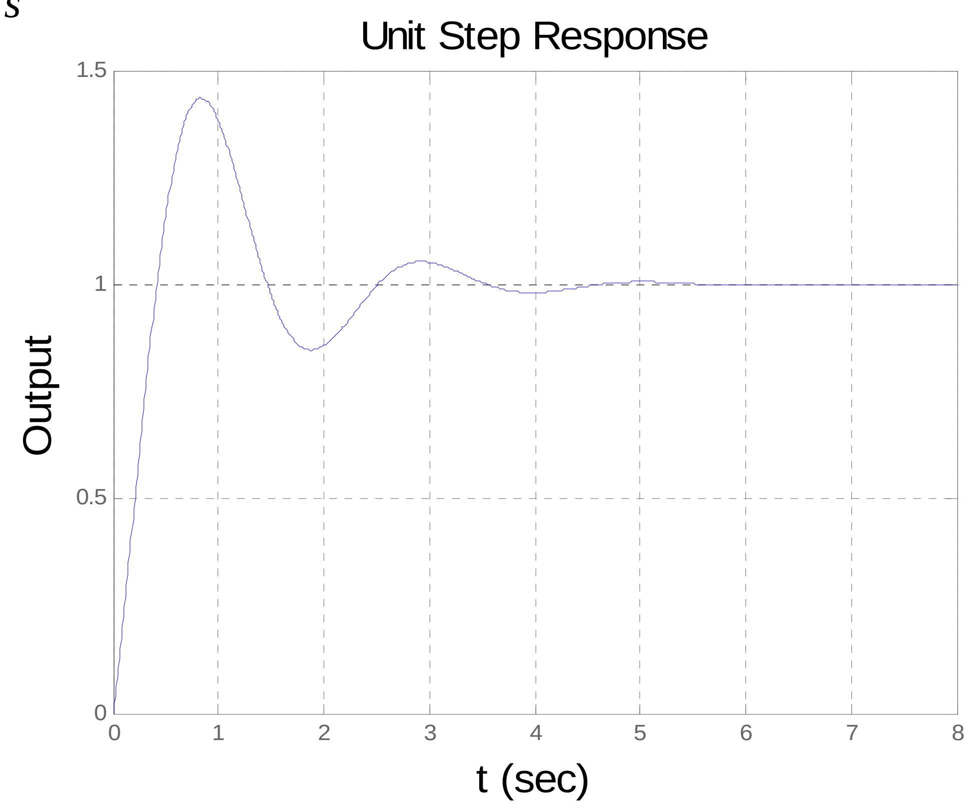
```
>>num=[8];
>>den=[50   100   2500];
>>sys=tf(num, den)
>> impulse(sys)
>> grid
>>title('      ')
>>xlabel('   ')
>>ylabel('   ')
```

Transient Response Analysis with MATLAB

$$\frac{Y(s)}{U(s)} = \frac{20s+100}{10s^2+20s+100} = \frac{2s+10}{s^2+2s+10}, \qquad U(s) = \frac{1}{s}$$

$$\therefore Y(s) = \frac{2s+10}{s^2+2s+10}\frac{1}{s} = \frac{2s+10}{s^3+2s^2+10s}$$

```
t=0:0.01:8;
num=[2 10];
den=[1 2 10];
sys=tf(num,den);
step(sys,t)
grid
title('Unit Step Response','Fontsize',15')
xlabel('t(sec)','Fontsize',15')
ylabel('Output','Fontsize',15')
```



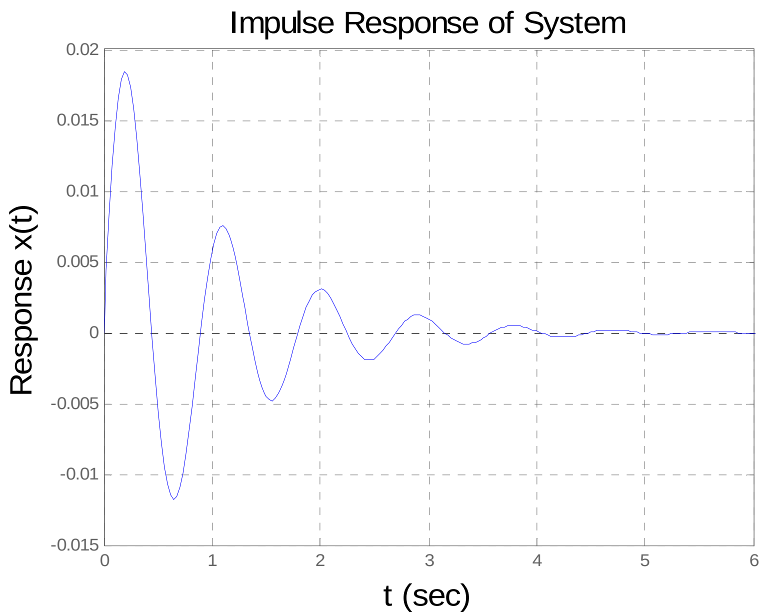
Impulse Input

ex)

