Mathematical Modeling of Dynamic Systems in State Space II

Transfer Function and State Equation

1.

$$\frac{z(s)}{u(s)} = \frac{3}{s+2} \qquad \begin{aligned} z(s)(s+2) &= 3u(s) \\ \dot{z}(t) + 2z(t) &= 3u(t) \end{aligned}$$

$$x_1 = z(t),\ \dot{x}_1 = \dot{z} = -2z(t)$$

$$\begin{bmatrix} \dot{x}_1 = -2x_1 + 3u(t) \\ y = x_1 \end{bmatrix}$$

2.

$$\frac{z(s)}{u(s)} = \frac{5s+3}{s+2} = \frac{5(s+2)-7}{s+2} = 5 - \frac{7}{s+2}$$

$$z(s) = 5u(s) - \frac{7}{s+2}u(s)$$

$$let\ z_1(s) = \frac{7}{s+2}u(s),\ \ \dot{z}_1 + 2z_1 = 7u$$

$$let\ \ x_1 = z_1, \begin{bmatrix} \dot{x}_1 = -2x_1 + 7u \\ y = -z_1 + 5u \\ \quad = -x_1 + 5u \end{bmatrix}$$

Another way,

$$\frac{z(s)}{u(s)} = \frac{5s+3}{s+2}$$

$$\frac{z(s)}{5s+3} = \frac{u(s)}{s+2} = Q(s)$$

$$\dot{q} + 2q = u(t)$$

$$\begin{aligned} z(t) &= 5\dot{q} + 3q \\ &= 5(-2q+u) + 3q \\ &= 5u - 7q \end{aligned}$$

$$let\; x_1 = q, \begin{bmatrix} \dot{x}_1 = -2x_1 + u \\ y = -7x_1 + 5u \end{bmatrix}$$

2^{nd} order systems

State Equation and Transfer Function

ex1) Consider a system defined by $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$

Choose state variables,

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

→ Phase variables (each subsequent state variable is defined to be the derivative of the previous state variable.)

Then we obtain, $\dot{x}_1 = x_2,$

$$\dot{x}_3 = \dddot{y} = -6x_1 - 11x_2 - 6x_3 + 6u$$

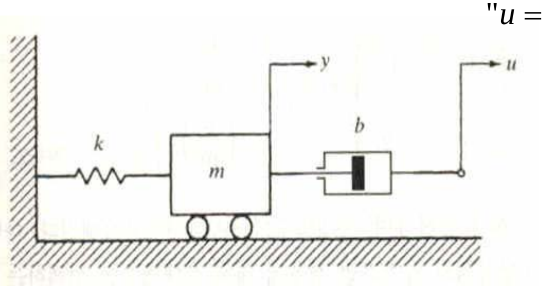
$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State Equation and Transfer Function

ex2) Consider a mechanical system,

$$m\ddot{y} = -ky - b(\dot{y} - \dot{u}), \quad \ddot{y} = -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u}$$

$"u = displacement"$



Choose state variables, $x_1 = y, \; x_2 = \dot{y}$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{b}{m}u \qquad \dot{x} = Ax + Bu$$

The right side includes $\dot{u}$ term. To explain the reason we should not include differentiation of u, assume $u = \delta(t)$ $(unit\ impulse\ \ function)$

$$x_2 = -\frac{k}{m}\int y\,dt - \frac{b}{m}y + \frac{k}{m}\delta(t)$$

x_2 includes $(k/m)\,\delta(t)$ term. It means $x_2(0) = \infty$ and cannot be accepted as a state variable.

That's why the standard form is $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \qquad \mathbf{y} = \mathbf{Cx} + \mathbf{Du}$

State Equation and Transfer Function

Method 1: Choose a state variable that includes *u*

To eliminate $\dot{u}$ term,

$$\ddot{y} = -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u} \rightarrow \ddot{y} - \frac{b}{m}\dot{u} = -\frac{k}{m}y - \frac{b}{m}\dot{y}$$

$$\frac{d}{dt}\left(\dot{y} - \frac{b}{m}u\right) = -\frac{k}{m}y - \frac{b}{m}\left(\dot{y} - \frac{b}{m}u\right) - \left(\frac{b}{m}\right)^2 u$$

So we choose state variables as, $x_1 = y,\ x_2 = \dot{y} - \frac{b}{m}u$

$$\dot{x}_2 = \ddot{y} - \frac{b}{m}\dot{u} = \left(-\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u}\right) - \frac{b}{m}\dot{u} = -\frac{k}{m}x_1 - \frac{b}{m}\left(x_2 + \frac{b}{m}u\right)$$

$$= -\frac{k}{m}x_1 - \frac{b}{m}x_2 - \left(\frac{b}{m}\right)^2 u$$

$\longrightarrow$ $\dot{u}$ term has been eliminated.

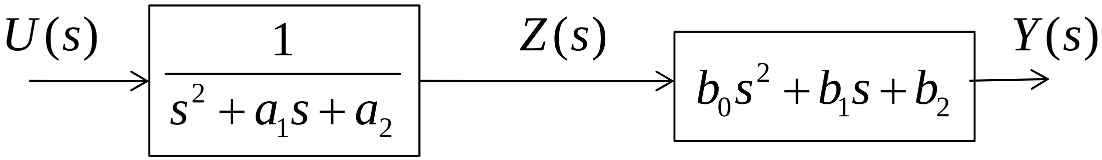
$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m} \\ -\left(\frac{b}{m}\right)^2 \end{bmatrix} u$$

State Equation and Transfer Function

Method 2: Include the input derivates in the output equation

Consider a second-order system, $\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} \rightarrow \boxed{\frac{Z(s)}{U(s)} = \frac{1}{s^2 + a_1 s + a_2}}, \quad \boxed{\frac{Y(s)}{Z(s)} = b_0 s^2 + b_1 s + b_2}$$



$$\ddot{z} + a_1\dot{z} + a_2 z = u$$
$$y = b_0\ddot{z} + b_1\dot{z} + b_2 z$$

$let, \quad x_1 = z, \quad x_2 = \dot{z}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + u$$

$$b_0\ddot{z} + b_1\dot{z} + b_2 z = b_0(-a_2 x_1 - a_1 x_2 + u) + b_1 x_2 + b_2 x_1 = y$$

$$\therefore \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -a_2 x_1 - a_1 x_2 + u$$
$$y = (b_2 - a_2 b_0)x_1 + (b_1 - a_1 b_0)x_2 + b_0 u$$

State Equation and Transfer Function

Method 2: Include the input derivates in the output equation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} b_2 - a_2 b_0 & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u$$

N-th order differential equation, $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y$

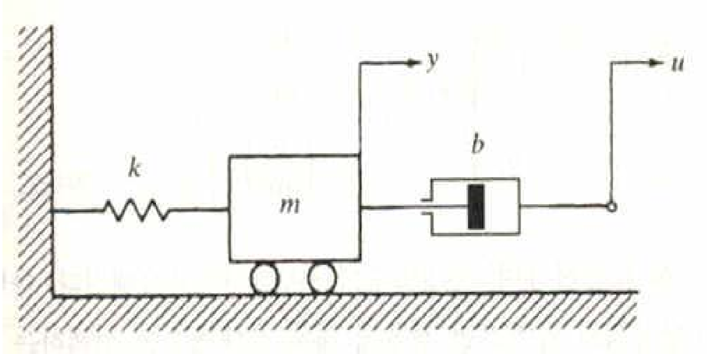
$$= b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} b_n - a_n b_0 & \vdots & b_{n-1} - a_{n-1} b_0 & \vdots & \cdots & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

State Equation and Transfer Function

ex2) Consider this mechanical system again,



$$m\ddot{y} = -ky - b(\dot{y} - \dot{u}), \quad m\ddot{y} + b\dot{y} + ky = b\dot{u}$$

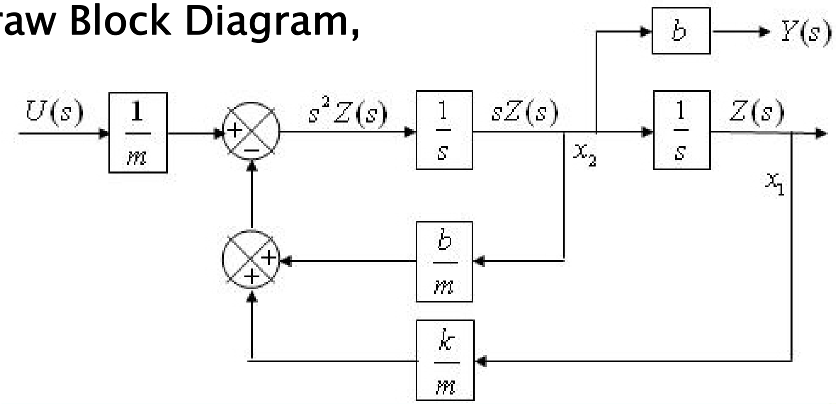
$$\frac{Y(s)}{U(s)} = \frac{bs}{ms^2 + bs + k}, \quad \frac{Z(s)}{U(s)} = \frac{1}{ms^2 + bs + k}, \quad \frac{Y(s)}{Z(s)} = bs$$