$$M = 50kg, \quad m = 0.01kg, \quad b = 100Ns/m, \quad k = 2500N/m, \quad v(0-) = 800m/s$$

$$X(s) = \frac{1}{50.01s^2 + 100s + 2500}\frac{50.01 \times 0.01 \times 800}{50.02} = \frac{7.9984}{50.01s^2 + 100s + 2500}$$

```
num=[7.9984];
den=[50.01 100 2500];
sys=tf(num,den);
impulse(sys)
grid
title('Impulse Response of System','Fontsize',15')
xlabel('t(sec)','Fontsize',15')
ylabel('Response x(t)','Fontsize',15')
```



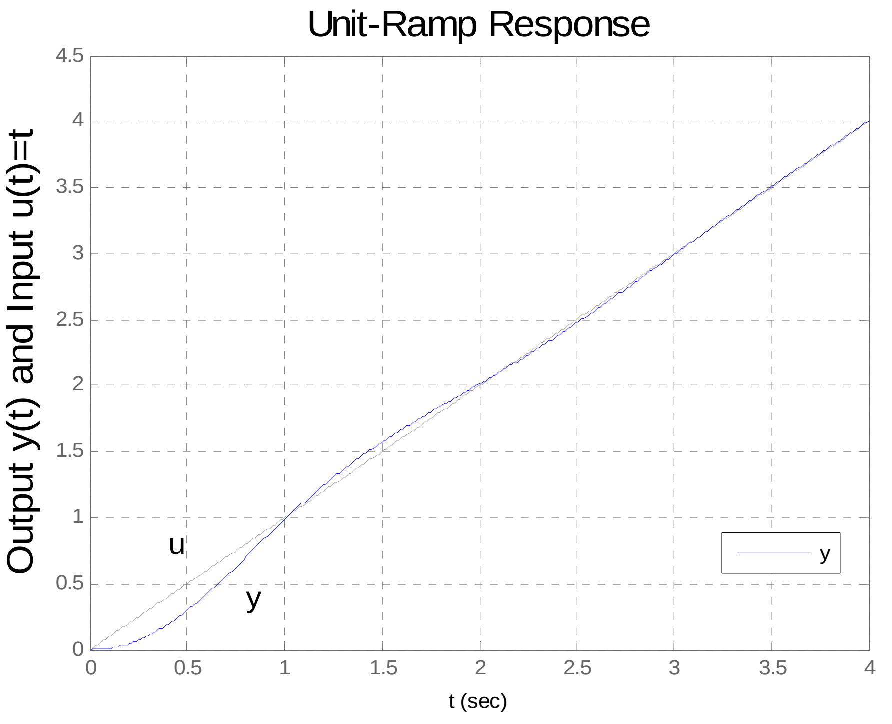
Ramp Response

$$\frac{Y(s)}{U(s)} = \frac{2s+10}{s^2+2s+10}$$

M=10kg, b=20Ns/m, k=100N/m

$$u(t): Unit\ ramp\ input, \quad u = \alpha t, \quad \alpha = 1$$

```
num=[2 10];
den=[1 2 10];
sys=tf(num,den);
t=0:0.01:4;
u=t;
lsim(sys,u,t)
grid
title('Unit-Ramp Response','Fontsize',15)
xlabel('t')
ylabel('Output y(t) and Input u(t)=t','Fontsize',15)
text(0.8,0.4,'y','Fontsize',12)
text(0.4,0.8,'u','Fontsize',12)
legend('y')
```



Partial Fraction Expansion with MATLAB

$$\frac{B(s)}{A(s)} = \frac{num}{den} = \frac{b(0)S^m + b(2)S^{m-1} + \cdots + b(m)}{a(0)S^n + a(2)S^{n-1} + \cdots + a(n)}$$

$$num = [b(0) \quad b(1) \quad \cdots \quad b(m)], \quad den = [a(0) \quad a(1) \quad \cdots \quad a(n)]$$

[r, p, k] = residue (num, den)

$$\frac{B(s)}{A(s)} = k(s) + \frac{r(1)}{s-p(1)} + \frac{r(2)}{s-p(2)} + \cdots + \frac{r(n)}{s-p(n)}$$

ex) $$\frac{B(s)}{A(s)} = \frac{s^4 + 8s^3 + 16s^2 + 9s + 6}{s^3 + 6s^2 + 11s + 6}$$

```
>>num=[1   8   16   9   6];
>>den=[1   6   11   6];
>>[r,p,k]=residue(num,den)
r=
        -6.0000
        -4.0000
         3.0000
p=
        -3.0000
        -2.0000
        -1.0000
k=
         1         2
```

$$= s + 2 + \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1}$$

End of lecture 4