$$\frac{Y(s)}{cU(s)} = \frac{bs}{ms^2 + bs + k}, \frac{Y(s)}{b(s)} = \frac{u(s)}{ms^2} = Q(s)$$

$$(ms^2 + bs + k)Z(s) = U(s), \quad bsZ(s) = Y(s)$$

$$s^2 Z(s) = \frac{1}{m}U(s) - \frac{b}{m}sZ(s) - \frac{k}{m}Z(s)$$

Draw Block Diagram,



State variables,

$$x_1 = z, \quad x_2 = \dot{z}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

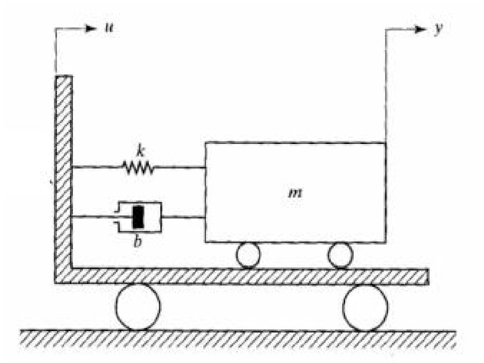
$$y = b\,x_2$$

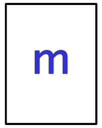
System Matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & b \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

State Equation and Transfer Function

ex) Consider a spring-mass-damper system





$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

$$or \quad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

$$Transfer\ Function = \frac{Y(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}$$

State Eq:

Output Eq:

State variables:

Rewrite equations:

$$b_0 = 0, \quad b_1 = b, \ b_2 = k \qquad b_2 - a_2 b_0 = \frac{k}{m} - \frac{k}{m} + 0 = \frac{k}{m}$$

$$a_1 = \frac{b}{m}, \quad a_2 = \frac{k}{m} \qquad b_1 - a_1 b_0 = \frac{b}{m} - \frac{b}{m} + 0 = \frac{b}{m}$$

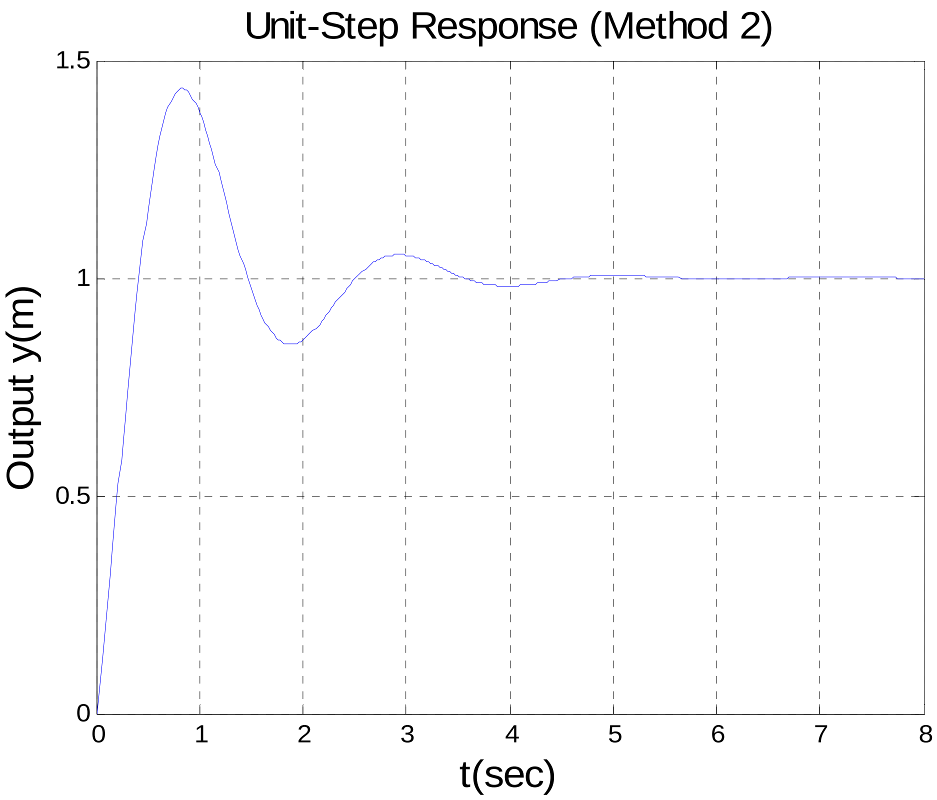
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{u}, \quad y = \begin{bmatrix} \frac{k}{m} & \frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Matlab Example

If, m=10kg, b=20N-s/m, k=100N/m

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 10 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

```
t=0:0.02:8;
A=[0 1;-10 -2];
B=[0;1];
C=[10 2];
D=[0];
sys=ss(A,B,C,D);
[y,t]=step(sys,t);
plot(t,y)
grid
title('Unit-Step Response (Method 2)','Fontsize',15)
xlabel('t(sec)','Fontsize',15)
ylabel('Output y(m)','Fontsize',15)
```



Transformation of Mathematical Models with MATLAB

$$\frac{Y(s)}{U(s)} = \frac{numerator\ \ polynomial\ in\ s}{denominator\ \ polynomial\ in\ s} = \frac{num}{den}$$

MATLAB command, `[A, B, C, D] = tf2ss(num,den)` **gives a state space representation.**

ex) Consider, $\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160}$

One of many possible state-space representations is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & 160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

```
>> num=[0 0 1 0];
>> den=[1 14 56 160];
>> [A,B,C,D]=tf2ss(num,den)
A =
   -14   -56  -160
     1     0     0
     0     1     0
B =
     1
     0
     0
C =
     0     1     0
D =
     0
```

Transformation of a State-Space Models into Another One

- $\dot{x} = Ax + Bu$
  $y = Cx + Du$
  can be written as,

$$P\dot{\hat{x}} = AP\hat{x} + Bu \qquad \dot{\hat{x}} = P^{-1}AP\hat{x} + P^{-1}Bu$$
$$y = CP\hat{x} + Du \quad \text{or} \quad y = CP\hat{x} + Du$$

  - Since infinitely many n x n matrices can be a transformation matrix P, there are infinitely many state-space models for a given system.

- Eigenvalues of an n x n matrix A are the roots of the characteristic equation.

$$\left|\lambda I - A\right| = 0$$

  The eigenvalues are also called the characteristic roots.

ex) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $\quad \left|\lambda I - A\right| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix} = \lambda^3 + 6\lambda^2 + 11\lambda + 6$

$$= (\lambda+1)(\lambda+2)(\lambda+3) = 0$$

Diagonalization of State Matrix A

Consider an n x n state matrix A :

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

If matrix A has distinct eigenvalues and the state vector x is transformed into another state vector z by use of a transformation matrix P ,

$$x = Pz, \ \ \textit{where} \ P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ {\lambda_1}^2 & {\lambda_2}^2 & \cdots & {\lambda_n}^2 \\ \vdots & \vdots & & \vdots \\ {\lambda_2}^{n-1} & {\lambda_2}^{n-1} & \cdots & {\lambda_n}^{n-1} \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$P^{-1}AP$ is a canonical matrix and each column of P is an eigenvector of matrix A

Jordan Canonical Form

If matrix A involves <u>multiple eigenvalues</u>, diagonalization is not possible but matrix A can be transformed into a Jordan Canonical Form.

Consider the 3 x 3 matrix A : $\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$

Assume that matrix A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ *where* $\lambda_1 = \lambda_2 \neq \lambda_3$

$(\lambda \mathrm{I} - \mathrm{A}) v_1 = 0 \qquad \mathrm{A} : 3 \times 3\, matrix, \qquad \lambda_1, \lambda_2, \lambda_3 \qquad v_1, v_2, v_3$

Case 1. $rank(\lambda_1 \mathrm{I} - \mathrm{A}) = 1$ can determine two eigenvectors $v_1,\ v_2$.

$$Av_1 = \lambda_1 v_1,\ Av_2 = \lambda_1 v_2,\ Av_3 = \lambda_3 v_3$$

$$\mathrm{A}\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad \mathrm{P}^{-1}\mathrm{AP} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Jordan Canonical Form

Case 2. $rank\left(\lambda_1 \mathrm{I} - \mathrm{A}\right) = 2$ can determine one eigenvector v_1 .

$$\mathrm{A}\, v_1 = \lambda_1 v_1, \qquad \mathrm{A}\, v_3 = \lambda_3 v_3$$

Find v_2 such that $\left|\mathrm{A} - \lambda_1 \mathrm{I}\right| v_2 = v_1 \quad \mathrm{A}\, v_2 = v_1 + \lambda_1 v_2$

$$\mathrm{A}\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\mathrm{x} = \mathrm{S}\,\mathrm{z} \quad where \quad \mathrm{S} = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1{}^2 & 2\lambda_1 & \lambda_3{}^2 \end{bmatrix}$ will yield $\mathrm{S}^{-1}\,\mathrm{AS} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \mathrm{J}$

This is in the Jordan Canonical Form.

• Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)

Case 3. Complex Roots

$$AP = P\Lambda$$

Note : Complex Roots, Complex State x

$$\dot{x} = \Lambda x + bu$$

$$y = Cx$$

→ Complex case의 diagonalization 방법 이용

$$\Lambda K = KJ$$

$$\Lambda = \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix} \qquad K = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{1}{2} & \frac{j}{2} \end{bmatrix} \qquad K^{-1} = \frac{2}{j}\begin{bmatrix} \frac{j}{2} & \frac{j}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$J = K^{-1}\Lambda K$$

$$= K^{-1}P^{-1}APK \qquad J = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

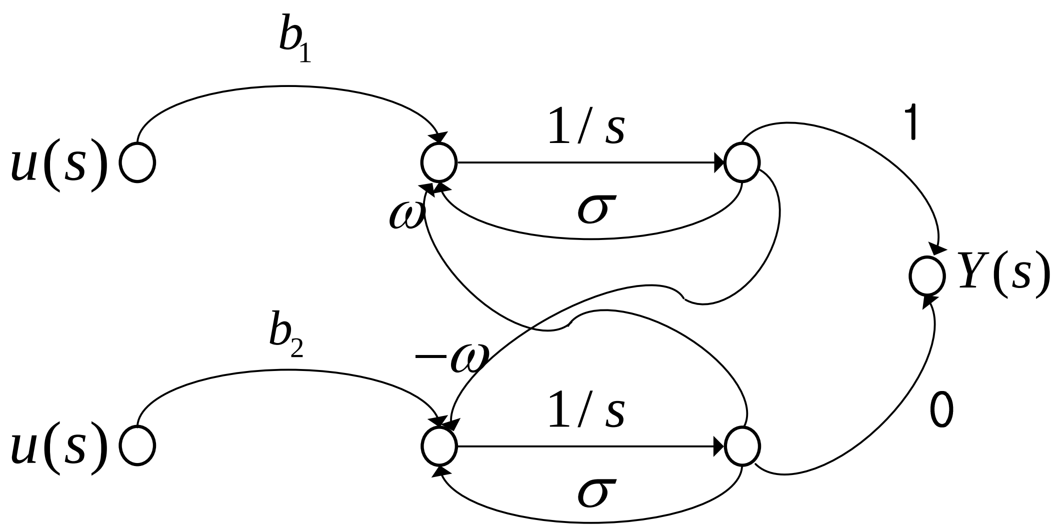
• Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)

Case 3. Complex Roots

Ex)

$$\dot{z} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} z$$



• Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)

Case 3. **Complex Roots**

Step 1 $\dot{x} = Ax + Bu$

Step 2 **let** $x = P\xi$

$$\dot{\xi} = \underbrace{P^{-1}AP}_{\Lambda}\,\xi + P^{-1}Bu$$

: diagonal

Step 3 **let** $\xi = Kz$

$$\dot{z} = \underbrace{K^{-1}\Lambda K}_{J}\,z + K^{-1}P^{-1}Bu$$

$$= \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$$\begin{cases} x = P\xi = PKz \\ \dot{z} = \underbrace{K^{-1}\Lambda K}_{J}\,z + K^{-1}P^{-1}bu \\ y = CPKz \end{cases}$$

End of Lecture 5(II)